Rigid versus non-rigid cyclic actions.

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Rigid versus non-rigid cyclic actions

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§0. Introduction

Let p denote an odd prime number, and let G_p be the cyclic group of order p. In this paper we study (locally) smooth G_p actions whose fixed point set consists of a codimension two component and an isolated point. Following [M1] we say that such a G_p action is of Type II_0. Of particular interest is the case where the underlying space is a closed manifold having the same cohomology (or the same homotopy type) as $\mathbb{C}P^n$. Such a space is called a cohomology $\mathbb{C}P^n$ or homotopy $\mathbb{C}P^n$ respectively.

When we study transformation groups, we often adopt the following approach. First we take a familiar action as a model and compute its invariants. Then we ask if a general action has similar invariants provided that the underlying space has a topological type similar to that of the model action. We take a linear G_p action of Type II_0 on $\mathbb{C}P^n$ as the model. The invariants which are of interest in this paper are Pontrjagin classes, tangential representations at fixed points, and defects [M1].

We explain why we study Type II_0 actions especially. For that we pose our problem in a more general setting. Suppose G_p acts (locally) smoothly on a cohomology $\mathbb{C}P^n$ denoted by X. Let $\{F_i\}_{i=1}^r$ be the set of fixed point components of the action. The Fixed Point Theorem of Bredon and Su [B, p. 382] says that

(0.1) Each F_i has the same cohomology ring as $\mathbb{C}P^{n_i}$ for some n_i with $\mathbb{Z}_p$ coefficients and $\sum (n_i + 1) = n + 1$,

(0.2) the restriction map from $H^*(X; \mathbb{Z}_p)$ to $H^*(F_i; \mathbb{Z}_p)$ is surjective.

Fix a generator x of $H^2(X; \mathbb{Z})$ and let x_i be its restriction to F_i. To simplify notation we regard x_i also as a class in $H^*(F_i; \mathbb{Q})$. Motivated by the linear model actions we make the

DEFINITION. A (locally) smooth G_p action on X is *algebraically standard* if

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the following three conditions are satisfied:

(0.3) $p(F_i)=(1+x_i^2)^{n_i+1}$ in $H^*(F_i;\mathbb{Q})$.

(0.4) $|x_i^{n_i}[F_i]|=D_X(F_i)=1$, the number $D_X(F_i)$ defined here is called the defect of F_i in X.

(0.5) The tangential representations of G_p at fixed points are of the linear type.

The concept of tangential representations of the linear type has been introduced by A. Hattori [Ha] (see also [Hs]). There only S^1 actions are treated, but the definition of the linear type can be adopted with no difficulties to G_p actions. In case of Type II_0 actions this means

(0.5)′ $T_fX=nv\mid q$ as real representations. Here T_fX is the tangential representation at the isolated fixed point f, v is the normal bundle to the codimension 2 fixed point component F, and $v\mid q$ is the restriction to a point q of F.

Remark. It follows quite easily from (0.2) that $D_X(F_i)\neq 0$. The term of algebraically standardness was introduced in [D1] for actions of Type II_0 in a slightly different way, two more conditions were given, i.e.

$$p(X)=(1+x^2)^{n+1},$$

$$c(v)=1\pm j^*x, \text{ where } j:F\to X \text{ is the inclusion map.}$$

A lemma shows that these two definitions are nevertheless equivalent.

It is natural to ask.

Question. Is a (locally) smooth G_p action on a homotopy $\mathbb{C}P^n$ algebraically standard?

In general, the answer is no. In fact, in [DM] we showed

THEOREM. *For any odd integer m and any odd $n\geqq 3$, there are infinitely many homotopy $\mathbb{C}P^n$'s with smooth G_m actions which satisfy neither* (0.3) *nor* (0.4).

As for (0.5), G_p actions of non-linear type with isolated fixed points are constructed in [P1, 2], [MT], [T], [DM]. In these counterexamples, and also in the ones in the last theorem, the dimensions of the fixed point components are small in comparison with the dimension of the ambient manifold. In fact, they are less than half of it. To the contrary one has the vague feeling that an action with a fixed point component of low codimension is restrictive (see [D1], [M1,2] for example). In this sense Type II_0 actions are the extreme case. Thus we are led to study Type II_0 actions.

Our first result is a rigidity theorem.

THEOREM A. *Let X be a cohomology $\mathbb{C}P^n$. There is a constant c_X, which depends only on the Pontrjagin class of X, such that any (locally) smooth G_p action of Type II_0 on X is algebraically standard if $p \geqq c_X$.*

This, and several other studies of cyclic actions on homotopy complex projective spaces, were motivated by Petrie's Conjecture [P1]. It may be stated as follows:

Suppose S^1 acts smoothly and effectively on a closed manifold X homotopy equivalent to $\mathbb{C}P^n$. Then $p(X) = (1 + x^2)^{n+1}$.

This conjecture has been verified in several special cases, and for a long bibliography see [D2]. In particular it holds if the action is semifree (which is equivalent to having two fixed point components) [Wg1], [Y]. Theorem A provides a new proof of Petrie's Conjecture for Type II_0 actions of S^1 (cf. [M2]). In fact, it follows from Theorem A and the above remark that we have

COROLLARY. *Let X be a homotopy $\mathbb{C}P^n$ with non-standard Pontrjagin class. Then $p \geqq c_X$ (c_X as in Theorem A) implies that X does not admit a (locally) smooth Type II_0 G_p action.*

The interesting aspect of this alternative proof is that it is merely based on the G Signature formula for elements of sufficiently large but finite prime order. In addition we do not need that the action is smooth, but only that it is locally smooth.

In contrast we have

THEOREM B. *Let $\varepsilon_n = 1$ if $n \equiv 3$ (mod 4) and $\varepsilon_n = 0$ otherwise. If $[(n-2)/4] - \varepsilon_n \geqq (p+1)/2$, then there are infinitely many homotopy $\mathbb{C}P^n$'s with smooth G_p action of Type II_0 such that (0.3) is not satisfied, in particular they are not algebraically standard. (The inequality holds if $n \geqq 2p + 8$).*

We note that a smooth free S^1 action on the standard sphere S^{2n-1} which restricts to a linear G_p action on S^{2n-1} produces a homotopy $\mathbb{C}P^n$ with a smooth G_p action of Type II_0 having the S^1 orbit space of S^{2n-1} as the fixed point component of codimension two (see §3 for details). The inequality in Theorem B is a sufficient condition for infinitely many such actions to exist (Theorem 3.2).

Theorem 3.2 is valid even when $p = 1$. It then says that there are infinitely many smooth free S^1 actions on S^{2n-1} provided that $n \geqq 8$ or $n = 6$. This almost proves a main theorem of Wang [Wg2] (in fact, Wang proves the existence for $n \geqq 7$). Our proof is quite different from his proof and much simpler.

Having both Theorems A and B in mind we make this

Conjecture. There exists a function $d(n)$ such that (locally) smooth Type II_0 actions of G_p on cohomology $\mathbb{C}P^n$'s are algebraically standard if $p \geqq d(n)$.

It would mean that c_X in Theorem A depends only on n. As supporting evidence we quote experience from Theorems A and B. Low dimensional evidence is also the main result of [D1].

THEOREM C ([D1]). *A (locally) smooth Type* II_0 *action of* G_p *on a cohomology* $\mathbb{C}P^n$ *is algebraically standard if*

(1) $n \leqq 3$

(2) $n = 4$ *and* $p \geqq 5$

(3) $n = 5$, $p \geqq 7$ *and the relative class number* $h_1(p)$ *is odd or* $p = 29$.

Some computer assisted computation in the spirit of [$D1$] *also show further cases of this theorem*:

(4) $n = 6$ *and* $7 \leqq p \leqq 43$

(5) $n = 7$ *and* $11 \leqq p \leqq 19$.

In spite of the estimate for $d(n)$ in Theorem B it seems reasonable to expect that $d(n)$ is approximately n, up to a small additive constant. This guess is based on the locally linear PL discussion of Theorem B in [D1].

This paper is organized as follows. In the first two sections we study rigidity phenomena of G_p actions, (0.4) and (0.5) in Section 1 and (0.3) in Section 2. In Section 3 we prove Theorem B. We also construct infinitely many smooth free S^1 actions on S^{2n-1} which restricts to linear G_p actions. In Section 4 we relate our results to one announced by Connolly-Weinberger [We].

§1. Rigidity of defects and tangential representations

As is well known the G signature theorem imposes a profound constraint on invariants of G actions. In this and the next section we observe to what extent it restricts our invariants of G_p actions of Type II_0. The G signature theorem holds for semifree tame actions ([Wal, 14B]), and for G_p actions of Type II_0 tameness is equivalent to local smoothness ([D1]). In the following we consider locally smooth G_p actions of Type II_0.

Roughly speaking our results say that those actions resemble the linear action of Type II_0 on $\mathbb{C}P^n$ provided that p is sufficiently large. Our argument works for a family of closed orientable even dimensional manifolds X satisfying the following three conditions:

(1.1) The second Betti number $\dim H^2(X;\mathbb{Q})$ of X is one. There is an element $x \in H^2(X;\mathbb{Z})$ which descends to a generator of $H^2(X;\mathbb{Z})/\text{torsion}$. We sometimes consider x as an element of $H^2(X;\mathbb{Q})$.

(1.2) If $\dim X = 2n$, then $x^n \neq 0$.

(1.3) The total Pontrjagin class of X is a polynomial of x in $H^*(X;\mathbb{Q})$.

This family contains $\mathbb{Z}_p$ ($\mathbb{Z}$ or $\mathbb{Q}$) cohomology $\mathbb{C}P^n$'s, i.e. closed smooth manifolds having the same cohomology ring as $\mathbb{C}P^n$ with $\mathbb{Z}_p$ ($\mathbb{Z}$ or $\mathbb{Q}$) coefficients. Examples of another type are non-singular algebraic hypersurfaces of $\mathbb{C}P^{n+1}$ ($n \geqq 3$).

Throughout this section X will denote a closed manifold of dimension $2n$ satisfying the above three conditions and we fix an element x in (1.1).

DEFINITION/OR CONVENTION 1.4. We choose an orientation class $[X]$ so that the defect $D(X)$ defined as $x^n[X]$ is non-negative. By (1.2) it is a positive integer.

Suppose X supports a locally smooth G_p action of Type II_0. We denote the fixed point component of codimension two by F and the isolated fixed point by f. The G signature formula is described in terms of local information around the fixed point set. We need more notations and conventions to write it down.

DEFINITION/OR CONVENTION 1.5. We choose an orientation class $[F]$ for F so that the defect $D_X(F) = j^*x^{n-1}[F]$ of F in X is a non-negative integer (cf. Introduction).

The orientations on X and F determine a unique orientation on the normal bundle ν of F in X such that their juxtaposition agrees with the orientation of X. Once ν is oriented, it can be regarded as a complex line bundle as usual. Let t^a denote the complex 1-dimensional representation of any subgroup G of S^1 such that $g \in G$ acts by multiplication with g^a. The identification of an appropriate element $g \in G_p$ with $\exp(2\pi i/p)$ in S^1 identifies G_p with a subgroup of S^1 such that

$$\nu \mid q = t^{-1} \quad \text{for} \quad q \in F. \tag{1.6}$$

We fix this identification of G_p with the subgroup of p-th roots of unity.

The tangential representation T_fX at the isolated fixed point f is oriented. As is well known one can put a complex structure on it such that the induced orientation agrees with the one given by the orientation of X, although such a complex structure is not unique. We choose integers m_j such that $1 \leqq |m_j| < p/2$ and

$$T_f X = \sum_{j=1}^{n} t^{m_j}. \tag{1.7}$$

As a step towards algebraical standardness, we pose an intermediate definition.

DEFINITION 1.8. We say that a locally smooth G_p action of Type II_0 is *weakly algebraically standard* if $D(X) = D_X(F) = 1$ (cf. (0.4)) and the m_j can be chosen to be 1 for every j (i.e. the action is of the linear type, cf. (0.5)).

Our main theorem in this section is

THEOREM 1.9. *Let X be a closed manifold of dimension $2n$ satisfying* (1.1), (1.2), *and* (1.3). *Define a_k by setting $\sum a_k x^{2k} = p(X)$. Then there is a constant b_X depending only on $\{a_k\}$, the Betti numbers of X, and the number of torsion elements of $H^*(X; \mathbb{Z})$ such that if $p \geqq b_X$, then any locally smooth G_p action of Type II_0 on X is weakly algebraically standard.*

The rest of this section is devoted to the proof of this theorem. The following lemma is proved in Lemma 2.8 of [M2].

LEMMA 1.10. (1) *$D(X)$ divides $D_X(F)$.*
(2) *$c_1(\nu) = D_X(F)/D(X) j^* x$, where $c_1(\nu)$ denotes the first Chern class of ν.*

Set $D_X(F)/D(X) = d$, $j^* x = \bar{x}$ and $\exp 2\pi i/p = z$. Since G_p is considered as a subgroup of S^1, z is an element of G_p. We apply the G signature theorem (see [HZ, p. 50]) together with (1.6), (1.7) and Lemma 1.10 (2) to get an identity

$$\mathrm{Sign}(z, X) = \frac{z^{-1} e^{2d\bar{x}} + 1}{z^{-1} e^{2d\bar{x}} - 1} L(F)[F] + \prod_{j=1}^{n} \frac{z^{m_j} + 1}{z^{m_j} - 1}. \tag{1.11}$$

Here $L(F)$ denotes the Hirzebruch L polynomial of F. One can easily verify that

$$\frac{z^{-1} e^{2d\bar{x}} + 1}{z^{-1} e^{2d\bar{x}} - 1} = 1 - \frac{2z}{z-1} \sum_{k=0}^{n-1} \left(\frac{e^{2d\bar{x}} - 1}{z - 1} \right)^k.$$

The sum in this expression is finite because $(2^{2d\bar{x}}-1)^j$ vanishes for $j \geqq n$ as $\dim F < 2n$. Moreover, Hirzebruch's signature formula states that

$$L(F)[F] = \operatorname{Sign} F.$$

Substituting both in (1.11) and multiplying the resulting identity by $(z-1)^n$, we get

$$0 = (z-1)^n(\operatorname{Sign}(z, X) - \operatorname{Sign} F) + 2z\left\{\sum_{k=0}^{n-1} (z-1)^{n-1-k}(e^{2d\bar{x}}-1)^k\right\}L(F)[F]$$
$$- (z-1)^n \prod_{j=1}^{n} (z^{m_j}+1)/(z^{m_j}-1). \quad (1.12)$$

We shall estimate the value of each term in this identity for a sufficiently large value of p. We begin with

LEMMA 1.13. (1) *Let s_X be the sum of all Betti numbers of X and the number of torsion elements in $H^*(X; \mathbb{Z})$. Then*

$$|\operatorname{Sign}(z, X) - \operatorname{Sign} F| \leqq s_X.$$

(2) $|(z-1)^n \prod (z^{m_j}+1)/(z^{m_j}-1)| < 2^n$.

(3) $|(z-1)^n \prod (z^{m_j}+1)/(z^{m_j}-1)| < 2^{n-1}(1+\tan^2 \pi/p)$ *unless* $|m_j| = 1$ *for every* j.

Proof. (1) It easily follows from the definition of the G signature that

$$|\operatorname{Sign}(z, X)| \leqq \dim_{\mathbb{C}} H^n(X; \mathbb{C}).$$

As for Sign F, it follows from the definition of the signature and the universal coefficient theorem that

$$|\operatorname{Sign} F| \leqq \dim_{\mathbb{R}} H^{n-1}(F; \mathbb{R}) \leqq \dim_{\mathbb{Z}_p} H^{n-1}(F; \mathbb{Z}_p).$$

On the other hand it is known (see p. 144 of [B]) that

$$\sum_{i \geqq n-1} \dim_{\mathbb{Z}_p} H^i(F; \mathbb{Z}_p) \leqq \sum_{i \geqq n-1} \dim_{\mathbb{Z}_p} H^i(X; \mathbb{Z}_p).$$

These show that

$$
\begin{aligned}
|\mathrm{Sign}(z, X) - \mathrm{Sign}\, F| &\leqq |\mathrm{Sign}(z, X)| + |\mathrm{Sign}\, F| \\
&\leqq \dim_{\mathbb{C}} H^n(X; \mathbb{C}) + \sum_{i \geqq n-1} \dim_{\mathbb{Z}_p} H^i(X; \mathbb{Z}_p) \leqq s_X.
\end{aligned}
$$

(2) Since $z = \exp 2\pi i/p$, we have

$$|(z^m - 1)/(z^m + 1)| = |\tan m\pi/p|.$$

Hence

$$
\begin{aligned}
&|(z-1)^n \prod (z^{m_j}+1)/(z^{m_j}-1)| \\
&\quad = |(z+1)^n (z-1)^n/(z+1)^n|\, |\prod (z^{m_j}+1)/(z^{m_j}-1)| \\
&\quad < 2^n \prod |\tan(\pi/p)/\tan(m_j\pi/p)|. \qquad (1.14)
\end{aligned}
$$

Remember that $|m_j|$ are chosen so that $1 \leqq |m_j| < p/2$. Since $\tan y$ is a monotone increasing function in the domain $|y| < \pi/2$, (2) follows from (1.14).

(3) Unless $|m_j| = 1$ for every j, we have

$$\prod |\tan \pi/p/\tan m_j\pi/p| \leqq |\tan \pi/p/\tan 2\pi/p|. \qquad (1.15)$$

Here $\tan 2\pi/p = 2\tan(\pi/p)/(1+\tan^2\pi/p)$. Now (3) follows from (1.14) and (1.15). Q.E.D.

We now consider the second summand of (1.12). By multiplicativity of the L polynomial we have

$$L(F) = j^* L(X) L(v)^{-1}.$$

By Lemma 1.10 (2) we have

$$L(v) = 1 + p_1(v)/3 = 1 + c_1(v)^2/3 = 1 + d^2\bar{x}^2/3.$$

On the other hand since $p(X)$ is a polynomial of x by assumption (1.3) and $L(X)$ is a polynomial of Pontrjagin classes, we may then write

$$L(X) = \sum_{k=0}^{[n/2]} A_k x^{2k} \qquad (A_0 = 1)$$

with $A_k \in \mathbb{Q}$. Consequently we have

$$L(F) = \left(\sum A_k \bar{x}^{2k}\right)\left(\sum (-d^2\bar{x}^2/3)^l\right). \tag{1.16}$$

Put this into the second summand of (1.12) and expand it with respect to d. Then, since $(\bar{x})^{n-1}[F] = D_X(F) = dD(X)$ by the definition of $D_X(F)$ and d, the exponent of d is at most n. In fact, we can express

$$\text{the second summand of (1.12)} = \sum_{j=0}^{n} B_j(z)d^j$$

with polynomials $B_j(z)$ of degree less than or equal to n. We shall collect properties of $B_j(z)$ in the following lemma. The proof is easy, so we leave it to the reader.

LEMMA 1.17. (1) *If* $p \geqq n+1$, *then* $B_j(z)$ *is described in terms of the coefficients of the polynomial* $p(X) = \sum a_k x^{2k}$.

(2) $B_j(1) = 0$ *for* $j < n$

(3) $B_n(1) = 2^n D(X)$.

We need one more lemma.

LEMMA 1.18. $d \neq 0$, *i.e. d is a positive integer.*

Proof. Suppose $d = 0$. Then, since $(\bar{x})^{n-1}[F] = D_X(F) = dD(X) = 0$ and $L(F)$ is a polynomial of $\bar{x}$ (see (1.16)), the first term in the right hand side of (1.11) vanishes. Hence identity (1.11) turns into

$$\text{Sign}(z, X) = \prod_{j=0}^{n} (z^{m_j} + 1)/(z^{m_j} - 1). \tag{1.19}$$

However this is impossible as observed below. The proof is essentially the same as in Theorem 7.1 of [AB].

Let $R(G_p)_z$ denote the ring localized at the prime ideal of $R(G_p)$ vanishing at z. Identity (1.19) implies that

$$\text{Sign}(G_p, X) = \prod_{j=0}^{n} (t^{m_j} + 1)/(t^{m_j} - 1) \text{ in } R(G_p)_z$$

where t denotes the standard complex 1-dimensional representation of G_p as before. Multiplying both sides by $\prod (t^{m_j} - 1)$, it turns out that the resultant identity in $R(G_p)_z$ comes from $R(G_p)$. Since the kernel of the natural map from $R(G_p)$ to $R(G_p)_z$ is the ideal generated by the cyclotomic polynomial $\sum_{j=1}^{p} t^j$, one concludes that

$$\operatorname{Sign}(G_p, X) \prod (t^{m_j} - 1) = \prod (t^{m_j} + 1) + h\left(\sum t^j\right) \text{ in } R(G_p)$$

with some integer h. Here we can evaluate this identity at $t = 1$. Then it reduces to

$$0 = 2^n + hp$$

which is impossible because p is odd. Q.E.D.

Proof of Theorem 1.9. We now fix the underlying manifold X. If we take p sufficiently large, then z converges to 1. We denote this by $z \approx 1$. By Lemma 1.13 (1) one can see that

$$\text{the first term of (1.12)} \approx 0. \tag{1.20}$$

On the other hand, the third term of (1.12) is bounded independent of the value of p by Lemma 1.13 (2). These together with Lemma 1.17 imply that the values of d are also bounded. Hence one can conclude by Lemma 1.17 that

$$\text{the second summand of (1.12)} \approx 2^n d^n D(X). \tag{1.21}$$

Suppose $|m_j| \geqq 2$ for some j. Then Lemma 1.13 (3) tells that since $\tan \pi/p \approx 0$, the absolute value of the third term of (1.12) converges to a value strictly less than 2^n. However this contradicts (1.12) together with (1.20) and (1.21) because $2^n d^n D(X) \geqq 2^n$ by Definition 1.4 and Lemma 1.18.

Thus we have established that $m_j = \pm 1$ for every j. Then

$$\text{the third term of (1.12)} \approx 2^n \prod m_j. \tag{1.22}$$

Since d and $D(X)$ are positive integers and $\prod m_j = \pm 1$, it follows from (1.20), (1.21), and (1.22) that

$$d = 1, \qquad D(X) = 1 \quad \text{and} \quad \prod m_j = 1.$$

Remember that there was an ambiguity of a choice of a complex structure on T_fX. The only constraint is that the orientation on T_fX induced from the complex structure agrees with the given one. As is easily seen, it is equivalent that the sign of $\prod m_j$ is positive. Hence we may assume $m_j = 1$ for each j. Q.E.D.

§2. Rigidity of Pontrjagin classes

Let X be the same as in (1.1). We use the notation of §1 freely. The following definition is consistent with that of the Introduction.

DEFINITION 2.1. We say that a locally smooth G_p action of Type II_0 on X is *algebraically standard* if it is weakly algebraically standard and $p(F) = (1 + \bar{x}^2)^n$ in $H^*(F; \mathbb{Q})$.

The purpose of this section is to prove

THEOREM 2.2. *Let a locally smooth G_p action of Type II_0 on X be weakly algebraically standard. Suppose the induced action of G_p on $H^n(X; \mathbb{Q})$ is trivial. Then the G_p action is algebraically standard provided that $p \geqq n + 2$.*

Remark 2.3. If X is a $\mathbb{Q}$-cohomology $\mathbb{C}P^n$, then G_p acts trivially on the cohomology because $\dim_{\mathbb{Q}} H^n(X; \mathbb{Q}) \leqq 1$. Generally, if we take a basis on the vector space $H^n(X; \mathbb{Q})$ coming from $H^n(X; \mathbb{Z})$, the induced action of G_p on $H^n(X; \mathbb{Q})$ gives a homomorphism from G_p to a general linear group $GL(r, \mathbb{Z})$ where $r = \dim_{\mathbb{Q}} H^n(X; \mathbb{Q})$. Therefore if $GL(r, \mathbb{Z})$ does not contain an element of order p, then the assumption is satisfied. This is the case if $p \geqq r + 2$. The proof is as follows. Diagonalize the image of $z = e^{2\pi i/p}$ over $\mathbb{C}$. Then the trace is a polynomial of z over $\mathbb{Z}$ with at most r factors of degree less than p. It must be an integer as the image of $z \in G_p$ lies in $GL(r, \mathbb{Z})$. Since a minimal polynomial of z over $\mathbb{Z}$ (or $\mathbb{Q}$) is the cyclotomic polynomial $\sum_{j=0}^{p-1} z^j$, the above polynomial does not contain any factor of z^j $(0<j<p)$. This means that the image of z is the identity matrix.

COROLLARY 2.4. *Let X and b_X be the same as in Theorem 1.9. Then a smooth G_p action of Type II_0 on X^{2n} is algebraically standard if $p \geqq c_X = \max\{b_X, n+2, \dim_{\mathbb{Q}} H^n(X; \mathbb{Q}) + 2\}$.*

Proof of Theorem 2.2. Triviality of the induced action of G_p on $H^n(X; \mathbb{Q})$ implies that $\mathrm{Sign}(z, X)$ is equal to $\mathrm{Sign}\, X$. Since the action is weakly algebraically standard, i.e. $d=1$ and $m_j=1$ for every j, (1.12) turns into

$$0=(z-1)^n(\operatorname{Sign} X-\operatorname{Sign} F) + 2z\left\{\sum_{k=0}^{n-1}(z-1)^{n-1-k}(e^{2\bar{x}}-1)^k\right\}L(F)[F]-(z+1)^n. \tag{2.5}$$

In this identity the coefficients of z^j are rational numbers and the degree of z is at most n. Because the minimal polynomial of z is of degree $p-1$, the coefficients of z^j must identically vanish as $p \geqq n+2$.

Look at the constant term in (2.5). Since it must be zero, we get

$$\operatorname{Sign} X-\operatorname{Sign} F=(-1)^n.$$

Putting this into (2.5), we have

$$2z\left\{\sum_{k=0}^{n-1}(z-1)^{n-1-k}(e^{2\bar{x}}-1)^k\right\}L(F)[F]=(1+z)^n-(1-z)^n. \tag{2.6}$$

Compare the coefficients of z^j inductively. The values of $(e^{2\bar{x}}-1)^kL(F)[F]$ are then uniquely determined for each k. They determine $L(F)$, because $L(F)$ is a polynomial of $\bar{x}$ (see (1.16)). On the other hand the linear G_p action of Type II_0 on $\mathbb{C}P^n$ also satisfies (2.6) and $F=\mathbb{C}P^{n+1}$ in that case. Hence one can conclude that $p(F)=(1+\bar{x}^2)^n$ in general. Q.E.D.

§3. Free smooth S^1 actions on lens spaces

Let X be a homotopy $\mathbb{C}P^m$ with a homotopy equivalence h from Y to $\mathbb{C}P^m$. Let $h^*\gamma$ be the pullback of the canonical line bundle γ over $\mathbb{C}P^m$ by h. Let $D(h^*\gamma)$ (resp. $S(h^*\gamma)$) denote the disk (resp. sphere) bundle of $h^*\gamma$. Then $S(h^*\gamma)$ is a homotopy sphere with a smooth S^1 action induced from the complex multiplication on fibers.

Suppose that

(3.1) $S(h^*\gamma)$ with the restricted G_p action is equivariantly diffeomorphic to the unit sphere S^{2m+1} of $\mathbb{C}^{m+1}$ with the linear G_p action of weight one (i.e. $S((m+1)t)$).

Remark. If $S(h^*\gamma)$ with the restricted G_p action is equivariantly diffeomorphic to S^{2m+1} with a linear G_p action, then it must be $S((m+1)t)$. In fact, the Reidemeister torsion of the orbit space $S(h^*\gamma)/G_p$ is the same as that of the lens space of weight one because it is fibered over a homotopy complex projective space Y (see [Wal, Prop. 14E.8(c)]). On the other hand linear lens spaces are distinguished by Reidemeister torsion invariants up to diffeomorphism (see [Mi, 12.7]).

If (3.1) is satisfied, then we can attach the unit disk D^{2m+2} of $\mathbb{C}^{m+1}$ with the linear G_p action of weight one equivariantly to $D(h^*\gamma)$ along the boundary. The resulting space turns out to be a homotopy $\mathbb{C}P^{m+1}$ with a smooth weakly algebraically standard G_p action of Type II_0 whose fixed point set consists of Y and the center of D^{2m+2}. Hence a pair (Y, h), which satisfies (3.1) but the total Pontrjagin class $p(Y)$ of Y is not of the same form as $p(\mathbb{C}P^m)$, yields a weakly algebraically standard but algebraically non-standard smooth G_p action of Type II_0. In this section we use classical surgery theory to find such pairs.

Let $L^m(p)$ denote the standard lens space defined as the orbit space of S^{2m+1} by the linear G_p action of weight one. There is a natural S^1 fiber bundle $\pi^m(p): L^m(p) \to \mathbb{C}P^m$. Let $\bar{Y}$ be the total space of the pullback of this S^1 bundle via h and let $\bar{h}$: $\bar{Y} \to L^m(p)$ be the induced map covering h. We note that h is a simple homotopy equivalence, hence so is $\bar{h}$ ([A]). To assign a pair $(\bar{Y}, \bar{h})$ to (Y, h) gives a map

$$\pi^m(p)^*: hS(\mathbb{C}P^m) \to hS(L^m(p)).$$

Here $hS(Z)$ denotes the set of simple homotopy smoothings of Z, namely it is the set of equivalence classes of pairs (W, g) such that W is a smooth manifold and g is a simple homotopy equivalence from W to Z. In case Z has a boundary ∂Z, a set $hS(Z, \partial Z)$ similar to $hS(Z)$ is defined. But it is required in addition that g restricts to a diffeomorphism on the boundary. The set $hS(Z)$ or $hS(Z, \partial Z)$ has a distinguished element defined as a pair of Z and the identity map on Z. The inverse image of the distinguished element in $hS(L^m(p))$ by $\pi^m(p)^*$ is called the kernel of $\pi^m(p)^*$, and it is denoted by $\operatorname{Ker} \pi^m(p)^*$.

Since $\bar{Y}$ is exactly the orbit space of $S(h^*\gamma)$ by the restricted G_p action, statement (3.1) is equivalent to: $\bar{Y}$ is diffeomorphic to $L^m(p)$. Thus we are led to study $\operatorname{Ker} \pi^m(p)^*$. Our aim is to find an element $(Y, h) \in \operatorname{Ker} \pi^m(p)^*$ having non-standard total Pontrjagin classes. A sufficient condition for such an element to exist is that $\operatorname{Ker} \pi^m(p)^*$ is infinite, because diffeomorphism types of homotopy $\mathbb{C}P^m$'s are distinguished by Pontrjagin classes up to finite ambiguity, and up to homotopy there are only two homotopy equivalences from Y to $\mathbb{C}P^m$. In the sequel we ask

Question. When is $\operatorname{Ker} \pi^m(p)^*$ infinite?

Our answer is the following.

THEOREM 3.2. *Suppose* $m \geqq 3$.
(1) *If* $p \geqq m+2$, *then* $\operatorname{Ker} \pi^m(p)^*$ *is finite.*
(2) *If* $[(m-1)/4] - \varepsilon_{m-1} \geqq (p+1)/2$, *then* $\operatorname{Ker} \pi^m(p)^*$ *is infinite, where* ε_j *is the same as in Theorem B of the Introduction and p may also be one.*

Remark 3.3. (1) One can ask the same question in the PL category. In this case a complete answer is obtained in [D1]. It says that the kernel of $\pi^m(p)^* : hPL(\mathbb{C}P^m) \to hPL(L^m(p))$ is infinite if and only if $p \leqq m+1$. There the classification theorem of $hPL(L^m(p))$ ([Wal, §14]) plays a role. In the smooth category, however, such a classification theorem is not known. Nevertheless it seems plausible to conjecture the same conclusion as in the PL case. One only would need some "additivity" for $\pi^m(p)^*$.

(2) Infiniteness of $\operatorname{Ker} \pi^m(p)^*$ means that $L^m(p)$ admits infinitely many smooth free S^1 actions. Since $L^m(1) = S^{2m+1}$, Theorem 3.2 (2) can be considered as an extended version of Wang's result [Wg2] as indicated in the Introduction.

The proof of Theorem 3.2 (1) is the same as that of the PL case. In fact, it is shown in [D1] that if $(Y, h) \in hPL(\mathbb{C}P^m)$ belongs to $\operatorname{Ker} \pi^m(p)^*$, then the PL Pontrjagin class $p(Y)$ is of the same form as $p(\mathbb{C}P^m)$ provided $p \geqq m+2$. The argument is also essentially the same as the proof of Theorem 2.2.

The rest of this section is devoted to the proof of Theorem 3.2 (2). We note that the sets $hS(\mathbb{C}P^m)$ and $hS(L^m(p))$ do not support natural group structures. This makes our problem difficult. To solve it we find suitable subsets of $hS(\mathbb{C}P^m)$ and $hS(L^m(p))$ which form abelian groups and on which $\pi^m(p)^*$ is a homomorphism. Then we estimate their ranks explicitly. The inequality in Theorem 3.2 (2) is a sufficient condition that the rank of the subset of $hS(\mathbb{C}P^m)$ is greater than that of $hS(L^m(p))$.

Fix k between 1 and m and let P_1 (resp. P_2) denote the submanifold of $\mathbb{C}P^m$ defined by the equations $w_j = 0$ for $k+1 \leqq j \leqq m$ (resp. $0 \leqq j \leqq k$) where w_j denotes the $j+1$th homogeneous coordinate of $\mathbb{C}P^m$. Let v_i denote small open tubular neighborhoods of P_i. Remove v_1 and v_2 from $\mathbb{C}P^m$ and denote the resulting space by P. Let Q denote the inverse image of P by $\pi^m(p)$. The following lemma is an easy consequence of the h-cobordism theorem.

LEMMA 3.4. *The manifold* P (*resp.* Q) *is diffeomorphic to the product of the* S^1 (*resp.* G_p) *orbit space of* $S^{2k+1} \times S^{2(m-k-1)+1}$ *and the unit interval, where the* S^1 (*resp.* G_p) *action on* $S^{2k+1} \times S^{2(m-k+1)+1}$ *is the diagonal one induced from complex multiplication on each factor.*

Since P (resp. Q) is diffeomorphic to a product of a closed smooth manifold and the unit interval, the set $hS(P, \partial P)$ (resp. $hS(Q, \partial Q)$) admits a natural abelian group structure (see §10 of [Wal]). Let $\pi:(Q, \partial Q) \to (P, \partial P)$ denote the restriction of $\pi^m(p)$. It is clear that π induces a homomorphism $\pi^*: hS(P, \partial P) \to hS(Q, \partial Q)$ with respect to the group structures given above.

Given an element (W, g) of $hS(P, \partial P)$, one can glue v_i to W along the boundary via the diffeomorphism $g \mid \partial W : \partial W \to \partial P$. This defines a map κ_P from $hS(P, \partial P)$ to $hS(\mathbb{C}P^m)$. Similarly we have a map κ_Q from $hS(Q, \partial Q)$ to $hS(L^m(p))$. These maps fit into the following commutative diagram:

$$\begin{array}{ccc} hS(P, \partial P) & \xrightarrow{\kappa_P} & hS(\mathbb{C}P^m) \\ \Big\downarrow{\scriptstyle \pi^*} & & \Big\downarrow{\scriptstyle \pi^m(p)^*} \\ hS(Q, \partial Q) & \xrightarrow{\kappa_Q} & hS(L^m(p)) \end{array}$$

The following lemma ensures that $\operatorname{Ker} \pi^m(p)^*$ is infinite if $\operatorname{Ker} \pi^*$ is infinite.

LEMMA 3.5. *The map κ_P is finite to one.*

Proof. The surgery exact sequence yields a diagram:

$$\begin{array}{ccccccc} 0 = L_{2m+1}(1) & \to & hS(P, \partial P) & \to & [P/\partial P, F/O] & \to & L_{2m}(1) \\ & & \Big\downarrow{\scriptstyle \kappa_P} & & \Big\downarrow{\scriptstyle q^*} & & \\ 0 = L_{2m+1}(1) & \to & hS(\mathbb{C}P^m) & \to & [\mathbb{C}P^m, F/O] & \to & L_{2m}(1) \end{array}$$

where q^* is induced by the quotient map $q: \mathbb{C}P^m \to \mathbb{C}P^m / v_1 \cup v_2 = P/\partial P$. The middle square in this diagram is commutative and q^* is a homomorphism. Therefore it suffices to show that the kernel of q^* is finite, in other words, q^* is injective when tensored by $\mathbb{Q}$. The following fact is well known.

Fact 3.6. *There are isomorphisms between these three groups:*

$$[Z, F/O] \otimes Q \to [Z, BO] \otimes \mathbb{Q} = \tilde{K}O(Z) \otimes Q \to \sum \tilde{H}^{4j}(Z; \mathbb{Q})$$

for any finite CW complex Z. In fact, the former map is induced from the natural map from F/O to BO and the latter one is the Pontrjagin character.

These isomorphisms are functorial, so the problem reduces to the injectivity of

$$q^* : \tilde{H}^{4j}(P/\partial P; \mathbb{Q}) = \tilde{H}^{4j}(\mathbb{C}P^m/v_1 \cup v_2; \mathbb{Q}) \to \tilde{H}^{4j}(\mathbb{C}P^m; \mathbb{Q}).$$

The cohomology exact sequence of the pair $(\mathbb{C}P^m, v_1 \cup v_2)$ yields an exact sequence:

$$\tilde{H}^{4j-1}(v_1 \cup v_2; \mathbb{Q}) \longrightarrow \tilde{H}^{4j}(\mathbb{C}P^m/v_1 \cup v_2; \mathbb{Q}) \xrightarrow{q^*} \tilde{H}^{4j}(\mathbb{C}P^m; \mathbb{Q}).$$

Here the left most term vanishes because v_i is homotopy equivalent to the complex projective space P_i. This proves the lemma. Q.E.D.

Now we shall estimate the rank of Ker π^*. The surgery exact sequence yields a commutative diagram:

$$\begin{tikzcd}
0 = L_{2m+1}(1) \arrow[r, "\omega_P"] & hS(P, \partial P) \arrow[r, "\pi_P"] \arrow[d, "\pi^*"] & {[P/\partial P, F/O]} \arrow[r, "\sigma_P"] \arrow[d, "\tau"] & L_{2m}(1) \\
L_{2m+2}(G_p) \arrow[r, "\omega_Q"] & hS(Q, \partial Q) \arrow[r, "\eta_Q"] & {[Q/\partial Q, F/O]} \arrow[r, "\sigma_Q"] & L_{2m+1}(G_p)
\end{tikzcd}$$

where τ is the map induced from π. Here all the terms are abelian groups and all the maps are homomorphisms. It follows that

$$rk \text{ Ker } \pi^* = rk\, hS(P, \partial P) - rk\pi^* \tag{3.7}$$

$$\begin{aligned} &rk\, hS(P, \partial P) \geqq rk[P/\partial P, F/O] - rk\, L_{2m}(1), \\ &rk\, \pi^* \leqq rk\, \eta_Q \circ \pi^* + rk\, \omega_Q = rk\, \tau \circ \eta_P + rk\, \omega_Q \leqq rk\, \tau + rk\, \omega_Q, \end{aligned} \tag{3.8}$$

where rk indicates the rank of an abelian group or a homomorphism. Replace the right hand side of (3.7) by (3.8). Then we get a lower bound of rk Ker π^*:

$$\begin{aligned} rk \ker \pi^* &\geqq rk[P/\partial P, F/O] - rkL_{2m}(1) - rk\tau - rk\omega_Q \\ &= rk \text{ Ker } \tau - rkL_{2m}(1) - rk\omega_Q. \end{aligned} \tag{3.9}$$

LEMMA 3.10. (1) $rkL_{2m}(1) = \{1 + (-1)^m\}/2$.
(2) $rk\, \omega_Q = (p-1)/2$.

(3) *If* $k \geqq m-k-1$, (*i.e.* $2k \geqq m-1$, *we may assume this without loss of generality*), *then* $rk\ \mathrm{Ker}\ \tau = [m/2] - [(k+1)/2]$.

Proof. (1) $L_{2m}(1)$ is isomorphic to $\mathbb{Z}$ if m is even and is isomorphic to $\mathbb{Z}_2$ if m is odd (see [Wal, §13]). This means (1).

(2) It is known that the rank of ω_Q is the same as that of the reduced L group of $L_{2m+2}(G_p)$ and the latter is $(p-1)/2$ (see §14E of [Wall]). This verifies (2).

(3) By Fact 3.6 $rk\ \mathrm{Ker}\ \tau$ agrees with the rank of the kernel of $\pi^* : \sum H^{4j}(P, \partial P; \mathbb{Q}) \to \sum H^{4j}(Q, \partial Q; \mathbb{Q})$. Remember that P is diffeomorphic to $(S^{2k+1} \times S^{2(m-k-1)+1})/S^1 \times I$ where I is the unit interval. The projection from P to the last two factors of the product gives rise to a fibration: $P \to \mathbb{C}P^{m-k-1} \times I$ with fiber S^{2k+1}. The Serre spectral sequence of this fibration (relative boundary) collapses because the fiber is a sphere of dimension greater than or equal to that of the base space by the assumption $k \geqq m-k-1$. It implies an isomorphism:

$$H^*(P, \partial P; \mathbb{Q}) \cong H^*(\mathbb{C}P^{m-k-1} \times (I, \partial I); \mathbb{Q}) \otimes H^*(S^{2k+1}; \mathbb{Q}).$$

On the other hand $H^*(\mathbb{C}P^{m-k-1} \times (I, \partial I) : \mathbb{Q})$ is isomorphic to $H^{*-1}(\mathbb{C}P^{m-k-1}; \mathbb{Q})$. Consequently we have an isomorphism

$$H^*(P, \partial P; \mathbb{Q}) \cong H^{*-1}(\mathbb{C}P^{m-k-1}; \mathbb{Q}) \otimes H^*(S^{2k+1}; \mathbb{Q}).$$

Similarly we have an isomorphism

$$H^*(Q, \partial Q; \mathbb{Q}) \cong H^{*-1}(L^{m-k-1}(p); \mathbb{Q}) \otimes H^*(S^{2k+1}; \mathbb{Q}).$$

Through these isomorphisms π^* splits into $\pi^{m-k-1}(p)^* \otimes id^*$ where id denotes the identity map on S^{2k+1}. Hence the kernel of π^* in degree $4j$ is given by $H^{4j-2k-2}(\mathbb{C}P^{m-k-1}; \mathbb{Q}) \otimes H^{2k+1}(S^{2k+1}; \mathbb{Q})$ through the above isomorphism. This shows that a kernel of rank one appears for each j between $[(k+1)/2]+1$ and $[m/2]$. That we may assume $k \geqq m-k-1$ follows as we could have exchanged P_1 and P_2 freely in, and just before, Lemma 3.4. In fact, it would just interchange k and $m-k-1$. This implies (3). Q.E.D.

This lemma and (3.9) show that

$$\begin{aligned} rk\ \mathrm{Ker}\ \pi^* &\geqq [m/2] - [(k+1)/2] - \{1 + (-1)^m]/2 - (p-1)/2 \\ &= [(m-1)/2] - [(k+1)/2] - (p-1)/2. \end{aligned}$$

Since $2k \geqq m-1$, we have

$$k+1 \geqq \langle (m+1)/2 \rangle$$

where $\langle v \rangle$ denotes the least integer greater than or equal to v. We take $k = \langle (m+1)/2 \rangle - 1$. Then it is sufficient for rk Ker π^* to be positive that

$$[(m-1)/2] - [\langle (m+1)/2 \rangle /2] \geqq (p+1)/2.$$

As easily observed the left hand side of this inequality reduces to $[(m-1)/4] - \varepsilon_{m+1}$. This completes the proof of Theorem 3.2 (2).

§4. Comparison with a result of Connolly–Weinberger

In this section we combine the preceding results to obtain deeper insight into smooth cyclic group actions on homotopy complex projective spaces.

Recently Connolly and Weinberger announced the following result.

THEOREM 4.1 (Connolly–Weinberger). *Let M be a closed submanifold of $\mathbb{C}P^n$ of codimension two. Then there exists a semifree smooth G_{2k} action on $\mathbb{C}P^n$ whose fixed point set consists of M and an isolated point if and only if M is a cohomology $\mathbb{C}P^{n-1}$ with $\mathbb{Z}_{2k}$ coefficient and defect $D_{\mathbb{C}P^n}(M) = 1$, where k is any integer.*

Remark 4.2. Their original statement (Corollary 2 in p. 276 of [We]) is false in its form. The above statement is the revised form which they communicated to the authors, cf. Zentralbratt 566, 57025.

A conclusion of Theorem 4.1 is that if the imbedded M is a homotopy $\mathbb{C}P^{n-1}$ and $D_{\mathbb{C}P^n}(M) = 1$, then for any prime number p there is a smooth G_p action of Type II_0 which fixes M.

On the other hand Theorem B says that if $(p+1)/2 \leqq [(n-2)/4] - \varepsilon_n$, then there is a homotopy $\mathbb{C}P^n X$ with a smooth G_p action of Type II_0 such that

(1) the fixed point component F of codimension two is a homotopy $\mathbb{C}P^{n-1}$ and $D_X(F) = 1$,

(2) $p(F)$ (resp. $p(X)$) is not of the same form as $p(\mathbb{C}P^{n-1})$ (resp. $p(\mathbb{C}P^n)$).

However Theorem A says that F is never fixed under any smooth G_p action on X if $p \geqq c_X$. This contrasts the above result of Connolly–Weinberger. Namely this shows that Theorem 4.1 holds only for submanifolds of the standard $\mathbb{C}P^n$.

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