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# Leafwise hyperbolicity of proper foliations

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### Introduction

A foliated manifold  $(M, \mathcal{F})$  is said to be *proper* if every leaf of  $\mathcal{F}$  is proper. A leaf L is proper if the relative topology of L in M coincides with the manifold topology of L. This is equivalent to requiring that each point  $x \in L$  lies in a foliation chart  $U \subset M$  such that  $L \cap U$  is a single plaque. Equivalently, the leaf L is not asymptotic to itself.

Proper foliated manifolds have been studied by various authors. For example, in arbitrary codimension, Millett [Mi] has organized the leaves of such foliations into a countable ordinal hierarchy that is completely analogous to the Epstein hierarchy for foliations with all leaves compact [Ep].

In codimension one, with smoothness class at least  $C^2$ , there is a more rigid hierarchy by integral *levels* [C-C1]. Leaves at a given level wind in on those at lower levels in a way reminiscent of the Poincaré-Bendixson theorem. In [C-C2], this hierarchy is combined with Millett's to prove that  $C^2$ -smoothness for proper foliated manifolds of codimension one implies  $C^{\infty}$ -smoothability.

In the work of Gabai [Ga1], [Ga2], [Ga3], proper foliated 3-manifolds occur that are Reebless and of *finite depth* (which means that there is an upper bound on the levels). Generally, these foliations may only be of class  $C^0$ , although frequently they are smooth.

In what follows,  $(M, \mathcal{F})$  will denote a  $C^2$ -foliated 3-manifold, where M is compact and orientable and  $\mathcal{F}$  is of codimension one and transversely orientable.

An interesting geometric problem is to find a Riemannian metric on M relative to which each leaf of  $\mathcal{F}$  is hyperbolic (i.e., has constant curvature -1). We say that such a metric is leafwise hyperbolic.

MAIN THEOREM. Let  $(M, \mathcal{F})$  be proper and assume that each component of  $\partial M$  is a leaf of  $\mathcal{F}$ . Then  $\exists$  a leafwise hyperbolic metric on M if and only if no leaf of  $\mathcal{F}$  is a torus or a sphere.

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COROLLARY Let  $(M, \mathcal{F})$  be proper and assume that each component of  $\partial M$  is either a leaf or is transverse to  $\mathcal{F}$ . If no compact leaf of  $\mathcal{F}$  is a torus, a sphere, a disk, or an annulus, then  $\exists$  a leafwise hyperbolic metric on M.

This corollary is immediate upon doubling  $(M, \mathcal{F})$  along the transverse boundary.

Gabai's proper foliations of knot complements  $M = S^3 \setminus N^0(k)$  are taut (that is, every leaf meets a closed transversal nontrivially) and admit a minimal genus spanning surface of the knot k as a leaf [Ga1, Theorem 5.5]. Together with the fact that  $H_1(M; \mathbb{Z}) \cong \mathbb{Z}$ , this implies that, for knots of genus greater than one, no compact leaf is a torus, a sphere, an annulus, or a disk. If the foliation can be chosen to be smooth, the corollary implies that it admits a leafwise hyperbolic metric.

The "only if" part of the Main Theorem is evident. The proof of the "if" part uses the Poincaré-Bendixson theory of totally proper leaves [C-C1, §6] and the smooth parametrization of hyperbolic structures by Fenchel-Nielsen coordinates [Th, § 5].

*Remark.* The hypothesis in [C-C1] that  $\partial M = \emptyset$  can be replaced with the hypothesis that  $\partial M$  is a union of leaves without affecting any of the theorems or proofs in that paper. We fix this hypothesis throughout.

## 1. Constructing a hyperbolic skeleton

In this section, unless otherwise specified,  $(M, \mathcal{F})$  need not be proper.

We fix a smooth, one dimensional foliation  $\mathscr{F}^{\perp}$  of M that is everywhere transverse to  $\mathscr{F}$ . There corresponds a decomposition

$$T(M) = T(\mathscr{F}) \oplus T(\mathscr{F}^{\perp})$$

into the respective sub-bundles of tangents to the two foliations. We will fix a Riemannian metric on  $T(\mathcal{F}^{\perp})$ . All metrics constructed on T(M) will be understood to induce this fixed metric on  $T(\mathcal{F}^{\perp})$  and to make the two summands orthogonal. Accordingly, in this and the following sections, the Riemannian metrics that we will be constructing will be metrics on  $T(\mathcal{F})$ .

Let  $O(\mathcal{F})$  denote the family of open,  $\mathcal{F}$ -saturated subsets of M and, for  $U \in O(\mathcal{F})$ , let  $\hat{U}$  denote the completion of U in any Riemannian metric inherited from a Riemannian metric on M. As is well known [Di], [C-C1], [H-H], [Go],  $\hat{U}$  is a manifold with boundary and the inclusion  $i: U \hookrightarrow M$  induces a  $C^2$  immersion  $\hat{i}: \hat{U} \to M$  that may identify some components of  $\partial \hat{U}$  pairwise. Then we obtain a

 $C^2$  foliation,  $\hat{i}^{-1}(\mathcal{F}) = \hat{\mathcal{F}}$ , of  $\hat{U}$  that is tangent to  $\partial \hat{U}$ , while  $\hat{i}^{-1}(\mathcal{F}^{\perp}) = \hat{\mathcal{F}}^{\perp}$  is a smooth, one dimensional foliation of  $\hat{U}$ , everywhere transverse to  $\hat{\mathcal{F}}$ .

Of special interest is the case in which  $\hat{U}$  is diffeomorphic to  $L \times [0, 1]$  in such a way that the leaves of  $\hat{\mathcal{F}}^{\perp}$  are identified with the intervals  $\{x\} \times [0, 1]$ ,  $\forall x \in L$ . Following our usage elsewhere (e.g., [C-C1]), we will say that U is a *foliated product*. We will also identify L with the leaf  $\hat{i}(L \times \{0\})$  of  $\mathcal{F}$ , which may or may not be the same as the leaf  $\hat{i}(L \times \{1\})$ .

DEFINITION. A closed subset  $X \subset M$  that is a finite union of leaves will be called a skeleton of  $(M, \mathcal{F})$  if each component of  $M \setminus X$  is a foliated product.

(1.1) LEMMA. If  $(M, \mathcal{F})$  is proper, it has a skeleton.

*Proof.* Choose a countable family  $\{L_i\}_{i=1}^{\infty}$  of leaves of  $\mathscr{F}$  such that  $\bigcup_{i=1}^{\infty} L_i$  is everywhere dense in M. Let  $X_k = \bigcup_{i=1}^k L_i$ ,  $\forall k \geq 1$ . Since  $(M, \mathscr{F})$  is proper, the closure  $\bar{X}_k$  of  $X_k$  is a finite union of leaves [C-C1, (4.7)]. Since  $\bigcup_{k=1}^{\infty} \bar{X}_k$  is everywhere dense in M, the components of  $M \setminus \bar{X}_k$  will all be foliated products, for k sufficiently large [Di, Proposition 1].

Our proof of the Main Theorem will proceed by finding, first, a metric making all leaves of the skeleton X hyperbolic and, then, by modifying the metric in each of the complementary foliated products. For the first step, the only property of the skeleton X that will be needed is that it is a compact union of finitely many leaves.

Let X be a compact, nonempty, union of finitely many leaves of  $\mathcal{F}$ , these being, of necessity, proper whether or not the foliation is proper (the closure of a nonproper leaf contains uncountably many leaves by the results in [C-C1, § 4 and § 5]). There will be an integer  $n \ge 0$  such that each leaf in X has level at most n and, for each of the integers  $k = 0, 1, \ldots, n$ , at least one leaf in X has level k. The leaves at level 0 are exactly the compact ones, and the leaves at level k > 0 are asymptotic to some of the leaves at each level strictly less than k, but to no others. For the general theory of levels, see [C-C1].

Let  $L \subset X$  be a leaf at level k > 0. The Poincaré-Bendixson theory of totally proper leaves applies [C-C1, §6] and gives a decomposition

$$L = A \cup B^1 \cup \cdots \cup B^r,$$

together with projections

$$p^j: B^j \to L^j$$

that are *spirals* onto leaves  $L^j \subset X$  at lower levels. Each  $p^j$  is a projection along leaves of  $\mathscr{F}^\perp$  and is a semi-covering with covering semigroup the positive integers  $\mathbb{Z}^+$ . Indeed, if  $y_0 \in B^j$  and  $p^j(y_0) = y \in L^j$ , there is a compact subarc of a leaf of  $\mathscr{F}^\perp$ , having endpoints  $y_0$  and y, which meets  $B^j$  in a sequence of points  $\{y_i\}_{i=0}^\infty$  that converges monotonically to y. The semigroup action of  $\mathbb{Z}^+$  on  $B^j$  is generated by the proper imbedding  $\pi: B^j \to B^j$  defined by  $\pi(y_0) = y_1$ . Finally, the submanifold A is compact and connected,  $A \cap B^j = \partial B^j$  is also a component of  $\partial A$ ,  $1 \le j \le r$ , and  $B^i \cap B^j = \emptyset$  if  $i \ne j$ .

The above action of  $\mathbb{Z}^+$  corresponds to an element of contracting holonomy of the leaf  $L^j$ . This element of holonomy determines a *compactly supported* cohomology class  $\alpha \in H^1_c(L^j; \mathbb{Z})$ . The proof that  $\alpha$  is compactly supported makes essential use of the hypothesis that  $(M, \mathcal{F})$  is of class  $C^2$ . The Poincaré dual of  $\alpha$  can be represented by a compact, connected, nonseparating submanifold  $N^j \subset L^j$  of codimension one, called the *juncture* of the spiral  $p^j$ . In our case, dim  $(L^j) = 2$ , so  $N^j$  is always a circle.

The geometric interpretation of the juncture  $N^j$  is as follows. The manifold  $B^j$  will be an "infinite repetition" of segments  $\{B_i^j\}_{i=0}^{\infty}$ , each diffeomorphic to a copy of the manifold  $L^j_*$  with two boundary components that is obtained by cutting  $L^j$  along  $N^j$ . If k > 1, then  $B_i^j$  and  $B_{i+k}^j$  will be disjoint,  $\forall i \geq 0$ , while  $B_i^j \cap B_{i+1}^j = N^j_{i+1}$  will be a common boundary component of these manifolds. We also let  $N^j_0$  denote  $\partial B^j$ , one of the two components of  $\partial B^j_0$ . Finally, the projection  $p^j$  will carry  $N^j_i$  diffeomorphically onto  $N^j$ ,  $\forall i \geq 0$ .

For all of the above, only the homology class of  $N^j$  in  $L^j$  matters, so we will always take the juncture to be a closed geodesic in a metric relativized from a metric on M.

Finally, it should be remarked that the compact manifold  $A \subset L$  can be chosen as large as desired. It follows that, if L is not topologically a plane or a cylinder, then A can be chosen to be neither a disk nor an annulus.

For more details concerning the Poincaré-Bendixson theory of totally proper leaves, the reader should consult [C-C1, § 6].

(1.2) PROPOSITION. Let X be a compact union of finitely many leaves of  $\mathcal{F}$ . Then, if no leaf in X is a torus or a sphere,  $\exists$  a Riemannian metric g on M and an open neighborhood W of X such that  $g \mid W$  is a leafwise hyperbolic for  $\mathcal{F} \setminus W$  and such that projection along the leaves of  $\mathcal{F}^{\perp} \mid W$  defines local isometries between the leaves of  $\mathcal{F} \mid W$ .

*Proof.* We proceed by induction on the levels of the leaves in X. The leaves at level 0 are compact and, by the hypothesis, each supports a hyperbolic metric. In standard fashion, one constructs a metric on M that relativizes to the chosen

hyperbolic metric on each leaf at level 0 in X. Arrange that, on the union  $W_0$  of suitable disjoint normal neighborhoods of these compact leaves (the normal fibers are open subarcs of leaves of  $\mathcal{F}^{\perp}$ ), the metric is the lift, via the projection, of the hyperbolic metric. Hence, the metric is leafwise hyperbolic in  $W_0$  and projection along the normal fibers defines local isometries between the leaves of  $\mathcal{F} \mid W_0$ .

Let  $X_k$  denote the union of all leaves in X at levels at most k. This is a compact set. Let  $W_k \subseteq M$  be an open neighborhood of  $X_k$  obtained as the (generally not disjoint) union of sufficiently small normal neighborhoods of the leaves in  $X_k$ . Inductively, assume that a metric on M has been found relative to which each leaf of  $\mathcal{F} \mid W_k$  has constant curvature -1 and such that the local projections along leaves of  $\mathcal{F}^\perp \mid W_k$  define local isometries between the leaves of  $\mathcal{F} \mid W_k$ . Let L be a leaf in X at level k+1. Choose a decomposition  $L = A \cup B^1 \cup \ldots \cup B^r$  with spirals  $p^j : B^j \to L^j$ ,  $1 \le j \le r$ . By the inductive hypothesis, each of the leaves  $L^j$  is hyperbolic and we can arrange, as remarked above, that the juncture  $N^j$  be a closed geodesic in  $L^j$ . The metric on each leaf  $L^j$  is lifted, via  $p^j$ , to a hyperbolic metric on  $B^j$  in which  $\partial B^j$  is a closed geodesic. Furthermore, this metric on  $B^j \cap W_k$  agrees with the one already defined.

An easy induction on the level of L (necessarily > 1) shows that this leaf has infinite genus, so the compact submanifold A can be chosen to be neither a disk nor an annulus. Decompose A into "pairs of pants". Each pair of pants has a hyperbolic metric, uniquely specified (up to isometries that are isotopic to the identity) by requiring each boundary circle to be a geodesic of specified length [Po]. Using this, one readily extends the metric on  $\bigcup_{j=1}^r B^j$  smoothly over A so as to produce a hyperbolic metric on the leaf L. One does this for each leaf  $L \subset X$  at level k+1. There is a neighborhood  $\tilde{W}$  of  $X_{k+1}$  of the desired type, obtained by extending  $W_k$  via disjoint, normal neighborhoods of the compact sets  $L \setminus W_k$ , and the metric extends over  $\tilde{W}$  to be as desired. Select the new neighborhood  $W_{k+1}$  of  $X_{k+1}$  by shrinking  $\tilde{W}$  slightly in the  $\mathcal{F}^\perp$  directions and then modify the metric in  $\tilde{W} \setminus W_{k+1}$  so that it extends  $C^2$ -smoothly to a Riemannian metric on M.

### 2. The metric on the foliated products

In this section, we assume that  $(M, \mathcal{F})$  is proper. By (1.1) and (1.2), we choose a skeleton  $X \subset M$  and a Riemannian metric on M that makes each leaf in X hyperbolic.

Let U be a component of  $M \setminus X$ . Fix the identification  $\hat{U} \cong L \times [0, 1]$ . Let

$$\hat{\rho}: L \times \{0\} \rightarrow L \times \{1\}$$

be projection along the leaves of  $\hat{\mathcal{F}}^{\perp}$ . That is to say,  $\hat{\rho}(x, 0) = (x, 1)$ . Let  $L = \hat{i}(L \times \{0\})$  and  $\tilde{L} = \hat{i}(L \times \{1\})$ . The projection  $\hat{\rho}$  defines a diffeomorphism

$$\rho: L \to \tilde{L}$$

that is generally not an isometry. For instance, it is quite possible that  $L = \tilde{L}$  and then, if L is not compact, it is impossible that  $\rho$  be the identity and generally it is impossible that it be an isometry. Consequently, we cannot expect  $\hat{\rho}$  to be an isometry either.

Recall the decomposition of L into the compact submanifold A and "arms"  $B^j$ ,  $1 \le j \le r$ , together with  $p^j: B^j \to L^j$ . Let  $\tilde{A} = \hat{i}(A \times \{1\})$ , let  $\tilde{B}^j = \hat{i}(\tilde{B}^j \times \{1\})$ ,  $1 \le j \le r$ , and let  $p^j$  denote both of the spirals  $B^j \to L^j$  and  $\tilde{B}^j \to L^j$ . The metrics on  $B^j$  and  $\tilde{B}^j$  have been lifted by  $p^j$  from a hyperbolic metric on  $L^j$ ,  $\forall j$ . If  $L = \tilde{L}$ , then, depending on the side of the leaf  $L^j$  on which the arm  $B^j$  spirals, either  $\rho$  or  $\rho^{-1}$  carries  $B^j$  into itself and generates the covering semigroup  $\mathbb{Z}^+$  on  $B^j$ . In any case,  $\rho$  carries  $B^j$  isometrically onto  $\tilde{B}^j$  and it follows that, in  $\hat{U}$  and relative to the metric pulled back by  $\hat{i}$ , the projection

$$\hat{\rho}: L \times \{0\} \rightarrow L \times \{1\}$$

carries  $B^j \times \{0\}$  isometrically onto  $B^j \times \{1\}$ ,  $1 \le j \le r$ . The problem alluded to in the previous paragraph is that the diffeomorphism  $\hat{\rho} \mid (A \times \{0\})$  of  $A \times \{0\}$  onto  $A \times \{1\}$  is not generally an isometry. It is true, however, that  $\hat{\rho}$  is an isometry of the boundary geodesics since these are also boundary geodesics for the arms.

Fix a decomposition of  $A \times \{0\}$  into pairs of pants, each bounded by closed geodesics. Then  $\hat{\rho}$  carries this to a decomposition of  $A \times \{1\}$  into pairs of pants, but the boundary circles that are not already components of  $\partial A \times \{1\}$  may not be geodesics. These circles can be replaced with closed geodesics in their free homotopy class, giving a decomposition of  $A \times \{1\}$  that is diffeomorphic to the first one. Finally, one smoothly alters  $\hat{\mathcal{F}}^{\perp} \mid (A \times [0, 1])$ , without changing it near  $A \times \{0, 1\}$  nor near  $\partial A \times [0, 1]$ , so that  $\hat{\rho}$  carries the decomposition of  $A \times \{0\}$  to the one of  $A \times \{1\}$ , although *not isometrically*. Remark that  $\mathcal{F}^{\perp}$  itself has been slightly modified, but in regions not affecting any preceding construction.

(2.1) LEMMA. The Riemannian metric on  $A \times \{0, 1\}$  extends to a metric g' on  $A \times [0, 1]$  that restricts to a hyperbolic metric  $g'_t$  on each level set  $A \times \{t\}$ . For t sufficiently near 0 or 1, respectively, the metric  $g'_t$  is independent of t. Finally,  $g'_t$  makes each component  $\Sigma \times \{t\}$  of  $\partial A \times \{t\}$  into a geodesic isometric to  $\Sigma$ ,  $0 \le t \le 1$ .

**Proof.** The space of isotopy classes of hyperbolic metrics on A, relative to which the components of  $\partial A$  are closed geodesics of specified lengths, is smoothly parametrized as a Euclidean space  $\mathbb{R}^N$ , in a way that depends on the choice of decomposition of A into pairs of pants. These coordinates, due to Fenchel and Nielsen [Th, § 5], are the logarithms of the lengths of the boundary components of the pairs of pants, other than the components of  $\partial A$ , together with real numbers that record relative "twisting" along common boundary geodesics. Since  $\hat{\rho}$  carries the decomposition of  $A \times \{0\}$  to that of  $A \times \{1\}$ , we can view the metrics on these manifolds as points  $(a_1, \ldots, a_N)$  and  $(b_1, \ldots, b_N)$  in the same Euclidean space  $\mathbb{R}^N$ . Choose a smooth path  $\sigma:[0, 1] \to \mathbb{R}^N$  that joins these points and is constant near 0 and 1. Then  $\sigma(t) = g'_t$  is a hyperbolic metric on  $A \times \{t\}$ ,  $0 \le t \le 1$ , and all assertions follow.

Should it happen that  $\hat{\mathcal{F}} \mid (A \times [0, 1])$  is the product foliation, we are done. This is not generally the case. Furthermore, if  $\Sigma$  is a component of  $\partial A$  and if F is a leaf of  $\hat{\mathcal{F}} \mid (A \times [0, 1])$ , the components of  $F \cap (\Sigma \times [0, 1])$  need not be closed.

Let  $\mathcal{H}'$  denote the product foliation of  $A \times [0, 1]$  and let  $\mathcal{H}$  denote the foliation of that manifold induced by  $\widehat{\mathcal{F}}$ . Then g' is a Riemannian metric on  $T(\mathcal{H}')$ . If  $v_i \in T_{(x,t)}(\mathcal{H})$ , i=1, 2, let  $v_i' \in T_{(x,t)}(\mathcal{H}')$  be the unique vector that differs from  $v_i$  by an element of  $T(\widehat{\mathcal{F}}^\perp)$ . Define a Riemannian metric g on  $T(\mathcal{H})$  by setting  $g(v_1, v_2) = g'(v_1', v_2')$ .

# (2.2) LEMMA. Under the metric g, each leaf of $\mathcal{H}$ has constant curvature -1.

*Proof.* For  $v_i$  and  $v_i'$  as above, it is evident that  $(v_1, v_2)$  is a g-orthonormal frame if and only if  $(v_1', v_2')$  is g'-orthonormal. It will be enough, therefore, to show that the respective curvature tensors satisfy the identity

$$(R(v_1, v_2)v_2)' = R'(v_1', v_2')v_2'.$$

Let  $\lambda: A \times [0, 1] \to A$  be the canonical projection. Set  $\omega_i = \lambda_*(v_i) = \lambda_*(v_i')$ , i = 1, 2. Let  $X_i$  be a smooth extension of  $\omega_i$  to a field on A. Let  $Y_i \in \Gamma(T(\mathcal{H}))$  and  $Y_i' \in \Gamma(T(\mathcal{H}'))$  be the unique elements that are  $\lambda$ -related to  $X_i$ , i = 1, 2. By integrability,  $[Y_1, Y_2] \in \Gamma(T(\mathcal{H}))$  and  $[Y_1', Y_2'] \in \Gamma(T(\mathcal{H}'))$ , and these fields are  $\lambda$ -related to  $[X_1, X_2]$ . It follows that  $[Y_1, Y_2]' = [Y_1', Y_2']$ . Let  $\nabla$  and  $\nabla'$  be the respective Levi-Civita connections (along the leaves of  $\mathcal{H}$  and  $\mathcal{H}'$ , respectively) for the metrics g and g'. The well known formula for the Levi-Civita connection [K-N, p. 160] involves only the metric and the bracket operation, hence

$$(\nabla_{Y_1}Y_2)'=\nabla'_{Y_1}Y_2'$$

and it follows that  $(R(v_1, v_2)v_2)' = R'(v_1', v_2')v_2'$ .

(2.3) LEMMA. Let H be a leaf of  $\mathcal{H}$ . Then each component of  $H \cap (\partial A \times [0, 1])$  is a geodesic for the metric  $g \mid H$  and  $\lambda$  defines a local isometry of this geodesic onto one of the closed geodesics bounding A.

*Proof.* Let J be a subarc of a component of  $H \cap (\partial A \times [0, 1])$ , small enough that  $\lambda$  carries it one-one into  $\partial A$ . Relative to g, let v be a unit tangent field along J. At each point of J, the corresponding vector v' has unit length relative to g'. Since  $\lambda$  is an isometry of each  $\partial A \times \{t\}$  (in the g' metric) onto  $\partial A$ , it follows that  $\lambda_*(v)$  is a unit tangent field along  $\lambda(J)$ . The field  $\lambda_*(v)$  lifts to a field Z on  $\lambda^{-1}(J)$  that is everywhere tangent to  $\mathcal{H}$ . The corresponding field Z', tangent to  $\mathcal{H}'$ , has unit g'-length and is everywhere tangent to curves that are geodesics in leaves of  $\mathcal{H}'$ , hence  $\nabla'_{Z'}Z' \equiv 0$ . Then, as in the proof of (2.2),  $(\nabla_Z Z)' = \nabla'_{Z'}Z' \equiv 0$ , so  $\nabla_Z Z \equiv 0$ . In particular, J is a geodesic segment for  $g \mid H$ .  $\square$ 

Remarks. Let  $g^j$  be the metric on  $B^j \times [0, 1]$  lifted from  $B^j$  via projection. Since  $\partial B^j \subset \partial A$  is a closed geodesic, the subarc J in the above proof has exactly the same geodesic structure relative to the metric  $g^j$  as it has relative to g.

We now build the metric g on  $\hat{U}$  (more precisely, on  $T(\hat{\mathcal{F}})$ ) by using the metric constructed above in  $A \times [0, 1]$  and using, in each arm  $B^j \times [0, 1]$ , the lifted metric  $g^j$ . The only problem is to make sure these definitions fit together smoothly (i.e.,  $C^2$ ). For this, it will be enough to verify leafwise smoothness.

(2.4) COROLLARY. The metric g, as defined above on  $\hat{U}$ , is of class  $C^2$  and leaf-wise hyperbolic.

*Proof.* Let H be a leaf of  $\mathcal{H}$  and let J be a small open subarc of  $H \cap (\partial A \times [0, 1])$ , as in the proof of (2.3). Let W be an open subset of H such that  $W \cap (\partial A \times [0, 1]) = J$ . Let  $W^+ = W \cap (A \times [0, 1])$  and  $W^- = W \cap (B^j \times [0, 1])$  and let  $\varphi^{\pm} : W^{\pm} \to \mathbb{H}^2$  be isometric mappings into the hyperbolic plane  $\mathbb{H}^2$ . Then  $\varphi^-(J)$  and  $\varphi^+(J)$  are hyperbolic line segments of the same length and, by standard hyperbolic geometry, there is an isometry  $\tau : \mathbb{H}^2 \to \mathbb{H}^2$  such that  $\tau(\varphi^-(J)) = \varphi^+(J)$ . Thus, the isometric imbeddings  $\tau \circ \varphi^- : W^- \to \mathbb{H}^2$  and  $\varphi^+ : W^+ \to \mathbb{H}^2$  fit together to define an isometric imbedding  $\tilde{\varphi} : W \to \mathbb{H}^2$ .  $\square$ 

(2.5) LEMMA. Outside of a compact subset of  $A \times ]0, 1[$ , the metric g is just the lift, via projections along the leaves of  $\hat{\mathcal{F}}^{\perp}$ , of the hyperbolic metric on  $\partial \hat{U}$  that is pulled back from the skeleton X via  $\hat{i}$ .

*Proof.* This is clear in the arms. In  $A \times [0, 1]$ , it follows from the relation between g and g' and the fact that, for t sufficiently near 0 (respectively, 1),  $g'_t$  is independent of t.  $\square$ 

For each component U of  $M \setminus X$ , we have produced a leafwise hyperbolic metric of class at least  $C^2$ . We claim that these metrics, together with the hyperbolic metric on X produced in (1.2), assemble to form the desired  $C^2$  metric on M. Indeed, the metric in (1.2) is of class  $C^2$  and, by (2.5), it agrees in a neighborhood of X with the metric we have just produced. Since the only points where differentiability could fail are the points of X, the proof of the Main Theorem is complete.

#### References

- [C-C1] J. CANTWELL and L. CONLON, Poincaré-Bendixson theory for leaves of codimension one, Trans. Amer. Math. Soc., 265 (1981), 181-209.
- [C-C2] J. CANTWELL and L. CONLON, Smoothability of proper foliations, Ann. Inst. Fourier, (to appear).
- [Di] P. DIPPOLITO, Codimension one foliations of closed manifolds, Ann. of Math., 107 (1978), 403-453.
- [Ep] D. B. A. EPSTEIN, Periodic flows on 3-manifolds, Ann. of Math., 95 (1972), 68-82.
- [Ga1] D. GABAI, Foliations and the topology of 3-manifolds, J. Diff. Geo., 18 (1983), 445-503.
- [Ga2] D. GABAI, Foliations and the topology of 3-manifolds II, J. Diff. Geo., 26 (1987), 461-478.
- [Ga3] D. GABAI, Foliations and the topology of 3-manifolds III, J. Diff. Geo., 26 (1987), 479-536.
- [Go] C. GODBILLON, Feuilletages, Études Geometrique, II, Publ. Inst. de Recherche Math. Avancée, Univ. Louis Pasteur, Strasbourg, 1986.
- [H-H] G. HECTOR and U. HIRSCH, Introduction to the Geometry of Foliations, Part B, Vieweg and Sohn, Braunschweig, 1983.
- [K-N] S. KOBAYASHI and K. NOMIZU, Foundations of Differential Geometry, Vol. I, Interscience Publishers, New York, 1963.
- [Mi] K. MILLETT, Generic properties of proper foliations, (I.H.E.S. preprint).
- [Po] V. POÉNARU, Rappels de géométrie hyperbolique en dimension 2 et généralités sur  $i: \mathcal{T} \times \mathcal{T} \to \mathbb{R}_+$ , Astérisque, 66–67 (1979), 33–35.
- [Th] W. THURSTON, The geometry and topology of 3-manifolds, Princeton Lecture Notes.

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