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Curves on surfaces of degree $2r-\delta$ in \mathbf{P}^r

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Introduction

In this paper we consider the problem of finding the values of d, g for which there exists a nonsingular irreducible and nondegenerate (i.e. not contained in a hyperplane) curve X of degree d and genus g in \mathbf{P}^r , the projective space over an algebraically closed \mathbf{k} of *arbitrary characteristic*.

This problem has been completely solved in \mathbf{P}^3 by Gruson–Peskin [GP] in the case $\text{char}(\mathbf{k})=0$, then extended to arbitrary characteristic by Hartshorne [Ha], and in \mathbf{P}^4 and \mathbf{P}^5 by Rathmann [Ra]. The approach of [GP], which has been generalized in [Ra], is divided into two parts. The first consists in constructing, on a quartic surface with a double line F , nonsingular curves of degree d and genus g for every (d, g) such that

$$0 \leq g \leq (d-1)^2/8.$$

A similar result has been proved by Mori [M] in complex projective 3-space for every d, g as above, and his construction has been extended in [Ra], proving the existence of smooth curves of degree d and genus g in \mathbf{P}^r lying on a K -3 surface when

$$0 \leq g \leq d^2/2(2r-2) - (r-1)/4.$$

The second part of the approach of [GP] is a detailed study of curves on a nonsingular cubic surface, which implies the existence result in the range

$$(d-1)^2/8 < g \leq d(d-3)/6.$$

We generalize the first construction of Gruson–Peskin and we prove the existence of nonsingular curves of degree d and genus g in \mathbf{P}^r for all $r \geq 6$ in a

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wide range of (d, g) (see statement below). Our curves are constructed on certain rational surfaces which are all embeddings of one and the same surface S' : this is the blow-up of \mathbf{P}^2 at nine points in general position. We exploit the rich geometry of S' very much in the same way as it is done in [GP] and [Ra], with the difference that, for technical reasons, we first work with the surface S obtained by blowing up nine points which are *not* in general position, but are base points of a generic pencil of cubics. Then we prove the main result using deformation theoretic arguments. The main consequence of our analysis of curves lying on the surface S is the following:

MAIN THEOREM. (i) *For every $r \geq 5$ there exists an embedding of S' as a nonsingular surface F^{2r-3} of degree $2r - 3$ in \mathbf{P}^r , and for every (d, g) such that*

$$0 \leq g \leq (d - r)^2 / 2(2r - 3)$$

there exists a nonsingular irreducible and nondegenerate curve X of degree d and genus g on F^{2r-3}

(ii) *For every $r \geq 7$ there exists an embedding of S' as a nonsingular surface F^{2r-4} of degree $2r - 4$ in \mathbf{P}^r , and for every (d, g) such that*

$$0 \leq g \leq (d - r)^2 / 2(2r - 4)$$

there exists a nonsingular irreducible and nondegenerate curve X of degree d and genus g on F^{2r-4} .

Clearly, the existence result for curves in \mathbf{P}^5 , contained in part (i) of the above theorem, follows from [Ra]. In more detail, the content of the paper is the following.

In section 1 we prove preliminary general results on the surface S which are repeatedly used in the paper. Precisely we give a criterion (proposition 1) for a linear system on S to be base point free and such that the associated map to projective space realizes S as a nonsingular surface. From this result we directly deduce an ampleness criterion which can also be found in [H].

In section 2 we introduce the notion of δ -system on S , which is a δ -tuple, $\delta \geq 3$, of elements of $\text{Pic}(S)$ satisfying certain conditions. This notion turns out to be a powerful tool in the study of curves lying on the surface S . The main result of this section (theorem 6) states that, if a δ -system exists on S , then S can be embedded in \mathbf{P}^r with degree $2r - \delta$ for all $r \geq 2\delta - 1$, in such a way that it contains smooth nondegenerate curves of degree d and genus g for all (d, g) such

that

$$(*) \quad 0 \leq g \leq (d-r)^2/2(2r-\delta).$$

Actually we prove a slightly better bound (see remark 1).

In section 3 we consider the problem of existence of δ -systems. It is easy to show that δ -systems do not exist for $\delta > 9$ (see remark 2). It is not difficult to find candidates for $3 \leq \delta \leq 9$, namely to find δ -tuples of classes of $\text{Pic}(S)$ which satisfy all but the last of the defining conditions. To prove that the last condition is also satisfied boils down to finding certain lists of elements of $\text{Pic}(S)$. These lists become increasingly long as δ grows, and this has forced us to consider the cases $\delta = 3, 4$ only, in which we are able to exhibit them. Via theorem 6, this proves a result which differs from the main theorem only in the fact that the surface S appears instead of S' in its statement. In remark 2 we also deduce the existence of smooth rational surfaces of degree $2r - \delta$ in \mathbf{P}^r , $r \geq \delta - 1$, for $5 \leq \delta \leq 9$.

In section 4 we show how to extend to S' most of the previous results concerning the surface S . Of course, this and the results quoted above imply the main theorem. We also discuss linear normality and the Brill-Noether map for the curves we have constructed.

Relying on the results of this paper, the first author has proved in [C] an asymptotic existence result for smooth nondegenerate curves in \mathbf{P}^r for *all* values of r , which essentially says that for $d \gg 0$ smooth curves of degree d and genus g exist when

$$0 \leq g \leq \varphi_r(d)$$

where $\varphi_r(d) \sim d^2/2(4r/3 - 1)$, improving a similar one of Rathmann [Ra].

After this work was completed we have become aware of a preprint of Parescu [P], where he claims the existence of smooth nondegenerate curves of degree d and genus g in \mathbf{P}^r for all $r \geq 5$ and all d, g such that

$$0 \leq g \leq (d-1)^2/2(2r-2).$$

His proof appears incomplete to us as it stands (on page 9, line -4, the maximum is not necessarily attained at an *integer*, as needed). From the argument of Parescu it seems to us that only a weaker bound, which is worse than ours for every d, g, r , can be deduced.

The second author would like to thank G. Pareschi for suggesting the proof of linear normality given in section 4, 2), and C. Procesi for a useful conversation on infinite reflection groups.

1. Preliminaries

As already stated in the introduction, we work over an algebraically closed field \mathbf{k} of arbitrary characteristic. We denote by S the surface obtained by blowing up nine points P_1, \dots, P_9 of \mathbf{P}^2 which are base points of a *generic* pencil of cubics; we let $\pi: S \rightarrow \mathbf{P}^2$ be the projection. Note that any cubic $C \subset \mathbf{P}^2$ containing P_1, \dots, P_9 is reduced, irreducible and with at most one node and no other singularity.

Let's denote by E_1, \dots, E_9 the exceptional curves (of the first kind) on S corresponding to P_1, \dots, P_9 , and by H the inverse image on S of a general line of \mathbf{P}^2 .

We identify an invertible sheaf on S with its class in $\text{Pic}(S)$. As a basis of $\text{Pic}(S)$ we take the classes $\mathbf{o}(H), \mathbf{o}(-E_1), \dots, \mathbf{o}(-E_9)$; we will sometimes denote an element of $\text{Pic}(S)$ by the 10-tuple of its coordinates with respect to this basis.

We have:

$$\omega_S = (-3, -1, \dots, -1) = \mathbf{o}(-C)$$

where C is the proper transform of a cubic through P_1, \dots, P_9 .

We will use without further mention the obvious fact that if D is an irreducible curve on S such that $(D, \omega_S) = 0$, then $D \in |-\omega_S|$.

We will freely use the notion of 1-connectedness of an effective divisor on a surface. We will also use without further notice the following vanishing theorem, referring the reader to [R] for the proof.

VANISHING THEOREM: *If D is an effective 1-connected divisor on a projective nonsingular surface F such that $h^1(F, \mathbf{o}_F) = 0$, then*

$$H^1(F, \mathbf{o}_F(-D)) = (0).$$

PROPOSITION 1. *Let D be an effective divisor on S .*

a) *If D is 1-connected and $(D, \omega_S) \leq -2$, then the linear system $|D|$ has no base points.*

b) *If D is 1-connected and $(D, \omega_S) \leq -3$, then $|D|$ has no base points, the morphism $\varphi_D: S \rightarrow \mathbf{P}(H^0(S, \mathbf{o}(D)))$ is an isomorphism of S onto its image, except possibly for the contraction of some exceptional curves, and the image $\varphi_D(S)$ is nonsingular. In particular, a general element of $|D|$ is irreducible and nonsingular.*

c) *If D is 1-connected and $(D, \omega_S) \leq -3$, then D is very ample if and only if $(D, E) > 0$ for every exceptional curve E .*

d) If $|\omega_S(D)|$ is not empty, contains an effective 1-connected divisor and $(D, \omega_S) \leq -3$, then $|D|$ is very ample.

Proof. a) For every $C \in |-\omega_S|$ we have an exact sequence

$$0 \rightarrow \omega_S(D) \rightarrow \mathcal{O}(D) \rightarrow \mathcal{O}_C(D) \rightarrow 0.$$

Since, by the connectedness of D , $h^1(S, \omega_S(D)) = h^1(S, \mathcal{O}(-D)) = 0$, we see that the restriction map $H^0(S, \mathcal{O}(D)) \rightarrow H^0(C, \mathcal{O}_C(D))$ is surjective. Therefore $|D|$ cuts a complete series on C of degree $(D, C) \geq 2$, hence without base points (recall that every $C \in |-\omega_S|$ is reduced and irreducible). It follows that $|D|$ has no base points on C . Since $\dim(|-\omega_S|) > 0$, the conclusion follows.

Proof of b) and c). By part a), $|D|$ has no base points. Let's denote by $|D - p|$ the linear system consisting of the curves of $|D|$ passing through p , for a given point p .

CLAIM 1. *Let p be any point of S ; if $|D - p|$ has a fixed part, then it consists of an exceptional curve E passing through p . Moreover $(D, E) = 0$, i.e. E is contracted to a point by the morphism φ_D .*

Proof of claim 1. The fixed divisor F of $|D - p|$ satisfies $(F, \omega_S) = -1$, because $|D|$ cuts a complete series on any $C \in |-\omega_S|$ and $|D - p|$ has codimension one in $|D|$. Therefore $F = E$ is reduced, irreducible and rational (because $|-\omega_S|$ cuts on E a series of dimension and degree one), hence it is an exceptional curve of the first kind. Moreover, since p is a fixed point of the linear series $|D - p|_C$ cut on C by $|D - p|$, and since $|D - p|_C$ has codimension at most one in $|D|_C$, necessarily E contains p . Since $|D|$ has no base points we have $(D, E) \geq 0$. If $(D, E) > 0$, then from the exact sequence

$$0 \rightarrow \mathcal{O}(D - E) \rightarrow \mathcal{O}(D) \rightarrow \mathcal{O}_E(D) \rightarrow 0$$

and from $h^0(S, \mathcal{O}(D - E)) = h^0(S, \mathcal{O}(D) \otimes \mathcal{I}_p) = h^0(S, \mathcal{O}(D)) - 1$ ($\mathcal{I}_p \subset \mathcal{O}_S$ the ideal sheaf of p) it follows that $|D|$ has base points on E ; this is a contradiction. This proves the claim.

As a consequence we have:

CLAIM 2. *If p and q are distinct points on S , $|D|$ does not separate p and q if and only if p and q are both contained in an exceptional curve E such that $(D, E) = 0$.*

Proof of claim 2. If p and q are not separated by $|D|$, then they cannot belong

to the same $C \in |-\omega_S|$ because $|D|_C$ is very ample on C . If the general curve of $|D-p|$ is reducible then, by claim 1, it contains an exceptional curve E and, by the same claim, this curve contains both p and q . If the general curve M of $|D-p|$ is irreducible, then it passes simply through p , because on the curve $C \in |-\omega_S|$ containing p , $|D-p|_C$ has codimension one in $|D|_C$, hence it cannot have $2p$ as a fixed divisor. Similarly M passes simply through q . If M has genus g , then the degree of $|D|_M$ is at least $2g+1$; it follows that $|D|_M$ is very ample, therefore p and q are separated by $|D|_M$, and this is a contradiction.

Next we prove the following

CLAIM 3. *If p does not belong to an exceptional curve E such that $(D, E) = 0$, then φ_D separates tangent vectors in p .*

Proof of claim 3. The general curve M of $|D-p|$ is irreducible and nonsingular in p . Since $2p$ is not a fixed divisor of $|D-p|_M$, because $|D-p|_M$ has codimension at most one in the complete and very ample $|D|_M$, the curves of $|D-p|$ are not all tangent to each other in p ; this proves the claim.

Finally we prove

CLAIM 4. *If E is an exceptional curve such that $(D, E) = 0$, then $\varphi_D(E)$ is a nonsingular point of the surface $\varphi_D(S)$. In particular $\varphi_D(S)$ is nonsingular.*

Proof of claim 4. On every curve $C \in |-\omega_S|$ the series $|D-E|_C$ coincides with the complete series $|D|_C - (E, C)$. Since we have $(D-E, \omega_S) = (D, \omega_S) + 1 \leq -2$. It follows that $|D-E|$ has no base points on C , hence $|D-E|$ has no base points. The surface $\varphi_{D-E}(S)$ can be regarded as the projection of $\varphi_D(S)$ from the point $q = \varphi_D(E)$. Letting $\mu = \text{mult}_q(\varphi_D(S))$, we have

$$\begin{aligned} \deg(\varphi_D(S)) - \mu &= \deg[\varphi_{D-E}(S)] \deg(\varphi_{D-E}) \\ &= (D-E, D-E) = (D, D) - 1 = \deg(\varphi_D(S)) - 1, \end{aligned}$$

it follows that $\mu = 1$, hence $\varphi_D(E)$ is nonsingular.

Assertions b) and c) of the proposition are clearly a consequence of claims 1), ..., 4).

d) Since $|D| = |\omega_S(D) + C|$, $C \in |-\omega_S|$, it follows that $|D|$ contains a 1-connected divisor D' . Since, by a), $|\omega_S(D)|$ has no base points, for every exceptional curve E we have $(\omega_S(D), E) \geq 0$, and therefore

$$(D', E) = (D, E) \geq -(\omega_S, E) = 1 > 0.$$

From c) it follows that $|D| = |D'|$ is very ample.

PROPOSITION 2. *Let D be an effective 1-connected divisor on S . Then every $D' \in |D|$ is 1-connected.*

Proof. Since D is effective we have $(D, \omega_S) \leq 0$. If $(D, \omega_S) = 0$ then $|D| = |-\omega_S|$ and the conclusion is clear. If $(D, \omega_S) = -1$ then $D = E + C_1 + \dots + C_h$, with E exceptional curve and $C_1, \dots, C_h \in |-\omega_S|$, and E is a fixed component of $|D|$. Then D' has a similar decomposition, hence it is 1-connected.

Suppose now that $(D, \omega_S) = -2$ and that $D' = A_1 + A_2$, with A_1, A_2 effective. Since $(A_i, \omega_S) \leq 0$ we have $-2 \leq (A_i, \omega_S) \leq 0$, $i = 1, 2$. If $(A_1, \omega_S) = 0$, $(A_2, \omega_S) = -2$ then $A_1 \in |-\omega_S|$ for some $h \geq 1$, hence $(A_1, A_2) = 2h > 0$. Similar conclusion we have if $(A_1, \omega_S) = -2$, $(A_2, \omega_S) = 0$. If $(A_1, \omega_S) = -1 = (A_2, \omega_S)$ then $A_1 = E_1 + C_1 + \dots + C_h$, $A_2 = E_2 + C'_1 + \dots + C'_k$, with E_1, E_2 exceptional curves and $C_1, \dots, C'_k \in |-\omega_S|$. We now have $(A_1, A_2) > 0$ except in the case $A_1 = E + C$, $A_2 = E$. But then $(D, D) = (D', D') = 0$, hence $|D|$, being base point free by proposition 1, part a), is composed with a pencil; moreover, since D is 1-connected, $|D|$ is a pencil. But then, since $|D| = |2E + C|$ and C moves in a pencil, we see that $2E$ is a fixed divisor of $|D|$, and this contradicts proposition 1, part a).

Finally assume that $(D, \omega_S) \leq -3$. Since $(C, \omega_S(-D)) \leq -3 < 0$, we have

$$h^2(S, \mathcal{O}(D)) = h^0(S, \omega_S(-D)) = 0.$$

From the Riemann–Roch theorem we therefore obtain

$$\dim(|D|) \geq ((D, D) - (D, \omega_S))/2;$$

since, by proposition 1, part a), $|D|$ is base point free, we have $(D, D) \geq 0$ hence $\dim(|D|) \geq 2$. therefore $|D|$ is not a pencil; moreover $|D|$ cannot be composed with a pencil because D is 1-connected. It follows that $(D, D) > 0$. Now the conclusion follows from lemma 2 of [R].

We will say that an effective class $\mathcal{O}(D) \in \text{Pic}(S)$, respectively a linear system $|D|$, is 1-connected if $|D|$ contains a 1-connected divisor, or equivalently, if every divisor of $|D|$ is 1-connected. The equivalence of the two conditions follows from proposition 2.

2. δ -systems on S

We will denote by \mathbf{G} the group of Cremona isometries of S , namely the group of automorphisms of $\text{Pic}(S)$ which

- 1) leave the semigroup of effective classes invariant,
- 2) preserve the intersection form $(,)$,

3) leave the canonical class ω_S invariant.

We will need the following elementary result.

LEMMA 3. *If D is an effective 1-connected divisor and $\sigma \in \mathbf{G}$, then $\sigma(\mathbf{o}(D)) = \mathbf{o}(\bar{D})$, where \bar{D} is an effective 1-connected divisor.*

Proof. Because of the defining property 1) of \mathbf{G} , an effective divisor \bar{D} such that $\sigma(\mathbf{o}(D)) = \mathbf{o}(\bar{D})$ exists. Suppose that $\bar{D} = \bar{A}_1 + \bar{A}_2$, with \bar{A}_1 and \bar{A}_2 effective. Then, again by 1), $\sigma^{-1}(\bar{A}_1) = \mathbf{o}(A_i)$, with A_i effective, $i = 1, 2$. Since $\mathbf{o}(D) = \mathbf{o}(A_1 + A_2)$, and D is 1-connected, from proposition 2 it follows that $(A_1, A_2) \geq 1$. Hence $(\bar{A}_1, \bar{A}_2) = (A_1, A_2) \geq 1$ and therefore \bar{D} is 1-connected.

With an abuse of notation we will often write $\sigma(D)$ instead of $\sigma(\mathbf{o}(D))$, for a divisor D on S and $\sigma \in \mathbf{G}$. We will denote by $g(D)$ the arithmetic genus of D , namely

$$g(D) := (D, D + \omega_S)/2 + 1.$$

We give the following definition.

DEFINITION. Let $\delta \geq 2$ be an integer. An ordered δ -tuple $\{D_0, D_1, \dots, D_{\delta-1}\}$ of effective classes in $\text{Pic}(S)$ is called a δ -system if the following conditions are satisfied:

- I) $(D_i, \omega_S) = -\delta$, $i = 0, \dots, \delta - 1$.
- II) $(D_i, D_i) = -\delta + 2i$, $i = 0, \dots, \delta - 1$.
- III) $D_0 - \omega_S, \dots, D_{\delta-2} - \omega_S, D_{\delta-1}$ are effective and 1-connected.
- IV) For every $0 \leq i, j \leq \delta - 1$, the number $(D_i, \sigma(D_j))$ assumes all integral values N such that

$$i + j + 2 - \delta \leq N,$$

when σ varies in \mathbf{G} .

Note that property IV) is clearly equivalent to the following:

- V) for every $0 \leq i \leq j \leq \delta - 1$, all integral values $N \geq i + j + 2 - \delta$ are assumed by $(\sigma(D_i), \rho(D_j))$ as σ, ρ vary in \mathbf{G} .

In this section we will investigate some of the consequences of the existence of a δ -system of divisors on S for some δ . In the next section we will construct such systems for $\delta = 3, 4$.

PROPOSITION 4. *Let $\{D_0, D_1, \dots, D_{\delta-1}\}$ be a δ -system on S , with $\delta \geq 3$.*

Then

i) for every $\alpha \geq 1$ the linear systems

$$|D_0 - \alpha\omega_S|, \dots, |D_{\delta-2} - \alpha\omega_S|, |D_{\delta-1} - (\alpha-1)\omega_S|$$

are base point free and with general member irreducible and nonsingular; moreover for every $\alpha \geq 2$ they are very ample.

ii) We have:

$$(D_1 - \alpha\omega_S, D_1 - \alpha\omega_S) = (2\alpha - 1)\delta + 2i,$$

$$g(D_1 - \alpha\omega_S) = (\alpha - 1)\delta + i + 1,$$

$$\dim(|D_i - \alpha\omega_S|) = \alpha\delta + i (=g(D_i - \alpha\omega_S) + \delta - 1)$$

for $0 \leq i \leq \delta - 2$ and for every $\alpha \geq 1$, and for $i = \delta - 1$ and for every $\alpha \geq 0$.

Proof. From the defining property III) it follows that each of the linear systems $|D_0 - \alpha\omega_S|, \dots, |D_{\delta-2} - \alpha\omega_S|, |D_{\delta-1} - (\alpha-1)\omega_S|$ contains a 1-connected effective divisor for all $\alpha \geq 1$ and has intersection number equal to $-\delta \leq -3$ with ω_S . Applying parts b) and d) of proposition 1 we obtain i). The expressions for $(D_i - \alpha\omega_S, D_i - \alpha\omega_S)$ and $g(D_i - \alpha\omega_S)$ are obvious. The $\dim(|D_i - \alpha\omega_S|)$ is computed using the Riemann–Roch theorem on S , noting that

$$h^1(S, \mathcal{O}(D_1 - \alpha\omega_S)) = 0 = h^2(S, \mathcal{O}(D_i - \alpha\omega_S)).$$

The first equality follows because $|D_i - \alpha\omega_S - \omega_S|$ contains a 1-connected divisor when α and i assume the indicated values. The second equality is obvious because

$$h^2(S, \mathcal{O}(D_1 - \alpha\omega_S)) = h^0(S, \mathcal{O}(-(D_1 - (\alpha+1)\omega_S)))$$

and $|D_1 - (\alpha+1)\omega_S| \neq \emptyset$. This proves ii).

PROPOSITION 5. Assume that a δ -system $\{D_0, D_1, \dots, D_{\delta-1}\}$ exists on S for some $\delta \geq 3$. For each $r \geq 2\delta - 1$ write $r = n\delta + i$, $i \in \{0, \dots, \delta - 1\}$, and let

$$H_r := D_i - n\omega_S.$$

Then H_r is very ample and for every (d, g) such that $0 \leq g \leq d - r - 1$, there exists

an irreducible and nonsingular curve $X \subset S$ such that

$$(H_r, X) = d, \quad g(X) = g$$

and

$$h^0(S, \mathcal{O}(H_r - X)) = 0.$$

Proof. Note that, since $r \geq 2\delta - 1$, we have $n \geq 2$ except in the case $r = 2\delta - 1$ when $n = 1$, $i = \delta - 1$, hence H_r is very ample by proposition 4. We can write in a unique way

$$g = (\alpha - 1)\delta + j + 1$$

for some $j \in \{0, \dots, \delta - 1\}$ and $\alpha \geq 0$ ($\alpha \geq 1$ if $0 \leq j \leq \delta - 2$ and $\alpha = 0$ if and only if $j = \delta - 1$). For every $\sigma \in \mathbf{G}$ we have

$$\begin{aligned} (H_r, \sigma(D_j - \alpha\omega_S)) &= (D_i - n\omega_S, \sigma(D_j) - \alpha\omega_S) \\ &= (D_i, \sigma(D_j)) + (\alpha + n)\delta \\ &= (D_i, \sigma(D_j)) + g + r + \delta - 1 - i - j. \end{aligned}$$

Since $d \geq g + r + 1$, we have

$$d - [g + r + \delta - 1 - i - j] \geq i + j + 2 - \delta,$$

hence, by the defining property IV), there exists $\sigma \in \mathbf{G}$ such that

$$(D_i, \sigma(D_j)) = d - [g + r + \delta - 1 - i - j],$$

equivalently such that

$$(H_r, \sigma(D_j - \alpha\omega_S)) = d.$$

Since by lemma 3 $|\sigma(D_j - \alpha\omega_S)|$ contains an effective 1-connected divisor, from proposition 1b) it follows that $|\sigma(D_j - \alpha\omega_S)|$ contains an irreducible and nonsingular curve X . From proposition 4 we deduce that this curve has genus

$$g(D_j - \alpha\omega_S) = (\alpha - 1)\delta + j + 1 = g.$$

By contradiction, let's assume that

$$(*) \quad h^0(S, \mathcal{O}(H_r - X)) \neq 0.$$

From (*) and $(H_r - X, \omega_S) = 0$ we deduce that $|H_r - X| = |-a\omega_S|$, $a \geq 0$. Therefore

$$|\sigma(D_j) - a\omega_S| = |X| = |H_r + a\omega_S| = |D_i + (a - n)\omega_S|.$$

and it follows that $i = j$ and $\sigma(D_j) = D_i$. Then:

$$-\delta + 2i = (D_i, D_i) = (D_i, \sigma(D_j)) \geq 2i + 2 - \delta,$$

a contradiction. This concludes the proof.

Using proposition 5 we can prove the following consequence of the existence of a δ -system on S for some $\delta \geq 3$.

THEOREM 6. *Assume that a δ -system $\{D_0, D_1, \dots, D_{\delta-1}\}$ exists on S for some $\delta \geq 3$ and let $r \geq 2\delta - 1$ be an integer. Then there exists an embedding of S as a nonsingular surface F of degree $2r - \delta$ in \mathbf{P}^r , and for every (d, g) such that*

$$0 \leq g \leq (d - r)^2 / 2(2r - \delta) \quad (1)$$

there exists a nonsingular irreducible and nondegenerate curve X of degree d and genus g on F .

Proof. Writing $r = n\delta + i$, the embedding of S is given by the very ample class $H_r = D_i - n\omega_S$ considered in proposition 5. If $0 \leq g \leq d - r - 1$ the assertion has already been proved (proposition 5). Note that if a nonsingular irreducible and nondegenerate curve Z lies on F then a general element $Z' \in |Z + H_F|$ (H_F a hyperplane section of F) is a nonsingular irreducible curve, by proposition 1b), and is obviously nondegenerate. If Z has $(\text{degree}, \text{genus}) = (d, g)$, then Z' has $(\text{degree}, \text{genus}) = (d', g')$ given by

$$(d', g') = (d + 2r - \delta, g + d + r - \delta).$$

The inverse of the transformation

$$d' = d + 2r + \delta \quad (2)$$

$$g' = g + d + r - \delta$$

is

$$d = d' - 2r + \delta \quad (3)$$

$$g = g' - d' + r.$$

Applying (2) s times we obtain the transformation

$$\begin{aligned} d^{(s)} &= d + s(2r - \delta) \\ g^{(s)} &= g + sd + s(s-1)(2r - \delta)/2 + s(r - \delta) \end{aligned} \quad (2^s)$$

whose inverse is:

$$\begin{aligned} d &= d^{(s)} - s(2r - \delta) \\ g &= g^{(s)} - sd^{(s)} + s(s+1)(2r - \delta)/2 - s(r - \delta). \end{aligned} \quad (3^s)$$

In the plane with coordinates d, g represent with integral points the couples (d, g) for which we want to prove the theorem. They fill the region R under the parabola K with equation

$$g = (d - r)^2 / 2(2r - \delta).$$

We know that the theorem is true for all the points strictly below the line L_0 with equation $g = d - r$ and above the d -axis, by proposition 5. Let's denote by V_0 the set of these points. The transformation (2) maps the d -axis into L_0 and maps L_0 into the line $L_1: g = 2d - 4r + \delta$. Therefore (2) maps V_0 in the set of points (d', g') such that

$$d' - r \leq g' < 2d' - 4r + \delta.$$

Since the theorem is also true in V_0 , we see that the transformation (2) and the remark made at the beginning allow us to extend the validity of the theorem to $V_0 \cup V_1$, where V_1 is defined by:

$$0 \leq g \leq 2d - 4r + \delta.$$

Note that the two lines L_0 and L_1 have in common the point $P_1 = (3r - \delta, 2r - \delta)$. For every $s \geq 1$ let's denote by L_s the line whose equation is

$$g = (s+1)d - s(s+3)(2r - \delta)/2 + s(r - \delta) - r,$$

and which is the image of L_0 under (2^s) . The line $g = 0$ will be denoted L_{-1} ; it is transformed by (2^s) into L_{s-1} .

By induction we deduce that for every $k \geq 0$ the theorem is true in

$V_0 \cup \dots \cup V_k$ where V_s is the region defined by the inequalities:

$$0 \leq g < (s+1)d - s(s+3)(2r-\delta)/2 + s(r-\delta) - r.$$

Note that $L_s \cap L_{s-1}$ is the point

$$P_s = ((2s+1)r - s\delta, s(s+1)(2r-\delta)/2).$$

In particular $P_0 = (r, 0)$. Note that the expression $g - (d-r)^2/2(2r-\delta)$ is zero in P_0 . If we prove that it is positive in all the points $P_s, s \geq 1$, then these points are above the parabola K . Since K is concave upwards, it follows that the segments $P_{s-1}P_s$ lie above K and the theorem is true. In P_s we have:

$$\begin{aligned} (d-r)^2/2(2r-\delta) &= (2sr - s\delta)^2/2(2r-\delta) \\ &= s^2(2r-\delta)^2/2(2r-\delta) = s^2(2r-\delta)/2. \end{aligned}$$

Therefore in P_s :

$$g - (d-r)^2/2(2r-\delta) = s(s+1)(2r-\delta)/2 - s^2(2r-\delta)/2 = s(2r-\delta)/2,$$

which is positive for all $s \geq 1$. This concludes the proof.

Remark 1. Note that the proof of theorem 6 actually shows the existence of curves X of degree d and genus g on S for all (d, g) located below the polygon whose vertices are the points P_s considered in the proof. This region is slightly larger than that defined by (1).

3. Existence of δ -systems for $\delta = 3, 4$.

In this section we discuss the existence of δ -systems on S . We will show that δ -systems exist for $\delta = 3, 4$.

The basis $\mathbf{o}(H), \mathbf{o}(-E_1), \dots, \mathbf{o}(-E_9)$ identifies $\text{Pic}(S)$ with \mathbf{Z}^{10} and the intersection form $(,)$ with the inner product $x_0^2 - \sum_1^9 x_i^2$. Consider the elements of $\text{Pic}(S)$:

$$\begin{aligned} r_1 &= (1, 1, 1, 1, 0, \dots, 0), \quad r_2 = (0, -1, 1, 0, \dots, 0), \\ r_3 &= (0, 0, -1, 1, 0, \dots, 0), \dots \\ r_8 &= (0, \dots, 0, -1, 1, 0), \quad r_9 = (0, \dots, 0, -1, 1). \end{aligned}$$

Letting:

$$f_i(x) = x + (x, r_i)r_i, \quad i = 1, \dots, 9,$$

we obtain elements $f_1, \dots, f_9 \in \mathbf{G}$. Recall that f_1, f_2, \dots, f_9 act on an element $(x_0, x_1, \dots, x_9) \in \text{Pic}(S)$ in the following way:

$$\begin{aligned} f_1(x_0, x_1, \dots, x_9) &= (x_0 + h, x_1 + h, x_2 + h, x_3 + h, x_4, \dots, x_9), \\ h &= x_0 - x_1 - x_2 - x_3. \end{aligned}$$

$$f_2(x_0, x_1, \dots, x_9) = (x_0, x_2, x_1, x_3, \dots, x_9),$$

$$\begin{aligned} f_3(x_0, x_1, \dots, x_9) &= (x_0, x_1, x_3, x_2, x_4, \dots, x_9), \\ &\dots \end{aligned}$$

$$f_9(x_0, x_1, \dots, x_9) = (x_0, x_1, \dots, x_7, x_9, x_8).$$

In particular note that combining f_2, \dots, f_9 we can obtain any permutation of x_1, \dots, x_9 . We will also consider, for every $z \in \omega_S^\perp$, the element $\tau_z \in \mathbf{G}$ defined as follows:

$$\tau_z(x) = x - (x, z)\omega_S + (x, \omega_S)z - (z, z)(x, \omega_S)\omega_S/2.$$

The following lemma generalizes lemma 1.4.1 of [Ra].

LEMMA 7. *Let $x, y \in \text{Pic}(S)$ be such that $(x, \omega_S) = (y, \omega_S) = -\delta$, for some $\delta \geq 1$, and such that $x - y = (u_0, u_1, \dots, u_g)$ satisfies either one of the following two conditions:*

$$\text{i) } u_1 = u_2, \quad u_3 = u_4, \quad u_5 = u_6, \quad u_7 = u_8.$$

$$\text{ii) } u_1 = u_2, \quad u_3 = u_4, \quad u_5 = u_6, \quad u_7 - u_8 = \delta.$$

Then $(\rho(y), \sigma(x))$ assumes all integer values $(y, x) + n\delta^2$, $n \geq 0$, as ρ, σ vary in \mathbf{G} .

Proof. It suffices to show that the conclusion holds taking $\rho = \text{identity}$ and $\sigma = \tau_z$, $z \in \omega_S^\perp$. For every $z \in \omega_S^\perp$ we have:

$$\begin{aligned} (y, \tau_z(x)) &= (y, x - (x, z)\omega_S + (x, \omega_S)z - (z, z)(x, \omega_S)\omega_S/2) \\ &= (y, x) - (x, z)(y, \omega_S) + (x, \omega_S)(y, z) - (z, z)(x, \omega_S)(y, \omega_S)/2 \\ &= (y, x) + \delta(x - y, z) - (z, z)\delta^2/2. \end{aligned}$$

Suppose that we are in case i). Then, taking

$$z = (0, a, -a, b, -b, c, -c, d, -d, 0)$$

we have $(x - y, z) = 0$ and $-(z, z)/2 = a^2 + b^2 + c^2 + d^2$ and the conclusion follows from the fact that every positive integer is the sum of four squares. Suppose now that we are in case ii). Taking

$$z = (0, a, -a, b, -b, c, -c, 0, 0, 0)$$

we obtain $(x - y, z) = 0$ and $-(z, z)/2 = a^2 + b^2 + c^2$. This takes care of the cases in which $n \equiv 1, 2, 3, 5, 6 \pmod{8}$, because every such integer n is the sum of three squares (see [5]). If $n \equiv 0, 4, 7 \pmod{8}$ then $n - 2 > 0$; we can write

$$n - 2 = a^2 + b^2 + c^2$$

and we take

$$z = (0, a, -a, b, -b, c, -c, 1, -1, 0).$$

We obtain $(x - y, z) = \delta$, and therefore:

$$(y, \tau_z(x)) = (y, x) + \delta^2 - (z, z)\delta^2/2 = (y, x) + \delta^2 + \delta^2(n - 1).$$

This concludes the proof.

The existence of a 3-system is our next result. One half of the computations of this theorem are already in [Ra], where the divisors D_0 , D_1 , D_2 of theorem 8 are also considered.

THEOREM 8. *The classes*

$$D_0 = (0, 0, 0, 0, 0, 0, 0, -1, -1, -1)$$

$$D_1 = (1, 1, -1, 0, 0, 0, 0, 0, 0, 0)$$

$$D_2 = (1, 0, 0, 0, 0, 0, 0, 0, 0, 0)$$

are a 3-system on S .

Proof. The defining properties I), II) and III) are obviously satisfied. Let's prove property V). For every $0 \leq i \leq j \leq 2$ and for every

$$i + j - 1 \leq N \leq i + j + 7$$

we will find elements of the form $\sigma(D_i)$, $\rho(D_j)$, $\sigma, \rho \in \mathbf{G}$, such that:

- 1) $(\sigma(D_i), \rho(D_j)) = N$,
- 2) $\sigma(D_i) - \rho(D_j)$ satisfies either one of conditions i), ii) of lemma 7.

Then property V), and the conclusion, will follow from lemma 7 applied to $x = \sigma(D_i)$ and $y = \rho(D_j)$. We give a list of such elements below. Each $\sigma(D_i)$ and $\rho(D_j)$ is obtained from D_i and D_j respectively by acting with a combination of the elements f_1, \dots, f_g of \mathbf{G} . The tables are the following:

$\sigma(D_0)$	$\rho(D_0)$	$(\sigma(D_0), \rho(D_0))$
$(0, -1, -1, 0, 0, 0, 0, 0, 0, -1)$	$(0, 0, 0, -1, -1, 0, 0, 0, 0, -1)$	-1
$(0, -1, -1, 0, 0, 0, 0, 0, 0, -1)$	$(1, 0, 0, 1, 1, -1, -1, 0, 0, 0)$	0
$(0, 0, 0, 0, 0, 0, 0, 0, -1, -1, -1)$	$(2, 1, 1, 0, 0, 0, 0, 0, -1, 2, 0)$	1
$(0, -1, -1, 0, 0, 0, 0, 0, 0, -1)$	$(1, 1, 1, 0, 0, -1, -1, 0, 0, 0)$	2
$(0, -1, -1, 0, 0, 0, 0, 0, 0, -1)$	$(2, 1, 1, -1, -1, 1, 1, 0, 0, 1)$	3
$(0, 0, 0, 0, 0, 0, 0, 0, -1, -1, -1)$	$(3, 1, 1, -1, -1, 1, 1, 1, 1, 2)$	4
$(2, 1, 1, -1, -1, 1, 1, 0, 0, 1)$	$(2, 1, 1, 1, 1, -1, -1, 0, 0, 1)$	5
$(2, 1, 1, 1, 1, -1, -1, 0, 0, 1)$	$(3, 1, 1, -1, -1, 1, 1, 1, 1, 2)$	6
$(1, -1, -1, 1, 1, 0, 0, 0, 0, 0)$	$(3, 1, 1, -1, -1, 1, 1, 1, 1, 2)$	7

$\sigma(D_0)$	$\rho(D_1)$	$(\sigma(D_0), \rho(D_1))$
$(0, 0, -1, -1, -1, 0, 0, 0, 0, 0)$	$(1, 1, 0, 0, 0, 0, 0, 0, 0, -1)$	0
$(0, -1, -1, 0, 0, 0, 0, 0, 0, -1)$	$(2, 1, 1, 1, 1, 0, 0, 0, 0, -1)$	1
$(2, 1, 1, 1, 0, 1, 1, -1, -1, 0)$	$(1, 0, 0, 0, -1, 0, 0, 0, 0, 1)$	2
$(2, 1, 1, 0, 0, 1, 1, -1, -1, 1)$	$(2, 1, 1, 1, 1, 0, 0, 0, 0, -1)$	3
$(1, -1, -1, 0, 0, 0, 0, 1, 1, 0)$	$(3, 1, 1, 1, 1, 0, 0, -1, 2, 1)$	4
$(1, -1, -1, 0, 0, 1, 1, 0, 0, 0)$	$(3, 1, 1, 1, 1, 0, 0, -1, 2, 1)$	5
$(3, 1, 1, -1, -1, 1, 1, 1, 1, 2)$	$(3, 1, 1, 1, 1, 0, 0, -1, 2, 1)$	6
$(2, 1, 1, 0, 0, 1, 1, -1, -1, 1)$	$(2, 0, 0, 1, 1, 0, 0, 1, 1, -1)$	7
$(3, 1, 1, -1, -1, 1, 1, 1, 1, 2)$	$(2, 1, 1, 1, 1, 0, 0, 0, 0, -1)$	8

$\sigma(D_0)$	$\rho(D_2)$	$(\sigma(D_0), \rho(D_2))$
$(0, -1, -1, -1, 0, 0, 0, 0, 0, 0)$	$(4, 0, 0, 1, 2, 2, 2, 1, 1, 0)$	1
$(0, -1, -1, -1, 0, 0, 0, 0, 0, 0)$	$(2, 1, 1, 0, 1, 0, 0, 0, 0, 0)$	2
$(0, -1, -1, -1, 0, 0, 0, 0, 0, 0)$	$(4, 1, 1, 1, 2, 2, 2, 0, 0, 0)$	3
$(0, -1, -1, -1, 0, 0, 0, 0, 0, 0)$	$(6, 1, 1, 2, 3, 3, 3, 1, 1, 0)$	4
$(0, -1, -1, -1, 0, 0, 0, 0, 0, 0)$	$(4, 2, 2, 1, 2, 1, 1, 0, 0, 0)$	5
$(0, -1, -1, -1, 0, 0, 0, 0, 0, 0)$	$(6, 3, 3, 0, 1, 2, 2, 2, 2, 0)$	6

$(0, -1, -1, -1, 0, 0, 0, 0, 0, 0)$	$(8, 3, 3, 1, 2, 4, 4, 2, 2, 0)$	7
$(0, -1, -1, -1, 0, 0, 0, 0, 0, 0)$	$(6, 3, 3, 2, 3, 1, 1, 1, 1, 0)$	8
$(0, -1, -1, -1, 0, 0, 0, 0, 0, 0)$	$(8, 4, 4, 1, 2, 3, 3, 2, 2, 0)$	9
$\sigma(D_1)$	$\rho(D_1)$	$(\sigma(D_1), \rho(D_1))$
$(1, 1, 0, -1, 0, 0, 0, 0, 0, 0)$	$(1, 0, -1, 0, 1, 0, 0, 0, 0, 0)$	1
$(1, 1, 0, 0, 0, 0, 0, 0, 0, -1)$	$(1, 0, -1, 0, 0, 0, 0, 0, 0, 1)$	2
$(2, 1, 1, 1, 1, 0, 0, 0, 0, -1)$	$(2, 0, 0, 0, 0, 1, 1, 1, 1, -1)$	3
$(2, 1, 1, 0, 0, 0, 0, 1, 1, -1)$	$(3, 1, 1, 0, 0, 1, 1, -1, 2, 1)$	4
$(2, 1, 1, 1, 1, 0, 0, 0, 0, -1)$	$(3, 1, 1, 0, 0, 1, 1, -1, 2, 1)$	5
$(2, 1, 1, 1, 1, 0, 0, 0, 0, -1)$	$(3, 0, 0, 0, 0, 1, 1, 2, 2, 0)$	6
$(3, 0, 0, 2, 2, 1, 1, 0, 0, 0)$	$(3, 0, 0, 0, 0, 1, 1, 2, 2, 0)$	7
$(3, 0, 0, 2, 2, 0, 0, 1, 1, 0)$	$(3, 1, 1, 0, 0, 1, 1, -1, 2, 1)$	8
$(3, 1, 1, 2, 2, 0, 0, 0, 0, 0)$	$(3, 0, 0, 0, 0, 1, 1, 2, 2, 0)$	9
$\sigma(D_1)$	$\rho(D_2)$	$(\sigma(D_1), \rho(D_2))$
$(1, 0, 0, -1, 0, 0, 0, 0, 0, 1)$	$(2, 0, 0, 1, 0, 1, 1, 0, 0, 0)$	2
$(1, 1, 0, -1, 0, 0, 0, 0, 0, 0)$	$(3, 1, 0, 1, 2, 1, 1, 0, 0, 0)$	3
$(3, 1, 1, 2, 2, 0, 0, 0, 0, 0)$	$(2, 1, 1, 0, 0, 0, 0, 0, 0, 1)$	4
$(2, 0, 0, 1, 1, 1, 1, 0, 0, -1)$	$(2, 1, 1, 0, 0, 0, 0, 0, 0, 1)$	5
$(3, 1, 1, 2, 2, 0, 0, 0, 0, 0)$	$(2, 0, 0, 0, 0, 1, 1, 0, 0, 1)$	6
$(3, 1, 1, 2, 2, 0, 0, 0, 0, 0)$	$(3, 1, 1, 0, 0, 1, 1, 0, 0, 2)$	7
$(2, 0, 0, 1, 1, 1, 1, 0, 0, -1)$	$(3, 1, 1, 0, 0, 0, 0, 1, 1, 2)$	8
$(3, 1, 1, 2, 2, 0, 0, 0, 0, 0)$	$(3, 0, 0, 0, 0, 1, 1, 1, 1, 2)$	9
$(3, 1, 1, 2, 2, 0, 0, 0, 0, 0)$	$(4, 1, 0, 0, 1, 1, 1, 1, 1, 3)$	10
$\sigma(D_2)$	$\rho(D_2)$	$(\sigma(D_2), \rho(D_2))$
$(1, 0, 0, 0, 0, 0, 0, 0, 0, 0)$	$(3, 1, 1, 1, 1, 0, 0, 0, 0, 2)$	3
$(1, 0, 0, 0, 0, 0, 0, 0, 0, 0)$	$(4, 1, 1, 1, 1, 1, 1, 0, 0, 3)$	4
$(1, 0, 0, 0, 0, 0, 0, 0, 0, 0)$	$(5, 1, 1, 1, 1, 1, 1, 1, 1, 4)$	5
$(2, 1, 1, 0, 0, 0, 0, 0, 0, 1)$	$(4, 1, 1, 1, 1, 1, 1, 0, 3, 0)$	6
$(2, 1, 1, 0, 0, 0, 0, 0, 0, 1)$	$(5, 1, 1, 1, 1, 1, 1, 1, 4, 1)$	7
$(3, 1, 1, 1, 1, 0, 0, 0, 0, 2)$	$(4, 1, 1, 1, 1, 1, 1, 0, 3, 0)$	8
$(3, 1, 1, 1, 1, 0, 0, 0, 0, 2)$	$(5, 1, 1, 1, 1, 1, 1, 1, 4, 1)$	9
$(4, 1, 1, 1, 1, 1, 1, 0, 0, 3)$	$(4, 1, 1, 1, 1, 1, 1, 0, 3, 0)$	10
$(4, 1, 1, 1, 1, 1, 1, 0, 0, 3)$	$(5, 1, 1, 1, 1, 1, 1, 1, 4, 1)$	11

This concludes the proof of theorem 7.

Now we will prove the existence of a 4-system on S .

THEOREM 9. *The classes*

$$D_0 = (0, -1, -1, -1, -1, 0, 0, 0, 0, 0)$$

$$D_1 = (1, -1, -1, 0, 0, 0, 0, 0, 0, 1)$$

$$D_2 = (1, 0, 0, 0, 0, 0, 0, 0, 0, -1)$$

$$D_3 = (2, 1, 1, 0, 0, 0, 0, 0, 0, 0)$$

are a 4-system on S .

Proof. Also in this case it is obvious that properties I), II) and III) are satisfied. We will proceed as in the proof of theorem 8: by applying lemma 7, for every $0 \leq i \leq j \leq 3$ and for every

$$i + j - 2 \leq N \leq i + j + 13$$

it will suffice to find elements of the form $\sigma(D_i), \rho(D_j), \sigma, \rho \in \mathbf{G}$, such that:

$$1) (\sigma(D_i), \rho(D_j)) = N,$$

2) $\sigma(D_i) - \rho(D_j)$ satisfies either one of conditions i), ii) of lemma 7. A list of such elements is given below:

$\sigma(D_0)$	$\rho(D_0)$	$(\sigma(D_0), \rho(D_0))$
$(0, -1, -1, -1, -1, 0, 0, 0, 0, 0)$	$(0, 0, 0, 1, -1, -1, -1, 0, 0, 0)$	-2
$(0, 0, 0, -1, -1, 0, -1, 0, -1, 0)$	$(2, 0, 0, -1, -1, 1, 0, 2, 1, 0)$	-1
$(0, -1, 1, -1, -1, 0, 0, 0, 0, 0)$	$(0, 0, 0, 0, 0, -1, -1, -1, -1, 0)$	0
$(0, 0, 0, -1, -1, 0, 0, 0, -1, -1)$	$(3, 0, 0, -1, -1, 1, 1, 2, 1, 2)$	1
$(1, -1, -1, 0, 0, 1, 1, 0, 0, -1)$	$(1, 1, 1, -1, -1, 0, 0, 0, 0, -1)$	2
$(0, -1, -1, 0, 0, 0, 0, 0, -1, -1)$	$(3, 0, 0, -1, -1, 1, 1, 2, 1, 2)$	3
$(1, -1, -1, 1, 1, 0, 1, 0, 0, -1)$	$(1, 1, 1, -1, -1, 0, 0, 0, 0, -1)$	4
$(0, 0, 0, 0, 0, -1, -1, 0, -1, -1)$	$(3, 0, 0, -1, -1, 1, 1, 2, 1, 2)$	5
$(1, 1, 1, -1, -1, 0, 0, 0, 0, -1)$	$(2, -1, -1, 0, 0, 1, 1, 0, 0, 2)$	6
$(2, 1, 0, 0, 0, -1, -1, 2, 1, 0)$	$(2, 0, -1, 0, 0, 1, 1, 0, -1, 2)$	7
$(1, 1, 1, -1, -1, 0, 0, 0, 0, -1)$	$(2, -1, -1, 1, 1, 0, 0, 0, 0, 2)$	8
$(1, 1, 1, -1, -1, 0, 0, 0, -1, 0)$	$(4, -1, -1, 1, 1, 1, 1, 2, 1, 3)$	9
$(1, 1, 1, -1, -1, 0, 0, 0, 0, -1)$	$(3, -1, -1, 2, 2, 1, 1, 0, 0, 1)$	10
$(1, 1, 1, -1, -1, 0, 0, 0, -1, 0)$	$(5, -1, -1, 1, 1, 2, 2, 3, 2, 2)$	11

$(1, 1, 1, -1, -1, 0, 0, 0, 0, -1)$	$(5, -1, -1, 1, 1, 2, 2, 2, 2, 3)$	12
$(1, 1, 1, 0, 0, -1, -1, 0, -1, 0)$	$(5, -1, -1, 1, 1, 2, 2, 3, 2, 2)$	13
$\sigma(D_0)$	$\rho(D_1)$	$(\sigma(D_0), \rho(D_1))$
$(1, 1, 1, -1, -1, 0, 0, 0, -1, 0)$	$(1, 0, 0, -1, -1, 0, 0, 1, 0, 0)$	-1
$(1, 1, 1, -1, -1, 0, 0, 0, 0, -1)$	$(1, 0, 0, -1, -1, 0, 0, 0, 0, 1)$	0
$(1, 1, 1, -1, -1, 0, 0, 0, -1, 0)$	$(1, 0, 0, 0, 0, -1, -1, 1, 0, 0)$	1
$(1, 1, 1, -1, -1, 0, 0, 0, 0, -1)$	$(1, 0, 0, 0, 0, -1, -1, 0, 0, 1)$	2
$(1, 1, 1, -1, -1, 0, 0, 0, -1, 0)$	$(1, -1, -1, 0, 0, 0, 0, 1, 0, 0)$	3
$(1, 1, 1, -1, -1, 0, 0, 0, 0, -1)$	$(1, -1, -1, 0, 0, 0, 0, 0, 0, 1)$	4
$(0, -1, -1, 0, 0, 0, 0, 0, -1, -1)$	$(3, 1, 1, 0, 0, 0, 0, 0, 3, 0)$	5
$(1, 0, 0, 1, 1, -1, -1, 0, -1, 0)$	$(3, 1, 1, 0, 0, 0, 0, 0, 3, 0)$	6
$(0, -1, -1, 0, 0, 0, 0, -1, 0, -1)$	$(4, 2, 2, 0, 0, 0, 0, 0, 1, 3)$	7
$(0, -1, -1, 0, 0, 0, 0, 0, -1, -1)$	$(4, 2, 2, 0, 0, 0, 0, 0, 3, 1)$	8
$(0, -1, -1, 0, 0, 0, 0, 0, -1, -1)$	$(5, 3, 3, 0, 0, 0, 0, 2, 1, 2)$	9
$(2, -1, -1, 1, 1, 0, 0, 0, 0, 2)$	$(5, 3, 3, 2, 2, 0, 0, 0, 0, 1)$	10
$(1, -1, -1, 0, 0, 1, 1, -1, 0, 0)$	$(5, 3, 3, 2, 2, 0, 0, 0, 1, 0)$	11
$(1, -1, -1, 0, 0, 1, 1, 0, 0, -1)$	$(5, 3, 3, 2, 2, 0, 0, 0, 0, 1)$	12
$(0, 0, 0, -1, -1, 0, 0, 0, -1, -1)$	$(10, 0, 0, 4, 4, 5, 5, 3, 2, 3)$	13
$(2, -1, -1, 0, 0, 1, 1, 0, 0, 2)$	$(5, 3, 3, 2, 2, 0, 0, 0, 0, 1)$	14
$\sigma(D_0)$	$\rho(D_2)$	$(\sigma(D_0), \rho(D_2))$
$(0, -1, -1, -1, -1, 0, 0, 0, 0, 0)$	$(1, 0, 0, 0, 0, 0, 0, 0, 0, -1)$	0
$(0, -1, -1, 0, 0, 0, 0, -1, 0, -1)$	$(2, 1, 1, 0, 0, 0, 0, 0, 1, -1)$	1
$(1, 0, 0, 1, 1, -1, -1, -1, 0, 0)$	$(2, 1, 1, 0, 0, 0, 0, 0, 1, -1)$	2
$(2, 0, 0, -1, -1, 0, 0, 1, 2, 1)$	$(2, 1, 1, 0, 0, 0, 0, 0, 1, -1)$	3
$(1, -1, -1, 1, 1, 0, 0, -1, 0, 0)$	$(2, 1, 1, 0, 0, 0, 0, 0, 1, -1)$	4
$(2, 0, 0, -1, -1, 0, 0, 2, 1, 1)$	$(2, 0, 0, 1, 1, 0, 0, 1, 0, -1)$	5
$(2, 0, 0, -1, -1, 1, 1, 2, 0, 0)$	$(2, 1, 1, 0, 0, 0, 0, -1, 1, 0)$	6
$(2, 0, 0, -1, -1, 0, 0, 1, 2, 1)$	$(3, 1, 1, 1, 1, 0, 0, -1, 0, 2)$	7
$(2, 0, 0, -1, -1, 0, 0, 2, 1, 1)$	$(3, 1, 1, 1, 1, 0, 0, -1, 2, 0)$	8
$(3, 1, 1, -1, -1, 0, 0, 2, 1, 2)$	$(3, 1, 1, 1, 1, 0, 0, -1, 2, 0)$	9
$(1, 1, 0, -1, -1, 0, 0, 1, 0, -1)$	$(5, 0, -1, 2, 2, 1, 1, 2, 1, 3)$	10
$(3, 0, 0, -1, -1, 1, 1, 2, 1, 2)$	$(3, 1, 1, 1, 1, 0, 0, -1, 2, 0)$	11
$(3, 0, 0, -1, -1, 1, 1, 2, 1, 2)$	$(4, 2, 2, 1, 1, 0, 0, -1, 2, 1)$	12
$(4, -1, -1, 1, 1, 1, 1, 2, 3, 1)$	$(4, 1, 1, 1, 1, 1, 1, -1, 0, 3)$	13
$(3, -1, -1, 0, 0, 1, 1, 2, 1, 2)$	$(4, 2, 2, 1, 1, 0, 0, -1, 2, 1)$	14
$(3, -1, -1, 0, 0, 1, 1, 2, 1, 2)$	$(6, 2, 2, 2, 2, 3, 3, 0, -1, 1)$	15

$\sigma(D_0)$	$\rho(D_3)$	$(\sigma(D_0), \rho(D_3))$
(1, 1, 1, -1, 0, 0, 0, -1, 0)	(3, 1, 1, 0, 0, 0, 0, 1, 0, 2)	1
(0, -1, -1, -1, -1, 0, 0, 0, 0)	(2, 1, 1, 0, 0, 0, 0, 0, 0, 0)	2
(1, 0, 0, -1, -1, 1, 1, 0, -1, 0)	(3, 1, 1, 0, 0, 0, 0, 1, 0, 2)	3
(0, -1, -1, 0, 0, 0, 0, 0, -1, -1)	(3, 1, 1, 0, 0, 0, 0, 1, 0, 2)	4
(1, -1, -1, 0, 0, 1, 1, 0, -1, 0)	(3, 1, 1, 0, 0, 0, 0, 1, 0, 2)	5
(1, -1, -1, 0, 0, 1, 1, 0, -1, 0)	(3, 1, 1, 0, 0, 0, 0, 2, 1, 0)	6
(1, -1, -1, 0, 0, 0, 1, -1, 0)	(3, 1, 1, 0, 1, 0, 0, 0, 2, 0)	7
(3, -1, -1, 0, 0, 1, 1, 2, 2, 1)	(2, 1, 1, 0, 0, 0, 0, 0, 0, 0)	8
(1, -1, -1, 1, 1, 0, 0, -1, 0)	(4, 2, 2, 0, 0, 0, 0, 2, 1, 1)	9
(0, -1, -1, -1, -1, 0, 0, 0, 0, 0)	(6, 3, 3, 2, 2, 2, 2, 0, 0, 0)	10
(1, -1, -1, 0, 0, 1, 1, 0, -1, 0)	(5, 2, 2, 1, 1, 0, 0, 3, 2, 0)	11
(1, -1, -1, 0, 0, 1, 1, 0, -1, 0)	(5, 2, 2, 1, 1, 0, 0, 0, 3, 2)	12
(0, 0, 0, -1, -1, 0, 0, -1, 0, -1)	(9, 3, 3, 4, 4, 0, 0, 3, 4, 2)	13
(2, -1, -1, 0, 0, 1, 1, 2, 0, 0)	(5, 2, 2, 1, 1, 0, 0, 0, 2, 3)	14
(1, 0, 0, -1, -1, 1, 1, 0, -1, 0)	(7, 3, 3, 2, 2, 0, 0, 1, 4, 2)	15
(4, -1, -1, 1, 1, 1, 1, 3, 1, 2)	(4, 2, 2, 0, 0, 1, 1, 0, 2, 0)	16

$\sigma(D_1)$	$\rho(D_1)$	$(\sigma(D_1), \rho(D_1))$
(1, -1, -1, 0, 0, 0, 0, 0, 0, 1)	(1, 0, 0, -1, -1, 0, 0, 0, 0, 1)	0
(1, -1, 0, 0, 0, 0, 0, -1, 0, 1)	(1, 0, 1, 0, 0, 0, 0, -1, 0, -1)	1
(1, -1, -1, 0, 0, 0, 0, 0, 0, 1)	(3, 1, 1, 0, 0, 0, 0, 0, 0, 3)	2
(2, 1, 0, 0, 0, 0, 0, 2, 0, -1)	(3, 1, 0, 0, 0, 0, 0, 1, 3, 0)	3
(2, 0, -1, 0, 0, 0, 0, 2, 0, 1)	(3, 1, 0, 0, 0, 0, 0, 1, 3, 0)	4
(2, 1, 0, 0, 0, 0, 0, 2, -1, 0)	(2, 0, -1, 0, 0, 0, 0, 0, 1, 2)	5
(4, 2, 2, 0, 0, 0, 0, 0, 1, 3)	(4, 0, 0, 2, 2, 0, 0, 0, 1, 3)	6
(1, -1, -1, 0, 0, 0, 0, 1, 0, 0)	(4, 2, 2, 0, 0, 0, 0, 1, 0, 3)	7
(1, -1, -1, 0, 0, 0, 0, 1, 0, 0)	(4, 2, 2, 0, 0, 0, 0, 0, 3, 1)	8
(3, 1, 1, 0, 0, 0, 0, 3, 0, 0)	(5, 3, 3, 2, 2, 0, 0, 0, 1, 0)	9
(2, 0, 0, 0, 0, 0, 0, 1, -1, 2)	(4, 2, 2, 0, 0, 0, 0, 1, 3, 0)	10
(4, 0, 0, 2, 2, 0, 0, 0, 1, 3)	(5, 3, 3, 2, 2, 0, 0, 0, 1, 0)	11
(3, 0, 0, 0, 0, 1, 1, 0, 0, 3)	(5, 3, 3, 2, 2, 0, 0, 0, 0, 1)	12
(4, 2, 2, 0, 0, 0, 0, 3, 0, 1)	(4, 0, 0, 2, 2, 0, 0, 0, 1, 3)	13
(2, 0, 0, 0, 0, -1, 0, 2, 0, 1)	(6, 2, 2, 0, 0, 3, 4, 0, 2, 1)	14
(3, 0, 0, 0, 0, 1, 1, 3, 0, 0)	(5, 3, 3, 2, 2, 0, 0, 0, 1, 0)	15

$\sigma(D_1)$	$\rho(D_2)$	$(\sigma(D_1), \rho(D_2))$
(1, -1, -1, 0, 0, 0, 0, 1, 0, 0)	(1, 0, 0, 0, 0, 0, 0, 0, -1, 0)	1
(1, -1, -1, 0, 0, 0, 0, 0, 0, 1)	(1, 0, 0, 0, 0, 0, 0, 0, 0, -1)	2
(2, 0, 0, 0, 0, 0, 0, 2, 1, -1)	(1, 0, 0, 0, 0, 0, 0, 0, -1, 0)	3
(2, 0, 0, 0, 0, 0, 0, 2, -1, 1)	(1, 0, 0, 0, 0, 0, 0, -1, 0, 0)	4
(2, 1, 0, 0, 0, 0, 0, 2, -1, 0)	(2, 1, 0, 1, 1, 0, 0, -1, 0, 0)	5
(3, 1, 1, 0, 0, 0, 0, 0, 0, 3)	(1, 0, 0, 0, 0, 0, 0, 0, 0, -1)	6
(3, 1, 1, 0, 0, 0, 0, 3, 0, 0)	(2, 1, 1, 0, 0, 0, 0, -1, 0, 1)	7
(3, 1, 0, 0, 0, 0, 0, 3, 0, 1)	(2, 1, 0, 1, 1, 0, 0, -1, 0, 0)	8
(3, 1, 1, 0, 0, 0, 0, 3, 0, 0)	(2, 0, 0, 1, 1, 0, 0, -1, 0, 1)	9
(3, 1, 1, 0, 0, 0, 0, 3, 0, 0)	(3, 1, 1, 1, 1, 0, 0, -1, 0, 2)	10
(5, 0, 0, 2, 2, 3, 3, 0, 1, 0)	(3, 1, 1, 1, 1, 0, 0, -1, 0, 2)	11
(3, 0, 0, 0, 0, 1, 1, 3, 0, 0)	(3, 1, 1, 1, 1, 0, 0, -1, 0, 2)	12
(3, 0, 0, 0, 0, 1, 0, 3, 0, 1)	(4, 2, 2, 1, 1, 2, 1, -1, 0, 0)	13
(5, 0, 0, 2, 2, 3, 3, 1, 0, 0)	(4, 2, 2, 1, 1, 0, 0, 2, 1, -1)	14
(5, 0, 0, 2, 2, 3, 3, 1, 0, 0)	(4, 2, 2, 0, 0, 1, 1, -1, 2, 1)	15
(9, 4, 4, 4, 4, 0, 0, 3, 1, 3)	(2, 0, 0, 0, 0, 1, 1, 1, -1, 0)	16

$\sigma(D_1)$	$\rho(D_3)$	$(\sigma(D_1), \rho(D_3))$
(1, 0, 0, -1, -1, 0, 0, 0, 0, 1)	(2, 1, 1, 0, 0, 0, 0, 0, 0, 0)	2
(1, -1, -1, 1, 0, 0, 0, 0, 0, 0)	(3, 1, 1, 2, 1, 0, 0, 0, 0, 0)	3
(1, -1, -1, 0, 0, 0, 0, 0, 0, 1)	(2, 1, 1, 0, 0, 0, 0, 0, 0, 0)	4
(1, 0, 0, -1, -1, 1, 0, 0, 0, 0)	(4, 1, 1, 1, 1, 1, 0, 0, 0, 3)	5
(3, 0, 0, 1, 1, 0, 0, 0, 0, 3)	(2, 1, 1, 0, 0, 0, 0, 0, 0, 0)	6
(3, 1, 1, 0, 0, 0, 0, 3, 0, 0)	(3, 1, 1, 0, 0, 0, 0, 0, 1, 2)	7
(5, 3, 3, 2, 2, 0, 0, 0, 1, 0)	(3, 1, 1, 0, 0, 0, 0, 0, 1, 2)	8
(3, 0, 0, 1, 1, 0, 0, 3, 0, 0)	(3, 1, 1, 0, 0, 0, 0, 0, 1, 2)	9
(3, 1, 1, 0, 0, 0, 0, 3, 0, 0)	(4, 1, 1, 1, 1, 0, 0, 0, 1, 3)	10
(1, -1, -1, 0, 0, 1, 0, 0, 0, 0)	(7, 3, 3, 2, 2, 2, 1, 0, 0, 4)	11
(3, 1, 1, 0, 0, 0, 0, 3, 0, 0)	(4, 0, 0, 1, 1, 1, 1, 0, 1, 3)	12
(4, 2, 2, 0, 0, 0, 0, 3, 0, 1)	(4, 0, 0, 1, 1, 1, 1, 0, 1, 3)	13
(5, 3, 3, 2, 2, 0, 0, 1, 0, 0)	(4, 1, 1, 0, 0, 1, 1, 0, 3, 1)	14
(5, 3, 3, 2, 2, 0, 0, 0, 1, 0)	(4, 0, 0, 1, 1, 1, 1, 0, 1, 3)	15
(5, 3, 3, 2, 2, 0, 0, 1, 0, 0)	(4, 0, 0, 1, 1, 1, 1, 0, 3, 1)	16
(9, 4, 4, 4, 4, 3, 3, 1, 0, 0)	(2, 0, 0, 0, 0, 0, 0, 1, 0, 1)	17

$\sigma(D_2)$	$\rho(D_2)$	$(\sigma(D_2), \rho(D_2))$
(1, 0, 0, 0, -1, 0, 0, 0, 0, 0)	(2, 1, 1, 1, 0, 0, 0, 0, 0, -1)	2
(2, 1, 1, 0, -1, 0, 0, 0, 0, 1)	(2, 1, 1, 1, 0, 0, 0, 0, 0, -1)	3
(1, 0, 0, 0, 0, 0, 0, 0, 0, -1)	(3, 2, 2, 0, 0, 0, 0, 0, 0, 1)	4
(2, 1, 1, 0, -1, 0, 0, 0, 0, 1)	(2, 1, 1, 1, 0, 0, 0, 0, 0, -1)	5
(2, 1, 1, 1, 0, 0, 0, 0, 0, -1)	(3, 0, 0, 2, 1, 0, 0, 0, 0, 2)	6
(3, 2, 2, 0, 0, 0, 0, 1, 0, 0)	(3, 0, 0, 0, 0, 0, 0, 2, 1, 2)	7
(3, 0, 0, 1, 0, 2, 2, 0, 0, 0)	(3, 0, 0, 1, 0, 0, 0, 2, 2, 0)	8
(6, 3, 3, 2, 2, 0, 0, 3, 0, 1)	(1, 0, 0, 0, 0, 0, 0, -1, 0, 0)	9
(6, 3, 3, 2, 2, 0, 0, 3, 0, 1)	(2, 0, 0, 1, 1, 0, 0, -1, 0, 1)	10
(5, 3, 3, 1, 1, 0, 0, 0, 1, 2)	(2, 0, 0, 0, 0, 1, 1, 0, 1, -1)	11
(6, 3, 3, 2, 2, 0, 0, 3, 1, 0)	(4, 1, 1, 2, 2, 0, 0, -1, 1, 2)	12
(3, 0, 0, 0, 0, 2, 2, 1, 0, 0)	(4, 2, 2, 1, 1, 0, 0, -1, 2, 1)	13
(6, 3, 3, 2, 2, 0, 0, 3, 0, 1)	(2, 0, 0, 0, 0, 1, 1, -1, 0, 1)	14
(4, 0, 0, 2, 2, 1, 1, 2, -1, 1)	(4, 2, 2, 0, 0, 1, 1, 1, 2, -1)	15
(6, 3, 3, 2, 2, 0, 0, 3, 0, 1)	(4, 1, 1, 0, 0, 2, 2, 1, 2, -1)	16
(6, 3, 3, 2, 2, 0, 0, 1, 0, 3)	(3, 0, 0, 0, 0, 2, 2, 1, 0, 0)	17

$\sigma(D_2)$	$\rho(D_3)$	$(\sigma(D_2), \rho(D_3))$
(2, 1, 1, 1, 0, 0, 0, 0, -1, 0)	(2, 0, 0, 1, 0, 0, 0, 1, 0, 0)	3
(2, 1, 1, 1, 0, 0, 0, 0, 0, -1)	(2, 0, 0, 1, 0, 0, 0, 0, 0, 1)	4
(2, 1, 1, 1, 0, 0, 0, 0, 0, -1)	(3, 1, 1, 1, 0, 0, 0, 0, 0, 2)	5
(3, 2, 2, 0, 0, 0, 0, 0, 0, 1)	(2, 0, 0, 1, 1, 0, 0, 0, 0, 0)	6
(2, 0, 0, 1, 0, 1, 1, 0, 0, -1)	(3, 1, 1, 1, 0, 0, 0, 0, 0, 2)	7
(3, 0, 0, 1, 0, 0, 0, 2, 2, 0)	(3, 1, 1, 1, 0, 0, 0, 0, 0, 2)	8
(1, 0, 0, 0, 0, 0, 0, 0, -1, 0)	(6, 2, 2, 2, 2, 0, 0, 0, 3, 3)	9
(4, 1, 1, 2, 2, 0, 0, 1, -1, 2)	(3, 1, 1, 0, 0, 0, 0, 0, 2, 1)	10
(3, 0, 0, 0, 0, 2, 2, 1, 0, 0)	(4, 1, 1, 1, 1, 0, 0, 1, 0, 3)	11
(4, 0, 0, 2, 2, 1, 1, 1, -1, 2)	(3, 1, 1, 0, 0, 0, 0, 0, 2, 1)	12
(5, 3, 3, 1, 1, 0, 0, 1, 0, 2)	(3, 0, 0, 0, 0, 1, 1, 2, 1, 0)	13
(4, 0, 0, 2, 2, 1, 1, 1, -1, 2)	(4, 2, 2, 1, 1, 0, 0, 0, 2, 0)	14
(6, 2, 2, 2, 2, 3, 3, 0, -1, 1)	(3, 1, 1, 0, 0, 0, 0, 2, 1, 0)	15
(4, 0, 0, 2, 2, 1, 1, 1, -1, 2)	(4, 2, 2, 0, 0, 1, 1, 0, 2, 0)	16
(13, 2, 2, 6, 6, 4, 4, 7, 2, 2)	(2, 0, 0, 0, 0, 0, 0, 1, 0, 1)	17
(5, 3, 3, 1, 1, 0, 0, 2, 0, 1)	(4, 0, 0, 1, 1, 2, 2, 0, 2, 0)	18

$\sigma(D_3)$	$\rho(D_3)$	$(\sigma(D_3), \rho(D_3))$
(2, 1, 1, 0, 0, 0, 0, 0, 0)	(2, 0, 0, 1, 1, 0, 0, 0, 0)	4
(3, 1, 1, 0, 0, 0, 0, 1, 0, 2)	(3, 1, 1, 0, 0, 0, 0, 2, 1, 0)	5
(3, 1, 1, 0, 0, 1, 1, 0, 0, 2, 0, 1)	(3, 1, 1, 0, 0, 0, 0, 0, 2, 1)	6
(3, 0, 0, 1, 1, 0, 0, 1, 0, 2)	(3, 1, 1, 0, 0, 0, 0, 2, 1, 0)	7
(3, 0, 0, 1, 1, 0, 0, 2, 0, 1)	(3, 1, 1, 0, 0, 0, 0, 0, 2, 1)	8
(5, 0, 0, 2, 2, 1, 1, 3, 0, 2)	(3, 0, 0, 0, 0, 1, 1, 0, 1, 2)	9
(3, 1, 1, 0, 0, 0, 0, 2, 0, 1)	(4, 1, 1, 0, 0, 2, 2, 0, 2, 0)	10
(5, 0, 0, 2, 2, 1, 1, 3, 0, 2)	(3, 1, 1, 0, 0, 0, 0, 0, 1, 2)	11
(3, 1, 1, 0, 0, 0, 0, 2, 0, 1)	(4, 0, 0, 1, 1, 2, 2, 0, 2, 0)	12
(5, 1, 1, 2, 2, 2, 3, 0, 0, 0)	(4, 0, 0, 1, 1, 0, 1, 1, 1, 3)	13
(5, 0, 0, 2, 2, 1, 1, 2, 0, 3)	(4, 2, 2, 0, 0, 1, 1, 2, 0, 0)	14
(5, 2, 2, 1, 1, 2, 3, 0, 0, 0)	(4, 0, 0, 1, 1, 0, 1, 1, 1, 3)	15
(5, 0, 0, 2, 2, 1, 1, 2, 0, 3)	(4, 1, 1, 0, 0, 2, 2, 0, 2, 0)	16
(11, 3, 3, 5, 5, 3, 4, 1, 0, 5)	(2, 0, 0, 0, 0, 0, 1, 1, 0, 0)	17
(5, 0, 0, 2, 2, 1, 1, 2, 0, 3)	(4, 2, 2, 0, 0, 1, 1, 0, 2, 0)	18
(13, 4, 4, 6, 6, 5, 5, 3, 2, 0)	(3, 0, 0, 1, 1, 0, 0, 2, 1, 0)	19

This concludes the proof of theorem 9.

We can now state the following theorem, which is a straightforward consequence of theorems 6, 8 and 9.

THEOREM 10. (i) *For every $r \geq 5$ there exists an embedding of S as a nonsingular surface F^{2r-3} of degree $2r - 3$ in \mathbf{P}^r , and for every (d, g) such that*

$$0 \leq g \leq (d - r)^2 / 2(2r - 3)$$

there exists a nonsingular irreducible and nondegenerate curve X of degree d and genus g on F^{2r-3} .

(ii) *For every $r \geq 7$ there exists an embedding of S as a nonsingular surface F^{2r-4} of degree $2r - 4$ in \mathbf{P}^r , and for every (d, g) such that*

$$0 \leq g \leq (d - r)^2 / 2(2r - 4)$$

there exists a nonsingular irreducible and nondegenerate curve X of degree d and genus g on F^{2r-4} .

Note that theorem 10 differs from our main theorem, as stated in the introduction, only in that the surface S appears instead of S' . In the next section we will show how to deduce the main theorem from theorem 10.

Remark 2. Arguing as in proposition 5 it is easy to see that if there exists a δ -tuple $D_0, \dots, D_{\delta-1}$ of classes of $\text{Pic}(S)$ satisfying conditions I), II), III) of the definition of δ -system, then S can be embedded in \mathbf{P}^r as a smooth linearly normal surface of degree $2r - \delta$ for all $r \geq \delta - 1$. The following is a list of such δ -tuples for $5 \leq \delta \leq 9$:

$\delta = 5$:

$$D_0 = (0, -1, -1, -1, 0, 0, 0, 0, 0, 0)$$

$$D_2 = (1, -1, -1, 0, 0, 0, 0, 0, 0, 0)$$

$$D_4 = (2, 1, 0, 0, 0, 0, 0, 0, 0, 0)$$

$$D_1 = (1, 1, -1, -1, 0, 0, 0, 0, 0, 0)$$

$$D_3 = (2, 1, 1, -1, 0, 0, 0, 0, 0, 0)$$

$\delta = 6$:

$$D_0 = (0, -1, -1, -1, -1, -1, -1, 0, 0, 0)$$

$$D_2 = (1, -1, -1, -1, -1, 0, 0, 0, 0, 0)$$

$$D_4 = (2, 1, -1, 0, 0, 0, 0, 0, 0, 0)$$

$$D_1 = (1, 1, -1, -1, -1, -1, 0, 0, 0, 0)$$

$$D_3 = (2, 1, 1, -1, -1, 0, 0, 0, 0, 0)$$

$$D_5 = (3, 2, 1, 0, 0, 0, 0, 0, 0, 0)$$

$\delta = 7$:

$$D_0 = (0, -1, -1, -1, -1, -1, -1, -1, 0, 0)$$

$$D_2 = (1, -1, -1, -1, -1, 0, 0, 0, 0, 0)$$

$$D_4 = (2, 1, -1, -1, 0, 0, 0, 0, 0, 0)$$

$$D_6 = (3, 2, 0, 0, 0, 0, 0, 0, 0, 0)$$

$$D_1 = (1, 1, -1, -1, -1, -1, -1, 0, 0, 0)$$

$$D_3 = (2, 1, 1, -1, -1, -1, 0, 0, 0, 0)$$

$$D_5 = (3, 2, 1, -1, 0, 0, 0, 0, 0, 0)$$

$\delta = 8$:

$$D_0 = (0, -1, -1, -1, -1, -1, -1, -1, -1, 0)$$

$$D_2 = (1, -1, -1, -1, -1, -1, 0, 0, 0, 0)$$

$$D_4 = (2, 1, -1, -1, -1, 0, 0, 0, 0, 0)$$

$$D_6 = (3, 2, -1, 0, 0, 0, 0, 0, 0, 0)$$

$$D_1 = (1, 1, -1, -1, -1, -1, -1, -1, 0, 0)$$

$$D_3 = (2, 1, 1, -1, -1, -1, -1, 0, 0, 0)$$

$$D_5 = (3, 2, 1, -1, -1, 0, 0, 0, 0, 0)$$

$$D_7 = (4, 3, 1, 0, 0, 0, 0, 0, 0, 0)$$

$\delta = 9$:

$$D_0 = (0, -1, -1, -1, -1, -1, -1, -1, -1, -1);$$

$$D_2 = (1, -1, -1, -1, -1, -1, -1, 0, 0, 0)$$

$$D_4 = (2, 1, -1, -1, -1, -1, 0, 0, 0, 0)$$

$$D_6 = (3, 2, -1, -1, 0, 0, 0, 0, 0, 0)$$

$$D_8 = (4, 3, 0, 0, 0, 0, 0, 0, 0, 0)$$

$$D_1 = (1, 1, -1, -1, -1, -1, -1, -1, -1, 0)$$

$$D_3 = (2, 1, 1, -1, -1, -1, -1, -1, -1, 0)$$

$$D_5 = (3, 2, 1, -1, -1, -1, 0, 0, 0, 0)$$

$$D_7 = (4, 3, 1, -1, 0, 0, 0, 0, 0, 0)$$

On the other hand it is clear that there are no such δ -tuples for $\delta \geq 10$: indeed, letting $D = |D_0 - \omega_S|$, $\varphi_D(S)$ is a smooth surface of degree δ in \mathbf{p}^δ with elliptic hyperplane sections.

4. Remarks

1) One of our main technical tools has been proposition 1, whose proof uses very strongly the geometrical properties of the surface S , particularly the fact that

the points P_1, \dots, P_9 are not in general position, but are base points of a generic pencil of cubics. It is pretty clear that proposition 1 cannot be generalized to a surface S' obtained by blowing up 9 points of \mathbf{P}^2 in general position. Nevertheless it is not difficult to see that our other main results generalize to S' . This can be done in the following way.

Let $P_1, \dots, P_9 \in \mathbf{P}^2$ be the points that define S , and let M be a general line through P_9 . In $\mathbf{P}^2 \times M$ denote by Γ the diagonal curve, whose support is $\{(p, p) : p \in M\}$. Let \mathbf{S} be the blow-up of $\mathbf{P}^2 \times M$ along $P_1 \times M \cup \dots \cup P_8 \times M \cup \Gamma$, and let $q : \mathbf{S} \rightarrow \mathbf{P}^2 \times M$ be the projection, and $\pi : \mathbf{S} \rightarrow M$ be the composition of q with the second projection $\mathbf{P}^2 \times M \rightarrow M$. Clearly π is a smooth family of projective surfaces, whose fibre over a point $p \in M$ is the surface $\mathbf{S}(p)$ obtained from \mathbf{P}^2 after blowing up P_1, \dots, P_8 and p . In particular $\mathbf{S}(P_9) = S$. Note that for all p in some open neighborhood of P_9 the points P_1, \dots, P_8, p are contained in a unique cubic curve C_p which is nonsingular, hence they are in general position.

In $\text{Pic}(\mathbf{S})$ consider the classes

$$\mathbf{o}(\mathbf{H}), \mathbf{o}(-\mathbf{E}_1), \dots, \mathbf{o}(-\mathbf{E}_8), \mathbf{o}(-\mathbf{E}_p),$$

where $\mathbf{H} = q^*(\mathbf{o}(1))$, and $\mathbf{E}_1, \dots, \mathbf{E}_8, \mathbf{E}_p$ are the exceptional surfaces coming from the curves $P_1 \times M, \dots, P_8 \times M, \Gamma$ respectively. It is clear that every element $\mathbf{o}(D)$ of $\text{Pic}(S)$, being a linear combination of $\mathbf{o}(H), \mathbf{o}(-E_1), \dots, \mathbf{o}(-E_9)$, extends to an element $\mathbf{o}(\mathbf{D}) \in \text{Pic}(\mathbf{S})$ which is the corresponding combination of the above classes; hence, by restriction, it defines a divisor class $\mathbf{o}(D_p) \in \text{Pic}(\mathbf{S}(p))$ for all $p \in M$.

Suppose that $\mathbf{o}(D) \in \text{Pic}(S)$ is such that

$$(*) \quad h^1(S, \mathbf{o}(D)) = 0 = h^2(S, \mathbf{o}(D)),$$

From the upper-semicontinuity principle it follows that there is an open neighborhood U_D of P_9 in M such that for all $p \in U_D$

$$h^1(S, \mathbf{o}(D_p)) = 0 = h^2(S, \mathbf{o}(D_p)).$$

If moreover the linear system $|D|$ has no base points and contains an irreducible and nonsingular curve, then, after possibly shrinking U_D , the same is true of $|D_p|$ for all $p \in U_D$. Indeed the base point freeness of $|D|$ implies that the natural map

$$\pi^*[\pi_*\mathbf{o}(\mathbf{D})] \rightarrow \mathbf{o}(\mathbf{D})$$

is surjective in a neighborhood of $\mathbf{S}(P_9) = S$. From the base change properties it then follows that $|D_p|$ is base point free for all p in that neighborhood. Condition

(*) implies that $\pi_*\mathbf{o}(\mathbf{D})$ is locally free of rank $h^0(S, \mathbf{o}(D))$ on some open set U containing P_9 . As a consequence we have that, if $X \in |D|$ is a general element, it can be extended to a relative effective Cartier divisor \mathbf{X} on $\pi^{-1}(U)$. And if X is a nonsingular curve, then it follows from the flatness of \mathbf{X} over U that the restriction X_p of \mathbf{X} to $\mathbf{S}(p)$ is also a nonsingular curve for all p in some open set $V \subset U$ containing P_9 .

Suppose in addition that $\mathbf{o}(D)$ is very ample; then it is easy to show that, after possibly shrinking U_D , $\mathbf{o}(D_p)$ is very ample for all $p \in U_D$. Indeed, on $\pi^{-1}(U_D)$ the natural map $\pi^*\pi_*\mathbf{o}(\mathbf{D}) \rightarrow \mathbf{o}(\mathbf{D})$ is surjective, hence it defines a U_D -morphism

$$\varphi: \pi^{-1}(U_D) \rightarrow \mathbf{P}(\pi_*\mathbf{o}(\mathbf{D})) =: \mathbf{P}$$

which restricts on every fibre $\mathbf{S}(p)$, $p \in U_D$, to the morphism

$$\varphi_p: \mathbf{S}(p) \rightarrow \mathbf{P}(H^0(\mathbf{S}(p), \mathbf{o}(D_p)))$$

defined by the linear system $|D_p|$. For $p = P_9$ this is a closed embedding, because $\mathbf{o}(D)$ is very ample; hence there is an open $V \subset U_D$ such that the restriction of φ to $\pi^{-1}(V)$ is finite and such that $\mathbf{o}_p \rightarrow \varphi_*\mathbf{o}_s$ is an isomorphism; equivalently φ is a closed embedding of $\pi^{-1}(V)$ in \mathbf{P} and this means that $\mathbf{o}(D_p)$ is very ample for all $p \in V$.

These remarks can be applied to $D = D_i - \alpha\omega_S$ to conclude that propositions 4 and 5 generalize to S' with no changes. As a consequence of this we have that theorem 6 is still true if we replace S by S' . Clearly lemma 7 extends to S' , and the proofs of theorems 8 and 9 extend word by word to S' . Consequently theorem 10 also extends. In particular *the main theorem, as stated in the introduction, is true.*

2) Suppose that D is a 1-connected effective divisor on a projective nonsingular surface F such that $h^1(F, \mathbf{o}_F) = 0$, and let H be a divisor on F such that $|D + H|$ contains an irreducible nonsingular curve C . From the exact sequence

$$0 \rightarrow \mathbf{o}_F(-D) \rightarrow \mathbf{o}_F(H) \rightarrow \mathbf{o}_C(H) \rightarrow 0$$

and from the Ramanujam' vanishing theorem (see section 1) it follows that

$$H^0(F, \mathbf{o}_F(H)) \cong H^0(C, \mathbf{o}_C(H)).$$

We apply this remark to the surface S , equipped with a δ -system (e.g. a 3-system or a 4-system), and we take H to be one of the very ample divisors H_r and C' any

of the curves X of degree d and genus g such that $0 \leq g \leq d - r - 1$, as described in proposition 5. It follows that the curves X of degree d and genus g constructed in theorem 6 satisfy $h^0(X, \mathcal{O}_X(H)) = r + 1$ (i.e. are “linearly normal”) if

$$d - r \leq g \leq (d - r)^2 / 2(2r - \delta).$$

In particular this applies to the curves of theorem 10 and, by upper-semicontinuity, to those of the main theorem which satisfy the corresponding inequalities for $\delta = 3, 4$.

3) Let $C \subset \mathbf{P}^r$ be a nonsingular irreducible and nondegenerate curve of degree n , $\mathcal{O}(H_C)$ the hyperplane section line bundle and ω_C the canonical bundle. Assume that $h^0(C, \mathcal{O}(H_C)) = r + 1$. The natural map

$$\mu_0(C): H^0(C, \mathcal{O}(H_C)) \otimes H^0(C, \omega_C(-H_C)) \rightarrow H^0(C, \omega_C)$$

is called the *Brill-Noether map* of $C \subset \mathbf{P}^r$.

The map $\mu_0(C)$ is relevant to the study of the scheme $W_n^r(C)$ of linear series of degree n and dimension at least r on C . In particular, the surjectivity of $\mu_0(C)$ is equivalent to the fact that $\mathcal{O}(H_C)$ is an isolated point of $W_n^r(C)$ with reduced structure. We will check this last property on some of the curves we have constructed.

Again suppose that the surface S is equipped with a δ -system, and let $F \subset \mathbf{P}^r$ be the nonsingular embedding of degree $2r - \delta$ of S given by theorem 6.

Let $X \subset F$ be a nonsingular nondegenerate curve of degree d and genus g as constructed in the proof of theorem 6. Assume that $g \geq 2d - 4r + \delta$ (with the notation of theorem 6, this means that (d, g) lies above the line L_1). Then it follows from the proof of theorem 6 that $X \in |C' + mH_r|$, for some $m \geq 2$ and for some irreducible, nonsingular and nondegenerate C' of degree d' and genus g' such that $0 \leq g' \leq d' - r - 1$. We will write $H_r = H$.

Since, by remark 2) above, $h^0(X, \mathcal{O}_X(H)) = r + 1$, we can consider the Brill-Noether map of $X \subset \mathbf{P}^r$. We claim that $\mu_0(X)$ is surjective. Indeed consider the following commutative diagram:

$$\begin{array}{ccc} \mu: H^0(F, \mathcal{O}_F(H)) \otimes H^0(F, \omega_F(X - H)) & \rightarrow & H^0(F, \omega_F(X)) \\ \downarrow & & \downarrow q \\ \mu_0(X): H^0(X, \mathcal{O}_X(H_X)) \otimes H^0(X, \omega_X(-H_X)) & \rightarrow & H^0(X, \omega_X). \end{array}$$

Since q is surjective, it suffices to show that μ is surjective, and for this purpose it is enough to show that the sheaf $\omega_F(X - H)$ is 0-regular with respect to

$\mathfrak{o}(H)$ (see [M]). This amounts to check that:

$$H^1(F, \omega_F(X - 2H)) = (0)$$

and

$$H^2(F, \omega_F(X - 3H)) = (0).$$

The first condition follows from the vanishing theorem because $|X - 2H| = |C' + (m - 2)H|$ contains a 1-connected divisor. The second condition is equivalent to

$$H^0(F, \mathfrak{o}_F(3H - X)) = (0),$$

which is true because

$$|3H - X| = |(3 - m)H - C'|,$$

and this is clearly empty if $m \geq 3$, and likewise empty for $m = 2$ because C' is nondegenerate.

Of course this remark applies to the curves of theorem 10, taking $\delta = 3$ or 4, and, by upper-semicontinuity, it extends to the curves of the main theorem.

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