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**Autor:** Cantwell, John / Conlon, Lawrence  
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## Leafwise hyperbolicity of proper foliations

JOHN CANTWELL<sup>1</sup> and LAWRENCE CONLON<sup>2</sup>

### Introduction

A foliated manifold  $(M, \mathcal{F})$  is said to be *proper* if every leaf of  $\mathcal{F}$  is proper. A leaf  $L$  is proper if the relative topology of  $L$  in  $M$  coincides with the manifold topology of  $L$ . This is equivalent to requiring that each point  $x \in L$  lies in a foliation chart  $U \subset M$  such that  $L \cap U$  is a single plaque. Equivalently, the leaf  $L$  is not asymptotic to itself.

Proper foliated manifolds have been studied by various authors. For example, in arbitrary codimension, Millett [Mi] has organized the leaves of such foliations into a countable ordinal hierarchy that is completely analogous to the Epstein hierarchy for foliations with all leaves compact [Ep].

In codimension one, with smoothness class at least  $C^2$ , there is a more rigid hierarchy by integral *levels* [C–C1]. Leaves at a given level wind in on those at lower levels in a way reminiscent of the Poincaré–Bendixson theorem. In [C–C2], this hierarchy is combined with Millett's to prove that  $C^2$ -smoothness for proper foliated manifolds of codimension one implies  $C^\infty$ -smoothability.

In the work of Gabai [Ga1], [Ga2], [Ga3], proper foliated 3-manifolds occur that are Reebless and of *finite depth* (which means that there is an upper bound on the levels). Generally, these foliations may only be of class  $C^0$ , although frequently they are smooth.

In what follows,  $(M, \mathcal{F})$  will denote a  $C^2$ -foliated 3-manifold, where  $M$  is compact and orientable and  $\mathcal{F}$  is of codimension one and transversely orientable.

An interesting geometric problem is to find a Riemannian metric on  $M$  relative to which each leaf of  $\mathcal{F}$  is *hyperbolic* (i.e., has constant curvature  $-1$ ). We say that such a metric is *leafwise hyperbolic*.

**MAIN THEOREM.** *Let  $(M, \mathcal{F})$  be proper and assume that each component of  $\partial M$  is a leaf of  $\mathcal{F}$ . Then  $\exists$  a leafwise hyperbolic metric on  $M$  if and only if no leaf of  $\mathcal{F}$  is a torus or a sphere.*

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**COROLLARY** *Let  $(M, \mathcal{F})$  be proper and assume that each component of  $\partial M$  is either a leaf or is transverse to  $\mathcal{F}$ . If no compact leaf of  $\mathcal{F}$  is a torus, a sphere, a disk, or an annulus, then  $\exists$  a leafwise hyperbolic metric on  $M$ .*

This corollary is immediate upon doubling  $(M, \mathcal{F})$  along the transverse boundary.

Gabai's proper foliations of knot complements  $M = S^3 \setminus N^0(k)$  are taut (that is, every leaf meets a closed transversal nontrivially) and admit a minimal genus spanning surface of the knot  $k$  as a leaf [Ga1, Theorem 5.5]. Together with the fact that  $H_1(M; \mathbb{Z}) \cong \mathbb{Z}$ , this implies that, for knots of genus greater than one, no compact leaf is a torus, a sphere, an annulus, or a disk. If the foliation can be chosen to be smooth, the corollary implies that it admits a leafwise hyperbolic metric.

The "only if" part of the Main Theorem is evident. The proof of the "if" part uses the Poincaré–Bendixson theory of totally proper leaves [C–C1, §6] and the smooth parametrization of hyperbolic structures by Fenchel–Nielsen coordinates [Th, § 5].

*Remark.* The hypothesis in [C–C1] that  $\partial M = \emptyset$  can be replaced with the hypothesis that  $\partial M$  is a union of leaves without affecting any of the theorems or proofs in that paper. *We fix this hypothesis throughout.*

## 1. Constructing a hyperbolic skeleton

In this section, unless otherwise specified,  $(M, \mathcal{F})$  need not be proper.

We fix a smooth, one dimensional foliation  $\mathcal{F}^\perp$  of  $M$  that is everywhere transverse to  $\mathcal{F}$ . There corresponds a decomposition

$$T(M) = T(\mathcal{F}) \oplus T(\mathcal{F}^\perp)$$

into the respective sub-bundles of tangents to the two foliations. We will fix a Riemannian metric on  $T(\mathcal{F}^\perp)$ . All metrics constructed on  $T(M)$  will be understood to induce this fixed metric on  $T(\mathcal{F}^\perp)$  and to make the two summands orthogonal. Accordingly, in this and the following sections, the Riemannian metrics that we will be constructing will be metrics on  $T(\mathcal{F})$ .

Let  $O(\mathcal{F})$  denote the family of open,  $\mathcal{F}$ -saturated subsets of  $M$  and, for  $U \in O(\mathcal{F})$ , let  $\hat{U}$  denote the completion of  $U$  in any Riemannian metric inherited from a Riemannian metric on  $M$ . As is well known [Di], [C–C1], [H–H], [Go],  $\hat{U}$  is a manifold with boundary and the inclusion  $i: U \hookrightarrow M$  induces a  $C^2$  immersion  $\hat{i}: \hat{U} \rightarrow M$  that may identify some components of  $\partial \hat{U}$  pairwise. Then we obtain a

$C^2$  foliation,  $\hat{i}^{-1}(\mathcal{F}) = \hat{\mathcal{F}}$ , of  $\hat{U}$  that is tangent to  $\partial\hat{U}$ , while  $\hat{i}^{-1}(\mathcal{F}^\perp) = \hat{\mathcal{F}}^\perp$  is a smooth, one dimensional foliation of  $\hat{U}$ , everywhere transverse to  $\hat{\mathcal{F}}$ .

Of special interest is the case in which  $\hat{U}$  is diffeomorphic to  $L \times [0, 1]$  in such a way that the leaves of  $\hat{\mathcal{F}}^\perp$  are identified with the intervals  $\{x\} \times [0, 1]$ ,  $\forall x \in L$ . Following our usage elsewhere (e.g., [C–C1]), we will say that  $U$  is a *foliated product*. We will also identify  $L$  with the leaf  $\hat{i}(L \times \{0\})$  of  $\mathcal{F}$ , which may or may not be the same as the leaf  $\hat{i}(L \times \{1\})$ .

**DEFINITION.** A closed subset  $X \subset M$  that is a finite union of leaves will be called a *skeleton* of  $(M, \mathcal{F})$  if each component of  $M \setminus X$  is a foliated product.

(1.1) **LEMMA.** *If  $(M, \mathcal{F})$  is proper, it has a skeleton.*

*Proof.* Choose a countable family  $\{L_i\}_{i=1}^\infty$  of leaves of  $\mathcal{F}$  such that  $\bigcup_{i=1}^\infty L_i$  is everywhere dense in  $M$ . Let  $X_k = \bigcup_{i=1}^k L_i$ ,  $\forall k \geq 1$ . Since  $(M, \mathcal{F})$  is proper, the closure  $\bar{X}_k$  of  $X_k$  is a finite union of leaves [C–C1, (4.7)]. Since  $\bigcup_{k=1}^\infty \bar{X}_k$  is everywhere dense in  $M$ , the components of  $M \setminus \bar{X}_k$  will all be foliated products, for  $k$  sufficiently large [Di, Proposition 1].

Our proof of the Main Theorem will proceed by finding, first, a metric making all leaves of the skeleton  $X$  hyperbolic and, then, by modifying the metric in each of the complementary foliated products. For the first step, the only property of the skeleton  $X$  that will be needed is that it is a compact union of finitely many leaves.

Let  $X$  be a compact, nonempty, union of finitely many leaves of  $\mathcal{F}$ , these being, of necessity, proper whether or not the foliation is proper (the closure of a nonproper leaf contains uncountably many leaves by the results in [C–C1, § 4 and § 5]). There will be an integer  $n \geq 0$  such that each leaf in  $X$  has level at most  $n$  and, for each of the integers  $k = 0, 1, \dots, n$ , at least one leaf in  $X$  has level  $k$ . The leaves at level 0 are exactly the compact ones, and the leaves at level  $k > 0$  are asymptotic to some of the leaves at each level strictly less than  $k$ , but to no others. For the general theory of levels, see [C–C1].

Let  $L \subset X$  be a leaf at level  $k > 0$ . The Poincaré–Bendixson theory of totally proper leaves applies [C–C1, §6] and gives a decomposition

$$L = A \cup B^1 \cup \dots \cup B^r,$$

together with projections

$$p^j : B^j \rightarrow L^j,$$

that are *spirals* onto leaves  $L^j \subset X$  at lower levels. Each  $p^j$  is a projection along leaves of  $\mathcal{F}^\perp$  and is a semi-covering with covering semigroup the positive integers  $\mathbb{Z}^+$ . Indeed, if  $y_0 \in B^j$  and  $p^j(y_0) = y \in L^j$ , there is a compact subarc of a leaf of  $\mathcal{F}^\perp$ , having endpoints  $y_0$  and  $y$ , which meets  $B^j$  in a sequence of points  $\{y_i\}_{i=0}^\infty$  that converges monotonically to  $y$ . The semigroup action of  $\mathbb{Z}^+$  on  $B^j$  is generated by the proper imbedding  $\pi: B^j \rightarrow B^j$  defined by  $\pi(y_0) = y_1$ . Finally, the submanifold  $A$  is compact and connected,  $A \cap B^j = \partial B^j$  is also a component of  $\partial A$ ,  $1 \leq j \leq r$ , and  $B^i \cap B^j = \emptyset$  if  $i \neq j$ .

The above action of  $\mathbb{Z}^+$  corresponds to an element of contracting holonomy of the leaf  $L^j$ . This element of holonomy determines a *compactly supported* cohomology class  $\alpha \in H_c^1(L^j; \mathbb{Z})$ . The proof that  $\alpha$  is compactly supported makes essential use of the hypothesis that  $(M, \mathcal{F})$  is of class  $C^2$ . The Poincaré dual of  $\alpha$  can be represented by a compact, connected, nonseparating submanifold  $N^j \subset L^j$  of codimension one, called the *junction* of the spiral  $p^j$ . In our case,  $\dim(L^j) = 2$ , so  $N^j$  is always a circle.

The geometric interpretation of the junction  $N^j$  is as follows. The manifold  $B^j$  will be an “infinite repetition” of segments  $\{B_i^j\}_{i=0}^\infty$ , each diffeomorphic to a copy of the manifold  $L_*^j$  with two boundary components that is obtained by cutting  $L^j$  along  $N^j$ . If  $k > 1$ , then  $B_i^j$  and  $B_{i+k}^j$  will be disjoint,  $\forall i \geq 0$ , while  $B_i^j \cap B_{i+1}^j = N_{i+1}^j$  will be a common boundary component of these manifolds. We also let  $N_0^j$  denote  $\partial B^j$ , one of the two components of  $\partial B_0^j$ . Finally, the projection  $p^j$  will carry  $N_i^j$  diffeomorphically onto  $N^j$ ,  $\forall i \geq 0$ .

For all of the above, only the homology class of  $N^j$  in  $L^j$  matters, so we will always take the junction to be a closed geodesic in a metric relativized from a metric on  $M$ .

Finally, it should be remarked that the compact manifold  $A \subset L$  can be chosen as large as desired. It follows that, if  $L$  is not topologically a plane or a cylinder, then  $A$  can be chosen to be neither a disk nor an annulus.

For more details concerning the Poincaré–Bendixson theory of totally proper leaves, the reader should consult [C–C1, § 6].

(1.2) PROPOSITION. *Let  $X$  be a compact union of finitely many leaves of  $\mathcal{F}$ . Then, if no leaf in  $X$  is a torus or a sphere,  $\exists$  a Riemannian metric  $g$  on  $M$  and an open neighborhood  $W$  of  $X$  such that  $g|_W$  is a leafwise hyperbolic for  $\mathcal{F} \setminus W$  and such that projection along the leaves of  $\mathcal{F}^\perp|_W$  defines local isometries between the leaves of  $\mathcal{F}|_W$ .*

*Proof.* We proceed by induction on the levels of the leaves in  $X$ . The leaves at level 0 are compact and, by the hypothesis, each supports a hyperbolic metric. In standard fashion, one constructs a metric on  $M$  that relativizes to the chosen

hyperbolic metric on each leaf at level 0 in  $X$ . Arrange that, on the union  $W_0$  of suitable disjoint normal neighborhoods of these compact leaves (the normal fibers are open subarcs of leaves of  $\mathcal{F}^\perp$ ), the metric is the lift, via the projection, of the hyperbolic metric. Hence, the metric is leafwise hyperbolic in  $W_0$  and projection along the normal fibers defines local isometries between the leaves of  $\mathcal{F} \mid W_0$ .

Let  $X_k$  denote the union of all leaves in  $X$  at levels at most  $k$ . This is a compact set. Let  $W_k \subseteq M$  be an open neighborhood of  $X_k$  obtained as the (generally not disjoint) union of sufficiently small normal neighborhoods of the leaves in  $X_k$ . Inductively, assume that a metric on  $M$  has been found relative to which each leaf of  $\mathcal{F} \mid W_k$  has constant curvature  $-1$  and such that the local projections along leaves of  $\mathcal{F}^\perp \mid W_k$  define local isometries between the leaves of  $\mathcal{F} \mid W_k$ . Let  $L$  be a leaf in  $X$  at level  $k + 1$ . Choose a decomposition  $L = A \cup B^1 \cup \dots \cup B^r$  with spirals  $p^j : B^j \rightarrow L^j$ ,  $1 \leq j \leq r$ . By the inductive hypothesis, each of the leaves  $L^j$  is hyperbolic and we can arrange, as remarked above, that the juncture  $N^j$  be a closed geodesic in  $L^j$ . The metric on each leaf  $L^j$  is lifted, via  $p^j$ , to a hyperbolic metric on  $B^j$  in which  $\partial B^j$  is a closed geodesic. Furthermore, this metric on  $B^j \cap W_k$  agrees with the one already defined.

An easy induction on the level of  $L$  (necessarily  $> 1$ ) shows that this leaf has infinite genus, so the compact submanifold  $A$  can be chosen to be neither a disk nor an annulus. Decompose  $A$  into ‘‘pairs of pants’’. Each pair of pants has a hyperbolic metric, uniquely specified (up to isometries that are isotopic to the identity) by requiring each boundary circle to be a geodesic of specified length [Po]. Using this, one readily extends the metric on  $\bigcup_{j=1}^r B^j$  smoothly over  $A$  so as to produce a hyperbolic metric on the leaf  $L$ . One does this for each leaf  $L \subset X$  at level  $k + 1$ . There is a neighborhood  $\tilde{W}$  of  $X_{k+1}$  of the desired type, obtained by extending  $W_k$  via disjoint, normal neighborhoods of the compact sets  $L \setminus W_k$ , and the metric extends over  $\tilde{W}$  to be as desired. Select the new neighborhood  $W_{k+1}$  of  $X_{k+1}$  by shrinking  $\tilde{W}$  slightly in the  $\mathcal{F}^\perp$  directions and then modify the metric in  $\tilde{W} \setminus W_{k+1}$  so that it extends  $C^2$ -smoothly to a Riemannian metric on  $M$ .

## 2. The metric on the foliated products

In this section, we assume that  $(M, \mathcal{F})$  is proper. By (1.1) and (1.2), we choose a skeleton  $X \subset M$  and a Riemannian metric on  $M$  that makes each leaf in  $X$  hyperbolic.

Let  $U$  be a component of  $M \setminus X$ . Fix the identification  $\hat{U} \cong L \times [0, 1]$ . Let

$$\hat{\rho} : L \times \{0\} \rightarrow L \times \{1\}$$

be projection along the leaves of  $\mathcal{F}^\perp$ . That is to say,  $\hat{\rho}(x, 0) = (x, 1)$ . Let  $L = \hat{i}(L \times \{0\})$  and  $\tilde{L} = \hat{i}(L \times \{1\})$ . The projection  $\hat{\rho}$  defines a diffeomorphism

$$\rho: L \rightarrow \tilde{L}$$

that is generally not an isometry. For instance, it is quite possible that  $L = \tilde{L}$  and then, if  $L$  is not compact, it is impossible that  $\rho$  be the identity and generally it is impossible that it be an isometry. Consequently, we cannot expect  $\hat{\rho}$  to be an isometry either.

Recall the decomposition of  $L$  into the compact submanifold  $A$  and ‘‘arms’’  $B^j$ ,  $1 \leq j \leq r$ , together with  $p^j: B^j \rightarrow L^j$ . Let  $\tilde{A} = \hat{i}(A \times \{1\})$ , let  $\tilde{B}^j = \hat{i}(B^j \times \{1\})$ ,  $1 \leq j \leq r$ , and let  $p^j$  denote both of the spirals  $B^j \rightarrow L^j$  and  $\tilde{B}^j \rightarrow L^j$ . The metrics on  $B^j$  and  $\tilde{B}^j$  have been lifted by  $p^j$  from a hyperbolic metric on  $L^j$ ,  $\forall j$ . If  $L = \tilde{L}$ , then, depending on the side of the leaf  $L^j$  on which the arm  $B^j$  spirals, either  $\rho$  or  $\rho^{-1}$  carries  $B^j$  into itself and generates the covering semigroup  $\mathbb{Z}^+$  on  $B^j$ . In any case,  $\rho$  carries  $B^j$  isometrically onto  $\tilde{B}^j$  and it follows that, in  $\hat{U}$  and relative to the metric pulled back by  $\hat{i}$ , the projection

$$\hat{\rho}: L \times \{0\} \rightarrow L \times \{1\}$$

carries  $B^j \times \{0\}$  isometrically onto  $B^j \times \{1\}$ ,  $1 \leq j \leq r$ . The problem alluded to in the previous paragraph is that the diffeomorphism  $\hat{\rho} | (A \times \{0\})$  of  $A \times \{0\}$  onto  $A \times \{1\}$  is not generally an isometry. It is true, however, that  $\hat{\rho}$  is an isometry of the boundary geodesics since these are also boundary geodesics for the arms.

Fix a decomposition of  $A \times \{0\}$  into pairs of pants, each bounded by closed geodesics. Then  $\hat{\rho}$  carries this to a decomposition of  $A \times \{1\}$  into pairs of pants, but the boundary circles that are not already components of  $\partial A \times \{1\}$  may not be geodesics. These circles can be replaced with closed geodesics in their free homotopy class, giving a decomposition of  $A \times \{1\}$  that is diffeomorphic to the first one. Finally, one smoothly alters  $\mathcal{F}^\perp | (A \times [0, 1])$ , without changing it near  $A \times \{0, 1\}$  nor near  $\partial A \times [0, 1]$ , so that  $\hat{\rho}$  carries the decomposition of  $A \times \{0\}$  to the one of  $A \times \{1\}$ , although *not isometrically*. Remark that  $\mathcal{F}^\perp$  itself has been slightly modified, but in regions not affecting any preceding construction.

(2.1) LEMMA. *The Riemannian metric on  $A \times \{0, 1\}$  extends to a metric  $g'$  on  $A \times [0, 1]$  that restricts to a hyperbolic metric  $g'_t$  on each level set  $A \times \{t\}$ . For  $t$  sufficiently near 0 or 1, respectively, the metric  $g'_t$  is independent of  $t$ . Finally,  $g'_t$  makes each component  $\Sigma \times \{t\}$  of  $\partial A \times \{t\}$  into a geodesic isometric to  $\Sigma$ ,  $0 \leq t \leq 1$ .*

*Proof.* The space of isotopy classes of hyperbolic metrics on  $A$ , relative to which the components of  $\partial A$  are closed geodesics of specified lengths, is smoothly parametrized as a Euclidean space  $\mathbb{R}^N$ , in a way that depends on the choice of decomposition of  $A$  into pairs of pants. These coordinates, due to Fenchel and Nielsen [Th, § 5], are the logarithms of the lengths of the boundary components of the pairs of pants, other than the components of  $\partial A$ , together with real numbers that record relative “twisting” along common boundary geodesics. Since  $\hat{\rho}$  carries the decomposition of  $A \times \{0\}$  to that of  $A \times \{1\}$ , we can view the metrics on these manifolds as points  $(a_1, \dots, a_N)$  and  $(b_1, \dots, b_N)$  in the same Euclidean space  $\mathbb{R}^N$ . Choose a smooth path  $\sigma: [0, 1] \rightarrow \mathbb{R}^N$  that joins these points and is constant near 0 and 1. Then  $\sigma(t) = g'_t$  is a hyperbolic metric on  $A \times \{t\}$ ,  $0 \leq t \leq 1$ , and all assertions follow.

Should it happen that  $\hat{\mathcal{F}}|_{(A \times [0, 1])}$  is the product foliation, we are done. This is not generally the case. Furthermore, if  $\Sigma$  is a component of  $\partial A$  and if  $F$  is a leaf of  $\hat{\mathcal{F}}|_{(A \times [0, 1])}$ , the components of  $F \cap (\Sigma \times [0, 1])$  need not be closed.

Let  $\mathcal{H}'$  denote the product foliation of  $A \times [0, 1]$  and let  $\mathcal{H}$  denote the foliation of that manifold induced by  $\hat{\mathcal{F}}$ . Then  $g'$  is a Riemannian metric on  $T(\mathcal{H}')$ . If  $v_i \in T_{(x,t)}(\mathcal{H})$ ,  $i = 1, 2$ , let  $v'_i \in T_{(x,t)}(\mathcal{H}')$  be the unique vector that differs from  $v_i$  by an element of  $T(\hat{\mathcal{F}}^\perp)$ . Define a Riemannian metric  $g$  on  $T(\mathcal{H})$  by setting  $g(v_1, v_2) = g'(v'_1, v'_2)$ .

(2.2) LEMMA. *Under the metric  $g$ , each leaf of  $\mathcal{H}$  has constant curvature  $-1$ .*

*Proof.* For  $v_i$  and  $v'_i$  as above, it is evident that  $(v_1, v_2)$  is a  $g$ -orthonormal frame if and only if  $(v'_1, v'_2)$  is  $g'$ -orthonormal. It will be enough, therefore, to show that the respective curvature tensors satisfy the identity

$$(R(v_1, v_2)v_2)' = R'(v'_1, v'_2)v'_2.$$

Let  $\lambda: A \times [0, 1] \rightarrow A$  be the canonical projection. Set  $\omega_i = \lambda_*(v_i) = \lambda_*(v'_i)$ ,  $i = 1, 2$ . Let  $X_i$  be a smooth extension of  $\omega_i$  to a field on  $A$ . Let  $Y_i \in \Gamma(T(\mathcal{H}))$  and  $Y'_i \in \Gamma(T(\mathcal{H}'))$  be the unique elements that are  $\lambda$ -related to  $X_i$ ,  $i = 1, 2$ . By integrability,  $[Y_1, Y_2] \in \Gamma(T(\mathcal{H}))$  and  $[Y'_1, Y'_2] \in \Gamma(T(\mathcal{H}'))$ , and these fields are  $\lambda$ -related to  $[X_1, X_2]$ . It follows that  $[Y_1, Y_2]' = [Y'_1, Y'_2]$ . Let  $\nabla$  and  $\nabla'$  be the respective Levi-Civita connections (along the leaves of  $\mathcal{H}$  and  $\mathcal{H}'$ , respectively) for the metrics  $g$  and  $g'$ . The well known formula for the Levi-Civita connection [K-N, p. 160] involves only the metric and the bracket operation, hence

$$(\nabla_{Y_1} Y_2)' = \nabla'_{Y'_1} Y'_2$$

and it follows that  $(R(v_1, v_2)v_2)' = R'(v'_1, v'_2)v'_2$ .  $\square$

(2.3) LEMMA. *Let  $H$  be a leaf of  $\mathcal{H}$ . Then each component of  $H \cap (\partial A \times [0, 1])$  is a geodesic for the metric  $g|_H$  and  $\lambda$  defines a local isometry of this geodesic onto one of the closed geodesics bounding  $A$ .*

*Proof.* Let  $J$  be a subarc of a component of  $H \cap (\partial A \times [0, 1])$ , small enough that  $\lambda$  carries it one-one into  $\partial A$ . Relative to  $g$ , let  $v$  be a unit tangent field along  $J$ . At each point of  $J$ , the corresponding vector  $v'$  has unit length relative to  $g'$ . Since  $\lambda$  is an isometry of each  $\partial A \times \{t\}$  (in the  $g'$  metric) onto  $\partial A$ , it follows that  $\lambda_*(v)$  is a unit tangent field along  $\lambda(J)$ . The field  $\lambda_*(v)$  lifts to a field  $Z$  on  $\lambda^{-1}(J)$  that is everywhere tangent to  $\mathcal{H}$ . The corresponding field  $Z'$ , tangent to  $\mathcal{H}'$ , has unit  $g'$ -length and is everywhere tangent to curves that are geodesics in leaves of  $\mathcal{H}'$ , hence  $\nabla'_{Z'} Z' \equiv 0$ . Then, as in the proof of (2.2),  $(\nabla_Z Z)' = \nabla'_{Z'} Z' \equiv 0$ , so  $\nabla_Z Z \equiv 0$ . In particular,  $J$  is a geodesic segment for  $g|_H$ .  $\square$

*Remarks.* Let  $g^j$  be the metric on  $B^j \times [0, 1]$  lifted from  $B^j$  via projection. Since  $\partial B^j \subset \partial A$  is a closed geodesic, the subarc  $J$  in the above proof has exactly the same geodesic structure relative to the metric  $g^j$  as it has relative to  $g$ .

We now build the metric  $g$  on  $\hat{U}$  (more precisely, on  $T(\hat{\mathcal{F}})$ ) by using the metric constructed above in  $A \times [0, 1]$  and using, in each arm  $B^j \times [0, 1]$ , the lifted metric  $g^j$ . The only problem is to make sure these definitions fit together smoothly (i.e.,  $C^2$ ). For this, it will be enough to verify leafwise smoothness.

(2.4) COROLLARY. *The metric  $g$ , as defined above on  $\hat{U}$ , is of class  $C^2$  and leaf-wise hyperbolic.*

*Proof.* Let  $H$  be a leaf of  $\mathcal{H}$  and let  $J$  be a small open subarc of  $H \cap (\partial A \times [0, 1])$ , as in the proof of (2.3). Let  $W$  be an open subset of  $H$  such that  $W \cap (\partial A \times [0, 1]) = J$ . Let  $W^+ = W \cap (A \times [0, 1])$  and  $W^- = W \cap (B^j \times [0, 1])$  and let  $\varphi^\pm: W^\pm \rightarrow \mathbb{H}^2$  be isometric mappings into the hyperbolic plane  $\mathbb{H}^2$ . Then  $\varphi^-(J)$  and  $\varphi^+(J)$  are hyperbolic line segments of the same length and, by standard hyperbolic geometry, there is an isometry  $\tau: \mathbb{H}^2 \rightarrow \mathbb{H}^2$  such that  $\tau(\varphi^-(J)) = \varphi^+(J)$ . Thus, the isometric imbeddings  $\tau \circ \varphi^-: W^- \rightarrow \mathbb{H}^2$  and  $\varphi^+: W^+ \rightarrow \mathbb{H}^2$  fit together to define an isometric imbedding  $\tilde{\varphi}: W \rightarrow \mathbb{H}^2$ .  $\square$

(2.5) LEMMA. *Outside of a compact subset of  $A \times ]0, 1[$ , the metric  $g$  is just the lift, via projections along the leaves of  $\hat{\mathcal{F}}^\pm$ , of the hyperbolic metric on  $\partial \hat{U}$  that is pulled back from the skeleton  $X$  via  $\hat{i}$ .*

*Proof.* This is clear in the arms. In  $A \times [0, 1]$ , it follows from the relation between  $g$  and  $g'$  and the fact that, for  $t$  sufficiently near 0 (respectively, 1),  $g'_t$  is independent of  $t$ .  $\square$

For each component  $U$  of  $M \setminus X$ , we have produced a leafwise hyperbolic metric of class at least  $C^2$ . We claim that these metrics, together with the hyperbolic metric on  $X$  produced in (1.2), assemble to form the desired  $C^2$  metric on  $M$ . Indeed, the metric in (1.2) is of class  $C^2$  and, by (2.5), it agrees in a neighborhood of  $X$  with the metric we have just produced. Since the only points where differentiability could fail are the points of  $X$ , the proof of the Main Theorem is complete.

### References

- [C–C1] J. CANTWELL and L. CONLON, *Poincaré–Bendixson theory for leaves of codimension one*, Trans. Amer. Math. Soc., 265 (1981), 181–209.
- [C–C2] J. CANTWELL and L. CONLON, *Smoothability of proper foliations*, Ann. Inst. Fourier, (to appear).
- [Di] P. DIPPOLITO, *Codimension one foliations of closed manifolds*, Ann. of Math., 107 (1978), 403–453.
- [Ep] D. B. A. EPSTEIN, *Periodic flows on 3-manifolds*, Ann. of Math., 95 (1972), 68–82.
- [Ga1] D. GABAI, *Foliations and the topology of 3-manifolds*, J. Diff. Geo., 18 (1983), 445–503.
- [Ga2] D. GABAI, *Foliations and the topology of 3-manifolds II*, J. Diff. Geo., 26 (1987), 461–478.
- [Ga3] D. GABAI, *Foliations and the topology of 3-manifolds III*, J. Diff. Geo., 26 (1987), 479–536.
- [Go] C. GODBILLON, *Feuilletages, Études Géométrique, II*, Publ. Inst. de Recherche Math. Avancée, Univ. Louis Pasteur, Strasbourg, 1986.
- [H–H] G. HECTOR and U. HIRSCH, *Introduction to the Geometry of Foliations, Part B*, Vieweg and Sohn, Braunschweig, 1983.
- [K–N] S. KOBAYASHI and K. NOMIZU, *Foundations of Differential Geometry, Vol. I*, Interscience Publishers, New York, 1963.
- [Mi] K. MILLETT, *Generic properties of proper foliations*, (I.H.E.S. preprint).
- [Po] V. POÉNARU, *Rappels de géométrie hyperbolique en dimension 2 et généralités sur  $i: \mathcal{T} \times \mathcal{T} \rightarrow \mathbb{R}_+$* , Astérisque, 66–67 (1979), 33–35.
- [Th] W. THURSTON, *The geometry and topology of 3-manifolds*, Princeton Lecture Notes.

*St. Louis University*  
*St. Louis, MO 63103*

and

*Washington University*  
*St. Louis, MO 63130*

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