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# Rigid versus non-rigid cyclic actions

KARL HEINZ DOVERMANN,\* MIKIYA MASUDA, and DONG YOUP SUH

### §0. Introduction

Let p denote an odd prime number, and let  $G_p$  be the cyclic group of order p. In this paper we study (locally) smooth  $G_p$  actions whose fixed point set consists of a codimension two component and an isolated point. Following [M1] we say that such a  $G_p$  action is of Type II<sub>0</sub>. Of particular interest is the case where the underlying space is a closed manifold having the same cohomology (or the same homotopy type) as  $\mathbb{C}P^n$ . Such a space is called a cohomology  $\mathbb{C}P^n$  or homotopy  $\mathbb{C}P^n$  respectively.

When we study transformation groups, we often adopt the following approach. First we take a familiar action as a model and compute its invariants. Then we ask if a general action has similar invariants provided that the underlying space has a topological type similar to that of the model action. We take a linear  $G_p$  action of Type II<sub>0</sub> on  $\mathbb{C}P^n$  as the model. The invariants which are of interest in this paper are Pontrjagin classes, tangential representations at fixed points, and defects [M1].

We explain why we study Type II<sub>0</sub> actions especially. For that we pose our problem in a more general setting. Suppose  $G_p$  acts (locally) smoothly on a cohomology  $\mathbb{C}P^n$  denoted by X. Let  $\{F_i\}_{i=1}^r$  be the set of fixed point components of the action. The Fixed Point Theorem of Bredon and Su [B, p. 382] says that

- (0.1) Each  $F_i$  has the same cohomology ring as  $\mathbb{C}P^{n_i}$  for some  $n_i$  with  $\mathbb{Z}_p$  coefficients and  $\sum (n_i + 1) = n + 1$ ,
  - (0.2) the restriction map from  $H^*(X; \mathbb{Z}_p)$  to  $H^*(F_i; \mathbb{Z}_p)$  is surjective.

Fix a generator x of  $H^2(X; \mathbb{Z})$  and let  $x_i$  be its restriction to  $F_i$ . To simplify notation we regard  $x_i$  also as a class in  $H^*(F_i; \mathbb{Q})$ . Motivated by the linear model actions we make the

DEFINITION. A (locally) smooth  $G_p$  action on X is algebraically standard if

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the following three conditions are satisfied:

- (0.3)  $p(F_i) = (1 + x_i^2)^{n_i + 1}$  in  $H^*(F_i; \mathbb{Q})$ .
- (0.4)  $|x_i^{n_i}[F_i]| = D_X(F_i) = 1$ , the number  $D_X(F_i)$  defined here is called the defect of  $F_i$  in X.
- (0.5) The tangential representations of  $G_p$  at fixed points are of the linear type.

The concept of tangential representations of the linear type has been introduced by A. Hattori [Ha] (see also [Hs]). There only  $S^1$  actions are treated, but the definition of the linear type can be adopted with no difficulties to  $G_p$  actions. In case of Type II<sub>0</sub> actions this means

(0.5)'  $T_fX = nv \mid q$  as real representations. Here  $T_fX$  is the tangential representation at the isolated fixed point f, v is the normal bundle to the codimension 2 fixed point component F, and  $v \mid q$  is the restriction to a point q of F.

*Remark*. It follows quite easily from (0.2) that  $D_X(F_i) \neq 0$ . The term of algebraically standardness was introduced in [D1] for actions of Type II<sub>0</sub> in a slightly different way, two more conditions were given, i.e.

$$p(X) = (1 + x^2)^{n+1},$$

$$c(v) = 1 \pm j^* x, \text{ where } j: F \to X \text{ is the inclusion map.}$$

A lemma shows that these two definitions are nevertheless equivalent.

It is natural to ask.

Question. Is a (locally) smooth  $G_p$  action on a homotopy  $\mathbb{C}P^n$  algebraically standard?

In general, the answer is no. In fact, in [DM] we showed

THEOREM. For any odd integer m and any odd  $n \ge 3$ , there are infinitely many homotopy  $\mathbb{C}P^n$ 's with smooth  $G_m$  actions which satisfy neither (0.3) nor (0.4).

As for (0.5),  $G_p$  actions of non-linear type with isolated fixed points are constructed in [P1, 2], [MT], [T], [DM]. In these counterexamples, and also in the ones in the last theorem, the dimensions of the fixed point components are small in comparison with the dimension of the ambient manifold. In fact, they are less than half of it. To the contrary one has the vague feeling that an action with a fixed point component of low codimension is restrictive (see [D1], [M1,2] for

example). In this sense Type  $II_0$  actions are the extreme case. Thus we are led to study Type  $II_0$  actions.

Our first result is a rigidity theorem.

THEOREM A. Let X be a cohomology  $\mathbb{C}P^n$ . There is a constant  $c_X$ , which depends only on the Pontrjagin class of X, such that any (locally) smooth  $G_p$  action of Type  $II_0$  on X is algebraically standard if  $p \ge c_X$ .

This, and several other studies of cyclic actions on homotopy complex projective spaces, were motivated by Petrie's Conjecture [P1]. It may be stated as follows:

Suppose  $S^1$  acts smoothly and effectively on a closed manifold X homotopy equivalent to  $\mathbb{C}P^n$ . Then  $p(X) = (1 + x^2)^{n+1}$ .

This conjecture has been verified in several special cases, and for a long bibliography see [D2]. In particular it holds if the action is semifree (which is equivalent to having two fixed point components) [Wg1], [Y]. Theorem A provides a new proof of Petrie's Conjecture for Type II<sub>0</sub> actions of  $S^1$  (cf. [M2]). In fact, it follows from Theorem A and the above remark that we have

COROLLARY. Let X be a homotopy  $\mathbb{C}P^n$  with non-standard Pontrjagin class. Then  $p \ge c_X$  ( $c_X$  as in Theorem A) implies that X does not admit a (locally) smooth Type  $II_0$   $G_p$  action.

The interesting aspect of this alternative proof is that it is merely based on the G Signature formula for elements of sufficiently large but finite prime order. In addition we do not need that the action is smooth, but only that it is locally smooth.

In contrast we have

THEOREM B. Let  $\varepsilon_n = 1$  if  $n \equiv 3 \pmod{4}$  and  $\varepsilon_n = 0$  otherwise. If  $[(n-2)/4] - \varepsilon_n \ge (p+1)/2$ , then there are infinitely many homotopy  $\mathbb{C}P^n$ 's with smooth  $G_p$  action of Type II<sub>0</sub> such that (0.3) is not satisfied, in particular they are not algebraically standard. (The inequality holds if  $n \ge 2p + 8$ ).

We note that a smooth free  $S^1$  action on the standard sphere  $S^{2n-1}$  which restricts to a linear  $G_p$  action on  $S^{2n-1}$  produces a homotopy  $\mathbb{C}P^n$  with a smooth  $G_p$  action of Type II<sub>0</sub> having the  $S^1$  orbit space of  $S^{2n-1}$  as the fixed point component of codimension two (see §3 for details). The inequality in Theorem B is a sufficient condition for infinitely many such actions to exist (Theorem 3.2).

Theorem 3.2 is valid even when p = 1. It then says that there are infinitely many smooth free  $S^1$  actions on  $S^{2n-1}$  provided that  $n \ge 8$  or n = 6. This almost proves a main theorem of Wang [Wg2] (in fact, Wang proves the existence for  $n \ge 7$ ). Our proof is quite different from his proof and much simpler.

Having both Theorems A and B in mind we make this

Conjecture. There exists a function d(n) such that (locally) smooth Type II<sub>0</sub> actions of  $G_p$  on cohomology  $\mathbb{C}P^n$ 's are algebraically standard if  $p \ge d(n)$ .

It would mean that  $c_X$  in Theorem A depends only on n. As supporting evidence we quote experience from Theorems A and B. Low dimensional evidence is also the main result of [D1].

THEOREM C ([D1]). A (locally) smooth Type  $II_0$  action of  $G_p$  on a cohomology  $\mathbb{C}P^n$  is algebraically standard if

- (1)  $n \le 3$
- (2) n = 4 and  $p \ge 5$
- (3) n = 5,  $p \ge 7$  and the relative class number  $h_1(p)$  is odd or p = 29.

Some computer assisted computation in the spirit of [D1] also show further cases of this theorem:

- (4) n = 6 and  $7 \le p \le 43$
- (5) n = 7 and  $11 \le p \le 19$ .

In spite of the estimate for d(n) in Theorem B it seems reasonable to expect that d(n) is approximately n, up to a small additive constant. This guess is based on the locally linear PL discussion of Theorem B in [D1].

This paper is organized as follows. In the first two sections we study rigidity phenomena of  $G_p$  actions, (0.4) and (0.5) in Section 1 and (0.3) in Section 2. In Section 3 we prove Theorem B. We also construct infinitely many smooth free  $S^1$  actions on  $S^{2n-1}$  which restricts to linear  $G_p$  actions. In Section 4 we relate our results to one announced by Connolly-Weinberger [We].

# §1. Rigidity of defects and tangential representations

As is well known the G signature theorem imposes a profound constraint on invariants of G actions. In this and the next section we observe to what extent it restricts our invariants of  $G_p$  actions of Type II<sub>0</sub>. The G signature theorem holds for semifree tame actions ([Wal, 14B]), and for  $G_p$  actions of Type II<sub>0</sub> tameness is equivalent to local smoothness ([D1]). In the following we consider locally smooth  $G_p$  actions of Type II<sub>0</sub>.

Roughly speaking our results say that those actions resemble the linear action of Type  $II_0$  on  $\mathbb{C}P^n$  provided that p is sufficiently large. Our argument works for a family of closed orientable even dimensional manifolds X satisfying the following three conditions:

- (1.1) The second Betti number dim  $H^2(X; \mathbb{Q})$  of X is one. There is an element  $x \in H^2(X; \mathbb{Z})$  which descends to a generator of  $H^2(X; \mathbb{Z})$ /torsion. We sometimes consider x as an element of  $H^2(X; \mathbb{Q})$ .
  - (1.2) If dim X = 2n, then  $x^n \neq 0$ .
  - (1.3) The total Pontrjagin class of X is a polynomial of x in  $H^*(X; \mathbb{Q})$ .

This family contains  $\mathbb{Z}_p$  ( $\mathbb{Z}$  or  $\mathbb{Q}$ ) cohomology  $\mathbb{C}P^n$ 's, i.e. closed smooth manifolds having the same cohomology ring as  $\mathbb{C}P^n$  with  $\mathbb{Z}_p$  ( $\mathbb{Z}$  or  $\mathbb{Q}$ ) coefficients. Examples of another type are non-singular algebraic hypersurfaces of  $\mathbb{C}P^{n+1}$  ( $n \ge 3$ ).

Throughout this section X will denote a closed manifold of dimension 2n satisfying the above three conditions and we fix an element x in (1.1).

DEFINITION/OR CONVENTION 1.4. We choose an orientation class [X] so that the defect D(X) defined as  $x^n[X]$  is non-negative. By (1.2) it is a positive integer.

Suppose X supports a locally smooth  $G_p$  action of Type II<sub>0</sub>. We denote the fixed point component of codimension two by F and the isolated fixed point by f. The G signature formula is described in terms of local information around the fixed point set. We need more notations and conventions to write it down.

DEFINITION/OR CONVENTION 1.5. We choose an orientation class [F] for F so that the defect  $D_X(F) = j^*x^{n-1}[F]$  of F in X is a non-negative integer (cf. Introduction).

The orientations on X and F determine a unique orientation on the normal bundle v of F in X such that their juxtaposition agrees with the orientation of X. Once v is oriented, it can be regarded as a complex line bundle as usual. Let  $t^a$  denote the complex 1-dimensional representation of any subgroup G of  $S^1$  such that  $g \in G$  acts by multiplication with  $g^a$ . The identification of an appropriate element  $g \in G_p$  with  $\exp(2\pi i/p)$  in  $S^1$  identifies  $G_p$  with a subgroup of  $S^1$  such that

$$v \mid q = t^{-1} \quad \text{for} \quad q \in F. \tag{1.6}$$

We fix this identification of  $G_p$  with the subgroup of p-th roots of unity.

The tangential representation  $T_fX$  at the isolated fixed point f is oriented. As

is well known one can put a complex structure on it such that the induced orientation agrees with the one given by the orientation of X, although such a complex structure is not unique. We choose integers  $m_j$  such that  $1 \le |m_j| < p/2$  and

$$T_f X = \sum_{j=1}^n t^{m_j}. (1.7)$$

As a step towards algebraical standardness, we pose an intermediate definition.

**DEFINITION** 1.8. We say that a locally smooth  $G_p$  action of Type II<sub>0</sub> is weakly algebraically standard if  $D(X) = D_X(F) = 1$  (cf. (0.4)) and the  $m_j$  can be chosen to be 1 for every j (i.e. the action is of the linear type, cf. (0.5)).

Our main theorem in this section is

THEOREM 1.9. Let X be a closed manifold of dimension 2n satisfying (1.1), (1.2), and (1.3). Define  $a_k$  by setting  $\sum a_k x^{2k} = p(X)$ . Then there is a constant  $b_X$  depending only on  $\{a_k\}$ , the Betti numbers of X, and the number of torsion elements of  $H^*(X; \mathbb{Z})$  such that if  $p \ge b_X$ , then any locally smooth  $G_p$  action of Type  $II_0$  on X is weakly algebraically standard.

The rest of this section is devoted to the proof of this theorem. The following lemma is proved in Lemma 2.8 of [M2].

LEMMA 1.10. (1) 
$$D(X)$$
 divides  $D_X(F)$ .  
(2)  $c_1(v) = D_X(F)/D(X)j^*x$ , where  $c_1(v)$  denotes the first Chern class of  $v$ .

Set  $D_X(F)/D(X) = d$ ,  $j^*x = \bar{x}$  and  $\exp 2\pi i/p = z$ . Since  $G_p$  is considered as a subgroup of  $S^1$ , z is an element of  $G_p$ . We apply the G signature theorem (see [HZ, p. 50]) together with (1.6), (1.7) and Lemma 1.10 (2) to get an identity

$$\operatorname{Sign}(z, X) = \frac{z^{-1}e^{2d\bar{x}} + 1}{z^{-1}e^{2d\bar{x}} - 1}L(F)[F] + \prod_{i=1}^{n} \frac{z^{m_i} + 1}{z^{m_i} - 1}.$$
 (1.11)

Here L(F) denotes the Hirzebruch L polynomial of F. One can easily verify that

$$\frac{z^{-1}e^{2d\bar{x}}+1}{z^{-1}e^{2d\bar{x}}-1}=1-\frac{2z}{z-1}\sum_{k=0}^{n-1}\left(\frac{e^{2d\bar{x}}-1}{z-1}\right)^k.$$

The sum in this expression is finite because  $(2^{2d\bar{x}}-1)^j$  vanishes for  $j \ge n$  as dim F < 2n. Moreover, Hirzebruch's signature formula states that

$$L(F)[F] = \operatorname{Sign} F.$$

Substituting both in (1.11) and multiplying the resulting identity by  $(z-1)^n$ , we get

$$0 = (z - 1)^{n} (\operatorname{Sign}(z, X) - \operatorname{Sign} F) + 2z \left\{ \sum_{k=0}^{n-1} (z - 1)^{n-1-k} (e^{2d\bar{x}} - 1)^{k} \right\} L(F)[F]$$
$$- (z - 1)^{n} \prod_{j=1}^{n} (z^{m_{j}} + 1) / (z^{m_{j}} - 1). \quad (1.12)$$

We shall estimate the value of each term in this identity for a sufficiently large value of p. We begin with

LEMMA 1.13. (1) Let  $s_X$  be the sum of all Betti numbers of X and the number of torsion elements in  $H^*(X; \mathbb{Z})$ . Then

$$|\operatorname{Sign}(z, X) - \operatorname{Sign} F| \leq s_X$$
.

(2) 
$$|(z-1)^n \prod (z^{m_i}+1)/(z^{m_i}-1)| < 2^n$$
.

(3)  $|(z-1)^n \prod (z^{m_j}+1)/(z^{m_j}-1)| < 2^{n-1}(1+\tan^2\pi/p)$  unless  $|m_j|=1$  for every j.

*Proof.* (1) It easily follows from the definition of the G signature that

$$|\operatorname{Sign}(z, X)| \leq \dim_{\mathbb{C}} H^n(X; \mathbb{C}).$$

As for Sign F, it follows from the definition of the signature and the universal coefficient theorem that

$$|\operatorname{Sign} F| \leq \dim_{\mathbb{R}} H^{n-1}(F; \mathbb{R}) \leq \dim_{\mathbb{Z}_p} H^{n-1}(F; \mathbb{Z}_p).$$

On the other hand it is known (see p. 144 of [B]) that

$$\sum_{i \geq n-1} \dim_{\mathbb{Z}_p} H^i(F; \mathbb{Z}_p) \leq \sum_{i \geq n-1} \dim_{\mathbb{Z}_p} H^i(X; \mathbb{Z}_p).$$

These show that

$$|\operatorname{Sign}(z, X) - \operatorname{Sign} F| \leq |\operatorname{Sign}(z, X)| + |\operatorname{Sign} F|$$

$$\leq \dim_{\mathbb{C}} H^{n}(X; \mathbb{C}) + \sum_{i \geq n-1} \dim_{\mathbb{Z}_{p}} H^{i}(X; \mathbb{Z}_{p}) \leq s_{X}.$$

(2) Since  $z = \exp 2\pi i/p$ , we have

$$|(z^m-1)/(z^m+1)| = |\tan m\pi/p|.$$

Hence

$$|(z-1)^{n} \prod (z^{m_{i}}+1)/(z^{m_{i}}-1)|$$

$$= |(z+1)^{n}(z-1)^{n}/(z+1)^{n}| |\prod (z^{m_{i}}+1)/(z^{m_{i}}-1)|$$

$$< 2^{n} \prod |\tan (\pi/p)/\tan(m_{i}\pi/p)|.$$
(1.14)

Remember that  $|m_j|$  are chosen so that  $1 \le |m_j| < p/2$ . Since  $\tan y$  is a monotone increasing function in the domain  $|y| < \pi/2$ , (2) follows from (1.14).

(3) Unless  $|m_i| = 1$  for every j, we have

$$\prod |\tan \pi/p/\tan m_j \pi/p| \le |\tan \pi/p/\tan 2\pi/p|. \tag{1.15}$$

Here  $\tan 2\pi/p = 2\tan (\pi/p)/(1 + \tan^2 \pi/p)$ . Now (3) follows from (1.14) and (1.15). Q.E.D.

We now consider the second summand of (1.12). By multiplicativity of the L polynomial we have

$$L(F) = j^*L(X)L(\nu)^{-1}.$$

By Lemma 1.10 (2) we have

$$L(v) = 1 + p_1(v)/3 = 1 + c_1(v)^2/3 = 1 + d^2\bar{x}^2/3.$$

On the other hand since p(X) is a polynomial of x by assumption (1.3) and L(X) is a polynomial of Pontrjagin classes, we may then write

$$L(X) = \sum_{k=0}^{\lfloor n/2 \rfloor} A_k x^{2k} \qquad (A_0 = 1)$$

with  $A_k \in \mathbb{Q}$ . Consequently we have

$$L(F) = \left(\sum A_k \bar{x}^{2k}\right) \left(\sum (-d^2 \bar{x}^2 / 3)^l\right). \tag{1.16}$$

Put this into the second summand of (1.12) and expand it with respect to d. Then, since  $(\bar{x})^{n-1}[F] = D_X(F) = dD(X)$  by the definition of  $D_X(F)$  and d, the exponent of d is at most n. In fact, we can express

the second summand of (1.12) = 
$$\sum_{j=0}^{n} B_j(z)d^j$$

with polynomials  $B_j(z)$  of degree less than or equal to n. We shall collect properties of  $B_j(z)$  in the following lemma. The proof is easy, so we leave it to the reader.

LEMMA 1.17. (1) If  $p \ge n+1$ , then  $B_j(z)$  is described in terms of the coefficients of the polynomial  $p(X) = \sum a_k x^{2k}$ .

- (2)  $B_j(1) = 0$  for j < n
- (3)  $B_n(1) = 2^n D(X)$ .

We need one more lemma.

LEMMA 1.18.  $d \neq 0$ , i.e. d is a positive integer.

*Proof.* Suppose d = 0. Then, since  $(\bar{x})^{n-1}[F] = D_X(F) = dD(X) = 0$  and L(F) is a polynomial of  $\bar{x}$  (see (1.16)), the first term in the right hand side of (1.11) vanishes. Hence identity (1.11) turns into

$$\operatorname{Sign}(z, X) = \prod_{i=0}^{n} (z^{m_i} + 1) / (z^{m_i} - 1). \tag{1.19}$$

However this is impossible as observed below. The proof is essentially the same as in Theorem 7.1 of [AB].

Let  $R(G_p)_z$  denote the ring localized at the prime ideal of  $R(G_p)$  vanishing at z. Identity (1.19) implies that

Sign
$$(G_p, X) = \prod_{j=0}^{n} (t^{m_j} + 1)/(t^{m_j} - 1)$$
 in  $R(G_p)_z$ 

where t denotes the standard complex 1-dimensional representation of  $G_p$  as before. Multiplying both sides by  $\prod (t^{m_j}-1)$ , it turns out that the resultant identity in  $R(G_p)_z$  comes from  $R(G_p)$ . Since the kernel of the natural map from  $R(G_p)$  to  $R(G_p)_z$  is the ideal generated by the cyclotomic polynomial  $\sum_{j=1}^p t^j$ , one concludes that

$$\operatorname{Sign}(G_p, X) \prod (t^{m_j} - 1) = \prod (t^{m_j} + 1) + h\left(\sum t^j\right) \text{ in } R(G_p)$$

with some integer h. Here we can evaluate this identity at t = 1. Then it reduces to

$$0 = 2^n + hp$$

which is impossible because p is odd. Q.E.D.

**Proof of Theorem** 1.9. We now fix the underlying manifold X. If we take p sufficiently large, then z converges to 1. We denote this by  $z \approx 1$ . By Lemma 1.13 (1) one can see that

the first term of 
$$(1.12) \approx 0$$
.  $(1.20)$ 

On the other hand, the third term of (1.12) is bounded independent of the value of p by Lemma 1.13 (2). These together with Lemma 1.17 imply that the values of d are also bounded. Hence one can conclude by Lemma 1.17 that

the second summand of 
$$(1.12) \approx 2^n d^n D(X)$$
. (1.21)

Suppose  $|m_j| \ge 2$  for some j. Then Lemma 1.13 (3) tells that since  $\tan \pi/p \approx 0$ , the absolute value of the third term of (1.12) converges to a value strictly less than  $2^n$ . However this contradicts (1.12) together with (1.20) and (1.21) because  $2^n d^n D(X) \ge 2^n$  by Definition 1.4 and Lemma 1.18.

Thus we have established that  $m_j = \pm 1$  for every j. Then

the third term of 
$$(1.12) \approx 2^n \prod m_j$$
. (1.22)

Since d and D(X) are positive integers and  $\prod m_j = \pm 1$ , it follows from (1.20), (1.21), and (1.22) that

$$d=1$$
,  $D(X)=1$  and  $\prod m_i=1$ .

Remember that there was an ambiguity of a choice of a complex structure on  $T_fX$ . The only constraint is that the orientation on  $T_fX$  induced from the complex structure agrees with the given one. As is easily seen, it is equivalent that the sign of  $\prod m_j$  is positive. Hence we may assume  $m_j = 1$  for each j. Q.E.D.

## §2. Rigidity of Pontrjagin classes

Let X be the same as in (1.1). We use the notation of 1 freely. The following definition is consistent with that of the Introduction.

DEFINITION 2.1. We say that a locally smooth  $G_p$  action of Type II<sub>0</sub> on X is algebraically standard if it is weakly algebraically standard and  $p(F) = (1 + \bar{x}^2)^n$  in  $H^*(F; \mathbb{Q})$ .

The purpose of this section is to prove

THEOREM 2.2. Let a locally smooth  $G_p$  action of Type  $II_0$  on X be weakly algebraically standard. Suppose the induced action of  $G_p$  on  $H^n(X; \mathbb{Q})$  is trivial. Then the  $G_p$  action is algebraically standard provided that  $p \ge n + 2$ .

Remark 2.3. If X is a  $\mathbb{Q}$ -cohomology  $\mathbb{C}P^n$ , then  $G_p$  acts trivially on the cohomology because  $\dim_{\mathbb{Q}} H^n(X;\mathbb{Q}) \leq 1$ . Generally, if we take a basis on the vector space  $H^n(X;\mathbb{Q})$  coming from  $H^n(X;\mathbb{Z})$ , the induced action of  $G_p$  on  $H^n(X;\mathbb{Q})$  gives a homomorphism from  $G_p$  to a general linear group  $GL(r,\mathbb{Z})$  where  $r = \dim_{\mathbb{Q}} H^n(X;\mathbb{Q})$ . Therefore if  $GL(r,\mathbb{Z})$  does not contain an element of order p, then the assumption is satisfied. This is the case if  $p \geq r + 2$ . The proof is as follows. Diagonalize the image of  $z = e^{2\pi i/p}$  over  $\mathbb{C}$ . Then the trace is a polynomial of z over  $\mathbb{Z}$  with at most r factors of degree less than p. It must be an integer as the image of  $z \in G_p$  lies in  $GL(r,\mathbb{Z})$ . Since a minimal polynomial of z over  $\mathbb{Z}$  (or  $\mathbb{Q}$ ) is the cyclotomic polynomial  $\sum_{j=0}^{p-1} z^j$ , the above polynomial does not contain any factor of  $z^j$  (0 < j < p). This means that the image of z is the identity matrix.

COROLLARY 2.4. Let X and  $b_X$  be the same as in Theorem 1.9. Then a smooth  $G_p$  action of Type  $II_0$  on  $X^{2n}$  is algebraically standard if  $p \ge c_X = \max\{b_X, n+2, \dim_{\mathbb{Q}} H^n(X; \mathbb{Q}) + 2\}$ .

*Proof of Theorem* 2.2. Triviality of the induced action of  $G_p$  on  $H^n(X; \mathbb{Q})$  implies that  $\operatorname{Sign}(z, X)$  is equal to  $\operatorname{Sign} X$ . Since the action is weakly algebraically

standard, i.e. d = 1 and  $m_i = 1$  for every j, (1.12) turns into

$$0 = (z - 1)^{n} (\operatorname{Sign} X - \operatorname{Sign} F)$$

$$+ 2z \left\{ \sum_{k=0}^{n-1} (z - 1)^{n-1-k} (e^{2\bar{x}} - 1)^{k} \right\} L(F)[F] - (z + 1)^{n}.$$
(2.5)

In this identity the coefficients of  $z^j$  are rational numbers and the degree of z is at most n. Because the minimal polynomial of z is of degree p-1, the coefficients of  $z^j$  must identically vanish as  $p \ge n+2$ .

Look at the constant term in (2.5). Since it must be zero, we get

$$\operatorname{Sign} X - \operatorname{Sign} F = (-1)^n.$$

Putting this into (2.5), we have

$$2z\left\{\sum_{k=0}^{n-1}(z-1)^{n-1-k}(e^{2\bar{x}}-1)^k\right\}L(F)[F] = (1+z)^n - (1-z)^n.$$
 (2.6)

Compare the coefficients of  $z^j$  inductively. The values of  $(e^{2\bar{x}}-1)^k L(F)[F]$  are then uniquely determined for each k. They determine L(F), because L(F) is a polynomial of  $\bar{x}$  (see (1.16)). On the other hand the linear  $G_p$  action of Type II<sub>0</sub> on  $\mathbb{C}P^n$  also satisfies (2.6) and  $F = \mathbb{C}P^{n+1}$  in that case. Hence one can conclude that  $p(F) = (1 + \bar{x}^2)^n$  in general. Q.E.D.

# §3. Free smooth $S^1$ actions on lens spaces

Let X be a homotopy  $\mathbb{C}P^m$  with a homotopy equivalence h from Y to  $\mathbb{C}P^m$ . Let  $h^*\gamma$  be the pullback of the canonical line bundle  $\gamma$  over  $\mathbb{C}P^m$  by h. Let  $D(h^*\gamma)$  (resp.  $S(h^*\gamma)$ ) denote the disk (resp. sphere) bundle of  $h^*\gamma$ . Then  $S(h^*\gamma)$  is a homotopy sphere with a smooth  $S^1$  action induced from the complex multiplication on fibers.

Suppose that

(3.1)  $S(h^*\gamma)$  with the restricted  $G_p$  action is equivariantly diffeomorphic to the unit sphere  $S^{2m+1}$  of  $\mathbb{C}^{m+1}$  with the linear  $G_p$  action of weight one (i.e. S((m+1)t)).

Remark. If  $S(h^*\gamma)$  with the restricted  $G_p$  action is equivariantly diffeomorphic to  $S^{2m+1}$  with a linear  $G_p$  action, then it must be S((m+1)t). In fact, the Reidemeister torsion of the orbit space  $S(h^*\gamma)/G_p$  is the same as that of the lens

space of weight one because it is fibered over a homotopy complex projective space Y (see [Wal, Prop. 14E.8(c)]). On the other hand linear lens spaces are distinguished by Reidemeister torsion invariants up to diffeomorphism (see [Mi, 12.7]).

If (3.1) is satisfied, then we can attach the unit disk  $D^{2m+2}$  of  $\mathbb{C}^{m+1}$  with the linear  $G_p$  action of weight one equivariantly to  $D(h^*\gamma)$  along the boundary. The resulting space turns out to be a homotopy  $\mathbb{C}P^{m+1}$  with a smooth weakly algebraically standard  $G_p$  action of Type II<sub>0</sub> whose fixed point set consists of Y and the center of  $D^{2m+2}$ . Hence a pair (Y, h), which satisfies (3.1) but the total Pontrjagin class p(Y) of Y is not of the same form as  $p(\mathbb{C}P^m)$ , yields a weakly algebraically standard but algebraically non-standard smooth  $G_p$  action of Type II<sub>0</sub>. In this section we use classical surgery theory to find such pairs.

Let  $L^m(p)$  denote the standard lens space defined as the orbit space of  $S^{2m+1}$  by the linear  $G_p$  action of weight one. There is a natural  $S^1$  fiber bundle  $\pi^m(p):L^m(p)\to \mathbb{C}P^m$ . Let  $\bar{Y}$  be the total space of the pullback of this  $S^1$  bundle via h and let  $\bar{h}:\bar{Y}\to L^m(p)$  be the induced map covering h. We note that h is a simple homotopy equivalence, hence so is  $\bar{h}$  ([A]). To assign a pair  $(\bar{Y},\bar{h})$  to (Y,h) gives a map

$$\pi^m(p)^*:hS(\mathbb{C}P^m)\to hS(L^m(p)).$$

Here hS(Z) denotes the set of simple homotopy smoothings of Z, namely it is the set of equivalence classes of pairs (W, g) such that W is a smooth manifold and g is a simple homotopy equivalence from W to Z. In case Z has a boundary  $\partial Z$ , a set  $hS(Z, \partial Z)$  similar to hS(Z) is defined. But it is required in addition that g restricts to a diffeomorphism on the boundary. The set hS(Z) or  $hS(Z, \partial Z)$  has a distinguished element defined as a pair of Z and the identity map on Z. The inverse image of the distinguished element in  $hS(L^m(p))$  by  $\pi^m(p)^*$  is called the kernel of  $\pi^m(p)^*$ , and it is denoted by  $\operatorname{Ker} \pi^m(p)^*$ .

Since  $\bar{Y}$  is exactly the orbit space of  $S(h^*\gamma)$  by the restricted  $G_p$  action, statement (3.1) is equivalent to:  $\bar{Y}$  is diffeomorphic to  $L^m(p)$ . Thus we are led to study  $\operatorname{Ker} \pi^m(p)^*$ . Our aim is to find an element  $(Y, h) \in \operatorname{Ker} \pi^m(p)^*$  having non-standard total Pontrjagin classes. A sufficient condition for such an element to exist is that  $\operatorname{Ker} \pi^m(p)^*$  is infinite, because diffeomorphism types of homotopy  $\mathbb{C}P^m$ 's are distinguished by Pontrjagin classes up to finite ambiguity, and up to homotopy there are only two homotopy equivalences from Y to  $\mathbb{C}P^m$ . In the sequel we ask

Question. When is Ker  $\pi^m(p)^*$  infinite?

Our answer is the following.

THEOREM 3.2. Suppose  $m \ge 3$ .

- (1) If  $p \ge m + 2$ , then  $\operatorname{Ker} \pi^m(p)^*$  is finite.
- (2) If  $[(m-1)/4] \varepsilon_{m-1} \ge (p+1)/2$ , then  $\operatorname{Ker} \pi^m(p)^*$  is infinite, where  $\varepsilon_j$  is the same as in Theorem B of the Introduction and p may also be one.
- Remark 3.3. (1) One can ask the same question in the PL category. In this case a complete answer is obtained in [D1]. It says that the kernel of  $\pi^m(p)^*:hPL(\mathbb{C}P^m)\to hPL(L^m(p))$  is infinite if and only if  $p\leq m+1$ . There the classification theorem of  $hPL(L^m(p))$  ([Wal, §14]) plays a role. In the smooth category, however, such a classification theorem is not known. Nevertheless it seems plausible to conjecture the same conclusion as in the PL case. One only would need some "additivity" for  $\pi^m(p)^*$ .
- (2) Infiniteness of Ker  $\pi^m(p)^*$  means that  $L^m(p)$  admits infinitely many smooth free  $S^1$  actions. Since  $L^m(1) = S^{2m+1}$ , Theorem 3.2 (2) can be considered as an extended version of Wang's result [Wg2] as indicated in the Introduction.

The proof of Theorem 3.2 (1) is the same as that of the PL case. In fact, it is shown in [D1] that if  $(Y, h) \in hPL(\mathbb{C}P^m)$  belongs to  $\operatorname{Ker} \pi^m(p)^*$ , then the PL Pontrjagin class p(Y) is of the same form as  $p(\mathbb{C}P^m)$  provided  $p \ge m + 2$ . The argument is also essentially the same as the proof of Theorem 2.2.

The rest of this section is devoted to the proof of Theorem 3.2 (2). We note that the sets  $hS(\mathbb{C}P^m)$  and  $hS(L^m(p))$  do not support natural group structures. This makes our problem difficult. To solve it we find suitable subsets of  $hS(\mathbb{C}P^m)$  and  $hS(L^m(p))$  which form abelian groups and on which  $\pi^m(p)^*$  is a homomorphism. Then we estimate their ranks explicitly. The inequality in Theorem 3.2 (2) is a sufficient condition that the rank of the subset of  $hS(\mathbb{C}P^m)$  is greater than that of  $hS(L^m(p))$ .

Fix k between 1 and m and let  $P_1$  (resp.  $P_2$ ) denote the submanifold of  $\mathbb{C}P^m$  defined by the equations  $w_j = 0$  for  $k+1 \le j \le m$  (resp.  $0 \le j \le k$ ) where  $w_j$  denotes the j+1th homogeneous coordinate of  $\mathbb{C}P^m$ . Let  $v_i$  denote small open tubular neighborhoods of  $P_i$ . Remove  $v_1$  and  $v_2$  from  $\mathbb{C}P^m$  and denote the resulting space by P. Let Q denote the inverse image of P by  $\pi^m(p)$ . The following lemma is an easy consequence of the h-cobordism theorem.

LEMMA 3.4. The manifold P (resp. Q) is diffeomorphic to the product of the  $S^1$  (resp.  $G_p$ ) orbit space of  $S^{2k+1} \times S^{2(m-k-1)+1}$  and the unit interval, where the  $S^1$  (resp.  $G_p$ ) action on  $S^{2k+1} \times S^{2(m-k+1)+1}$  is the diagonal one induced from complex multiplication on each factor.

Since P (resp. Q) is diffeomorphic to a product of a closed smooth manifold and the unit interval, the set  $hS(P, \partial P)$  (resp.  $hS(Q, \partial Q)$ ) admits a natural abelian group structure (see §10 of [Wal]). Let  $\pi:(Q, \partial Q) \to (P, \partial P)$  denote the restriction of  $\pi^m(p)$ . It is clear that  $\pi$  induces a homomorphism  $\pi^*:hS(P, \partial P) \to hS(Q, \partial Q)$  with respect to the group structures given above.

Given an element (W, g) of  $hS(P, \partial P)$ , one can glue  $v_i$  to W along the boundary via the diffeomorphism  $g \mid \partial W : \partial W \to \partial P$ . This defines a map  $\kappa_P$  from  $hS(P, \partial P)$  to  $hS(\mathbb{C}P^m)$ . Similarly we have a map  $\kappa_Q$  from  $hS(Q, \partial Q)$  to  $hS(L^m(p))$ . These maps fit into the following commutative diagram:

$$hS(P, \partial P) \xrightarrow{\kappa_P} hS(C\mathbb{P}^m)$$

$$\downarrow^{\pi^*} \qquad \downarrow^{\pi^m(p)^*}$$

$$hS(Q, \partial Q) \xrightarrow{\kappa_Q} hS(L^m(p))$$

The following lemma ensures that  $\operatorname{Ker} \pi^m(p)^*$  is infinite if  $\operatorname{Ker} \pi^*$  is infinite.

LEMMA 3.5. The map  $\kappa_P$  is finite to one.

*Proof.* The surgery exact sequence yields a diagram:

$$0 = L_{2m+1}(1) \to hS(P, \partial P) \to [P/\partial P, F/O] \to L_{2m}(1)$$

$$\downarrow^{\kappa_P} \qquad \qquad \downarrow^{q^*}$$

$$0 = L_{2m+1}(1) \to hS(\mathbb{C}P^m) \to [\mathbb{C}P^m, F/O] \to L_{2m}(1)$$

where  $q^*$  is induced by the quotient map  $q: \mathbb{C}P^m \to \mathbb{C}P^m/v_1 \cup v_2 = P/\partial P$ . The middle square in this diagram is commutative and  $q^*$  is a homomorphism. Therefore it suffices to show that the kernel of  $q^*$  is finite, in other words,  $q^*$  is injective when tensored by  $\mathbb{Q}$ . The following fact is well known.

Fact 3.6. There are isomorphisms between these three groups:

$$[Z, F/O] \otimes Q \rightarrow [Z, BO] \otimes \mathbb{Q} = \tilde{K}O(Z) \otimes Q \rightarrow \sum \tilde{H}^{4j}(Z; \mathbb{Q})$$

for any finite CW complex Z. In fact, the former map is induced from the natural map from F/O to BO and the latter one is the Pontrjagin character.

These isomorphisms are functorial, so the problem reduces to the injectivity of

$$q^*: \tilde{H}^{4j}(P/\partial P; \mathbb{Q}) = \tilde{H}^{4j}(\mathbb{C}P^m/\nu_1 \cup \nu_2; \mathbb{Q}) \to \tilde{H}^{4j}(\mathbb{C}P^m; \mathbb{Q}).$$

The cohomology exact sequence of the pair  $(\mathbb{C}P^m, v_1 \cup v_2)$  yields an exact sequence:

$$\tilde{H}^{4j-1}(v_1 \cup v_2; \mathbb{Q}) \longrightarrow \tilde{H}^{4j}(\mathbb{C}P^m/v_1 \cup v_2; \mathbb{Q}) \xrightarrow{q^*} \tilde{H}^{4j}(\mathbb{C}P^m; \mathbb{Q}).$$

Here the left most term vanishes because  $v_i$  is homotopy equivalent to the complex projective space  $P_i$ . This proves the lemma. Q.E.D.

Now we shall estimate the rank of Ker  $\pi^*$ . The surgery exact sequence yields a commutative diagram:

$$0 = L_{2m+1}(1) \xrightarrow{\omega_P} hS(P, \partial P) \xrightarrow{\pi_P} [P/\partial P, F/O] \xrightarrow{\sigma_P} L_{2m}(1)$$

$$\downarrow^{\pi^*} \qquad \qquad \downarrow^{\tau}$$

$$L_{2m+2}(G_p) \xrightarrow{\omega_Q} hS(Q, \partial Q) \xrightarrow{\eta_Q} [Q/\partial Q, F/O] \xrightarrow{\sigma_Q} L_{2m+1}(G_p)$$

where  $\tau$  is the map induced from  $\pi$ . Here all the terms are abelian groups and all the maps are homomorphisms. It follows that

$$rk \operatorname{Ker} \pi^* = rk hS(P, \partial P) - rk\pi^*$$
(3.7)

$$rk \, hS(P, \, \partial P) \ge rk[P/\partial P, \, F/O] - rk \, L_{2m}(1),$$

$$rk \, \pi^* \le rk \, \eta_O \circ \pi^* + rk \, \omega_Q = rk \, \tau \circ \eta_P + rk \, \omega_Q \le rk \, \tau + rk \, \omega_Q,$$

$$(3.8)$$

where rk indicates the rank of an abelian group or a homomorphism. Replace the right hand side of (3.7) by (3.8). Then we get a lower bound of rk Ker  $\pi^*$ :

$$rk \ker \pi^* \ge rk[P/\partial P, F/O] - rkL_{2m}(1) - rk\tau - rk\omega_Q$$

$$= rk \operatorname{Ker} \tau - rkL_{2m}(1) - rk\omega_Q. \tag{3.9}$$

LEMMA 3.10. (1) 
$$rkL_{2m}(1) = \{1 + (-1)^m\}/2$$
. (2)  $rk \omega_Q = (p-1)/2$ .

(3) If  $k \ge m - k - 1$ , (i.e.  $2k \ge m - 1$ , we may assume this without loss of generality), then  $rk \operatorname{Ker} \tau = [m/2] - [(k+1)/2]$ .

*Proof.* (1)  $L_{2m}(1)$  is isomorphic to  $\mathbb{Z}$  if m is even and is isomorphic to  $\mathbb{Z}_2$  if m is odd (see [Wal, §13]). This means (1).

(2) It is known that the rank of  $\omega_Q$  is the same as that of the reduced L group of  $L_{2m+2}(G_p)$  and the latter is (p-1)/2 (see §14E of [Wall]). This verifies (2).

(3) By Fact 3.6 rk Ker  $\tau$  agrees with the rank of the kernel of  $\pi^*: \sum H^{4j}(P, \partial P; \mathbb{Q}) \to \sum H^{4j}(Q, \partial Q; \mathbb{Q})$ . Remember that P is diffeomorphic to  $(S^{2k+1} \times S^{2(m-k-1)+1})/S^1 \times I$  where I is the unit interval. The projection from P to the last two factors of the product gives rise to a fibration:  $P \to \mathbb{C}P^{m-k-1} \times I$  with fiber  $S^{2k+1}$ . The Serre spectral sequence of this fibration (relative boundary) collapses because the fiber is a sphere of dimension greater than or equal to that of the base space by the assumption  $k \ge m-k-1$ . It implies an isomorphism:

$$H^*(P, \partial P; \mathbb{Q}) \cong H^*(\mathbb{C}P^{m-k-1} \times (I, \partial I); \mathbb{Q}) \otimes H^*(S^{2k+1}; \mathbb{Q}).$$

On the other hand  $H^*(\mathbb{C}P^{m-k-1}\times(I,\partial I);\mathbb{Q})$  is isomorphic to  $H^{*-1}(\mathbb{C}P^{m-k-1};\mathbb{Q})$ . Consequently we have an isomorphism

$$H^*(P, \partial P; \mathbb{Q}) \cong H^{*-1}(\mathbb{C}P^{m-k-1}; \mathbb{Q}) \otimes H^*(S^{2k+1}; \mathbb{Q}).$$

Similarly we have an isomorphism

$$H^*(Q, \partial Q; \mathbb{Q}) \cong H^{*-1}(L^{m-k-1}(p); \mathbb{Q}) \otimes H^*(S^{2k+1}; \mathbb{Q}).$$

Through these isomorphisms  $\pi^*$  splits into  $\pi^{m-k-1}(p)^* \otimes id^*$  where id denotes the identity map on  $S^{2k+1}$ . Hence the kernel of  $\pi^*$  in degree 4j is given by  $H^{4j-2k-2}(\mathbb{C}P^{m-k-1};\mathbb{Q}) \otimes H^{2k+1}(S^{2k+1};\mathbb{Q})$  through the above isomorphism. This shows that a kernel of rank one appears for each j between [(k+1)/2]+1 and [m/2]. That we may assume  $k \geq m-k-1$  follows as we could have exchanged  $P_1$  and  $P_2$  freely in, and just before, Lemma 3.4. In fact, it would just interchange k and m-k-1. This implies (3). Q.E.D.

This lemma and (3.9) show that

rk Ker 
$$\pi^* \ge [m/2] - [(k+1)/2] - \{1 + (-1)^m]/2 - (p-1)/2$$
  
=  $[(m-1)/2] - [(k+1)/2] - (p-1)/2$ .

Since  $2k \ge m - 1$ , we have

$$k+1 \ge \langle (m+1)/2 \rangle$$

where  $\langle v \rangle$  denotes the least integer greater than or equal to v. We take  $k = \langle (m+1)/2 \rangle - 1$ . Then it is sufficient for rk Ker  $\pi^*$  to be positive that

$$[(m-1)/2] - [\langle (m+1)/2 \rangle/2] \ge (p+1)/2.$$

As easily observed the left hand side of this inequality reduces to  $[(m-1)/4] - \varepsilon_{m+1}$ . This completes the proof of Theorem 3.2 (2).

## §4. Comparison with a result of Connolly-Weinberger

In this section we combine the preceding results to obtain deeper insight into smooth cyclic group actions on homotopy complex projective spaces.

Recently Connolly and Weinberger announced the following result.

THEOREM 4.1 (Connolly-Weinberger). Let M be a closed submanifold of  $\mathbb{C}P^n$  of codimension two. Then there exists a semifree smooth  $G_{2k}$  action on  $\mathbb{C}P^n$  whose fixed point set consists of M and an isolated point if and only if M is a cohomology  $\mathbb{C}P^{n-1}$  with  $\mathbb{Z}_{2k}$  coefficient and defect  $D_{\mathbb{C}P^n}(M) = 1$ , where k is any integer.

Remark 4.2. Their original statement (Corollary 2 in p. 276 of [We]) is false in its form. The above statement is the revised form which they communicated to the authors, cf. Zentralbratt 566, 57025.

A conclusion of Theorem 4.1 is that if the imbedded M is a homotopy  $\mathbb{C}P^{n-1}$  and  $D_{\mathbb{C}P^n}(M)=1$ , then for any prime number p there is a smooth  $G_p$  action of Type II<sub>0</sub> which fixes M.

On the other hand Theorem B says that if  $(p+1)/2 \le [(n-2)/4] - \varepsilon_n$ , then there is a homotopy  $\mathbb{C}P^nX$  with a smooth  $G_p$  action of Type II<sub>0</sub> such that

- (1) the fixed point component F of codimension two is a homotopy  $\mathbb{C}P^{n-1}$  and  $D_X(F)=1$ ,
- (2) p(F) (resp. p(X)) is not of the same form as  $p(\mathbb{C}P^{n-1})$  (resp.  $p(\mathbb{C}P^n)$ ). However Theorem A says that F is never fixed under any smooth  $G_p$  action on X if  $p \ge c_X$ . This contrasts the above result of Connolly-Weinberger. Namely this shows that Theorem 4.1 holds only for submanifolds of the standard  $\mathbb{C}P^n$ .

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