Comment. Math. Helvetici 64 (1989) 256–268 0010-2571/89/020256-13$1.50 + 0.20/0

Calibrated geometries in Grassmann manifolds

HERMAN GLUCK, FRANK MORGAN AND WOLFGANG ZILLER

We take up here the search for the subvarieties of Lie groups and homogeneous spaces which can be shown to be volume minimizing in their homology classes by applying the method of calibrated geometries to the invariant differential forms.

As a first step, we study the real Grassmann manifolds G_kR^n of oriented k-planes through the origin in n-space. Using normalized Euler forms and other calibrations, we prove

THEOREM. *If k is an even integer* ≥ 4, *then in the Grassmann manifold* G_kR^n, *each subgrassmannian in the sequence*

$$G_kR^{k+1} \subset G_kR^{k+2} \subset \cdots \subset G_kR^{n-1}$$

is volume minimizing in its homology class. Moreover, any other minimizer in the same homology class is congruent to it.

Remarks. 1) When $k = 2$, the above submanifolds are all complex submanifolds of the Kähler manifold G_2R^n, and are well known to minimize volume in their homology classes; see Wirtinger [Wi] 1936 and Federer [Fe] 1969. Uniqueness fails because there are other complex submanifolds in these homology classes. The same argument shows that in the case of complex Grassmann manifolds G_kC^n, all the subgrassmannians minimize volume in their homology classes.

2) When $k = 4$, the G_4R^m are quaternionic submanifolds of the quaternionic Kähler manifold G_4R^n, and hence minimize volume in their homology classes by the work of Berger [Be] 1972. He established an analog of Wirtinger's inequality for a quaternionic form. He then used this to prove that if a manifold is quaternionically Kähler, and hence admits a closed quaternionic form, then the normalized powers of that form calibrate all quaternionic submanifolds, and show them to be volume minimizing in their homology classes. In particular,

quaternionic projective subspaces HP^m in HP^n are homologically volume minimizing.

3) When k is odd, each G_kR^m bounds over the reals in G_kR^n. We note in §4 how this follows directly from a presentation of the real cohomology of the Grassmann manifold. But these submanifolds do not necessarily bound over the integers. For example, G_3R^5 does not bound over the integers in G_3R^6, but twice it does.

4) Using the isometry of G_kR^n with $G_{n-k}R^n$, we can expand the above theorem to cover the submanifolds

$$G_1R^{n-k+1} \subset G_2R^{n-k+2} \subset \cdots \subset G_{k-1}R^{n-1}$$

of G_kR^n, according to the value of $n-k$.

5) Examples of totally geodesic submanifolds which minimize volume in their homology classes in other symmetric spaces were provided by A. T. Fomenko [Fo] in 1972 by different methods. For example, he established the existence of homologically volume minimizing round 3-spheres in all simply connected compact Lie groups. Also, he proved that in quaternionic projective space HP^n, each HP^m is volume minimizing in its homology class, by a method quite different from Berger's.

6) Using a method essentially equivalent to that of calibrations, D. C. Thi [Th] gave in 1977 other examples of homologically volume minimizing subvarieties. Some of these were not totally geodesic; others had singularities.

7) H. Tasaki [Ta] 1985, using the method of calibrations, has given examples of homologically volume minimizing cycles in Lie groups. Recently, he has observed that our methods show equally well that in quaternionic Grassmann manifolds G_kH^n, the subgrassmannians G_kH^m are uniquely volume minimizing in their homology classes, with no restriction on the value of k. Note that these quaternionic Grassmann manifolds are not quaternionically Kähler, and hence are not covered by Berger's theorem.

8) Using the method of calibrations to identify the homologically volume minimizing submanifolds of the unit tangent bundle of the round 3-sphere, it was shown in [G–Z] in 1986 that a unit vector field on the 3-sphere has minimum

volume if and only if it is tangent to a Hopf fibration. This unit tangent bundle is a homogeneous space, but not symmetric.

Consider the following table of real Grassmann manifolds, displaying the various inclusions.

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \\
\cup & & \cup & & \cup & & \cup & & \\
G_4R^5 & \subset & G_4R^6 & \subset & G_4R^7 & \subset & G_4R^8 & \subset & \cdots \\
\cup & & \cup & & \cup & & \cup & & \\
G_3R^4 & \subset & G_3R^5 & \subset & G_3R^6 & \subset & G_3R^7 & \subset & \cdots \\
\cup & & \cup & & \cup & & \cup & & \\
G_2R^3 & \subset & G_2R^4 & \subset & G_2R^5 & \subset & G_2R^6 & \subset & \cdots \\
\cup & & \cup & & \cup & & \cup & & \\
G_1R^2 & \subset & G_1R^3 & \subset & G_1R^4 & \subset & G_1R^5 & \subset & \cdots .
\end{array}$$

GUIDING QUESTION. *In the above table, which little Grassmann manifolds are homologically volume minimizing in which larger ones?*

The present theorem answers this question for inclusions which go across rows or up columns.

The first case not covered by this theorem is whether or not G_2R^4 minimizes volume in its homology class in G_3R^6. The answer is No, and reveals the simplest example of a subgrassmannian which is homologically nontrivial over the reals, yet does not minimize volume in its homology class. There is another 4-cycle in G_3R^6, homeomorphic to the suspension of RP^3, which lies in the same homology class as G_2R^4 but has smaller volume. This involves us in the calibrated geometry of the first Pontryagin form and the search for natural generators of the rational homotopy of the Grassmann manifolds, and will be dealt with in a separate paper.

This work has been partially supported by grants from the National Science Foundation.

1. Calibrated geometries

Let V be a finite dimensional real vector space with an inner product, and μ a d-form on V (that is, an alternating, d-fold multilinear map to the reals). Suppose that μ satisfies the inequality

$$\mu(v_1 \wedge v_2 \wedge \cdots \wedge v_d) \leq \mathrm{vol}\,(v_1 \wedge v_2 \wedge \cdots \wedge v_d), \tag{$*$}$$

with equality occurring somewhere. Then we say that μ is a *calibration* on V, and that it *calibrates* those oriented d-planes in V for whose ordered bases equality holds in $(*)$.

Now let X be a Riemannian manifold, and suppose that μ is a smooth closed d-form satisfying the inequality $(*)$ at every point of X. Let M be a smooth compact oriented d-dimensional submanifold of X for which $(*)$ is an equality whenever $v_1 \wedge v_2 \wedge \cdots \wedge v_d$ is tangent to M. Then this submanifold must be volume minimizing in its homology class in X. For if M' is another d-dimensional submanifold in the same homology class, then

$$\mathrm{vol}\, M = \int_M \mu = \int_{M'} \mu \leq \mathrm{vol}\, M',$$

by equality in $(*)$, Stokes' theorem and inequality in $(*)$, respectively. We say that *μ calibrates M*. If $\mathrm{vol}\, M = \mathrm{vol}\, M'$, then μ also calibrates M'. In this way, the d-form μ determines a family (possibly empty) of d-dimensional submanifolds of X, each of which minimizes volume in its homology class. A form μ satisfying $(*)$ and admitting equality at some point of X is called a *calibration* on X.

To distinguish the two situations above, we will sometimes say that μ calibrates *infinitesimally* certain oriented d-planes in the vector space V, and that it calibrates *globally* certain d-dimensional submanifolds of the Riemannian manifold X.

The standard example is provided by the powers of the Kähler form ω on a Kähler manifold X. For each integer s, $1 \leq s \leq \dim_C X$, the $2s$-form $\mu = \omega^s/s!$ is closed and satisfies $(*)$ by the Wirtinger inequality [Wi]. The corresponding submanifolds of X are just the canonically oriented complex submanifolds of complex dimension s. Many more examples are provided by Reese Harvey and Blaine Lawson in their beautiful foundational essay [H–L], in which they concentrate mostly on calibrated geometries in R^n given by differential forms with constant coefficients. But they call attention in part V.1 to the question of determining the geometries on Grassmann manifolds calibrated by the invariant forms which represent the universal characteristic classes. See also [Ha] and [Mo] and [Mo 2].

2. Calibrations on tensor products

The Grassmann manifold G_kR^n is a symmetric space,

$$G_kR^n = SO(n)/SO(k)\times SO(n-k),$$

and inherits, from the bi-invariant metric on $SO(n)$, a Riemannian metric and an action of $SO(n)$ as isometries. The $SO(n)$-invariant differential forms on G_kR^n are automatically closed, with precisely one in each cohomology class [Wo].

We can express the cohomology in yet a simpler way. Let $m = n - k$. Let K be an oriented k-plane in $k + m$ space, hence a point in the Grassmann manifold G_kR^{k+m}. Then the tangent space to G_kR^{k+m} at K is isomorphic to $R^k \otimes R^m$, and the isotropy subgroup $SO(k)\times SO(m)$ of K acts on this tensor product by acting on the individual factors. The $SO(k)\times SO(m)$ invariant alternating multilinear forms on $R^k \otimes R^m$ correspond in one–one fashion to the $SO(k+m)$-invariant differential forms on G_kR^{k+m}, and hence to the cohomology classes of the Grassmann manifold.

We will consider, for each choice of p-dimensional subspace R^p in R^m, the *p-fold column subspace* $R^k \otimes R^p$ of $R^k \otimes R^m$. When $R^k \otimes R^m$ is identified with the tangent space to G_kR^{k+m} at a point, the p-fold column subspaces are tangent to the subgrassmannians G_kR^{k+p}.

We suppose henceforth that k is even and that an orientation of R^k has been fixed.

If $e_1, \ldots, e_k$ is an ordered basis for R^k consistent with this orientation, and if $f_1, \ldots, f_p$ is a randomly ordered basis for R^p, we give $R^k \otimes R^p$ the orientation determined by the ordered basis

$$e_1 \otimes f_1, \ldots, e_k \otimes f_1, \ldots, e_1 \otimes f_p, \ldots, e_k \otimes f_p.$$

Because k is even, this orientation doesn't depend on an orientation for R^p. If the orientation on R^k is reversed, the orientation on $R^k \otimes R^p$ is reversed if p is odd, but is unchanged if p is even. We call this the *canonical orientation* for $R^k \otimes R^p$.

We will construct in this section a family of $SO(k)\times SO(m)$ invariant kp-forms λ_p on $R^k \otimes R^m$ which calibrate these canonically oriented p-fold column subspaces, and which, for $k \geq 4$, calibrate nothing else.

To begin the construction, let J be a complex structure on R^k consistent with its given orientation, that is, an orthogonal transformation $J\colon R^k \to R^k$ such that $J^2 = -I$, and such that the orientation on R^k associated with this complex structure agrees with the given one. For each such J, define a complex structure of the same name on $R^k \otimes R^m$ by the formula $J(u \otimes v) = J(u) \otimes v$. Let ω_J denote the corresponding Kähler form on $R^k \otimes R^m$. Then consider on this space the kp-form $\omega_J{}^{rp}/(rp)!$, where $k = 2r$.

We now define a twisted average of these powers of Kähler forms over the space of all possible complex structures J on R^k, equivalently over the group $O(k)$, as follows. Fix the complex structure J on R^k. Then for each g in $O(k)$, consider the corresponding complex structure $g^{-1}Jg$ on R^k, and by our convention, also on $R^k \otimes R^m$. Let $g^*(\omega_J) = \omega_{g^{-1}Jg}$. Then define the *p-fold column form* λ_p to be the average over all g in $O(k)$ of the forms

$$(\det g)^p\, g^*(\omega_J{}^{rp}/(rp)!), \qquad r = k/2.$$

PROPOSITION 2.1. *For even k, the p-fold column form λ_p is an $SO(k) \times SO(m)$ invariant kp-form on $R^k \otimes R^m$ which calibrates the canonically oriented p-fold column subspaces, and, when $k \geq 4$, nothing else.*

Remark. For $k = 2$, λ_1 is the Kähler form, and λ_p is proportional to $(\lambda_1)^p$. For $k = 4$, λ_1 was identified as a calibration of type $(3, 3)$ in [D–H–M, Chapt. 3].

Proof. Note that for each choice of complex structure J on R^k, the Kähler form ω_J on $R^k \otimes R^m$ is $O(m)$-invariant. Then the averaged form λ_p is certainly $SO(k) \otimes O(m)$ invariant, but, because of the twisting factor $(\det g)^p$, it is $O(k) \otimes O(m)$ invariant only when p is even.

By Wirtinger's inequality, the form $\omega_J{}^{rp}/(rp)!$ calibrates those oriented kp-planes in $R^k \otimes R^m$ which are canonically oriented complex rp-planes with respect to J. Then λ_p, as a twisted average of such forms over $O(k)$, calibrates those oriented kp-planes in $R^k \otimes R^m$ which are complex rp-planes with respect to all of the complex structures $g^{-1}Jg$ as g ranges over $O(k)$, and which for even p are canonically oriented for all g in $O(k)$, but for odd p are canonically oriented only when g lies in $SO(k)$, and are "canonically misoriented" when g lies in $O(k) - SO(k)$.

The canonically oriented p-fold column spaces have exactly this property, and hence are calibrated by λ_p. We must show that for $k \geq 4$, the form λ_p calibrates nothing else.

Let $e_1, e_2, \ldots, e_k$ be an orthonormal basis for R^k, let R^2 denote the subspace spanned by e_1 and e_2, and R^{k-2} its orthogonal complement. Suppose P is an oriented kp-plane in $R^k \otimes R^m$ calibrated by λ_p. Let $v + w$ be a vector in P, with v in $R^2 \otimes R^m$ and w in $R^{k-2} \otimes R^m$. Let g be the transformation in $O(k) - SO(k)$ which changes the sign of e_1, but is the identity on its orthogonal complement.

Let J denote the complex structure on R^k such that $J(e_1)=e_2$, $J(e_3)=e_4$, etc. And let it also denote the corresponding complex structure on $R^k \otimes R^m$. Since P is calibrated by λ_p, it is closed under application of J, and hence contains $J(v+w)=J(v)+J(w)$. But it is also closed under application of $g^{-1}Jg$, and hence must contain $g^{-1}Jg(v+w)=-J(v)+J(w)$. Hence P contains $J(v)$, and therefore also v. A similar argument applies to any two-dimensional subspace R^2 of R^k.

Now write any vector in P in the form $v_1+v_2+\cdots+v_k$, where v_i lies in $e_i \otimes R^m$. We may assume $v_1 \neq 0$. Since $k \geq 3$, we conclude as above that P contains v_1+v_2, and also v_1+v_3, and also v_2+v_3. Hence P contains v_1. Write $v_1=e_1 \otimes w$, with w in R^m. But there is a complex structure J on R^k taking e_1 to any desired e_i. Since P is closed under the induced action of J on $R^k \otimes R^m$, it must also contain each $e_i \otimes w$. Thus P must contain the single column space $R^k \otimes w$. Splitting this column off from P and iterating the argument, P must itself be a p-fold column subspace, as claimed, completing the proof of the proposition.

Remark. It is interesting to compare the p-fold column forms λ_p with the powers λ^p of the Euler k-form λ.

For all even k, we easily see that λ_1 is proportional to λ, as follows. The Euler form λ is, up to multiplication by a constant, the unique $SO(k)\times SO(m)$ invariant k-form on $R^k \otimes R^m$ with the property that if g lies in $O(k)-SO(k)$, then $g^*\lambda=-\lambda$. This follows because the other invariant forms (Pontryagin forms, and dual Euler forms in the case that m is also even) are all $O(k)$ invariant. Our column form λ_1 has the same symmetry as the Euler form λ, and so must be proportional to it.

But in general, λ_p is not proportional to λ^p. This first happens in G_4R^8, where λ_2 is not proportional to λ^2.

The power λ^p of the Euler form λ is obtained by first averaging a form and then raising that average to the pth power. The p-fold column form λ_p is obtained by starting with the same form, but first raising it to the p^{th} power and then averaging this power. So they differ by an interchange in the order of carrying out these two processes.

3. Proof of the main theorem

With Proposition 2.1 in hand, we immediately obtain the existence part of the main theorem.

PROPOSITION 3.1. *If k is an even integer, then the subgrassmannian G_kR^{k+p} of G_kR^{k+m} is volume minimizing in its homology class.*

Proof. As mentioned earlier, when we view $R^k \otimes R^m$ as the tangent space to G_kR^{k+m} at a point, the p-fold column subspaces $R^k \otimes R^p$ are tangent to the subgrassmannians G_kR^{k+p} through that point. By Proposition 2.1 these linear subspaces are calibrated by the $SO(k) \times SO(m)$ invariant linear kp-form λ_p on $R^k \otimes R^m$. This linear form then determines an $SO(k+m)$ invariant differential kp-form λ_p on G_kR^{k+m}, which by homogeneity calibrates the subgrassmannians at every point, thus showing them to be volume minimizing in their homology class.

The uniqueness part of the main theorem will follow by combining Proposition 2.1 with the next result.

PROPOSITION 3.2. *Let k be an integer ≥ 2. Let M^{kp} be a kp-dimensional subvariety of G_kR^{k+m}, tangent at each point to a subgrassmannian of the form G_kR^{k+p}. Then M^{kp} must itself be one of these subgrassmannians.*

When $k=1$, the grassmannians and subgrassmannians are round spheres and great subspheres. Because great subspheres through a point exist in all possible directions, the requirement on a subvariety, of being tangent to a great subsphere at each of its points, is no restriction at all. So the proposition is clearly false in this case.

The first case in which the proposition is true is

$$k=2, \qquad p=1, \qquad m-p=1, \qquad m=2,$$

that is, for surfaces in G_2R^4 tangent at each point to a subgrassmannian of the form G_2R^3. It is easy to see the truth visually in this case: the grassmannian G_2R^4 is isometric to $S^2 \times S^2$; subgrassmannians of the form G_2R^3 appear in this product as graphs of orientation preserving isometries; and a surface in $S^2 \times S^2$ tangent at each point to the graph of an orientation preserving isometry is itself clearly the graph of such an isometry.

Before giving the proof in general, it will be convenient to set up local coordinates in the Grassmann manifold G_kR^{k+m}. Let K be an oriented k-plane through the origin in $k+m$ space, and hence an element of G_kR^{k+m}, and $K^\perp$ its orthogonal complement. Let $e_1, \ldots, e_k$ and $f_1, \ldots, f_m$ be orthonormal bases for K and $K^\perp$, respectively. Define a map

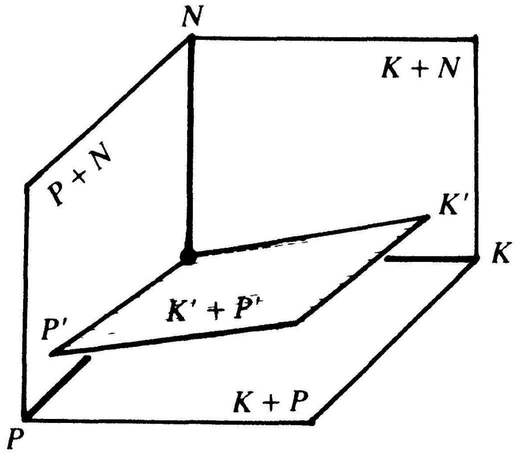
$$c: R^k \otimes R^m \to G_kR^{k+m}$$

which takes $\sum a_{ij}e_i \otimes f_j$ to the oriented k-plane spanned by the ordered basis

$$e_i + a_{i1}f_1 + \cdots + a_{im}f_m, \qquad 1 \leq i \leq k.$$

This map parametrizes a large neighborhood of K in G_kR^{k+m}, consisting of all oriented k-planes K' whose orthogonal projection to K is nonsingular and orientation preserving.

In the following diagram of R^{k+m}, we show the k-plane K spanned by $e_1, \ldots, e_k$, while its orthogonal complement is split up into the p-plane P spanned by $f_1, \ldots, f_p$ and the $m-p$ plane N spanned by $f_{p+1}, \ldots, f_m$. Suppose K' is an oriented k-plane in R^{k+m} close to K, as shown in the figure. Any $k+p$ plane in R^{k+m} containing K' and close to $K+P$ will meet $P+N$ in a p-plane P' close to P.



Suppose now that a subgrassmannian of the form G_kR^{k+p} is given containing K' and lying close to $G_k(K+P)$. Then it must be of the form $G_k(K'+P')$, with P' as above. We want to characterize the tangent kp-plane to $G_k(K'+P')$ at K' in terms of the local coordinates in $R^k \otimes R^m$. We assert

(3.3) *The tangent kp-plane to a G_kR^{k+p} near $G_k(K+P)$ is the graph of a linear map of the form $Id \otimes L: R^k \otimes R^p \to R^k \otimes R^{m-p}$, where L is a linear map from $R^p \to R^{m-p}$.*

We verify (3.3) as follows. Let K' be spanned by the basis

$$e_1 + u_1 + v_1, \ldots, e_k + u_k + v_k,$$

where

on

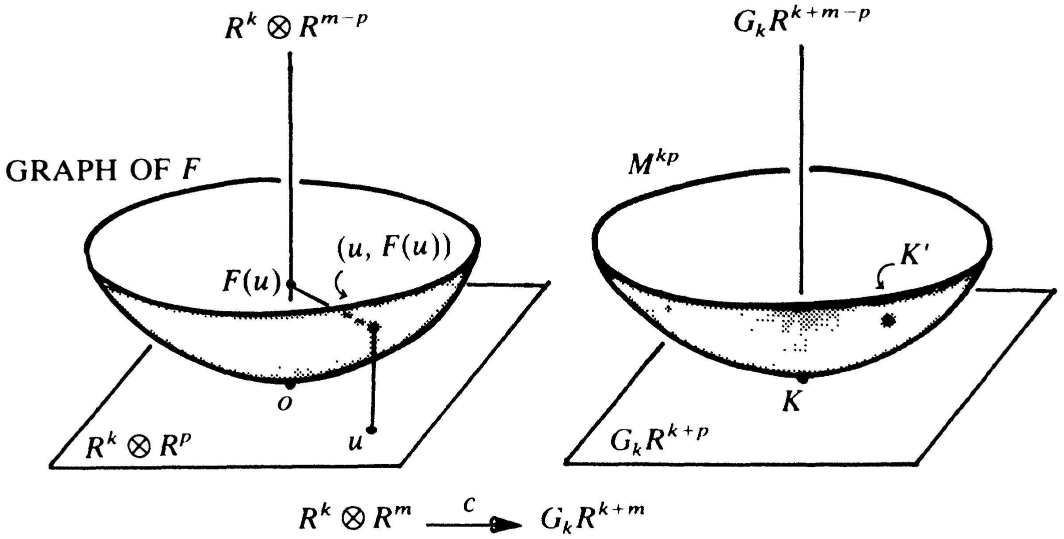
$$u_i = a_{i1}f_1 + \cdots + a_{ip}f_p$$

and

$$v_i = a_{i,p+1}f_{p+1} + \cdots + a_{im}f_m, \qquad 1 \leq i \leq k.$$

If we move out from K' along a curve in the subgrassmannian $G_k(K'+P')$, then each of the velocity vectors $u_1' + v_1', \ldots, u_k' + v_k'$ must lie in P'. But P', lying close to P in $P + N$, is the graph of a linear map $L: P \to N$. So $v_i' = L(u_i')$ for $1 \le i \le k$. Hence in local coordinates as above, the tangent kp-plane to this subgrassmannian is the graph of the linear map $Id \otimes L$, as claimed.

Proof of Proposition 3.2. Let M^{kp} be a kp-dimensional subvariety of $G_k R^{k+m}$, tangent at each point to a subgrassmannian of the form $G_k R^{k+p}$. We can suppose that M^{kp} passes through the point K, and is tangent there to the subgrassmannian $G_k(K + P)$. Using local coordinates in $R^k \otimes R^m$, the subvariety M^{kp} appears there as the graph of a smooth function $F: R^k \otimes R^p \to R^k \otimes R^{m-p}$ with $F(0) = 0$ and $dF_0 = 0$.



LEMMA 3.4. *Let $k \ge 2$. Let $F: R^k \otimes R^p \to R^k \otimes R^{m-p}$ be a smooth function with $F(0) = 0$, $dF_0 = 0$ and $dF_u = Id \otimes L_u$, where $L_u: R^p \to R^{m-p}$ is a linear map, which may depend on u. Then $F \equiv 0$.*

The primitive case occurs when $k = 2$, $p = 1$, $m - p = 1$ and $m = 2$. We have $u = u(x, y)$, $v = v(x, y)$ with Jacobian matrix

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{pmatrix} B(x, y) & 0 \\ 0 & B(x, y) \end{pmatrix}$$

and initial conditions $u(0, 0) = 0$, $v(0, 0) = 0$ and $B(0, 0) = 0$. Now $\partial u/\partial y = 0$ implies $u = u(x)$, which implies $B(x, y) = \partial u/\partial x$ is a function of x alone. Likewise $\partial v/\partial x = 0$ implies $v = v(y)$, which implies $B(x, y) = \partial v/\partial y$ is a function of y alone. Thus B must be constant, and in view of the initial condition, we have

$B \equiv 0$. Hence $u = u(x, y)$ and $v = v(x, y)$ are also constant, and again in view of the initial conditions, we have $u \equiv 0$ and $v \equiv 0$.

The proof in general is no different.

Now our subvariety M^{kp} appears in $R^k \otimes R^m$ as the graph of a function F which, because of (3.3), satisfies the hypotheses of the above lemma. Hence $F \equiv 0$ and M^{kp} coincides in a neighborhood of K with the subgrassmannian $G_k(K + P) \cong G_k R^{k+p}$. Since the corresponding statement is true in a neighborhood of each point of M^{kp}, this subvariety must itself be one of these subgrassmannians, completing the proof of Proposition 3.2.

PROPOSITION 3.5. *If k is an even integer ≥ 4, then the subgrassmannians $G_k R^{k+p}$ inside $G_k R^{k+m}$ are uniquely volume minimizing in their homology classes.*

Let M^{kp} be a kp-dimensional subvariety of $G_k R^{k+m}$ which also minimizes volume in the homology class of the subgrassmannian $G_k R^{k+p}$. Then it too must be calibrated by the p-fold column form λ_p at every point. By Proposition 2.1, when k is an even integer ≥ 4, the form λ_p calibrates the p-fold column spaces and nothing else. But these are just the tangent spaces to the subgrassmannians of the form $G_k R^{k+p}$. Thus M^{kp} must be tangent at each of its points to such a subgrassmannian, and therefore by Proposition 3.2 coincides with one of them, as claimed.

This completes the proof of the main theorem.

4. The case when k is odd

It remains to see that if k is odd and $n \geq k + 2$, then $G_k R^{n-1}$ bounds in $G_k R^n$ over the reals. To do this, we first describe the cohomology ring of $G_k R^n$ with real coefficients, in terms of generators and relations. We thank Steve Costenoble for explaining this to us. See [Bo].

Some of the Pontryagin classes $p_1, p_2, \ldots$ and (if k is even) the Euler class e_k of the canonical k-plane bundle over $G_k R^n$ appear among the generators of $H^*(G_k R^n; R)$. The Pontryagin class p_i has dimension $4i$, the Euler class e_k has dimension k. Some of the Pontryagin classes $\bar{p}_1, \bar{p}_2, \ldots$ and (if $n - k$ is even) the Euler class $\bar{e}_{n-k}$ of the canonical n–k plane bundle over $G_k R^n$ provide most of the remaining generators. The relations among these generators all follow from the fact that the sum of the two canonical bundles is trivial. When k and n–k are both odd, there is an extra generator x_{n-1} of dimension $n - 1$, whose square is zero.

The actual presentations are as follows, with four cases, according to the parity of k and n.

$$H^*G_{2r}R^{2r+2s} = R[p_1, \ldots, p_{r-1}, e_{2r}, \bar{p}_1, \ldots, \bar{p}_{s-1}, \bar{e}_{2s}]/I,$$

where I is the ideal of relations generated by $e_{2r}\bar{e}_{2s} = 0$ and the relations in

$$(1 + p_1 + \cdots + p_{r-1} + e_{2r}^2)(1 + \bar{p}_1 + \cdots + \bar{p}_{s-1} + \bar{e}_{2s}^2) = 1.$$

$$H^*G_{2r}R^{2r+2s+1} = R[p_1, \ldots, p_{r-1}, e_{2r}, \bar{p}_1, \ldots, \bar{p}_s]/I,$$

where I is the ideal generated by the relations in

$$(1 + p_1 + \cdots + p_{r-1} + e_{2r}^2)(1 + \bar{p}_1 + \cdots + \bar{p}_s) = 1.$$

$$H^*G_{2r+1}R^{2r+2s+1} = R[p_1, \ldots, p_r, \bar{p}_1, \ldots, \bar{p}_{s-1}, \bar{e}_{2s})/I,$$

where I is generated by the relations in

$$(1 + p_1 + \cdots + p_r)(1 + \bar{p}_1 + \cdots + \bar{p}_{s-1} + \bar{e}_{2s}^2) = 1.$$

$$H^*G_{2r+1}R^{2r+2s+2} = E[x_{2r+2s+1}] \otimes R[p_1, \ldots, p_r, \bar{p}_1, \ldots, \bar{p}_s]/I$$

where the first factor is the exterior algebra on a single generator of dimension $2r + 2s + 1$, and where I is generated by the relations in

$$(1 + p_1 + \cdots + p_r)(1 + \bar{p}_1 + \cdots + \bar{p}_s) = 1.$$

Remark. Note from the form of the preceding relations that the dual Pontryagin classes $\bar{p}_1, \bar{p}_2, \ldots$ can be solved for in terms of the original Pontryagin classes and both the original and dual Euler classes. But the dual Euler class can not be expressed in terms of the remaining generators: it provides genuinely new information.

We now look at these presentations for the cohomology of G_kR^n, and use them to show that this space has no cohomology (and hence no homology) in dimension $k(n-1-k)$, where the submanifold G_kR^{n-1} resides.

So let k be odd, and first assume that n is odd. Then the dimension of G_kR^{n-1} is odd. But in this case the cohomology of G_kR^n is zero in odd dimensions, as a glance at the table shows. Hence G_kR^{n-1} bounds in G_kR^n over the reals.

Now assume that n is even. Write $k = 2r + 1$ and $n = 2r + 2s + 2$ to conform with the notation of the table. The subgrassmannian G_kR^{n-1} has dimension $(2r+1)2s$, while the bigger Grassmannian G_kR^n has dimension $(2r+1)(2s+1)$. The codimension is $2r+1$. If the cohomology of G_kR^n were nonzero in this codimension, then by Poincaré duality it would also be nonzero in dimension $2r+1$. But a glance at the last case in the table shows that all the generators of $H^*(G_kR^n; R)$ are even dimensional, with the single exception of $x_{2r+2s+1}$. Hence $2r + 2s + 1$ is the first odd dimension in which one finds nonzero cohomology. But $s \geq 1$, since $n \geq k + 2$. Thus the cohomology is zero in dimension $2r+1$. Hence G_kR^{n-1} bounds in G_kR^n over the reals.

This completes the case when k is odd.

REFERENCES

[Be] MARCEL BERGER, *Du côté de chez Pu,* Ann. scient. Éc. Norm. Sup. *5* (1972), 1–44.

[Bo] ARMAND BOREL, *Sur la cohomologie des espaces fibrés principaux et des espaces homogènes des groupes de Lie compacts,* Ann. of Math. *57* (1953), 115–207.

[D-H-M] JIRI DADOK, REESE HARVEY and FRANK MORGAN, *Calibrations on* R^8, to appear in Trans. Amer. Math. Soc.

[Fe] HERBERT FEDERER, *Geometric measure theory,* Springer-Verlag, N.Y. (1969).

[Fo] A. T. FOMENKO, *Minimal compacta in Riemannian manifolds and Reifenberg's conjecture,* Math. USSR Izvest. *6* (1972), 1037–1066.

[G-Z] HERMAN GLUCK and WOLFGANG ZILLER, *On the volume of a unit vector field on the three-sphere,* Comment. Math. Helv. *61* (1986), 177–192.

[Ha] REESE HARVEY, *Calibrated geometries,* Proc. Int. Cong. Math., Warsaw (1983), 797–808.

[H-L] REESE HARVEY and H. BLAINE LAWSON JR., *Calibrated geometries,* Acta Math. *147* (1981), 47–157.

[La] H. BLAINE LAWSON, JR., *The stable homology of a flat torus,* Math. Scand. *36* (1975), 49–73.

[Mo] FRANK MORGAN, *Area minimizing surfaces, faces of grassmannians and calibrations,* to appear in Amer. Math. Monthly.

[Mo 2] FRANK MORGAN, *Calibrations and new singularities in area-minimizing surfaces: a survey,* Proc. Conference Problemes Variationnels, Paris (1988), to appear.

[Ta] H. TASAKI, *Certain minimal or homologically volume minimizing submanifolds in compact symmetric spaces,* Tsukuba Math. J. *9* (1985), 117–131.

[Th] D. C. THI, *Minimal real currents on compact Riemannian manifolds,* Math. USSR Izvest. *11* (1977), 807–820.

[Wi] W. WIRTINGER, *Eine Determinantenidentität und ihre Anwendung auf analytische Gebilde und Hermitesche Massbestimmung,* Monatsh. f. Math. u Physik *44* (1936), 343–365.

[Wo] JOSEPH WOLF, *Spaces of constant curvature,* 2nd ed., Berkeley (1972).

University of Pennsylvania
Williams College
University of Pennsylvania

Received May 28, 1987