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## Curves on surfaces of degree $2r-\delta$ in $\mathbf{P}^r$

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### Introduction

In this paper we consider the problem of finding the values of  $d, g$  for which there exists a nonsingular irreducible and nondegenerate (i.e. not contained in a hyperplane) curve  $X$  of degree  $d$  and genus  $g$  in  $\mathbf{P}^r$ , the projective space over an algebraically closed  $\mathbf{k}$  of *arbitrary characteristic*.

This problem has been completely solved in  $\mathbf{P}^3$  by Gruson–Peskine [GP] in the case  $\text{char}(\mathbf{k})=0$ , then extended to arbitrary characteristic by Hartshorne [Ha], and in  $\mathbf{P}^4$  and  $\mathbf{P}^5$  by Rathmann [Ra]. The approach of [GP], which has been generalized in [Ra], is divided into two parts. The first consists in constructing, on a quartic surface with a double line  $F$ , nonsingular curves of degree  $d$  and genus  $g$  for every  $(d, g)$  such that

$$0 \leq g \leq (d-1)^2/8.$$

A similar result has been proved by Mori [M] in complex projective 3-space for every  $d, g$  as above, and his construction has been extended in [Ra], proving the existence of smooth curves of degree  $d$  and genus  $g$  in  $\mathbf{P}^r$  lying on a  $K3$  surface when

$$0 \leq g \leq d^2/2(2r-2) - (r-1)/4.$$

The second part of the approach of [GP] is a detailed study of curves on a nonsingular cubic surface, which implies the existence result in the range

$$(d-1)^2/8 < g \leq d(d-3)/6.$$

We generalize the first construction of Gruson–Peskine and we prove the existence of nonsingular curves of degree  $d$  and genus  $g$  in  $\mathbf{P}^r$  for all  $r \geq 6$  in a

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wide range of  $(d, g)$  (see statement below). Our curves are constructed on certain rational surfaces which are all embeddings of one and the same surface  $S'$ : this is the blow-up of  $\mathbf{P}^2$  at nine points in general position. We exploit the rich geometry of  $S'$  very much in the same way as it is done in [GP] and [Ra], with the difference that, for technical reasons, we first work with the surface  $S$  obtained by blowing up nine points which are *not* in general position, but are base points of a generic pencil of cubics. Then we prove the main result using deformation theoretic arguments. The main consequence of our analysis of curves lying on the surface  $S$  is the following:

**MAIN THEOREM.** (i) *For every  $r \geq 5$  there exists an embedding of  $S'$  as a nonsingular surface  $F^{2r-3}$  of degree  $2r-3$  in  $\mathbf{P}^r$ , and for every  $(d, g)$  such that*

$$0 \leq g \leq (d-r)^2/2(2r-3)$$

*there exists a nonsingular irreducible and nondegenerate curve  $X$  of degree  $d$  and genus  $g$  on  $F^{2r-3}$ .*

(ii) *For every  $r \geq 7$  there exists an embedding of  $S'$  as a nonsingular surface  $F^{2r-4}$  of degree  $2r-4$  in  $\mathbf{P}^r$ , and for every  $(d, g)$  such that*

$$0 \leq g \leq (d-r)^2/2(2r-4)$$

*there exists a nonsingular irreducible and nondegenerate curve  $X$  of degree  $d$  and genus  $g$  on  $F^{2r-4}$ .*

Clearly, the existence result for curves in  $\mathbf{P}^5$ , contained in part (i) of the above theorem, follows from [Ra]. In more detail, the content of the paper is the following.

In section 1 we prove preliminary general results on the surface  $S$  which are repeatedly used in the paper. Precisely we give a criterion (proposition 1) for a linear system on  $S$  to be base point free and such that the associated map to projective space realizes  $S$  as a nonsingular surface. From this result we directly deduce an ampleness criterion which can also be found in [H].

In section 2 we introduce the notion of  $\delta$ -system on  $S$ , which is a  $\delta$ -tuple,  $\delta \geq 3$ , of elements of  $\text{Pic}(S)$  satisfying certain conditions. This notion turns out to be a powerful tool in the study of curves lying on the surface  $S$ . The main result of this section (theorem 6) states that, if a  $\delta$ -system exists on  $S$ , then  $S$  can be embedded in  $\mathbf{P}^r$  with degree  $2r-\delta$  for all  $r \geq 2\delta-1$ , in such a way that it contains smooth nondegenerate curves of degree  $d$  and genus  $g$  for all  $(d, g)$  such

that

$$(*) \quad 0 \leq g \leq (d - r)^2 / 2(2r - \delta).$$

Actually we prove a slightly better bound (see remark 1).

In section 3 we consider the problem of existence of  $\delta$ -systems. It is easy to show that  $\delta$ -systems do not exist for  $\delta > 9$  (see remark 2). It is not difficult to find candidates for  $3 \leq \delta \leq 9$ , namely to find  $\delta$ -tuples of classes of  $\text{Pic}(S)$  which satisfy all but the last of the defining conditions. To prove that the last condition is also satisfied boils down to finding certain lists of elements of  $\text{Pic}(S)$ . These lists become increasingly long as  $\delta$  grows, and this has forced us to consider the cases  $\delta = 3, 4$  only, in which we are able to exhibit them. Via theorem 6, this proves a result which differs from the main theorem only in the fact that the surface  $S$  appears instead of  $S'$  in its statement. In remark 2 we also deduce the existence of smooth rational surfaces of degree  $2r - \delta$  in  $\mathbf{P}^r$ ,  $r \geq \delta - 1$ , for  $5 \leq \delta \leq 9$ .

In section 4 we show how to extend to  $S'$  most of the previous results concerning the surface  $S$ . Of course, this and the results quoted above imply the main theorem. We also discuss linear normality and the Brill-Noether map for the curves we have constructed.

Relying on the results of this paper, the first author has proved in [C] an asymptotic existence result for smooth nondegenerate curves in  $\mathbf{P}^r$  for *all* values of  $r$ , which essentially says that for  $d \gg 0$  smooth curves of degree  $d$  and genus  $g$  exist when

$$0 \leq g \leq \varphi_r(d)$$

where  $\varphi_r(d) \sim d^2 / 2(4r/3 - 1)$ , improving a similar one of Rathmann [Ra].

After this work was completed we have become aware of a preprint of Pasarescu [P], where he claims the existence of smooth nondegenerate curves of degree  $d$  and genus  $g$  in  $\mathbf{P}^r$  for all  $r \geq 5$  and all  $d, g$  such that

$$0 \leq g \leq (d - 1)^2 / 2(2r - 2).$$

His proof appears incomplete to us as it stands (on page 9, line -4, the maximum is not necessarily attained at an *integer*, as needed). From the argument of Pasarescu it seems to us that only a weaker bound, which is worse than ours for every  $d, g, r$ , can be deduced.

The second author would like to thank G. Pareschi for suggesting the proof of linear normality given in section 4, 2), and C. Procesi for a useful conversation on infinite reflection groups.

## 1. Preliminaries

As already stated in the introduction, we work over an algebraically closed field  $\mathbf{k}$  of arbitrary characteristic. We denote by  $S$  the surface obtained by blowing up nine points  $P_1, \dots, P_9$  of  $\mathbf{P}^2$  which are base points of a *generic* pencil of cubics; we let  $\pi: S \rightarrow \mathbf{P}^2$  be the projection. Note that any cubic  $C \subset \mathbf{P}^2$  containing  $P_1, \dots, P_9$  is reduced, irreducible and with at most one node and no other singularity.

Let's denote by  $E_1, \dots, E_9$  the exceptional curves (of the first kind) on  $S$  corresponding to  $P_1, \dots, P_9$ , and by  $H$  the inverse image on  $S$  of a general line of  $\mathbf{P}^2$ .

We identify an invertible sheaf on  $S$  with its class in  $\text{Pic}(S)$ . As a basis of  $\text{Pic}(S)$  we take the classes  $\mathbf{o}(H), \mathbf{o}(-E_1), \dots, \mathbf{o}(-E_9)$ ; we will sometimes denote an element of  $\text{Pic}(S)$  by the 10-tuple of its coordinates with respect to this basis.

We have:

$$\omega_S = (-3, -1, \dots, -1) = \mathbf{o}(-C)$$

where  $C$  is the proper transform of a cubic through  $P_1, \dots, P_9$ .

We will use without further mention the obvious fact that if  $D$  is an irreducible curve on  $S$  such that  $(D, \omega_S) = 0$ , then  $D \in |-\omega_S|$ .

We will freely use the notion of 1-connectedness of an effective divisor on a surface. We will also use without further notice the following vanishing theorem, referring the reader to [R] for the proof.

**VANISHING THEOREM:** *If  $D$  is an effective 1-connected divisor on a projective nonsingular surface  $F$  such that  $h^1(F, \mathbf{o}_F) = 0$ , then*

$$H^1(F, \mathbf{o}_F(-D)) = (0).$$

**PROPOSITION 1.** *Let  $D$  be an effective divisor on  $S$ .*

a) *If  $D$  is 1-connected and  $(D, \omega_S) \leq -2$ , then the linear system  $|D|$  has no base points.*

b) *If  $D$  is 1-connected and  $(D, \omega_S) \leq -3$ , then  $|D|$  has no base points, the morphism  $\varphi_D: S \rightarrow \mathbf{P}(H^0(S, \mathbf{o}(D)))$  is an isomorphism of  $S$  onto its image, except possibly for the contraction of some exceptional curves, and the image  $\varphi_D(S)$  is nonsingular. In particular, a general element of  $|D|$  is irreducible and nonsingular.*

c) *If  $D$  is 1-connected and  $(D, \omega_S) \leq -3$ , then  $D$  is very ample if and only if  $(D, E) > 0$  for every exceptional curve  $E$ .*

d) If  $|\omega_S(D)|$  is not empty, contains an effective 1-connected divisor and  $(D, \omega_S) \leq -3$ , then  $|D|$  is very ample.

*Proof.* a) For every  $C \in |-\omega_S|$  we have an exact sequence

$$0 \rightarrow \omega_S(D) \rightarrow \mathbf{o}(D) \rightarrow \mathbf{o}_C(D) \rightarrow 0.$$

Since, by the connectedness of  $D$ ,  $h^1(S, \omega_S(D)) = h^1(S, \mathbf{o}(-D)) = 0$ , we see that the restriction map  $H^0(S, \mathbf{o}(D)) \rightarrow H^0(C, \mathbf{o}_C(D))$  is surjective. Therefore  $|D|$  cuts a complete series on  $C$  of degree  $(D, C) \geq 2$ , hence without base points (recall that every  $C \in |-\omega_S|$  is reduced and irreducible). It follows that  $|D|$  has no base points on  $C$ . Since  $\dim(|-\omega_S|) > 0$ , the conclusion follows.

Proof of b) and c). By part a),  $|D|$  has no base points. Let's denote by  $|D - p|$  the linear system consisting of the curves of  $|D|$  passing through  $p$ , for a given point  $p$ .

**CLAIM 1.** *Let  $p$  be any point of  $S$ ; if  $|D - p|$  has a fixed part, then it consists of an exceptional curve  $E$  passing through  $p$ . Moreover  $(D, E) = 0$ , i.e.  $E$  is contracted to a point by the morphism  $\varphi_D$ .*

*Proof of claim 1.* The fixed divisor  $F$  of  $|D - p|$  satisfies  $(F, \omega_S) = -1$ , because  $|D|$  cuts a complete series on any  $C \in |-\omega_S|$  and  $|D - p|$  has codimension one in  $|D|$ . Therefore  $F = E$  is reduced, irreducible and rational (because  $|-\omega_S|$  cuts on  $E$  a series of dimension and degree one), hence it is an exceptional curve of the first kind. Moreover, since  $p$  is a fixed point of the linear series  $|D - p|_C$  cut on  $C$  by  $|D - p|$ , and since  $|D - p|_C$  has codimension at most one in  $|D|_C$ , necessarily  $E$  contains  $p$ . Since  $|D|$  has no base points we have  $(D, E) \geq 0$ . If  $(D, E) > 0$ , then from the exact sequence

$$0 \rightarrow \mathbf{o}(D - E) \rightarrow \mathbf{o}(D) \rightarrow \mathbf{o}_E(D) \rightarrow 0$$

and from  $h^0(S, \mathbf{o}(D - E)) = h^0(S, \mathbf{o}(D) \otimes \mathbf{I}_p) = h^0(S, \mathbf{o}(D)) - 1$  ( $\mathbf{I}_p \subset \mathbf{o}_s$  the ideal sheaf of  $p$ ) it follows that  $|D|$  has base points on  $E$ ; this is a contradiction. This proves the claim.

As a consequence we have:

**CLAIM 2.** *If  $p$  and  $q$  are distinct points on  $S$ ,  $|D|$  does not separate  $p$  and  $q$  if and only if  $p$  and  $q$  are both contained in an exceptional curve  $E$  such that  $(D, E) = 0$ .*

*Proof of claim 2.* If  $p$  and  $q$  are not separated by  $|D|$ , then they cannot belong

to the same  $C \in |-\omega_S|$  because  $|D|_C$  is very ample on  $C$ . If the general curve of  $|D - p|$  is reducible then, by claim 1, it contains an exceptional curve  $E$  and, by the same claim, this curve contains both  $p$  and  $q$ . If the general curve  $M$  of  $|D - p|$  is irreducible, then it passes simply through  $p$ , because on the curve  $C \in |-\omega_S|$  containing  $p$ ,  $|D - p|_C$  has codimension one in  $|D|_C$ , hence it cannot have  $2p$  as a fixed divisor. Similarly  $M$  passes simply through  $q$ . If  $M$  has genus  $g$ , then the degree of  $|D|_M$  is at least  $2g + 1$ ; it follows that  $|D|_M$  is very ample, therefore  $p$  and  $q$  are separated by  $|D|_M$ , and this is a contradiction.

Next we prove the following

**CLAIM 3.** *If  $p$  does not belong to an exceptional curve  $E$  such that  $(D, E) = 0$ , then  $\varphi_D$  separates tangent vectors in  $p$ .*

*Proof of claim 3.* The general curve  $M$  of  $|D - p|$  is irreducible and nonsingular in  $p$ . Since  $2p$  is not a fixed divisor of  $|D - p|_M$ , because  $|D - p|_M$  has codimension at most one in the complete and very ample  $|D|_M$ , the curves of  $|D - p|$  are not all tangent to each other in  $p$ ; this proves the claim.

Finally we prove

**CLAIM 4.** *If  $E$  is an exceptional curve such that  $(D, E) = 0$ , then  $\varphi_D(E)$  is a nonsingular point of the surface  $\varphi_D(S)$ . In particular  $\varphi_D(S)$  is nonsingular.*

*Proof of claim 4.* On every curve  $C \in |-\omega_S|$  the series  $|D - E|_C$  coincides with the complete series  $|D|_C - (E, C)$ . Since we have  $(D - E, \omega_S) = (D, \omega_S) + 1 \leq -2$ . It follows that  $|D - E|$  has no base points on  $C$ , hence  $|D - E|$  has no base points. The surface  $\varphi_{D-E}(S)$  can be regarded as the projection of  $\varphi_D(S)$  from the point  $q = \varphi_D(E)$ . Letting  $\mu = \text{mult}_q((\varphi_D(S)))$ , we have

$$\begin{aligned} \deg(\varphi_D(S)) - \mu &= \deg[\varphi_{D-E}(S)] \deg(\varphi_{D-E}) \\ &= (D - E, D - E) = (D, D) - 1 = \deg(\varphi_D(S)) - 1, \end{aligned}$$

it follows that  $\mu = 1$ , hence  $\varphi_D(E)$  is nonsingular.

Assertions b) and c) of the proposition are clearly a consequence of claims 1), . . . , 4).

d) Since  $|D| = |\omega_S(D) + C|$ ,  $C \in |-\omega_S|$ , it follows that  $|D|$  contains a 1-connected divisor  $D'$ . Since, by a),  $|\omega_S(D)|$  has no base points, for every exceptional curve  $E$  we have  $(\omega_S(D), E) \geq 0$ , and therefore

$$(D', E) = (D, E) \geq -(\omega_S, E) = 1 > 0.$$

From c) it follows that  $|D| = |D'|$  is very ample.

**PROPOSITION 2.** *Let  $D$  be an effective 1-connected divisor on  $S$ . Then every  $D' \in |D|$  is 1-connected.*

*Proof.* Since  $D$  is effective we have  $(D, \omega_S) \leq 0$ . If  $(D, \omega_S) = 0$  then  $|D| = \{-\omega_S\}$  and the conclusion is clear. If  $(D, \omega_S) = -1$  then  $D = E + C_1 + \dots + C_h$ , with  $E$  exceptional curve and  $C_1, \dots, C_h \in \{-\omega_S\}$ , and  $E$  is a fixed component of  $|D|$ . Then  $D'$  has a similar decomposition, hence it is 1-connected.

Suppose now that  $(D, \omega_S) = -2$  and that  $D' = A_1 + A_2$ , with  $A_1, A_2$  effective. Since  $(A_i, \omega_S) \leq 0$  we have  $-2 \leq (A_i, \omega_S) \leq 0$ ,  $i = 1, 2$ . If  $(A_1, \omega_S) = 0$ ,  $(A_2, \omega_S) = -2$  then  $A_1 \in \{-h\omega_S\}$  for some  $h \geq 1$ , hence  $(A_1, A_2) = 2h > 0$ . Similar conclusion we have if  $(A_1, \omega_S) = -2$ ,  $(A_2, \omega_S) = 0$ . If  $(A_1, \omega_S) = -1 = (A_2, \omega_S)$  then  $A_1 = E_1 + C_1 + \dots + C_h$ ,  $A_2 = E_2 + C'_1 + \dots + C'_k$ , with  $E_1, E_2$  exceptional curves and  $C_1, \dots, C'_k \in \{-\omega_S\}$ . We now have  $(A_1, A_2) > 0$  except in the case  $A_1 = E + C$ ,  $A_2 = E$ . But then  $(D, D) = (D', D') = 0$ , hence  $|D|$ , being base point free by proposition 1, part a), is composed with a pencil; moreover, since  $D$  is 1-connected,  $|D|$  is a pencil. But then, since  $|D| = |2E + C|$  and  $C$  moves in a pencil, we see that  $2E$  is a fixed divisor of  $|D|$ , and this contradicts proposition 1, part a).

Finally assume that  $(D, \omega_S) \leq -3$ . Since  $(C, \omega_S(-D)) \leq -3 < 0$ , we have

$$h^2(S, \mathbf{o}(D)) = h^0(S, \omega_S(-D)) = 0.$$

From the Riemann–Roch theorem we therefore obtain

$$\dim(|D|) \geq ((D, D) - (D, \omega_S))/2;$$

since, by proposition 1, part a),  $|D|$  is base point free, we have  $(D, D) \geq 0$  hence  $\dim(|D|) \geq 2$ . therefore  $|D|$  is not a pencil; moreover  $|D|$  cannot be composed with a pencil because  $D$  is 1-connected. It follows that  $(D, D) > 0$ . Now the conclusion follows from lemma 2 of [R].

We will say that an effective class  $\mathbf{o}(D) \in \text{Pic}(S)$ , respectively a linear system  $|D|$ , is 1-connected if  $|D|$  contains a 1-connected divisor, or equivalently, if every divisor of  $|D|$  is 1-connected. The equivalence of the two conditions follows from proposition 2.

## 2. $\delta$ -systems on $S$

We will denote by  $\mathbf{G}$  the group of Cremona isometries of  $S$ , namely the group of automorphisms of  $\text{Pic}(S)$  which

- 1) leave the semigroup of effective classes invariant,
- 2) preserve the intersection form  $(,)$ ,

3) leave the canonical class  $\omega_S$  invariant.

We will need the following elementary result.

LEMMA 3. *If  $D$  is an effective 1-connected divisor and  $\sigma \in \mathbf{G}$ , then  $\sigma(\mathbf{o}(D)) = \mathbf{o}(\bar{D})$ , where  $\bar{D}$  is an effective 1-connected divisor.*

*Proof.* Because of the defining property 1) of  $\mathbf{G}$ , an effective divisor  $\bar{D}$  such that  $\sigma(\mathbf{o}(D)) = \mathbf{o}(\bar{D})$  exists. Suppose that  $\bar{D} = \bar{A}_1 + \bar{A}_2$ , with  $\bar{A}_1$  and  $\bar{A}_2$  effective. Then, again by 1),  $\sigma^{-1}(\bar{A}_1) = \mathbf{o}(A_i)$ , with  $A_i$  effective,  $i = 1, 2$ . Since  $\mathbf{o}(D) = \mathbf{o}(A_1 + A_2)$ , and  $D$  is 1-connected, from proposition 2 it follows that  $(A_1, A_2) \geq 1$ . Hence  $(\bar{A}_1, \bar{A}_2) = (A_1, A_2) \geq 1$  and therefore  $\bar{D}$  is 1-connected.

With an abuse of notation we will often write  $\sigma(D)$  instead of  $\sigma(\mathbf{o}(D))$ , for a divisor  $D$  on  $S$  and  $\sigma \in \mathbf{G}$ . We will denote by  $g(D)$  the arithmetic genus of  $D$ , namely

$$g(D) := (D, D + \omega_S)/2 + 1.$$

We give the following definition.

DEFINITION. Let  $\delta \geq 2$  be an integer. An ordered  $\delta$ -tuple  $\{D_0, D_1, \dots, D_{\delta-1}\}$  of effective classes in  $\text{Pic}(S)$  is called a  $\delta$ -system if the following conditions are satisfied:

- I)  $(D_i, \omega_S) = -\delta$ ,  $i = 0, \dots, \delta - 1$ .
- II)  $(D_i, D_i) = -\delta + 2i$ ,  $i = 0, \dots, \delta - 1$ .
- III)  $D_0 - \omega_S, \dots, D_{\delta-2} - \omega_S, D_{\delta-1}$  are effective and 1-connected.
- IV) For every  $0 \leq i, j \leq \delta - 1$ , the number  $(D_i, \sigma(D_j))$  assumes all integral values  $N$  such that

$$i + j + 2 - \delta \leq N,$$

when  $\sigma$  varies in  $\mathbf{G}$ .

Note that property IV) is clearly equivalent to the following:

V) for every  $0 \leq i \leq j \leq \delta - 1$ , all integral values  $N \geq i + j + 2 - \delta$  are assumed by  $(\sigma(D_i), \rho(D_j))$  as  $\sigma, \rho$  vary in  $\mathbf{G}$ .

In this section we will investigate some of the consequences of the existence of a  $\delta$ -system of divisors on  $S$  for some  $\delta$ . In the next section we will construct such systems for  $\delta = 3, 4$ .

PROPOSITION 4. *Let  $\{D_0, D_1, \dots, D_{\delta-1}\}$  be a  $\delta$ -system on  $S$ , with  $\delta \geq 3$ .*

Then

i) for every  $\alpha \geq 1$  the linear systems

$$|D_0 - \alpha\omega_S|, \dots, |D_{\delta-2} - \alpha\omega_S|, |D_{\delta-1} - (\alpha-1)\omega_S|$$

are base point free and with general member irreducible and nonsingular; moreover for every  $\alpha \geq 2$  they are very ample.

ii) We have:

$$(D_1 - \alpha\omega_S, D_1 - \alpha\omega_S) = (2\alpha - 1)\delta + 2i,$$

$$g(D_1 - \alpha\omega_S) = (\alpha - 1)\delta + i + 1,$$

$$\dim(|D_i - \alpha\omega_S|) = \alpha\delta + i \quad (= g(D_i - \alpha\omega_S) + \delta - 1)$$

for  $0 \leq i \leq \delta - 2$  and for every  $\alpha \geq 1$ , and for  $i = \delta - 1$  and for every  $\alpha \geq 0$ .

*Proof.* From the defining property III) it follows that each of the linear systems  $|D_0 - \alpha\omega_S|, \dots, |D_{\delta-2} - \alpha\omega_S|, |D_{\delta-1} - (\alpha-1)\omega_S|$  contains a 1-connected effective divisor for all  $\alpha \geq 1$  and has intersection number equal to  $-\delta \leq -3$  with  $\omega_S$ . Applying parts b) and d) of proposition 1 we obtain i). The expressions for  $(D_i - \alpha\omega_S, D_i - \alpha\omega_S)$  and  $g(D_i - \alpha\omega_S)$  are obvious. The  $\dim(|D_i - \alpha\omega_S|)$  is computed using the Riemann–Roch theorem on  $S$ , noting that

$$h^1(S, \mathbf{o}(D_1 - \alpha\omega_S)) = 0 = h^2(S, \mathbf{o}(D_1 - \alpha\omega_S)).$$

The first equality follows because  $|D_i - \alpha\omega_S - \omega_S|$  contains a 1-connected divisor when  $\alpha$  and  $i$  assume the indicated values. The second equality is obvious because

$$h^2(S, \mathbf{o}(D_1 - \alpha\omega_S)) = h^0(S, \mathbf{o}(-(D_1 - (\alpha+1)\omega_S)))$$

and  $|D_1 - (\alpha+1)\omega_S| \neq \emptyset$ . This proves ii).

**PROPOSITION 5.** *Assume that a  $\delta$ -system  $\{D_0, D_1, \dots, D_{\delta-1}\}$  exists on  $S$  for some  $\delta \geq 3$ . For each  $r \geq 2\delta - 1$  write  $r = n\delta + i$ ,  $i \in \{0, \dots, \delta - 1\}$ , and let*

$$H_r := D_i - n\omega_S.$$

*Then  $H_r$  is very ample and for every  $(d, g)$  such that  $0 \leq g \leq d - r - 1$ , there exists*

an irreducible and nonsingular curve  $X \subset S$  such that

$$(H_r, X) = d, \quad g(X) = g$$

and

$$h^0(S, \mathbf{o}(H_r - X)) = 0.$$

*Proof.* Note that, since  $r \geq 2\delta - 1$ , we have  $n \geq 2$  except in the case  $r = 2\delta - 1$  when  $n = 1$ ,  $i = \delta - 1$ , hence  $H_r$  is very ample by proposition 4. We can write in a unique way

$$g = (\alpha - 1)\delta + j + 1$$

for some  $j \in \{0, \dots, \delta - 1\}$  and  $\alpha \geq 0$  ( $\alpha \geq 1$  if  $0 \leq j \leq \delta - 2$  and  $\alpha = 0$  if and only if  $g = 0$ ). For every  $\sigma \in \mathbf{G}$  we have

$$\begin{aligned} (H_r, \sigma(D_j - \alpha\omega_S)) &= (D_i - n\omega_S, \sigma(D_j) - \alpha\omega_S) \\ &= (D_i, \sigma(D_j)) + (\alpha + n)\delta \\ &= (D_i, \sigma(D_j)) + g + r + \delta - 1 - i - j. \end{aligned}$$

Since  $d \geq g + r + 1$ , we have

$$d - [g + r + \delta - 1 - i - j] \geq i + j + 2 - \delta,$$

hence, by the defining property IV), there exists  $\sigma \in \mathbf{G}$  such that

$$(D_i, \sigma(D_j)) = d - [g + r + \delta - 1 - i - j],$$

equivalently such that

$$(H_r, \sigma(D_j - \alpha\omega_S)) = d.$$

Since by lemma 3  $|\sigma(D_j - \alpha\omega_S)|$  contains an effective 1-connected divisor, from proposition 1b) it follows that  $|\sigma(D_j - \alpha\omega_S)|$  contains an irreducible and nonsingular curve  $X$ . From proposition 4 we deduce that this curve has genus

$$g(D_j - \alpha\omega_S) = (\alpha - 1)\delta + j + 1 = g.$$

By contradiction, let's assume that

$$(*) \quad h^0(S, \mathbf{o}(H_r - X)) \neq 0.$$

From  $(*)$  and  $(H_r - X, \omega_S) = 0$  we deduce that  $|H_r - X| = |-a\omega_S|$ ,  $a \geq 0$ . Therefore

$$|\sigma(D_j) - a\omega_S| = |X| = |H_r + a\omega_S| = |D_i + (a - n)\omega_S|.$$

and it follows that  $i = j$  and  $\sigma(D_j) = D_i$ . Then:

$$-\delta + 2i = (D_i, D_i) = (D_i, \sigma(D_j)) \geq 2i + 2 - \delta,$$

a contradiction. This concludes the proof.

Using proposition 5 we can prove the following consequence of the existence of a  $\delta$ -system on  $S$  for some  $\delta \geq 3$ .

**THEOREM 6.** *Assume that a  $\delta$ -system  $\{D_0, D_1, \dots, D_{\delta-1}\}$  exists on  $S$  for some  $\delta \geq 3$  and let  $r \geq 2\delta - 1$  be an integer. Then there exists an embedding of  $S$  as a nonsingular surface  $F$  of degree  $2r - \delta$  in  $\mathbf{P}^r$ , and for every  $(d, g)$  such that*

$$0 \leq g \leq (d - r)^2/2(2r - \delta) \quad (1)$$

*there exists a nonsingular irreducible and nondegenerate curve  $X$  of degree  $d$  and genus  $g$  on  $F$ .*

*Proof.* Writing  $r = n\delta + i$ , the embedding of  $S$  is given by the very ample class  $H_r = D_i - n\omega_S$  considered in proposition 5. If  $0 \leq g \leq d - r - 1$  the assertion has already been proved (proposition 5). Note that if a nonsingular irreducible and nondegenerate curve  $Z$  lies on  $F$  then a general element  $Z' \in |Z + H_F|$  ( $H_F$  a hyperplane section of  $F$ ) is a nonsingular irreducible curve, by proposition 1b), and is obviously nondegenerate. If  $Z$  has (degree, genus) =  $(d, g)$ , then  $Z'$  has (degree, genus) =  $(d', g')$  given by

$$(d', g') = (d + 2r - \delta, g + d + r - \delta).$$

The inverse of the transformation

$$\begin{aligned} d' &= d + 2r + \delta \\ g' &= g + d + r - \delta \end{aligned} \quad (2)$$

is

$$\begin{aligned} d &= d' - 2r + \delta \\ g &= g' - d' + r. \end{aligned} \quad (3)$$

Applying (2)  $s$  times we obtain the transformation

$$\begin{aligned} d^{(s)} &= d + s(2r - \delta) \\ g^{(s)} &= g + sd + s(s-1)(2r - \delta)/2 + s(r - \delta) \end{aligned} \tag{2<sup>s</sup>}$$

whose inverse is:

$$\begin{aligned} d &= d^{(s)} - s(2r - \delta) \\ g &= g^{(s)} - sd^{(s)} + s(s+1)(2r - \delta)/2 - s(r - \delta). \end{aligned} \tag{3<sup>s</sup>}$$

In the plane with coordinates  $d, g$  represent with integral points the couples  $(d, g)$  for which we want to prove the theorem. They fill the region  $R$  under the parabola  $K$  with equation

$$g = (d - r)^2/2(2r - \delta).$$

We know that the theorem is true for all the points strictly below the line  $L_0$  with equation  $g = d - r$  and above the  $d$ -axis, by proposition 5. Let's denote by  $V_0$  the set of these points. The transformation (2) maps the  $d$ -axis into  $L_0$  and maps  $L_0$  into the line  $L_1: g = 2d - 4r + \delta$ . Therefore (2) maps  $V_0$  in the set of points  $(d', g')$  such that

$$d' - r \leq g' < 2d' - 4r + \delta.$$

Since the theorem is also true in  $V_0$ , we see that the transformation (2) and the remark made at the beginning allow us to extend the validity of the theorem to  $V_0 \cup V_1$ , where  $V_1$  is defined by:

$$0 \leq g \leq 2d - 4r + \delta.$$

Note that the two lines  $L_0$  and  $L_1$  have in common the point  $P_1 = (3r - \delta, 2r - \delta)$ . For every  $s \geq 1$  let's denote by  $L_s$  the line whose equation is

$$g = (s+1)d - s(s+3)(2r - \delta)/2 + s(r - \delta) - r,$$

and which is the image of  $L_0$  under (2<sup>s</sup>). The line  $g = 0$  will be denoted  $L_{-1}$ ; it is transformed by (2<sup>s</sup>) into  $L_{s-1}$ .

By induction we deduce that for every  $k \geq 0$  the theorem is true in

$V_0 \cup \dots \cup V_k$  where  $V_s$  is the region defined by the inequalities:

$$0 \leq g < (s+1)d - s(s+3)(2r-\delta)/2 + s(r-\delta) - r.$$

Note that  $L_s \cap L_{s-1}$  is the point

$$P_s = ((2s+1)r - s\delta, s(s+1)(2r-\delta)/2).$$

In particular  $P_0 = (r, 0)$ . Note that the expression  $g - (d-r)^2/2(2r-\delta)$  is zero in  $P_0$ . If we prove that it is positive in all the points  $P_s \geq 1$ , then these points are above the parabola  $K$ . Since  $K$  is concave upwards, it follows that the segments  $P_{s-1}P_s$  lie above  $K$  and the theorem is true. In  $P_s$  we have:

$$\begin{aligned} (d-r)^2/2(2r-\delta) &= (2sr - s\delta)^2/2(2r-\delta) \\ &= s^2(2r-\delta)^2/2(2r-\delta) = s^2(2r-\delta)/2. \end{aligned}$$

Therefore in  $P_5$ :

$$g - (d-r)^2/2(2r-\delta) = s(s+1)(2r-\delta)/2 - s^2(2r-\delta)/2 = s(2r-\delta)/2,$$

which is positive for all  $s \geq 1$ . This concludes the proof.

*Remark 1.* Note that the proof of theorem 6 actually shows the existence of curves  $X$  of degree  $d$  and genus  $g$  on  $S$  for all  $(d, g)$  located below the polygon whose vertices are the points  $P_s$  considered in the proof. This region is slightly larger than that defined by (1).

### 3. Existence of $\delta$ -systems for $\delta = 3, 4$ .

In this section we discuss the existence of  $\delta$ -systems on  $S$ . We will show that  $\delta$ -systems exist for  $\delta = 3, 4$ .

The basis  $\mathbf{o}(H), \mathbf{o}(-E_1), \dots, \mathbf{o}(-E_9)$  identifies  $\text{Pic}(S)$  with  $\mathbf{Z}^{10}$  and the intersection form  $(,)$  with the inner product  $x_0^2 - \sum_1^9 x_i^2$ . Consider the elements of  $\text{Pic}(S)$ :

$$r_1 = (1, 1, 1, 1, 0, \dots, 0), \quad r_2 = (0, -1, 1, 0, \dots, 0),$$

$$r_3 = (0, 0, -1, 1, 0, \dots, 0), \dots$$

$$r_8 = (0, \dots, 0, -1, 1, 0), \quad r_9 = (0, \dots, 0, -1, 1).$$

Letting:

$$f_i(x) = x + (x, r_i)r_i, \quad i = 1, \dots, 9,$$

we obtain elements  $f_1, \dots, f_g \in \mathbf{G}$ . Recall that  $f_1, f_2, \dots, f_9$  act on an element  $(x_0, x_1, \dots, x_9) \in \text{Pic}(S)$  in the following way:

$$f_1(x_0, x_1, \dots, x_9) = (x_0 + h, x_1 + h, x_2 + h, x_3 + h, x_4, \dots, x_9),$$

$$h = x_0 - x_1 - x_2 - x_3.$$

$$f_2(x_0, x_1, \dots, x_9) = (x_0, x_2, x_1, x_3, \dots, x_9),$$

$$f_3(x_0, x_1, \dots, x_9) = (x_0, x_1, x_3, x_2, x_4, \dots, x_9),$$

...

$$f_g(x_0, x_1, \dots, x_9) = (x_0, x_1, \dots, x_7, x_9, x_8).$$

In particular note that combining  $f_2, \dots, f_9$  we can obtain any permutation of  $x_1, \dots, x_9$ . We will also consider, for every  $z \in \omega_S^\perp$ , the element  $\tau_z \in \mathbf{G}$  defined as follows:

$$\tau_z(x) = x - (x, z)\omega_S + (x, \omega_S)z - (z, z)(x, \omega_S)\omega_S/2.$$

The following lemma generalizes lemma 1.4.1 of [Ra].

LEMMA 7. *Let  $x, y \in \text{Pic}(S)$  be such that  $(x, \omega_S) = (y, \omega_S) = -\delta$ , for some  $\delta \geq 1$ , and such that  $x - y = (u_0, u_1, \dots, u_g)$  satisfies either one of the following two conditions:*

- i)  $u_1 = u_2, u_3 = u_4, u_5 = u_6, u_7 = u_8$ .
- ii)  $u_1 = u_2, u_3 = u_4, u_5 = u_6, u_7 - u_8 = \delta$ .

*Then  $(\rho(y), \sigma(x))$  assumes all integer values  $(y, x) + n\delta^2$ ,  $n \geq 0$ , as  $\rho, \sigma$  vary in  $\mathbf{G}$ .*

*Proof.* It suffices to show that the conclusion holds taking  $\rho = \text{identity}$  and  $\sigma = \tau_z$ ,  $z \in \omega_S^\perp$ . For every  $z \in \omega_S^\perp$  we have:

$$\begin{aligned} (y, \tau_z(x)) &= (y, x - (x, z)\omega_S + (x, \omega_S)z - (z, z)(x, \omega_S)\omega_S/2) \\ &= (y, x) - (x, z)(y, \omega_S) + (x, \omega_S)(y, z) - (z, z)(x, \omega_S)(y, \omega_S)/2 \\ &= (y, x) + \delta(x - y, z) - (z, z)\delta^2/2. \end{aligned}$$

Suppose that we are in case i). Then, taking

$$z = (0, a, -a, b, -b, c, -c, d, -d, 0)$$

we have  $(x - y, z) = 0$  and  $-(z, z)/2 = a^2 + b^2 + c^2 + d^2$  and the conclusion follows from the fact that every positive integer is the sum of four squares. Suppose now that we are in case ii). Taking

$$z = (0, a, -a, b, -b, c, -c, 0, 0, 0)$$

we obtain  $(x - y, z) = 0$  and  $-(z, z)/2 = a^2 + b^2 + c^2$ . This takes care of the cases in which  $n \equiv 1, 2, 3, 5, 6 \pmod{8}$ , because every such integer  $n$  is the sum of three squares (see [5]). If  $n \equiv 0, 4, 7 \pmod{8}$  then  $n - 2 > 0$ ; we can write

$$n - 2 = a^2 + b^2 + c^2$$

and we take

$$z = (0, a, -a, b, -b, c, -c, 1, -1, 0).$$

We obtain  $(x - y, z) = \delta$ , and therefore:

$$(y, \tau_z(x)) = (y, x) + \delta^2 - (z, z)\delta^2/2 = (y, x) + \delta^2 + \delta^2(n - 1).$$

This concludes the proof.

The existence of a 3-system is our next result. One half of the computations of this theorem are already in [Ra], where the divisors  $D_0$ ,  $D_1$ ,  $D_2$  of theorem 8 are also considered.

#### THEOREM 8. *The classes*

$$D_0 = (0, 0, 0, 0, 0, 0, 0, -1, -1, -1)$$

$$D_1 = (1, 1, -1, 0, 0, 0, 0, 0, 0, 0)$$

$$D_2 = (1, 0, 0, 0, 0, 0, 0, 0, 0, 0)$$

*are a 3-system on  $S$ .*

*Proof.* The defining properties I), II) and III) are obviously satisfied. Let's prove property V). For every  $0 \leq i \leq j \leq 2$  and for every

$$i + j - 1 \leq N \leq i + j + 7$$

we will find elements of the form  $\sigma(D_i)$ ,  $\rho(D_j)$ ,  $\sigma, \rho \in \mathbf{G}$ , such that:

- 1)  $(\sigma(D_i), \rho(D_j)) = N$ ,
- 2)  $\sigma(D_i) - \rho(D_j)$  satisfies either one of conditions i), ii) of lemma 7.

Then property V), and the conclusion, will follow from lemma 7 applied to  $x = \sigma(D_i)$  and  $y = \rho(D_j)$ . We give a list of such elements below. Each  $\sigma(D_i)$  and  $\rho(D_j)$  is obtained from  $D_i$  and  $D_j$  respectively by acting with a combination of the elements  $f_1, \dots, f_g$  of  $\mathbf{G}$ . The tables are the following:

$\sigma(D_0)$	$\rho(D_0)$	$(\sigma(D_0), \rho(D_0))$
$(0, -1, -1, 0, 0, 0, 0, 0, 0, -1)$	$(0, 0, 0, -1, -1, 0, 0, 0, 0, -1)$	-1
$(0, -1, -1, 0, 0, 0, 0, 0, 0, -1)$	$(1, 0, 0, 1, 1, -1, -1, 0, 0, 0)$	0
$(0, 0, 0, 0, 0, 0, -1, -1, -1)$	$(2, 1, 1, 0, 0, 0, 0, -1, 2, 0)$	1
$(0, -1, -1, 0, 0, 0, 0, 0, 0, -1)$	$(1, 1, 1, 0, 0, -1, -1, 0, 0, 0)$	2
$(0, -1, -1, 0, 0, 0, 0, 0, 0, -1)$	$(2, 1, 1, -1, -1, 1, 1, 0, 0, 1)$	3
$(0, 0, 0, 0, 0, 0, -1, -1, -1)$	$(3, 1, 1, -1, -1, 1, 1, 1, 1, 2)$	4
$(2, 1, 1, -1, -1, 1, 1, 0, 0, 1)$	$(2, 1, 1, 1, 1, -1, -1, 0, 0, 1)$	5
$(2, 1, 1, 1, -1, -1, 0, 0, 1)$	$(3, 1, 1, -1, -1, 1, 1, 1, 1, 2)$	6
$(1, -1, -1, 1, 1, 0, 0, 0, 0, 0)$	$(3, 1, 1, -1, -1, 1, 1, 1, 1, 2)$	7

$\sigma(D_0)$	$\rho(D_1)$	$(\sigma(D_0), \rho(D_1))$
$(0, 0, -1, -1, -1, 0, 0, 0, 0, 0)$	$(1, 1, 0, 0, 0, 0, 0, 0, 0, -1)$	0
$(0, -1, -1, 0, 0, 0, 0, 0, 0, -1)$	$(2, 1, 1, 1, 1, 0, 0, 0, 0, -1)$	1
$(2, 1, 1, 1, 0, 1, 1, -1, -1, 0)$	$(1, 0, 0, 0, -1, 0, 0, 0, 0, 1)$	2
$(2, 1, 1, 0, 0, 1, 1, -1, -1, 1)$	$(2, 1, 1, 1, 1, 0, 0, 0, 0, -1)$	3
$(1, -1, -1, 0, 0, 0, 0, 1, 1, 0)$	$(3, 1, 1, 1, 1, 0, 0, -1, 2, 1)$	4
$(1, -1, -1, 0, 0, 1, 1, 0, 0, 0)$	$(3, 1, 1, 1, 1, 0, 0, -1, 2, 1)$	5
$(3, 1, 1, -1, -1, 1, 1, 1, 1, 2)$	$(3, 1, 1, 1, 1, 0, 0, -1, 2, 1)$	6
$(2, 1, 1, 0, 0, 1, 1, -1, -1, 1)$	$(2, 0, 0, 1, 1, 0, 0, 1, 1, -1)$	7
$(3, 1, 1, -1, -1, 1, 1, 1, 1, 2)$	$(2, 1, 1, 1, 1, 0, 0, 0, 0, -1)$	8

$\sigma(D_0)$	$\rho(D_2)$	$(\sigma(D_0), \rho(D_2))$
$(0, -1, -1, -1, 0, 0, 0, 0, 0, 0)$	$(4, 0, 0, 1, 2, 2, 2, 1, 1, 0)$	1
$(0, -1, -1, -1, 0, 0, 0, 0, 0, 0)$	$(2, 1, 1, 0, 1, 0, 0, 0, 0, 0)$	2
$(0, -1, -1, -1, 0, 0, 0, 0, 0, 0)$	$(4, 1, 1, 1, 2, 2, 2, 0, 0, 0)$	3
$(0, -1, -1, -1, 0, 0, 0, 0, 0, 0)$	$(6, 1, 1, 2, 3, 3, 3, 1, 1, 0)$	4
$(0, -1, -1, -1, 0, 0, 0, 0, 0, 0)$	$(4, 2, 2, 1, 2, 1, 1, 0, 0, 0)$	5
$(0, -1, -1, -1, 0, 0, 0, 0, 0, 0)$	$(6, 3, 3, 0, 1, 2, 2, 2, 2, 0)$	6

$(0, -1, -1, -1, 0, 0, 0, 0, 0, 0)$	$(8, 3, 3, 1, 2, 4, 4, 2, 2, 0)$	7
$(0, -1, -1, -1, 0, 0, 0, 0, 0, 0)$	$(6, 3, 3, 2, 3, 1, 1, 1, 1, 0)$	8
$(0, -1, -1, -1, 0, 0, 0, 0, 0, 0)$	$(8, 4, 4, 1, 2, 3, 3, 2, 2, 0)$	9
$\sigma(D_1)$	$\rho(D_1)$	$(\sigma(D_1),$ $\rho(D_1))$
$(1, 1, 0, -1, 0, 0, 0, 0, 0, 0)$	$(1, 0, -1, 0, 1, 0, 0, 0, 0, 0)$	1
$(1, 1, 0, 0, 0, 0, 0, 0, 0, -1)$	$(1, 0, -1, 0, 0, 0, 0, 0, 0, 1)$	2
$(2, 1, 1, 1, 1, 0, 0, 0, 0, -1)$	$(2, 0, 0, 0, 1, 1, 1, 1, -1)$	3
$(2, 1, 1, 0, 0, 0, 0, 1, 1, -1)$	$(3, 1, 1, 0, 0, 1, 1, -1, 2, 1)$	4
$(2, 1, 1, 1, 1, 0, 0, 0, 0, -1)$	$(3, 1, 1, 0, 0, 1, 1, -1, 2, 1)$	5
$(2, 1, 1, 1, 1, 0, 0, 0, 0, -1)$	$(3, 0, 0, 0, 1, 1, 2, 2, 0)$	6
$(3, 0, 0, 2, 2, 1, 1, 0, 0, 0)$	$(3, 0, 0, 0, 1, 1, 2, 2, 0)$	7
$(3, 0, 0, 2, 2, 0, 0, 1, 1, 0)$	$(3, 1, 1, 0, 0, 1, 1, -1, 2, 1)$	8
$(3, 1, 1, 2, 2, 0, 0, 0, 0, 0)$	$(3, 0, 0, 0, 1, 1, 2, 2, 0)$	9
$\sigma(D_1)$	$\rho(D_2)$	$(\sigma(D_1),$ $\rho(D_2))$
$(1, 0, 0, -1, 0, 0, 0, 0, 0, 1)$	$(2, 0, 0, 1, 0, 1, 1, 0, 0, 0)$	2
$(1, 1, 0, -1, 0, 0, 0, 0, 0, 0)$	$(3, 1, 0, 1, 2, 1, 1, 0, 0, 0)$	3
$(3, 1, 1, 2, 2, 0, 0, 0, 0, 0)$	$(2, 1, 1, 0, 0, 0, 0, 0, 0, 1)$	4
$(2, 0, 0, 1, 1, 1, 1, 0, 0, -1)$	$(2, 1, 1, 0, 0, 0, 0, 0, 0, 1)$	5
$(3, 1, 1, 2, 2, 0, 0, 0, 0, 0)$	$(2, 0, 0, 0, 1, 1, 0, 0, 0, 1)$	6
$(3, 1, 1, 2, 2, 0, 0, 0, 0, 0)$	$(3, 1, 1, 0, 0, 1, 1, 0, 0, 2)$	7
$(2, 0, 0, 1, 1, 1, 1, 0, 0, -1)$	$(3, 1, 1, 0, 0, 0, 0, 1, 1, 2)$	8
$(3, 1, 1, 2, 2, 0, 0, 0, 0, 0)$	$(3, 0, 0, 0, 1, 1, 1, 1, 1, 2)$	9
$(3, 1, 1, 2, 2, 0, 0, 0, 0, 0)$	$(4, 1, 0, 0, 1, 1, 1, 1, 3)$	10
$\sigma(D_2)$	$\rho(D_2)$	$(\sigma(D_2),$ $\rho(D_2))$
$(1, 0, 0, 0, 0, 0, 0, 0, 0, 0)$	$(3, 1, 1, 1, 0, 0, 0, 0, 0, 2)$	3
$(1, 0, 0, 0, 0, 0, 0, 0, 0, 0)$	$(4, 1, 1, 1, 1, 1, 0, 0, 0, 3)$	4
$(1, 0, 0, 0, 0, 0, 0, 0, 0, 0)$	$(5, 1, 1, 1, 1, 1, 1, 1, 1, 4)$	5
$(2, 1, 1, 0, 0, 0, 0, 0, 0, 1)$	$(4, 1, 1, 1, 1, 1, 1, 0, 3, 0)$	6
$(2, 1, 1, 0, 0, 0, 0, 0, 0, 1)$	$(5, 1, 1, 1, 1, 1, 1, 1, 4, 1)$	7
$(3, 1, 1, 1, 0, 0, 0, 0, 0, 2)$	$(4, 1, 1, 1, 1, 1, 1, 0, 3, 0)$	8
$(3, 1, 1, 1, 1, 0, 0, 0, 0, 2)$	$(5, 1, 1, 1, 1, 1, 1, 1, 4, 1)$	9
$(4, 1, 1, 1, 1, 1, 0, 0, 0, 3)$	$(4, 1, 1, 1, 1, 1, 1, 0, 3, 0)$	10
$(4, 1, 1, 1, 1, 1, 0, 0, 0, 3)$	$(5, 1, 1, 1, 1, 1, 1, 1, 4, 1)$	11

This concludes the proof of theorem 7.

Now we will prove the existence of a 4-system on  $S$ .

**THEOREM 9.** *The classes*

$$D_0 = (0, -1, -1, -1, -1, 0, 0, 0, 0, 0)$$

$$D_1 = (1, -1, -1, 0, 0, 0, 0, 0, 0, 1)$$

$$D_2 = (1, 0, 0, 0, 0, 0, 0, 0, 0, -1)$$

$$D_3 = (2, 1, 1, 0, 0, 0, 0, 0, 0, 0)$$

are a 4-system on  $S$ .

*Proof.* Also in this case it is obvious that properties I), II) and III) are satisfied. We will proceed as in the proof of theorem 8: by applying lemma 7, for every  $0 \leq i \leq j \leq 3$  and for every

$$i + j - 2 \leq N \leq i + j + 13$$

it will suffice to find elements of the form  $\sigma(D_i)$ ,  $\rho(D_j)$ ,  $\sigma, \rho \in \mathbf{G}$ , such that:

- 1)  $(\sigma(D_i), \rho(D_j)) = N$ ,
- 2)  $\sigma(D_i) - \rho(D_j)$  satisfies either one of conditions i), ii) of lemma 7. A list of such elements is given below:

$\sigma(D_0)$	$\rho(D_0)$	$(\sigma(D_0), \rho(D_0))$
$(0, -1, -1, -1, -1, 0, 0, 0, 0, 0)$	$(0, 0, 0, 1, -1, -1, -1, 0, 0, 0)$	-2
$(0, 0, 0, -1, -1, 0, -1, 0, -1, 0)$	$(2, 0, 0, -1, -1, 1, 0, 2, 1, 0)$	-1
$(0, -1, 1, -1, -1, 0, 0, 0, 0, 0)$	$(0, 0, 0, 0, 0, -1, -1, -1, -1, 0)$	0
$(0, 0, 0, -1, -1, 0, 0, 0, -1, -1)$	$(3, 0, 0, -1, -1, 1, 1, 2, 1, 2)$	1
$(1, -1, -1, 0, 0, 1, 1, 0, 0, -1)$	$(1, 1, 1, -1, -1, 0, 0, 0, 0, -1)$	2
$(0, -1, -1, 0, 0, 0, 0, -1, -1, -1)$	$(3, 0, 0, -1, -1, 1, 1, 2, 1, 2)$	3
$(1, -1, -1, 1, 1, 0, 1, 0, 0, -1)$	$(1, 1, 1, -1, -1, 0, 0, 0, 0, -1)$	4
$(0, 0, 0, 0, 0, -1, -1, 0, -1, -1)$	$(3, 0, 0, -1, -1, 1, 1, 2, 1, 2)$	5
$(1, 1, 1, -1, -1, 0, 0, 0, 0, -1)$	$(2, -1, -1, 0, 0, 1, 1, 0, 0, 2)$	6
$(2, 1, 0, 0, 0, -1, -1, 2, 1, 0)$	$(2, 0, -1, 0, 0, 1, 1, 0, -1, 2)$	7
$(1, 1, 1, -1, -1, 0, 0, 0, 0, -1)$	$(2, -1, -1, 1, 1, 0, 0, 0, 0, 2)$	8
$(1, 1, 1, -1, -1, 0, 0, 0, -1, 0)$	$(4, -1, -1, 1, 1, 1, 1, 2, 1, 3)$	9
$(1, 1, 1, -1, -1, 0, 0, 0, 0, -1)$	$(3, -1, -1, 2, 2, 1, 1, 0, 0, 1)$	10
$(1, 1, 1, -1, -1, 0, 0, 0, -1, 0)$	$(5, -1, -1, 1, 1, 2, 2, 3, 2, 2)$	11

$(1, 1, 1, -1, -1, 0, 0, 0, 0, -1)$	$(5, -1, -1, 1, 1, 2, 2, 2, 2, 3)$	12
$(1, 1, 1, 0, 0, -1, -1, 0, -1, 0)$	$(5, -1, -1, 1, 1, 2, 2, 3, 2, 2)$	13

$\sigma(D_0)$	$\rho(D_1)$	$(\sigma(D_0), \rho(D_1))$
$(1, 1, 1, -1, -1, 0, 0, 0, -1, 0)$	$(1, 0, 0, -1, -1, 0, 0, 1, 0, 0)$	-1
$(1, 1, 1, -1, -1, 0, 0, 0, 0, -1)$	$(1, 0, 0, -1, -1, 0, 0, 0, 0, 1)$	0
$(1, 1, 1, -1, -1, 0, 0, 0, -1, 0)$	$(1, 0, 0, 0, 0, -1, -1, 1, 0, 0)$	1
$(1, 1, 1, -1, -1, 0, 0, 0, 0, -1)$	$(1, 0, 0, 0, 0, -1, -1, 0, 0, 1)$	2
$(1, 1, 1, -1, -1, 0, 0, 0, -1, 0)$	$(1, -1, -1, 0, 0, 0, 0, 1, 0, 0)$	3
$(1, 1, 1, -1, -1, 0, 0, 0, 0, -1)$	$(1, -1, -1, 0, 0, 0, 0, 0, 0, 1)$	4
$(0, -1, -1, 0, 0, 0, 0, 0, -1, -1)$	$(3, 1, 1, 0, 0, 0, 0, 0, 3, 0)$	5
$(1, 0, 0, 1, 1, -1, -1, 0, -1, 0)$	$(3, 1, 1, 0, 0, 0, 0, 0, 3, 0)$	6
$(0, -1, -1, 0, 0, 0, 0, -1, 0, -1)$	$(4, 2, 2, 0, 0, 0, 0, 0, 1, 3)$	7
$(0, -1, -1, 0, 0, 0, 0, 0, -1, -1)$	$(4, 2, 2, 0, 0, 0, 0, 0, 3, 1)$	8
$(0, -1, -1, 0, 0, 0, 0, 0, -1, -1)$	$(5, 3, 3, 0, 0, 0, 0, 2, 1, 2)$	9
$(2, -1, -1, 1, 1, 0, 0, 0, 0, 2)$	$(5, 3, 3, 2, 2, 0, 0, 0, 0, 1)$	10
$(1, -1, -1, 0, 0, 1, 1, -1, 0, 0)$	$(5, 3, 3, 2, 2, 0, 0, 0, 1, 0)$	11
$(1, -1, -1, 0, 0, 1, 1, 0, 0, -1)$	$(5, 3, 3, 2, 2, 0, 0, 0, 0, 1)$	12
$(0, 0, 0, -1, -1, 0, 0, 0, -1, -1)$	$(10, 0, 0, 4, 4, 5, 5, 3, 2, 3)$	13
$(2, -1, -1, 0, 0, 1, 1, 0, 0, 2)$	$(5, 3, 3, 2, 2, 0, 0, 0, 0, 1)$	14

$\sigma(D_0)$	$\rho(D_2)$	$(\sigma(D_0), \rho(D_2))$
$(0, -1, -1, -1, -1, 0, 0, 0, 0, 0)$	$(1, 0, 0, 0, 0, 0, 0, 0, 0, -1)$	0
$(0, -1, -1, 0, 0, 0, 0, -1, 0, -1)$	$(2, 1, 1, 0, 0, 0, 0, 0, 1, -1)$	1
$(1, 0, 0, 1, 1, -1, -1, -1, 0, 0)$	$(2, 1, 1, 0, 0, 0, 0, 0, 1, -1)$	2
$(2, 0, 0, -1, -1, 0, 0, 1, 2, 1)$	$(2, 1, 1, 0, 0, 0, 0, 0, 1, -1)$	3
$(1, -1, -1, 1, 1, 0, 0, -1, 0, 0)$	$(2, 1, 1, 0, 0, 0, 0, 0, 1, -1)$	4
$(2, 0, 0, -1, -1, 0, 0, 2, 1, 1)$	$(2, 0, 0, 1, 1, 0, 0, 1, 0, -1)$	5
$(2, 0, 0, -1, -1, 1, 1, 2, 0, 0)$	$(2, 1, 1, 0, 0, 0, 0, -1, 1, 0)$	6
$(2, 0, 0, -1, -1, 0, 0, 1, 2, 1)$	$(3, 1, 1, 1, 1, 0, 0, -1, 0, 2)$	7
$(2, 0, 0, -1, -1, 0, 0, 2, 1, 1)$	$(3, 1, 1, 1, 1, 0, 0, -1, 2, 0)$	8
$(3, 1, 1, -1, -1, 0, 0, 2, 1, 2)$	$(3, 1, 1, 1, 1, 0, 0, -1, 2, 0)$	9
$(1, 1, 0, -1, -1, 0, 0, 1, 0, -1)$	$(5, 0, -1, 2, 2, 1, 1, 2, 1, 3)$	10
$(3, 0, 0, -1, -1, 1, 1, 2, 1, 2)$	$(3, 1, 1, 1, 1, 0, 0, -1, 2, 0)$	11
$(3, 0, 0, -1, -1, 1, 1, 2, 1, 2)$	$(4, 2, 2, 1, 1, 0, 0, -1, 2, 1)$	12
$(4, -1, -1, 1, 1, 1, 1, 2, 3, 1)$	$(4, 1, 1, 1, 1, 1, -1, 0, 3)$	13
$(3, -1, -1, 0, 0, 1, 1, 2, 1, 2)$	$(4, 2, 2, 1, 1, 0, 0, -1, 2, 1)$	14
$(3, -1, -1, 0, 0, 1, 1, 2, 1, 2)$	$(6, 2, 2, 2, 2, 3, 3, 0, -1, 1)$	15

$\sigma(D_0)$	$\rho(D_3)$	$(\sigma(D_0), \rho(D_3))$
(1, 1, 1, -1, 0, 0, 0, -1, 0)	(3, 1, 1, 0, 0, 0, 0, 1, 0, 2)	1
(0, -1, -1, -1, -1, 0, 0, 0, 0)	(2, 1, 1, 0, 0, 0, 0, 0, 0, 0)	2
(1, 0, 0, -1, -1, 1, 1, 0, -1, 0)	(3, 1, 1, 0, 0, 0, 0, 1, 0, 2)	3
(0, -1, -1, 0, 0, 0, 0, 0, -1, -1)	(3, 1, 1, 0, 0, 0, 0, 1, 0, 2)	4
(1, -1, -1, 0, 0, 1, 1, 0, -1, 0)	(3, 1, 1, 0, 0, 0, 0, 1, 0, 2)	5
(1, -1, -1, 0, 0, 1, 1, 0, -1, 0)	(3, 1, 1, 0, 0, 0, 0, 2, 1, 0)	6
(1, -1, -1, 0, 0, 0, 1, -1, 0)	(3, 1, 1, 0, 1, 0, 0, 0, 2, 0)	7
(3, -1, -1, 0, 0, 1, 1, 2, 2, 1)	(2, 1, 1, 0, 0, 0, 0, 0, 0, 0)	8
(1, -1, -1, 1, 1, 0, 0, -1, 0)	(4, 2, 2, 0, 0, 0, 0, 2, 1, 1)	9
(0, -1, -1, -1, -1, 0, 0, 0, 0, 0)	(6, 3, 3, 2, 2, 2, 2, 0, 0, 0)	10
(1, -1, -1, 0, 0, 1, 1, 0, -1, 0)	(5, 2, 2, 1, 1, 0, 0, 3, 2, 0)	11
(1, -1, -1, 0, 0, 1, 1, 0, -1, 0)	(5, 2, 2, 1, 1, 0, 0, 0, 3, 2)	12
(0, 0, 0, -1, -1, 0, 0, -1, 0, -1)	(9, 3, 3, 4, 4, 0, 0, 3, 4, 2)	13
(2, -1, -1, 0, 0, 1, 1, 2, 0, 0)	(5, 2, 2, 1, 1, 0, 0, 0, 2, 3)	14
(1, 0, 0, -1, -1, 1, 1, 0, -1, 0)	(7, 3, 3, 2, 2, 0, 0, 1, 4, 2)	15
(4, -1, -1, 1, 1, 1, 1, 3, 1, 2)	(4, 2, 2, 0, 0, 1, 1, 0, 2, 0)	16

$\sigma(D_1)$	$\rho(D_1)$	$(\sigma(D_1), \rho(D_1))$
(1, -1, -1, 0, 0, 0, 0, 0, 0, 1)	(1, 0, 0, -1, -1, 0, 0, 0, 0, 1)	0
(1, -1, 0, 0, 0, 0, 0, -1, 0, 1)	(1, 0, 1, 0, 0, 0, 0, -1, 0, -1)	1
(1, -1, -1, 0, 0, 0, 0, 0, 0, 1)	(3, 1, 1, 0, 0, 0, 0, 0, 0, 3)	2
(2, 1, 0, 0, 0, 0, 0, 2, 0, -1)	(3, 1, 0, 0, 0, 0, 0, 1, 3, 0)	3
(2, 0, -1, 0, 0, 0, 0, 2, 0, 1)	(3, 1, 0, 0, 0, 0, 0, 1, 3, 0)	4
(2, 1, 0, 0, 0, 0, 0, 2, -1, 0)	(2, 0, -1, 0, 0, 0, 0, 0, 1, 2)	5
(4, 2, 2, 0, 0, 0, 0, 0, 1, 3)	(4, 0, 0, 2, 2, 0, 0, 0, 1, 3)	6
(1, -1, -1, 0, 0, 0, 0, 1, 0, 0)	(4, 2, 2, 0, 0, 0, 0, 1, 0, 3)	7
(1, -1, -1, 0, 0, 0, 0, 1, 0, 0)	(4, 2, 2, 0, 0, 0, 0, 0, 3, 1)	8
(3, 1, 1, 0, 0, 0, 0, 3, 0, 0)	(5, 3, 3, 2, 2, 0, 0, 0, 1, 0)	9
(2, 0, 0, 0, 0, 0, 0, 1, -1, 2)	(4, 2, 2, 0, 0, 0, 0, 1, 3, 0)	10
(4, 0, 0, 2, 2, 0, 0, 0, 1, 3)	(5, 3, 3, 2, 2, 0, 0, 0, 1, 0)	11
(3, 0, 0, 0, 0, 1, 1, 0, 0, 3)	(5, 3, 3, 2, 2, 0, 0, 0, 0, 1)	12
(4, 2, 2, 0, 0, 0, 0, 3, 0, 1)	(4, 0, 0, 2, 2, 0, 0, 0, 1, 3)	13
(2, 0, 0, 0, 0, -1, 0, 2, 0, 1)	(6, 2, 2, 0, 0, 3, 4, 0, 2, 1)	14
(3, 0, 0, 0, 0, 1, 1, 3, 0, 0)	(5, 3, 3, 2, 2, 0, 0, 0, 1, 0)	15

$\sigma(D_1)$	$\rho(D_2)$	$(\sigma(D_1), \rho(D_2))$
(1, -1, -1, 0, 0, 0, 0, 1, 0, 0)	(1, 0, 0, 0, 0, 0, 0, -1, 0)	1
(1, -1, -1, 0, 0, 0, 0, 0, 0, 1)	(1, 0, 0, 0, 0, 0, 0, 0, -1)	2
(2, 0, 0, 0, 0, 0, 2, 1, -1)	(1, 0, 0, 0, 0, 0, 0, -1, 0)	3
(2, 0, 0, 0, 0, 0, 2, -1, 1)	(1, 0, 0, 0, 0, 0, -1, 0, 0)	4
(2, 1, 0, 0, 0, 0, 2, -1, 0)	(2, 1, 0, 1, 1, 0, 0, -1, 0, 0)	5
(3, 1, 1, 0, 0, 0, 0, 0, 0, 3)	(1, 0, 0, 0, 0, 0, 0, 0, 0, -1)	6
(3, 1, 1, 0, 0, 0, 0, 3, 0, 0)	(2, 1, 1, 0, 0, 0, 0, -1, 0, 1)	7
(3, 1, 0, 0, 0, 0, 3, 0, 1)	(2, 1, 0, 1, 1, 0, 0, -1, 0, 0)	8
(3, 1, 1, 0, 0, 0, 3, 0, 0)	(2, 0, 0, 1, 1, 0, 0, -1, 0, 1)	9
(3, 1, 1, 0, 0, 0, 3, 0, 0)	(3, 1, 1, 1, 1, 0, 0, -1, 0, 2)	10
(5, 0, 0, 2, 2, 3, 3, 0, 1, 0)	(3, 1, 1, 1, 1, 0, 0, -1, 0, 2)	11
(3, 0, 0, 0, 1, 1, 3, 0, 0)	(3, 1, 1, 1, 1, 0, 0, -1, 0, 2)	12
(3, 0, 0, 0, 1, 0, 3, 0, 1)	(4, 2, 2, 1, 1, 2, 1, -1, 0, 0)	13
(5, 0, 0, 2, 2, 3, 3, 1, 0, 0)	(4, 2, 2, 1, 1, 0, 0, 2, 1, -1)	14
(5, 0, 0, 2, 2, 3, 3, 1, 0, 0)	(4, 2, 2, 0, 0, 1, 1, -1, 2, 1)	15
(9, 4, 4, 4, 4, 0, 0, 3, 1, 3)	(2, 0, 0, 0, 1, 1, 1, -1, 0)	16

$\sigma(D_1)$	$\rho(D_3)$	$(\sigma(D_1), \rho(D_3))$
(1, 0, 0, -1, -1, 0, 0, 0, 0, 1)	(2, 1, 1, 0, 0, 0, 0, 0, 0, 0)	2
(1, -1, -1, 1, 0, 0, 0, 0, 0, 0)	(3, 1, 1, 2, 1, 0, 0, 0, 0, 0)	3
(1, -1, -1, 0, 0, 0, 0, 0, 0, 1)	(2, 1, 1, 0, 0, 0, 0, 0, 0, 0)	4
(1, 0, 0, -1, -1, 1, 0, 0, 0, 0)	(4, 1, 1, 1, 1, 0, 0, 0, 0, 3)	5
(3, 0, 0, 1, 1, 0, 0, 0, 0, 3)	(2, 1, 1, 0, 0, 0, 0, 0, 0, 0)	6
(3, 1, 1, 0, 0, 0, 0, 3, 0, 0)	(3, 1, 1, 0, 0, 0, 0, 0, 1, 2)	7
(5, 3, 3, 2, 2, 0, 0, 0, 1, 0)	(3, 1, 1, 0, 0, 0, 0, 0, 1, 2)	8
(3, 0, 0, 1, 1, 0, 0, 3, 0, 0)	(3, 1, 1, 0, 0, 0, 0, 0, 1, 2)	9
(3, 1, 1, 0, 0, 0, 0, 3, 0, 0)	(4, 1, 1, 1, 1, 0, 0, 0, 1, 3)	10
(1, -1, -1, 0, 0, 1, 0, 0, 0, 0)	(7, 3, 3, 2, 2, 1, 0, 0, 4)	11
(3, 1, 1, 0, 0, 0, 0, 3, 0, 0)	(4, 0, 0, 1, 1, 1, 0, 1, 3)	12
(4, 2, 2, 0, 0, 0, 0, 3, 0, 1)	(4, 0, 0, 1, 1, 1, 0, 1, 3)	13
(5, 3, 3, 2, 2, 0, 0, 1, 0, 0)	(4, 1, 1, 0, 0, 1, 1, 0, 3, 1)	14
(5, 3, 3, 2, 2, 0, 0, 0, 1, 0)	(4, 0, 0, 1, 1, 1, 1, 0, 1, 3)	15
(5, 3, 3, 2, 2, 0, 0, 1, 0, 0)	(4, 0, 0, 1, 1, 1, 1, 0, 3, 1)	16
(9, 4, 4, 4, 4, 3, 3, 1, 0, 0)	(2, 0, 0, 0, 0, 1, 0, 1, 0, 1)	17

$\sigma(D_2)$	$\rho(D_2)$	$(\sigma(D_2), \rho(D_2))$
(1, 0, 0, 0, -1, 0, 0, 0, 0, 0)	(2, 1, 1, 1, 0, 0, 0, 0, 0, -1)	2
(2, 1, 1, 0, -1, 0, 0, 0, 0, 1)	(2, 1, 1, 1, 0, 0, 0, 0, 0, -1)	3
(1, 0, 0, 0, 0, 0, 0, 0, 0, -1)	(3, 2, 2, 0, 0, 0, 0, 0, 0, 1)	4
(2, 1, 1, 0, -1, 0, 0, 0, 0, 1)	(2, 1, 1, 1, 0, 0, 0, 0, 0, -1)	5
(2, 1, 1, 1, 0, 0, 0, 0, 0, -1)	(3, 0, 0, 2, 1, 0, 0, 0, 0, 2)	6
(3, 2, 2, 0, 0, 0, 0, 1, 0, 0)	(3, 0, 0, 0, 0, 0, 0, 2, 1, 2)	7
(3, 0, 0, 1, 0, 2, 2, 0, 0, 0)	(3, 0, 0, 1, 0, 0, 0, 2, 2, 0)	8
(6, 3, 3, 2, 2, 0, 0, 3, 0, 1)	(1, 0, 0, 0, 0, 0, 0, -1, 0, 0)	9
(6, 3, 3, 2, 2, 0, 0, 3, 0, 1)	(2, 0, 0, 1, 1, 0, 0, -1, 0, 1)	10
(5, 3, 3, 1, 1, 0, 0, 0, 1, 2)	(2, 0, 0, 0, 0, 1, 1, 0, 1, -1)	11
(6, 3, 3, 2, 2, 0, 0, 3, 1, 0)	(4, 1, 1, 2, 2, 0, 0, -1, 1, 2)	12
(3, 0, 0, 0, 0, 2, 2, 1, 0, 0)	(4, 2, 2, 1, 1, 0, 0, -1, 2, 1)	13
(6, 3, 3, 2, 2, 0, 0, 3, 0, 1)	(2, 0, 0, 0, 0, 1, 1, -1, 0, 1)	14
(4, 0, 0, 2, 2, 1, 1, 2, -1, 1)	(4, 2, 2, 0, 0, 1, 1, 1, 2, -1)	15
(6, 3, 3, 2, 2, 0, 0, 3, 0, 1)	(4, 1, 1, 0, 0, 2, 2, 1, 2, -1)	16
(6, 3, 3, 2, 2, 0, 0, 1, 0, 3)	(3, 0, 0, 0, 0, 2, 2, 1, 0, 0)	17

$\sigma(D_2)$	$\rho(D_3)$	$(\sigma(D_2), \rho(D_3))$
(2, 1, 1, 1, 0, 0, 0, 0, -1, 0)	(2, 0, 0, 1, 0, 0, 0, 1, 0, 0)	3
(2, 1, 1, 1, 0, 0, 0, 0, 0, -1)	(2, 0, 0, 1, 0, 0, 0, 0, 0, 1)	4
(2, 1, 1, 1, 0, 0, 0, 0, 0, -1)	(3, 1, 1, 1, 0, 0, 0, 0, 0, 2)	5
(3, 2, 2, 0, 0, 0, 0, 0, 0, 1)	(2, 0, 0, 1, 1, 0, 0, 0, 0, 0)	6
(2, 0, 0, 1, 0, 1, 1, 0, 0, -1)	(3, 1, 1, 1, 0, 0, 0, 0, 0, 2)	7
(3, 0, 0, 1, 0, 0, 0, 2, 2, 0)	(3, 1, 1, 1, 0, 0, 0, 0, 0, 2)	8
(1, 0, 0, 0, 0, 0, 0, 0, -1, 0)	(6, 2, 2, 2, 2, 0, 0, 0, 3, 3)	9
(4, 1, 1, 2, 2, 0, 0, 1, -1, 2)	(3, 1, 1, 0, 0, 0, 0, 0, 2, 1)	10
(3, 0, 0, 0, 0, 2, 2, 1, 0, 0)	(4, 1, 1, 1, 1, 0, 0, 1, 0, 3)	11
(4, 0, 0, 2, 2, 1, 1, 1, -1, 2)	(3, 1, 1, 0, 0, 0, 0, 0, 2, 1)	12
(5, 3, 3, 1, 1, 0, 0, 1, 0, 2)	(3, 0, 0, 0, 0, 1, 1, 2, 1, 0)	13
(4, 0, 0, 2, 2, 1, 1, 1, -1, 2)	(4, 2, 2, 1, 1, 0, 0, 0, 2, 0)	14
(6, 2, 2, 2, 2, 3, 3, 0, -1, 1)	(3, 1, 1, 0, 0, 0, 0, 2, 1, 0)	15
(4, 0, 0, 2, 2, 1, 1, 1, -1, 2)	(4, 2, 2, 0, 0, 1, 1, 0, 2, 0)	16
(13, 2, 2, 6, 6, 4, 4, 7, 2, 2)	(2, 0, 0, 0, 0, 0, 1, 0, 1, 0)	17
(5, 3, 3, 1, 1, 0, 0, 2, 0, 1)	(4, 0, 0, 1, 1, 2, 2, 0, 2, 0)	18

$\sigma(D_3)$	$\rho(D_3)$	$(\sigma(D_3), \rho(D_3))$
(2, 1, 1, 0, 0, 0, 0, 0, 0, 0)	(2, 0, 0, 1, 1, 0, 0, 0, 0, 0)	4
(3, 1, 1, 0, 0, 0, 0, 1, 0, 2)	(3, 1, 1, 0, 0, 0, 0, 2, 1, 0)	5
(3, 1, 1, 0, 0, 1, 1, 0, 0, 2, 0, 1)	(3, 1, 1, 0, 0, 0, 0, 0, 2, 1)	6
(3, 0, 0, 1, 1, 0, 0, 1, 0, 2)	(3, 1, 1, 0, 0, 0, 0, 2, 1, 0)	7
(3, 0, 0, 1, 1, 0, 0, 2, 0, 1)	(3, 1, 1, 0, 0, 0, 0, 0, 2, 1)	8
(5, 0, 0, 2, 2, 1, 1, 3, 0, 2)	(3, 0, 0, 0, 1, 1, 0, 1, 2)	9
(3, 1, 1, 0, 0, 0, 0, 2, 0, 1)	(4, 1, 1, 0, 0, 2, 2, 0, 2, 0)	10
(5, 0, 0, 2, 2, 1, 1, 3, 0, 2)	(3, 1, 1, 0, 0, 0, 0, 1, 2)	11
(3, 1, 1, 0, 0, 0, 0, 2, 0, 1)	(4, 0, 0, 1, 1, 2, 2, 0, 2, 0)	12
(5, 1, 1, 2, 2, 2, 3, 0, 0, 0)	(4, 0, 0, 1, 1, 0, 1, 1, 1, 3)	13
(5, 0, 0, 2, 2, 1, 1, 2, 0, 3)	(4, 2, 2, 0, 0, 1, 1, 2, 0, 0)	14
(5, 2, 2, 1, 1, 2, 3, 0, 0, 0)	(4, 0, 0, 1, 1, 0, 1, 1, 1, 3)	15
(5, 0, 0, 2, 2, 1, 1, 2, 0, 3)	(4, 1, 1, 0, 0, 2, 2, 0, 2, 0)	16
(11, 3, 3, 5, 5, 3, 4, 1, 0, 5)	(2, 0, 0, 0, 0, 1, 1, 0, 0)	17
(5, 0, 0, 2, 2, 1, 1, 2, 0, 3)	(4, 2, 2, 0, 0, 1, 1, 0, 2, 0)	18
(13, 4, 4, 6, 6, 5, 5, 3, 2, 0)	(3, 0, 0, 1, 1, 0, 0, 2, 1, 0)	19

This concludes the proof of theorem 9.

We can now state the following theorem, which is a straightforward consequence of theorems 6, 8 and 9.

**THEOREM 10.** (i) *For every  $r \geq 5$  there exists an embedding of  $S$  as a nonsingular surface  $F^{2r-3}$  of degree  $2r-3$  in  $\mathbf{P}^r$ , and for every  $(d, g)$  such that*

$$0 \leq g \leq (d-r)^2/2(2r-3)$$

*there exists a nonsingular irreducible and nondegenerate curve  $X$  of degree  $d$  and genus  $g$  on  $F^{2r-3}$ .*

(ii) *For every  $r \geq 7$  there exists an embedding of  $S$  as a nonsingular surface  $F^{2r-4}$  of degree  $2r-4$  in  $\mathbf{P}^r$ , and for every  $(d, g)$  such that*

$$0 \leq g \leq (d-r)^2/2(2r-4)$$

*there exists a nonsingular irreducible and nondegenerate curve  $X$  of degree  $d$  and genus  $g$  on  $F^{2r-4}$ .*

Note that theorem 10 differs from our main theorem, as stated in the introduction, only in that the surface  $S$  appears instead of  $S'$ . In the next section we will show how to deduce the main theorem from theorem 10.

**Remark 2.** Arguing as in proposition 5 it is easy to see that if there exists a  $\delta$ -tuple  $D_0, \dots, D_{\delta-1}$  of classes of  $\text{Pic}(S)$  satisfying conditions I), II), III) of the definition of  $\delta$ -system, then  $S$  can be embedded in  $\mathbf{P}^r$  as a smooth linearly normal surface of degree  $2r - \delta$  for all  $r \geq \delta - 1$ . The following is a list of such  $\delta$ -tuples for  $5 \leq \delta \leq 9$ :

$\delta = 5$ :

$$\begin{aligned} D_0 &= (0, -1, -1, -1, 0, 0, 0, 0, 0, 0) \\ D_2 &= (1, -1, -1, 0, 0, 0, 0, 0, 0, 0) \\ D_4 &= (2, 1, 0, 0, 0, 0, 0, 0, 0, 0) \end{aligned}$$

$$\begin{aligned} D_1 &= (1, 1, -1, -1, 0, 0, 0, 0, 0, 0) \\ D_3 &= (2, 1, 1, -1, 0, 0, 0, 0, 0, 0) \end{aligned}$$

$\delta = 6$ :

$$\begin{aligned} D_0 &= (0, -1, -1, -1, -1, -1, -1, 0, 0, 0) \\ D_2 &= (1, -1, -1, -1, -1, 0, 0, 0, 0, 0) \\ D_4 &= (2, 1, -1, 0, 0, 0, 0, 0, 0, 0) \end{aligned}$$

$$\begin{aligned} D_1 &= (1, 1, -1, -1, -1, -1, 0, 0, 0, 0) \\ D_3 &= (2, 1, 1, -1, -1, 0, 0, 0, 0, 0) \\ D_5 &= (3, 2, 1, 0, 0, 0, 0, 0, 0, 0) \end{aligned}$$

$\delta = 7$ :

$$\begin{aligned} D_0 &= (0, -1, -1, -1, -1, -1, -1, -1, 0, 0) \\ D_2 &= (1, -1, -1, -1, -1, -1, 0, 0, 0, 0) \\ D_4 &= (2, 1, -1, -1, 0, 0, 0, 0, 0, 0) \\ D_6 &= (3, 2, 0, 0, 0, 0, 0, 0, 0, 0) \end{aligned}$$

$$\begin{aligned} D_1 &= (1, 1, -1, -1, -1, -1, -1, 0, 0, 0) \\ D_3 &= (2, 1, 1, -1, -1, -1, 0, 0, 0, 0) \\ D_5 &= (3, 2, 1, -1, 0, 0, 0, 0, 0, 0) \end{aligned}$$

$\delta = 8$ :

$$\begin{aligned} D_0 &= (0, -1, -1, -1, -1, -1, -1, -1, -1, 0) \\ D_2 &= (1, -1, -1, -1, -1, -1, -1, 0, 0, 0) \\ D_4 &= (2, 1, -1, -1, -1, 0, 0, 0, 0, 0) \\ D_6 &= (3, 2, -1, 0, 0, 0, 0, 0, 0, 0) \end{aligned}$$

$$\begin{aligned} D_1 &= (1, 1, -1, -1, -1, -1, -1, -1, 0, 0) \\ D_3 &= (2, 1, 1, -1, -1, -1, -1, 0, 0, 0) \\ D_5 &= (3, 2, 1, -1, -1, 0, 0, 0, 0, 0) \\ D_7 &= (4, 3, 1, 0, 0, 0, 0, 0, 0, 0) \end{aligned}$$

$\delta = 9$ :

$$\begin{aligned} D_0 &= (0, -1, -1, -1, -1, -1, -1, -1, -1, -1); \\ D_2 &= (1, -1, -1, -1, -1, -1, -1, 0, 0, 0) \\ D_4 &= (2, 1, -1, -1, -1, -1, 0, 0, 0, 0) \\ D_6 &= (3, 2, -1, -1, 0, 0, 0, 0, 0, 0) \\ D_8 &= (4, 3, 0, 0, 0, 0, 0, 0, 0, 0) \end{aligned}$$

$$\begin{aligned} D_1 &= (1, 1, -1, -1, -1, -1, -1, -1, -1, 0) \\ D_3 &= (2, 1, 1, -1, -1, -1, -1, -1, 0, 0) \\ D_5 &= (3, 2, 1, -1, -1, -1, 0, 0, 0, 0) \\ D_7 &= (4, 3, 1, -1, 0, 0, 0, 0, 0, 0) \end{aligned}$$

On the other hand it is clear that there are no such  $\delta$ -tuples for  $\delta \geq 10$ : indeed, letting  $D = |D_0 - \omega_S|$ ,  $\varphi_D(S)$  is a smooth surface of degree  $\delta$  in  $\mathbf{P}^{\delta}$  with elliptic hyperplane sections.

#### 4. Remarks

1) One of our main technical tools has been proposition 1, whose proof uses very strongly the geometrical properties of the surface  $S$ , particularly the fact that

the points  $P_1, \dots, P_9$  are not in general position, but are base points of a generic pencil of cubics. It is pretty clear that proposition 1 cannot be generalized to a surface  $S'$  obtained by blowing up 9 points of  $\mathbf{P}^2$  in general position. Nevertheless it is not difficult to see that our other main results generalize to  $S'$ . This can be done in the following way.

Let  $P_1, \dots, P_9 \in \mathbf{P}^2$  be the points that define  $S$ , and let  $M$  be a general line through  $P_9$ . In  $\mathbf{P}^2 \times M$  denote by  $\Gamma$  the diagonal curve, whose support is  $\{(p, p) : p \in M\}$ . Let  $S$  be the blow-up of  $\mathbf{P}^2 \times M$  along  $P_1 \times M \cup \dots \cup P_8 \times M \cup \Gamma$ , and let  $q : S \rightarrow \mathbf{P}^2 \times M$  be the projection, and  $\pi : S \rightarrow M$  be the composition of  $q$  with the second projection  $\mathbf{P}^2 \times M \rightarrow M$ . Clearly  $\pi$  is a smooth family of projective surfaces, whose fibre over a point  $p \in M$  is the surface  $S(p)$  obtained from  $\mathbf{P}^2$  after blowing up  $P_1, \dots, P_8$  and  $p$ . In particular  $S(P_9) = S$ . Note that for all  $p$  in some open neighborhood of  $P_9$  the points  $P_1, \dots, P_8, p$  are contained in a unique cubic curve  $C_p$  which is nonsingular, hence they are in general position.

In  $\text{Pic}(S)$  consider the classes

$$\mathbf{o}(\mathbf{H}), \mathbf{o}(-\mathbf{E}_1), \dots, \mathbf{o}(-\mathbf{E}_8), \mathbf{o}(-\mathbf{E}_p),$$

where  $\mathbf{H} = q^*(\mathbf{o}(1))$ , and  $\mathbf{E}_1, \dots, \mathbf{E}_8, \mathbf{E}_p$  are the exceptional surfaces coming from the curves  $P_1 \times M, \dots, P_8 \times M, \Gamma$  respectively. It is clear that every element  $\mathbf{o}(D)$  of  $\text{Pic}(S)$ , being a linear combination of  $\mathbf{o}(H), \mathbf{o}(-E_1), \dots, \mathbf{o}(-E_9)$ , extends to an element  $\mathbf{o}(\mathbf{D}) \in \text{Pic}(S)$  which is the corresponding combination of the above classes; hence, by restriction, it defines a divisor class  $\mathbf{o}(D_p) \in \text{Pic}(S(p))$  for all  $p \in M$ .

Suppose that  $\mathbf{o}(D) \in \text{Pic}(S)$  is such that

$$(*) \quad h^1(S, \mathbf{o}(D)) = 0 = h^2(S, \mathbf{o}(D)),$$

From the upper-semicontinuity principle it follows that there is an open neighborhood  $U_D$  of  $P_9$  in  $M$  such that for all  $p \in U_D$

$$h^1(S, \mathbf{o}(D_p)) = 0 = h^2(S, \mathbf{o}(D_p)).$$

If moreover the linear system  $|D|$  has no base points and contains an irreducible and nonsingular curve, then, after possibly shrinking  $U_D$ , the same is true of  $|D_p|$  for all  $p \in U_D$ . Indeed the base point freeness of  $|D|$  implies that the natural map

$$\pi^*[\pi_* \mathbf{o}(\mathbf{D})] \rightarrow \mathbf{o}(\mathbf{D})$$

is surjective in a neighborhood of  $S(P_9) = S$ . From the base change properties it then follows that  $|D_p|$  is base point free for all  $p$  in that neighborhood. Condition

(\*) implies that  $\pi_*\mathbf{o}(\mathbf{D})$  is locally free of rank  $h^0(S, \mathbf{o}(D))$  on some open set  $U$  containing  $P_9$ . As a consequence we have that, if  $X \in |D|$  is a general element, it can be extended to a relative effective Cartier divisor  $\mathbf{X}$  on  $\pi^{-1}(U)$ . And if  $X$  is a nonsingular curve, then it follows from the flatness of  $\mathbf{X}$  over  $U$  that the restriction  $X_p$  of  $\mathbf{X}$  to  $\mathbf{S}(p)$  is also a nonsingular curve for all  $p$  in some open set  $V \subset U$  containing  $P_9$ .

Suppose in addition that  $\mathbf{o}(D)$  is very ample; then it is easy to show that, after possibly shrinking  $U_D$ ,  $\mathbf{o}(D_p)$  is very ample for all  $p \in U_D$ . Indeed, on  $\pi^{-1}(U_D)$  the natural map  $\pi^*\pi_*\mathbf{o}(\mathbf{D}) \rightarrow \mathbf{o}(\mathbf{D})$  is surjective, hence it defines a  $U_D$ -morphism

$$\varphi : \pi^{-1}(U_D) \rightarrow \mathbf{P}(\pi_*\mathbf{o}(\mathbf{D})) =: \mathbf{P}$$

which restricts on every fibre  $\mathbf{S}(p)$ ,  $p \in U_D$ , to the morphism

$$\varphi_p : \mathbf{S}(p) \rightarrow \mathbf{P}(H^0(\mathbf{S}(p), \mathbf{o}(D_p)))$$

defined by the linear system  $|D_p|$ . For  $p = P_9$  this is a closed embedding, because  $\mathbf{o}(D)$  is very ample; hence there is an open  $V \subset U_D$  such that the restriction of  $\varphi$  to  $\pi^{-1}(V)$  is finite and such that  $\mathbf{o}_p \rightarrow \varphi_*\mathbf{o}_S$  is an isomorphism; equivalently  $\varphi$  is a closed embedding of  $\pi^{-1}(V)$  in  $\mathbf{P}$  and this means that  $\mathbf{o}(D_p)$  is very ample for all  $p \in V$ .

These remarks can be applied to  $D = D_i - \alpha\omega_S$  to conclude that propositions 4 and 5 generalize to  $S'$  with no changes. As a consequence of this we have that theorem 6 is still true if we replace  $S$  by  $S'$ . Clearly lemma 7 extends to  $S'$ , and the proofs of theorems 8 and 9 extend word by word to  $S'$ . Consequently theorem 10 also extends. In particular *the main theorem, as stated in the introduction, is true.*

2) Suppose that  $D$  is a 1-connected effective divisor on a projective nonsingular surface  $F$  such that  $h^1(F, \mathbf{o}_F) = 0$ , and let  $H$  be a divisor on  $F$  such that  $|D + H|$  contains an irreducible nonsingular curve  $C$ . From the exact sequence

$$0 \rightarrow \mathbf{o}_F(-D) \rightarrow \mathbf{o}_F(H) \rightarrow \mathbf{o}_C(H) \rightarrow 0$$

and from the Ramanujam' vanishing theorem (see section 1) it follows that

$$H^0(F, \mathbf{o}_F(H)) \cong H^0(C, \mathbf{o}_C(H)).$$

We apply this remark to the surface  $S$ , equipped with a  $\delta$ -system (e.g. a 3-system or a 4-system), and we take  $H$  to be one of the very ample divisors  $H$ , and  $C'$  any

of the curves  $X$  of degree  $d$  and genus  $g$  such that  $0 \leq g \leq d - r - 1$ , as described in proposition 5. It follows that the curves  $X$  of degree  $d$  and genus  $g$  constructed in theorem 6 satisfy  $h^0(X, \mathbf{o}_X(H)) = r + 1$  (i.e. are “linearly normal”) if

$$d - r \leq g \leq (d - r)^2/2(2r - \delta).$$

In particular this applies to the curves of theorem 10 and, by upper-semicontinuity, to those of the main theorem which satisfy the corresponding inequalities for  $\delta = 3, 4$ .

3) Let  $C \subset \mathbf{P}^r$  be a nonsingular irreducible and nondegenerate curve of degree  $n$ ,  $\mathbf{o}(H_C)$  the hyperplane section line bundle and  $\omega_C$  the canonical bundle. Assume that  $h^0(C, \mathbf{o}(H_C)) = r + 1$ . The natural map

$$\mu_0(C) : H^0(C, \mathbf{o}(H_C)) \otimes H^0(C, \omega_C(-H_C)) \rightarrow H^0(C, \omega_C)$$

is called the *Brill-Noether map* of  $C \subset \mathbf{P}^r$ .

The map  $\mu_0(C)$  is relevant to the study of the scheme  $W'_n(C)$  of linear series of degree  $n$  and dimension at least  $r$  on  $C$ . In particular, the surjectivity of  $\mu_0(C)$  is equivalent to the fact that  $\mathbf{o}(H_C)$  is an isolated point of  $W'_n(C)$  with reduced structure. We will check this last property on some of the curves we have constructed.

Again suppose that the surface  $S$  is equipped with a  $\delta$ -system, and let  $F \subset \mathbf{P}^r$  be the nonsingular embedding of degree  $2r - \delta$  of  $S$  given by theorem 6.

Let  $X \subset F$  be a nonsingular nondegenerate curve of degree  $d$  and genus  $g$  as constructed in the proof of theorem 6. Assume that  $g \geq 2d - 4r + \delta$  (with the notation of theorem 6, this means that  $(d, g)$  lies above the line  $L_1$ ). Then it follows from the proof of theorem 6 that  $X \in |C' + mH_r|$ , for some  $m \geq 2$  and for some irreducible, nonsingular and nondegenerate  $C'$  of degree  $d'$  and genus  $g'$  such that  $0 \leq g' \leq d' - r - 1$ . We will write  $H_r = H$ .

Since, by remark 2) above,  $h^0(X, \mathbf{o}_X(H)) = r + 1$ , we can consider the Brill-Noether map of  $X \subset \mathbf{P}^r$ . We claim that  $\mu_0(X)$  is surjective. Indeed consider the following commutative diagram:

$$\begin{array}{ccc} \mu : H^0(F, \mathbf{o}_F(H)) \otimes H^0(F, \omega_F(X - H)) & \rightarrow & H^0(F, \omega_F(X)) \\ \downarrow & & \downarrow q \\ \mu_0(X) : H^0(X, \mathbf{o}_X(H_X)) \otimes H^0(X, \omega_X(-H_X)) & \rightarrow & H^0(X, \omega_X). \end{array}$$

Since  $q$  is surjective, it suffices to show that  $\mu$  is surjective, and for this purpose it is enough to show that the sheaf  $\omega_F(X - H)$  is 0-regular with respect to

$\mathbf{o}(H)$  (see [M]). This amounts to check that:

$$H^1(F, \omega_F(X - 2H)) = (0)$$

and

$$H^2(F, \omega_F(X - 3H)) = (0).$$

The first condition follows from the vanishing theorem because  $|X - 2H| = |C' + (m - 2)H|$  contains a 1-connected divisor. The second condition is equivalent to

$$H^0(F, \mathbf{o}_F(3H - X)) = (0),$$

which is true because

$$|3H - X| = |(3 - m)H - C'|,$$

and this is clearly empty if  $m \geq 3$ , and likewise empty for  $m = 2$  because  $C'$  is nondegenerate.

Of course this remark applies to the curves of theorem 10, taking  $\delta = 3$  or 4, and, by upper-semicontinuity, it extends to the curves of the main theorem.

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