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Objekttyp: **Article**

Zeitschrift: **Commentarii Mathematici Helvetici**

Band (Jahr): **64 (1989)**

PDF erstellt am: **30.04.2024**

Persistenter Link: https://doi.org/10.5169/seals-48944

Comment. Math. Helvetici 64 (1989) 236–255 0010-2571/89/020236-20$1.50 + 0.20/0

Link concordance and algebraic closure of groups*

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The aim of this paper is to prove several results aimed at an understanding of the classification of links under concordance.

In [O], Kent Orr defines a sequence of (based) link concordance invariants $o_k(L)$, $2 \le k \le \omega$ which vanish on boundary links. Subsequently, Tim Cochran [C] showed that these invariants vanished, if $k < \infty$ and dimension $L > 1$. When $n = 1$, their vanishing is equivalent to the vanishing of the Milnor $\bar{\mu}$-invariants [O][1], [C]. Thus only $o_\omega(L)$ survives as a possible new non-zero invariant. Two important problems arise immediately. First of all, $o_\omega(L)$ is an element of $\pi_{n+2}(K_\omega^m)$, where $n = \dim L$, m = number of components of L, and K_ω^m is a space constructed from the Eilenberg–MacLane space $K(\hat{F}, 1)$, where $\hat{F}$ is the nilpotent completion of the free group F of rank m (see below for definition), by attaching 2-cells to kill a basis of $F \subset \hat{F}$. The homotopy groups of K_ω^m are not well-understood, although it is known from results of Bousfield [B] that $\pi_2(K_\omega^m)$ is uncountable and $\pi_1(K_\omega^m) = 0$. It is conjectured by Bousfield that $\pi_3(K_\omega^m)$ is also uncountable, which would allow the *possibility* of finding 1-dimensional links with vanishing $\bar{\mu}$-invariants but not concordant to a boundary link. This leads us to the second problem. It is not known which elements of $\pi_{n+2}(K_\omega^m)$ are realizable as $o_\omega(L)$ for some link L.

In this paper we construct a new space K_∞^m in an analogous manner to K_ω^m. Instead of $\hat{F}$ we use an intermediate group $\bar{F}$, where $F \subset \bar{F} \subset \hat{F}$, the *algebraic closure* of F in $\hat{F}$. This notion was first discussed by Gutierrez [G]. There is an obvious map $K_\infty^m \to K_\omega^m$ and we see that $o_\omega(L)$ can be lifted to a link concordance invariant $\theta_\infty(L)$ in $\pi_{n+2}(K_\infty^m)$, vanishing on boundary links. The main result is that any element of $\pi_{n+2}(K_\infty^m)$ can be realized as $\theta_\infty(L)$ for some L – in dimension >1 we must, however, allow more general links L where the components are not necessarily spheres. Actually, $\pi_{n+2}(K_\infty^m)$ is interpreted as "ω concordance" classes of (based) links, where ω-concordance is the natural generalization of concordance appropriate to our more general notion of link. An important consequence is that $\pi_3(K_\infty^m) \ne 0$ will now immediately imply the existence of a 1-dimensional link with vanishing $\bar{\mu}$-invariants but not concordant to a boundary link (or even a sublink of a homology boundary link – see below).

* Partially supported by NSF.

In [C] Tim Cochran shows that a sublink of a homology boundary link has vanishing o^m_ω. More generally he defines the notion of an E-group and shows that any link whose group maps to an E-group in a certain non-trivial manner has vanishing o^m_ω. We will show that any such link satisfying a, probably vacuous, extra homological condition is, in fact, concordant to a sublink of a homology boundary link. Thus the question of whether a link is concordant to a sublink of a homology boundary link is reduced to an algebraic property of its group and it is possible that every link group (with vanishing $\bar{\mu}$-invariants in dimension one) satisfies this property.

1. We begin by setting up the principal geometric constructions which will be needed in the proofs of the theorems. We will work entirely in the smooth category (with corners).

DEFINITION. Let $L_0, L_1 \subseteq M$ be properly imbedded compact submanifolds. A *concordance* from L_0 to L_1 is a properly imbedded compact submanifold $V \subseteq I \times M$ such that

(i) $V \cap (t \times M) = t \times L_t$, for $t = 0, 1$;

(ii) $(V, 0 \times L_0, 1 \times L_1)$ is diffeomorphic to $(I \times L_0, 0 \times L_0, 1 \times L_0)$ and

(iii) $V \cap (I \times \partial M) = I \times \partial L_0$.

A properly imbedded codimension two submanifold $L \subseteq M$ is *based* if for every component L_i of L a *meridian* $\mu_i \in \pi_1(M - L)$ is chosen, i.e. μ_i is represented by a loop in $M - L$ which bounds an imbedded 2-disk intersecting L_i in exactly one point. A *based concordance* between based submanifolds $(L_0, \{\mu_i\})$ and $(L_1, \{\mu_i'\})$ is a concordance V such that $i_{0*}(\mu_i) = i_{1*}(\mu_i')$, for all i, where $i_t : M - L_t \to (I \times M) - V$ is the inclusion. (This definition implicitly assumes an arc in $(I \times M) - V$ connecting the base-points of $M - L_0$ and $M - L_1$)

LEMMA 1. *Let $(L_0, \{\mu_i\})$ be a properly imbedded compact based n-dimensional submanifold $(n \geq 1)$ of a connected manifold M and suppose $\rho : \pi_1(M - L_0) \to H$ is a homomorphism satisfying*:

(a) *H is finitely generated*

(b) *H is generated by* Image ρ *and $\langle \rho(K) \rangle$ where $K =$* Kernel $\{\pi_1(M - L_0) \to \pi_1(M)\}$ *and $\langle\ \rangle$ indicates normal closure.*

Then there is a based concordance $V \subseteq I \times M$ from $(L_0, \{\mu_i\})$ to $(L_1, \{\mu_i'\})$, and a homomorphism $\bar{\rho} : \pi_1((I \times M) - V) \to H$ satisfying:

(i) $\bar{\rho} \cdot i_0 = \rho$

(ii) *$\bar{\rho}$ is onto*

(iii) *i_1 is onto*

where $i_r : \pi_1(M - L_r) \to \pi_1((I \times M) - V)$ is induced by inclusion $(r = 0, 1)$.

Proof. We will create the concordance by adding handles of index 1 and 2 to $I \times (M - L_0)$, along $1 \times (M - L_0)$, which cancel when viewed as handles added to $I \times M$. The concordance V corresponds to $I \times L_0 \subseteq I \times M$. Let $X = M - L_0$.

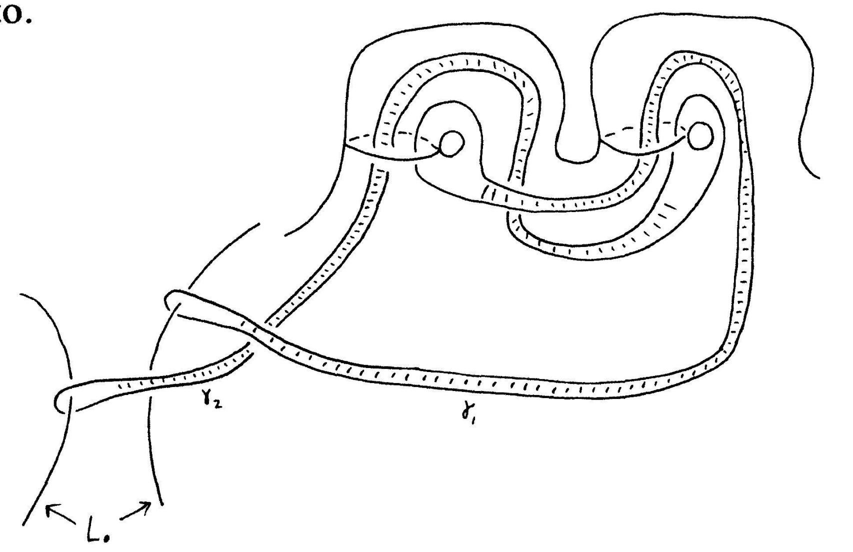
Note that $K = \langle \{\mu_j\} \rangle$. If $x_j = \rho(\mu_j)$, then it follows from (a) and (b) that there exists a finite set of elements $y_i = w_i x_{r_i} w_i^{-1} \in H$, $w_i \in H$, which, together with Image ρ, generate H. for each y_i add a handle of index 1, along $1 \times X \subseteq 1 \times M$, to create Y_0. Note that $\pi_1(Y_0 - (I \times L_0))$ is a free product of $\pi_1(X)$ and a free group with one generator, $\bar{y}_i$, for each y_i. Let $X_0 = Y_0 - (I \times L_0)$.

Now we define element $\bar{w}_i \in \pi_1(X_0)$ as follows. Write w_i as a formal word in the $\{y_j\}$ and elements $\{\rho(g_s)\}$ for some choice of $\{g_s\} \subseteq \pi_1(X)$. In this word replace y_j by $\bar{y}_j$ and $\rho(g_s)$ by g_s; the resulting element of $\pi_1(X_0)$ will be $\bar{w}_i$. Note that ρ extends to a homomorphism $\rho_0 : \pi_1(X_0) \to H$ by setting $\rho_0(\bar{y}_i) = y_i$. Then ρ_0 is onto and $\rho_0(\bar{w}_i) = w_i$.

Set $\xi_i = \bar{w}_i \mu_{r_i} \bar{w}_i^{-1} \in \pi_1(Y_0)$. These elements can be represented by meridian curves about the component of $1 \times L_0$ corresponding to μ_{r_i} in the component of ∂X_0 meeting $1 \times M$. These curves bound obvious disjoint disks in ∂Y_0. It follows that we can choose disjoint simple closed curves γ_i in ∂X_0 representing $\xi_i \bar{y}_i^{-1}$ which, in ∂Y_0, are isotopic (as a link in ∂Y_0) to a collection of curves which exactly cancel the 1-handles.

We define Y to be the result of adding handles of index 2 to Y_0 along the curves $\{\gamma_i\}$, using the normal framing which insures that Y is diffeomorphic to $I \times M$. Note that $\pi_1(Y - (I \times L_0))$ is naturally isomorphic to $\pi_1(X_0)$ with the added relations: $\bar{y}_i = \bar{w}_i \mu_{r_i} \bar{w}_i^{-1}$. Since $\rho_0(\bar{y}_i) = y_i = w_i x_{r_i} w_i^{-1} = \rho_0(\bar{w}_i \mu_{r_i} \bar{w}_i^{-1})$, ρ_0 induces a homomorphism $\rho_1 : \pi_1(Y - I \times L_0) \to H$. Since ρ_0 is onto, so is ρ_1.

Since Y is diffeomorphic to $I \times M$, we can now let V be the image of $I \times L_0 \subseteq Y$ under such a diffeomorphism, and $L_1 \subseteq M$, the image of $1 \times L_0$ in the appropriate component of ∂Y. Since $(I \times M) - V$ is obtained from $M - L_1$ by the addition of handles of index $n + 1$ and $n + 2$, and $n \geq 1$, we see that $\pi_1(M - L_1) \to \pi_1((I \times M) - V)$ is onto.



To see that we actually obtain a based concordance notice that we can choose curves representing $\{\mu_i\}$ which miss the attaching spheres of the handles added to $I \times M$. Therefore these curves also represent meridians $\{\mu_i^1\}$ of L_1 and it is clear that $(L_0, \{\mu_i\})$ is concordant to $(L_1, \{\mu_i^1\})$.

LEMMA 2. *Let $(L, \{\mu_i\})$ be a based compact proper n-dimensional submanifold of M and $\tau: L \to M - L$ defined by translation along a normal vector field. Suppose $\rho: \pi_1(M - L) \to H$ is an epimorphism satisfying $\rho \circ \tau_* = 0$. (Note that τ_* is only defined on the individual components of L and then requires a choice of paths connecting the base-points of each component to that of $M - L$. But the condition $\rho \circ \tau_* = 0$ is independent of these choices.)*

Suppose $\rho(\mu_r)$ is conjugate to $\rho(\mu_s)$, for some r, s. Then there exists a cobordism $V \subseteq I \times M$ from L to L^1, where $L^1 = L$ except that $L_r \cup L_s$ is replaced with a connected sum $L_r \# L_s$. V consists of $\{I \times L_i : i \neq r, s\}$ together with a boundary connected sum $I \times L_r \natural I \times L_s$.

Furthermore ρ extends to a homomorphism $\bar{\rho}: \pi_1((I \times M) - V) \to H$ and $\bar{\rho} \circ \bar{\tau}_* = 0$, where $\bar{\tau}: V \to (I \times M) - V$ is defined by translating along a normal vector field to V. The induced $\rho^1: \pi_1(M - L^1) \to H$ is again onto.

Proof. The construction of V is standard, using a path γ connecting L_r to L_s, whose interior is disjoint from L. We only need to choose γ carefully to obtain $\bar{\rho}$. To do this we take a slightly different approach.

First add a handle of index 1 to $I \times M$ along $1 \times M$ to obtain y_0, so that the two attaching $(n+2)$ balls D_1, D_2 intersection L_r, L_s, respectively, along standard n-balls $d_1 \subseteq D_1$, $d_2 \subseteq D_2$. Using a path from $\dot{d}_1$ to $\dot{d}_2$ in the boundary of the handle, we see that Y_0 contains a cobordism V of the desired type. $\pi_1(Y_0 - V)$ is obtained from $\pi_1(M - L)$ by adding a generator y and a relation $\mu_r = y\mu_s y^{-1}$. More precisely, we choose paths γ_1, γ_2 from the base-point of $M - L$ to $\dot{d}_1$, $\dot{d}_2$ and use these paths to define the meridians μ_r, μ_s. The generator y is represented by the path $\gamma = \gamma_1 \cdot \sigma \cdot \gamma_2^{-1}$, where σ runs along the handle from D_1 to D_2. Since $\rho(\mu_r) = g\rho(\mu_s)g^{-1}$, for some $g \in H$, we can define $\rho_0: \pi_1(Y_0 - V) \to H$ by setting $\rho_0(y) = g$.

The final step is to attach a handle of index 2 to $Y_0 - V$ to cancel the 1-handle. This can be done by choosing, as the attaching curve, any path of the form $\sigma_1 \cdot \sigma \cdot \sigma_2^{-1}$ where σ_1, σ_2 are paths in $M - L$ from the base-point of $M - L$ to the end-points of σ. The effect on $\pi_1(Y_0 - V)$ is to kill the element $\alpha_1 y \alpha_2^{-1}$, where α_i is represented by $\sigma_i \gamma_i^{-1}$. Thus we can define $\bar{\rho}$ only if $\rho(\alpha_1)g = \rho(\alpha_2)$. But since σ_1, σ_2 and, therefore, α_1 and α_2 can be chosen arbitrarily and $g \in \text{Image}\, \rho$, since ρ is assumed onto, we may complete this step.

That $\bar{\rho}\circ\bar{\tau}_* = 0$ is an immediate consequence of $\rho\circ\tau_* = 0$ and that $\pi_1(L)\to\pi_1(V)$ is onto. That ρ' is onto follows from the fact that $\pi_1(M - L^1)\to\pi_1((I\times M) - V$ is onto.

2. We now discuss the notion of "algebraic closure" in the context of groups, as adapted from [G]. Let F be a fixed free group of finite rank m. For any group G, we denote by $G_k \subseteq G$ the k-th term of the lower central series (i.e. $G_1 = G$, $G_{k+1} = [G, G_k]$ and $G_\omega = \bigcap_k G_k$), and $\hat{G} = \lim_k G/G_k$ the nilpotent completion of G. If G is a subgroup of $\hat{F}$, we define $\bar{G} \supseteq G$.

DEFINITION. For any group G, a word $w \in G * F(y_1, \ldots, y_k)$, where $F(y_1, \ldots, y_k)$ is the free group on the letters $y_1, \ldots, y_k$ is *contractible* (over G) if $w \mapsto 1$ under the projection $G * F(y_1, \ldots, y_k) \to F(y_1, \ldots, y_k)$.

LEMMA 3. *Suppose* $w_1, \ldots, w_k \in \hat{F} * F(y_1, \ldots, y_k)$ *are contractible. Then there exist unique elements* $g_1, \ldots, g_k \in \hat{F}$ *satisfying the equations*: $g_i = w_i(g_1, \ldots, g_k)$, $i = 1, \ldots, k$; *i.e.* $\rho(w_i) = g_i$ *where* $\rho : \hat{F} * F(y_1, \ldots, y_k) \to \hat{F}$ *is defined to be the identity on* $\hat{F}$ *and* $\rho(y_i) = g_i$.

Proof. We show, for any r, that there exist $g_{ir} \in F$ such that $g_{ir} \equiv w_i(g_{1r}, \ldots, g_{kr}) \bmod F_r$ and that g_{ir} is unique mod F_r. The lemma will then follow by taking $g_i = \lim_r g_{ir}$.

We proceed by induction on r; the case $r = 1$ is trivial. The crucial observation is that, for any contractible $w(g_1, \ldots, g_k)$ over a group G, the element $w(g_1, \ldots, g_k) \in G \bmod G_r$ depends only on $\{g_i\} \bmod G_{r-1}$. We can write $w(y_1, \ldots, y_k)$ as a product of conjugates of elements of $G \subset G * F(y_1, \ldots, y_k)$. Suppose $\tau g \tau^{-1}$ is one of these conjugates, after making the substitution $y_i \mapsto g_i$. Changing each g_i by an element of G_{r-1} will change τ by an element of G_{r-1} also and, therefore, $\tau g \tau^{-1}$ by an element of G_r.

Now suppose $\{g_{ir}\}$ are given as hypothesized. Set $g_{ir+1} = w_i(g_{1r}, \ldots, g_{kr})$. Then $w_i(g_{1r+1}, \ldots, g_{kr+1}) \equiv w_i(g_{1r}, \ldots, g_{kr}) \bmod F_{r+1}$, because of the previous observation, since $g_{ir} \equiv g_{ir+1} \bmod F_r$, and so $\{g_{ir+1}\}$ are solutions mod F_{r+1}. On the other hand, if $\{g'_{ir+1}\}$ is another set of solutions, then $g'_{ir+1} \equiv g_{ir} \bmod F_r$, by uniqueness of $\{g_{ir}\}$, and so, modulo F_{r+1}, $g'_{ir+1} \equiv w_i(g'_{ir+1}, \ldots, g'_{kr+1}) \equiv w(g_{1r}, \ldots, g_{kr}) = g_{ir+1}$.

DEFINITION. Let G be a subgroup of $\hat{F}$. We say G is *algebraically closed* if, for any contractible $w_i(y_1, \ldots, y_k)$ over G $(i = 1, \ldots, k)$, the solutions $\{g_i\}$ of the equations: $g_i = w_i(g_1, \ldots, g_k)$ also lie in G. Equivalently, we may say that G

is contained properly in no group $H \subseteq F$ satisfying:

(i) H is generated by G plus a finite number of elements.

(ii) H is normally generated by G

Note that the intersection of algebraically closed subgroups is algebraically closed. For any subgroup $G \subseteq \hat{F}$, we define the *algebraic closure* $\bar{G} \supseteq G$ to be the intersection of all algebraically closed subgroups containing G.

LEMMA 4. *For any $G \subseteq \hat{F}$, the elements of $\bar{G}$ are exactly those which belong to a set $y_1 = g_1, \ldots, y_k = g_k$ of solutions of equations $y_i = w_i(y_1, \ldots, y_k)$; $i = 1, \ldots, k$, where w_i are contractible over G.*

Proof. Let $\tilde{G}$ be the set of such elements. It is clear that $\bar{G} \subseteq \tilde{G}$ so we only need show that $\tilde{G}$ is an algebraically closed subgroup containing G.

Suppose $g_1, g_2 \in \tilde{G}$. We may assume that they are the first two members of a solution set $g_1, \ldots, g_k$. If we add to the system the equation $y_{k+1} = w_1(y_1, \ldots, y_k)w_2(y_1, \ldots, y_k)$ then the solution is $g_{k+1} = g_1g_2$, clearly, and w_1w_2 is contractible. If we add the equation $y_{k+2} = w_1(y_1, \ldots, y_k)^{-1}$, then $g_{k+2} = g_1^{-1}$ and w_1^{-1} is contractible. This shows $\tilde{G}$ is a group.

For any $g \in G$, the single equation $y = ygy^{-1}$ has solution $y = g$ and ygy^{-1} is contractible. This shows $G \subseteq \tilde{G}$.

Finally we need to show $\tilde{G}$ is algebraically closed. Suppose $y_i = w_i(y_1, \ldots, y_k)$ is a system of equations over $\tilde{G}$. The words w_i involve a finite set of elements $h_1, \ldots, h_r \in \tilde{G}$. We may assume that these make up a solution set $z_i = h_i$ of a system of equations $z_i = v_i(z_1, \ldots, z_r)$ over G. Define a new set of words (over G) $w_i'(y_1, \ldots, y_k, z_1, \ldots, z_r)$ by replacing each occurrence of h_j in $w_i(y_1, \ldots, y_k)$ with the word $v_j(z_1, \ldots, z_r)$. If $\{w_i\}$ and $\{v_i\}$ are contractible, then so are $\{w_i'\}$. Now consider the system of equations over G: $y_i = w_i'(y_1, \ldots, y_k, z_1, \ldots, z_r)$, $z_i = v_i(z_1, \ldots, z_r)$. The solutions lie in $\tilde{G}$, by the definition of $\tilde{G}$, and it is clear that $z_i = h_i$. But now it is clear that any solution for y_i is the same for the original system $y_i = w_i(y_1, \ldots, y_k)$.

COROLLARY. *If G is countable, so is $\bar{G}$, and so $\bar{G} \neq \hat{F}$.*

We will need the following result of Bousfield [B].

PROPOSITION 1. *For any group G such that $H_1(G)$ is finitely generated, the natural map $G/G_n \to \hat{G}/(\hat{G})_n$ is an isomorphism. (G_n is the n-th term of the lower central series of G and $\hat{G} = \lim_n G/G_n$ the nilpotent completion).*

We will have need to consider automorphisms of $\bar{F}$, the algebraic closure of F in $\hat{F}$, obtained by conjugating a basis $x_1, \ldots, x_m$ of F.

PROPOSITION 2. *Let* $g_1, \ldots, g_m \in \hat{F}$. *Then there is a unique automorphism* Φ *of* $\hat{F}$ *such that* $\varphi(x_i) = g_i x_i g_i^{-1}$, $i = 1, \ldots, m$. *If* $g_i \in \bar{F}$, *then* $\Phi(\bar{F}) = \bar{F}$.

Proof. If $g_i = (g_{ir})$, $g_{ir} \in F/F_r$, then $x_i \mapsto g_{ir} x_i g_{ir}^{-1}$ defines an automorphism of F/F_r (any endomorphism of a nilpotent group N which induces the identity on N/N_2 is an automorphism). The limit of these automorphisms is the desired Φ-uniqueness is clear.

If every $g_i \in \bar{F}$, then $\Phi(F) \subseteq \bar{F}$. Since $\Phi(\bar{F})$ is the algebraic closure of $\Phi(F)$, we see that $\Phi(\bar{F}) \subseteq \bar{F}$. To prove $\bar{F} \subseteq \Phi(\bar{F})$ it suffices to show $F \subseteq \Phi(\bar{F})$. Let G be the subgroup generated by $\{x_i, g_i\}$. Since $g_i \in \bar{F}$, it follows that G is normally generated by $\{x_i\}$, and, therefore, by $\{g_i x_i g_i^{-1}\}$. This shows $G \subseteq \overline{\Phi(F)} = \Phi(\bar{F})$.

We will need the following lemma:

LEMMA 5. *If* $g \in \hat{F}$ *and* $[g, x_i] = 1$, *then* g *is a power of* x_i.

Proof. We apply the Magnus expansion. Let Λ be the ring of power series in m *non-commuting* variables $t_1, \ldots, t_m$. A homomorphism $P: F \to \Lambda_0$, where $\Lambda_0 \subseteq \Lambda$ is the multiplicative group of power series with constant term 1, is defined by $P(x_i) = 1 + t_i$, $P(x_i^{-1}) = \sum_{r=0}^{\infty} (-1)^r t_i^r$. According to Magnus [M] P is an imbedding and $P^{-1}(\Lambda_q) = F_q$, where Λ_q is the subgroup of power series whose non-constant terms are all of degree $\geq q$. It follows easily that P extends to an imbedding $\hat{P}: \hat{F} \to \Lambda_0$ such that $\hat{P}^{-1}(\Lambda_q) = \hat{F}_q$, using Proposition 1.

Suppose $\hat{P}(g) = 1 + G + \tilde{G}$, $\hat{P}(h) = 1 + H + \tilde{H}$, where $g, h \in \hat{F}$; G, H are non-zero homogeneous polynomials and $\tilde{G}$, $\tilde{H}$ are sums of terms of degree $> \deg G$, $\deg H$, respectively. Then an easy computation shows that $\hat{P}[g, h] = 1 + GH - HG + K$, where the terms of K have degree $> \deg G + \deg H$. Therefore, $\hat{P}[g, x_i] = 1 - Gt_i - t_i G + K$ and $[g, x_i] = 1$ implies that $Gt_i = t_i G$. It follows that $G = at_i^k$ for some $a \neq 0$, $k > 0$. If $k > 1$, then $g \in (\hat{F})_2$ and it is easy to see that $\hat{P}(g)$ reduces to 1 if we let the variables commute. Thus either $k = 1$ or $g = 1$.

But now replace g by gx_i^{-a} in the above argument and we see that $gx_i^{-a} \in (\hat{F})_2$ and so $g = x_i^a$.

3. We now introduce the spaces and link invariant, following the work of Orr [O], except replacing his use of the nilpotent completion $\hat{F}$ with the algebraic closure $\bar{F}$.

Consider the Eilenberg–MacLane complexes $K(\bar{F}, 1)$ and $K(F, 1)$ together with the map $K(F, 1) \to K(\bar{F}, 1)$ corresponding to the inclusion $F \subseteq \bar{F}$. Let K_∞ be the mapping cone. Alternatively we may represent the elements $x_i \in F \subseteq \bar{F}$ by maps $S^1 \to K(\bar{F}, 1)$ and define K_∞ by attaching 2-cells. Note the obvious map $K_\infty \to K_\omega$ where K_ω is the Orr space, using $\hat{F}$.

PROPOSITION 3. *K_∞ is 1-connected.*

Proof. This corresponds to the observation that $\bar{F}$ is normally generated by F, which is obvious from the definition.

Let $\mathscr{H}$ be the group of automorphisms Φ of $\bar{F}$ which satisfy $\Phi(x_i) = g_i x_i g_i^{-1}$, for some $g_i \in \bar{F}$ – see Proposition 2. (It is easy to see that these automorphisms form a group). We associate to every element Φ of $\mathscr{H}$ a (homotopy class of) homotopy equivalence $\check{\Phi}: K_\infty \to K_\infty$, to give a homotopy group action i.e. $\widecheck{\Phi \circ \Psi} \simeq \check{\Phi} \circ \check{\Psi}$, $(\widecheck{\Phi^{-1}}) \simeq (\check{\Phi})^{-1}$, $\check{1} \simeq 1$.

First we consider the associated homotopy equivalence Φ^1 of $K(\bar{F}, 1)$, defining a homotopy group action. For the attaching map $\gamma_i : S^1 \to K(\bar{F}, 1)$, corresponding to $x_i \in F$, we have $\Phi^1 \circ \gamma_i$ representing $g_i x_i g_i^{-1} \in \bar{F}$. We extend Φ^1 over the 2-cells D_i attached by γ_i as follows. On the annulus $d_i - D_i^0$, where D_i^0 is an interior disk, let Φ be given by homotopy, in $K(\bar{F}, 1)$, from $\Phi^1 \circ \gamma_i$ to γ_i. We can then extend $\Phi \mid \dot{D}_i^0$ to a homeomorphism of D_i^0 onto D_i.

We now discuss the notion of (based) link which is appropriate to this context. It will be considerably more general than the usual spherical links, in higher dimensions, but will correspond precisely to the special case of links with vanishing $\bar{\mu}$-invariants, in dimension one. There will be a corresponding notion of concordance more general than the usual notion (which has already been used in §1). This general type of concordance was introduced in [C] and [O].

First we present some lemmas.

LEMMA 6. *Let $(L, \{\mu_i\})$ a proper bounded submanifold of codimension two with trivial normal bundle of the simply-connected manifold M, with components $L_1, \ldots, L_m$.*

Consider the homomorphism $\mu : F \to \pi$ defined by $\mu(x_i) = \mu_i$. Let $\tau_i : L_i \to X$ be the map defined by translation along the unique normal vector field for which $\tau_{i*} : H_1(L_i) \to H_1(X)$ is zero. Then μ induces an isomorphism $F/F_q \approx \pi/\pi_q$ if and only if $\tau_{i*}(\pi_1(L_i)) \subseteq \pi_{q-1}$, for every $i = 1, \ldots, m$.

Remark. We leave it to the reader to show the uniqueness of the asserted normal vector field. We will refer to it as the *unlinked normal field.*

Proof. See Theorem 5 of [O][1].

DEFINITION. Let $L \subseteq M$ as in Lemma 6. If H is any group and $\rho : \pi \to H$ a homomorphism, we will say that ρ *kills longitudes* if $\rho \circ \tau_{i*}$ is trivial for $i = 1, \ldots, m$. (See the remark in the statement of Lemma 2).

LEMMA 7. *Suppose $L \subseteq M$ as above, except that L_i may be disconnected, and $\{\mu_{ij}\}$ a choice of meridians, one for each component L_{ij} of L_i. Let $\rho : \pi \to \hat{F}$ be a* *homomorphism such that $\rho(\mu_{ij})$ is a conjugate of x_i. Then ρ kills longitudes and is uniquely determined by the elements $\{\rho(\mu_{ij})\}$.*

If each L_i is connected, then $\tau_{i*}(\pi_1(L_i)) \subseteq \pi_\omega$ and Kernel $\rho = \pi_\omega$.

LEMMA 8. *Suppose $L \subseteq M$ as above, with L_i connected and μ_i a meridian for L_i for each $i = 1, \ldots, m$. Let $\bar{x}_i$ be any conjugate of x_i in $\bar{F}$. If $\tau_{i*}(\pi_1(L_i)) \subseteq \pi_\omega$, for every i, then there exists a unique homomorphism $\rho: \pi \to \bar{F}$ such that $\rho(\mu_i) = \bar{x}_i$.*

Proof of Lemma 7. Suppose λ is a longitude i.e. $\lambda \in \tau_{ij*}(\pi_1(L_{ij}))$ for some i, j. Then, for some conjugate λ' of λ, we have $[\lambda', \mu_{ij}] = 1$. Applying ρ, we have $[\rho(\lambda), x_i'] = 1$, where x_i' is some conjugate of x_i. Applying Lemma 5, we conclude that $\rho(\lambda)$ is conjugate to a power of x_i. But since longitudes are null-homologous in $M - L$, it follows that $\rho(\lambda) = 1$.

Suppose $\rho_1, \rho_2: \pi \to \hat{F}$ and $\rho_1(\mu_{ij}) = \rho_2(\mu_{ij})$. Since π is normally generated by $\{\mu_{ij}\}$, we only need show that $\rho_1(g\mu_{ij}g^{-1}) = \rho_2(g\mu_{ij}g^{-1})$, for any $g \in \pi$. Since $\hat{F}$ is residually nilpotent we only need show this mod $\hat{F}_q$ for any q. We proceed by induction on q. For $q = 2$, it is clear. For the inductive step we may assume $\rho_1(g) \equiv \rho_2(g) \bmod \hat{F}_q$ for any $g \in \pi$. But now we have:

$$\begin{aligned}\rho_1(g\mu_{ij}g^{-1}) &= \rho_1([g, \mu_{ij}]\mu_{ij}) = [\rho_1(g), \rho_1(\mu_{ij})]\rho_1(\mu_{ij})\\ &\equiv [\rho_2(g), \rho_1(\mu_{ij})]\rho_1(\mu_{ij}) \bmod \hat{F}_{q+1}.\end{aligned}$$

But

$$[\rho_2(g), \rho_1(\mu_{ij})]\rho_1(\mu_{ij}) = [\rho_2(g), \rho_2(\mu_{ij})]\rho_2(\mu_{ij}) = \rho_2(g\mu_{ij}g^{-1}).$$

The final assertion of Lemma 7 will follows from Lemma 6 as follows. The composition $F/F_q \to \pi/\pi_q \to \hat{F}/\hat{F}_q$ induced by μ and ρ is the natural map, which, by Proposition 1, is an isomorphism. Thus $F/F_q \to \pi/\pi_q$ is a monomorphism. On the other hand, it is an epimorphism since π is normally generated by the $\{\mu_i\}$.

Proof of Lemma 8. By Lemma 6, we see that there is an isomorphism $\sigma: \hat{F} \xrightarrow{\approx} \hat{\pi}$ such that $\sigma(x_i) = \mu_i$. Consider the composition $\hat{\rho} = \sigma^{-1} \circ \rho$, where $\rho: \pi \to \hat{\pi}$ is the natural map. Since π is finitely generated and normally generated by $\{\mu_i\}$ it follows that $\hat{\rho}(\pi) \subseteq \bar{F}$.

By Proposition 2, there is an automorphism Φ of $\bar{F}$ such that $\Phi(x_i) = \bar{x}_i$. Now set $\rho = \Phi \circ \hat{\rho}$. Uniqueness follows from Lemma 7.

DEFINITION. A *link* L^n in S^{n+2} of *multiplicity* m is a collection of m disjoint connected closed oriented submanifolds $L_1, \ldots, L_m$ with trivial normal bundles such that $\tau_{i*}(\pi_1(L_i)) \subseteq \pi_1(S^{n+2}-L)_\omega$ where $\tau_i : L_i \to S^{n+2}-L$ is defined by translation along the unlinked normal field. The link is *based* if we are given, in addition, a meridian $\mu_i \in \pi_1(S^{n+2}-L)$ for each L_i.

We will associate to any based link of dimension n an element of $\pi_{n+2}(K_\infty)$. By Lemma 8, the meridians $\{\mu_i\}$ define a unique homomorphism $\rho : \pi_1(S^{n+2}-L) \to \bar{F}$ such that $\rho(\mu_i) = x_i$. Let X denote the complement of an open tubular neighborhood of L. We choose a map $f : X \to K(\bar{F}, 1)$ representing ρ which is specified on ∂X as follows. A tubular neighborhood T_i of L_i is diffeomorphic to $L_i \times D^2$ and this diffeomorphism is determined by insisting that the corresponding normal frame is consistent with the orientation of L_i and contains unlinked normal fields. Thus X is identified with $L \times S^1$.

CLAIM 1. *f can be chosen so that $f \mid \partial T_i = e_i \circ p$, where $p : \partial T_i \approx L_i \times S^1 \to S^1$ is projection on the second factor and $e_i : S^1 \to K(\bar{F}, 1)$ represents x_i.*

Since, by Lemma 7, ρ kills longitudes, it follows that $f \mid L_i \times *$ is null-homotopic. Since $* \times S^1$ is (freely) homotopic in X to μ_i, $f \mid * \times S^1$ is homotopic to a representative of x_i. Because the target space is aspherical, the claim follows.

Uniqueness of f is described as follows:

CLAIM 2. *Any two choice of f representing ρ and satisfying Claim 1 are homotopic via a homotopy f_t such that $f_t \mid \partial T = f_0 \circ F_t \mid \partial T$ where F_t is a deformation of $T = \bigcup_i T_i$ and $F_t \mid T_i \approx L_i \times D_2$ has the form $\phi_t \circ p : p$ is projection on the second factor and ϕ_t is a diffeotopy of D_2.*

Let $A \supseteq \partial X$ be the connected subspace of X formed by connecting each ∂T_i to the base-point of X with an arc γ_i used to determined the element μ_i. If the two choices of f, f_0 and f_1, agree on A, then they are homotopic rel A. Thus Claim 2 reduces to finding a homotopy $g_t : A \to K(\bar{F}, 1)$ from $f_0 \mid A$ to $f_1 \mid A$ of the asserted type.

Since $f_{0*}(\mu_i) = x_i = f_{1*}(\mu_i)$ it follows that $f_0 \mid \gamma_i$ and $f_1 \mid \gamma_i$ differ by an element $\alpha_i \in \pi_1(K(\bar{F}, 1)) = \bar{F}$ satisfying $\alpha_i x_i \alpha_i^{-1} = x_i$. By Lemma 5, α_i is a power of x_i. If we now choose $F_t \mid T_i$ to be the deformation of $T_i \approx L_i \times D^2$ which rotates the second coordinate a total angle of $2\pi a_i$, then $f_0 \circ F_t$ extends to a homotopy g_t so that $g_1 \mid \gamma_i \simeq f_1 \mid \gamma_i$ rel $\dot{\gamma}_i$ and $g_1 \mid \partial X = f_1 \mid \partial X$. So $g_1 \simeq f_1 \mid A$ and the claim is proved.

Now we extend the map $f: X \to K(\bar{F}, 1)$ to a map $\bar{f}: S^{n+2} \to K_\infty$ in a canonical manner, assuming f satisfies Claim 1, by defining $\bar{f} \mid T_i \approx L_i \times D^2$ to be projection on the second factor followed by a homeomorphism onto the 2-disk attached to $K(\bar{F}, 1)$ along x_i. The homotopy closs of $\bar{f}$ depends only on the based link since any homotopy between two choices of f of the type described in Claim 2 extends to a homotopy between the corresponding choices of $\bar{f}$.

Thus we have associated to any *based* link $\bar{L} = (L, \{\mu_i\})$ an element $\theta(\bar{L}) \in \pi_{n+2}(K_\infty)$. Suppose $\{\mu_i\}$ and $\{\mu_i'\}$ are two choices of meridians for the link L and $f: X \to K(\bar{F}, 1)$ is the map constructed for $\bar{L} = (L, \{\mu_i\})$. Now $\mu_i' = \alpha_i \mu_i \alpha_i^{-1}$ for some $\alpha_i \in \pi_1(X)$ and so $f_*(\mu_i') = g_i x_i g_i^{-1}$, where $g_i = f_*(\alpha_i)$. Let $\Phi \in \mathscr{H}$ be the automorphism of $\bar{F}$ defined by $\Phi(x_i) = g_i x_i g_i^{-1}$ and $\breve{\Phi}$ the self-homotopy equivalence of K_∞ defined by Φ. Then it is not hard to see that $\breve{\Phi} \circ \bar{f}$ represents $\theta(\bar{L}')$ where $\bar{L}'$ is the based link $(L, \{\mu_i'\})$.

Thus the class of $\theta(\bar{L})$ in $\pi_{n+2}(K_\infty)/\mathscr{H}$ is a well-defined invariant of L, which we denote by $\theta(L)$.

4. We now investigate the invariance of $\theta(\bar{L})$ and $\theta(L)$ under based and unbased concordance.

DEFINITION. An *ω-concordance* between two links $L, L' \subseteq S^{n+2}$ of multiplicity m is a collection V of m disjoint connected oriented proper submanifolds $V_i \subseteq I \times S^{n+2}$ with trivial normal bundles such that $V_i \cap (0 \times S^{n+2}) = L_i$, $V_i \cap (1 \times S^{n+2}) = L_i'$ and $\tau_{i*}\pi_1(V_i) \subseteq \pi_1((I \times S^{n+2}) - V)_\omega$, where $\tau_i: V_i \to (I \times S^{n+2}) - V$ is defined by translation along the unlinked vector field.

Note that if $\pi_1(L_i) \to \pi_1(V_i)$ is onto, for every i, the conditions on τ_{i*} are automatically satisfied. In particular, a concordance is an ω-concordance.

If $(L, \{\mu_i\})$ and $(L', \{\mu_i'\})$ are based links, a *based ω-concordance* is a concordance V such that $i_{0*}(\mu_i) \equiv i_{1*}(\mu_i') \bmod \pi_1((I \times S^{n+2}) - V)_\omega$ for every i, where $i_0: S^{n+2} - L \to (I \times S^{n+2}) - V$ and $i_1: S^{n+2} - L' \to (I \times S^{n+2}) - V$ are the inclusions (an arc in $(I \times S^{n+2}) - V$ connecting the base-points of $S^{n+2} - L$ and $S^{n+2} - L'$ is understood).

PROPOSITION 4. *If $\bar{L}$, $\bar{L}'$ are based ω-concordant links, then $\theta(\bar{L}) = \theta(\bar{L}')$. If L, L' are ω-concordant links then $\theta(L) = \theta(L')$.*

Proof. Let $\pi = \pi_1(S^{n+2} - L)$, $\pi' = \pi_1(S^{n+2} - L')$ and $G = \pi_1((I \times S^{n+2}) - V)$. We also have meridians $\{\mu_i\}$, $\{\mu_i'\}$ for L, L' such that $\bar{\mu}_i = i_*(\mu_i) = i_*'(\mu_i')$ in G/G_ω where $i: S^{n+2} - L \to (I \times S^{n+2}) - V$ and $i': S^{n+2} - L' \to (I \times S^{n+2}) - V$ are inclusions. We can construct $\bar{\rho}: G \to \bar{F}$ by Lemma 8, and a representative map $g: Y \to K(\bar{F}, 1)$, where Y is the complement of an open tubular neighborhood of

V in $I \times S^{n+2}$ satisfying the analogue of Claim 1 for $\{\mu_i\}$, using Lemma 7. Then $g \mid X = f$ and $g' \mid X' = f'$ satisfy Claim 1 for $\bar{L}$ and $\bar{L}'$ and the first assertion of the Proposition follows. If $\tau'_*(\mu'_i) \neq \bar{\mu}_i$ it is still true that $i'_*(\mu'_i) = \alpha_i \mu_i \alpha_i^{-1}$ for some $\alpha_i \in G$, and so $f'_*(\mu'_i) = g_i x_i g_i^{-1}$, where $g_i = \bar{\rho}(\alpha_i) \in \bar{F}$. Then it is not hard to see that $\phi \circ \bar{f}'$ represents $\theta(\bar{L}')$, where $\bar{L}' = (L', \{\mu'_i\})$ and Φ^{-1} is the automorphism of $\bar{F}$ defined by $x_i \mapsto g_i x_i g_i^{-1}$. Thus $\theta(\bar{L}') = \Phi_* \theta(\bar{L})$ which proves the Proposition.

DEFINITION. A link $L \subseteq S^{n+2}$ is a *boundary link* if, for each i, $L_i = \partial V_i$ where $\{V_i\}$ are disjoint compact orientable submanifolds of S^{n+2}.

COROLLARY. *If L is a boundary link, then $\theta(L) = 0$.*

Proof. We may assume L_i is connected. An ω-concordance with the trivial link is obtained by pushing int V_i into $I \times S^{n+2}$ to obtain V'_i satisfying $V'_i \cap (0 \times S^{n+2}) = L_i$, $V'_i \cap (1 + S^{n+2}) = D_i$ an interior disk. Then $\{V''_i = V'_i - D_i\}$ is an ω-concordance from L to $\{\dot{D}_i\}$, since the map $\pi_1(V''_i) \to \pi_1((I \times S^{n+2}) - V'')$ is trivial, for each i.

5. We show that the invariants give a complete classification of links up to ω-concordance.

THEOREM 1. *Two based links $\bar{L}$ and $\bar{L}'$ are based ω-concordant if and only if $\theta(\bar{L}) = \theta(\bar{L}')$. Two links L and L' are ω-concordant if and only if $\theta(L) = \theta(L')$.*

Proof. Suppose $\bar{L} = (L, \{\mu_i\})$ and $\bar{L}' = (L', \{\mu'_i\})$. Let $F: I \times S^{n+2} \to K_\infty$ be a homotopy between the maps f, f' representing $\theta(\bar{L})$, $\theta(\bar{L}')$. We can make F transverse regular on the midpoints of the 2-cells in K_∞. The result of this construction is a collection V of framed proper submanifolds $V_1, \ldots, V_m$ of $I \times S^{n+2}$ with trivial normal bundles such that $V_i \cap (0 \times S^{n+2}) = L_i$, $V_i \cap (1 \times S^{n+2}) = L'_i$ and a homomorphism $\bar{\rho}: \pi_1((I \times S^{n+2}) - V) \to \bar{F}$ extending $\rho: \pi_1(S^{n+2} - L) \to \bar{F}$ and $\rho': \pi_1(S^{n+2} - L') \to \bar{F}$ so that, for any meridian $\bar{\mu}_i$ of any component of V_i, $\bar{\rho}(\bar{\mu}_i)$ is conjugate to x_i (there is an arc connecting the base-points $S^{n+2} - L$, $S^{n+2} - L'$ and $(I \times S^{n+2}) - V$ which maps to the base-points of $K(\bar{F}, 1)$.) We refer to $(V, \bar{\rho})$ a *cobordism.* If each V_i were connected we would apply Lemma 7 to conclude that V is a based ω-concordance. In case some V_i is disconnected we will show how to replace $(V, \bar{\rho})$ by a cobordism with fewer components.

Suppose V_i is disconnected and ν, ν' are meridians for two different components of V_i, then $\bar{\rho}(\nu') = g\bar{\rho}(\nu)g^{-1}$ for some $g \in \bar{F}$. If $g \in \bar{\rho}(\pi_1((I \times S^{n+2}) - V))$ we can apply Lemma 2 to replace V by a new cobordism V' which coincides with V except that two components of V_i have been replaced by their connected sum. If $g \notin \bar{\rho}(\pi_1((I \times S^{n+2}) - V))$ we will apply Lemma 1. By the definition of algebraic closure, there is a finitely-generated subgroup H of $\bar{F}$, normally generated by F which contains $\bar{\rho}(\pi_1((I \times S^{n+2}) - V))$ and g. By Lemma 1, there is a based concordance W from V to V', (where the chosen meridians of V include v, v') such that $\bar{\rho}$ extends to *epimorphisms* from the fundamental groups of the complements of W and V' to H. In particular V' is a cobordism from L to L'. Since $\bar{\rho}(v)$ and $\bar{\rho}(v')$ are conjugate in H, this will be true for meridians of the corresponding components of V'. So now we can apply Lemma 2 to V'.

Now suppose L, L' are given meridians to yield based links $\bar{L}$, $\bar{L}'$. By assumption, we have $\Phi_*(\theta(\bar{L})) = \theta(\bar{L}')$ where $\Phi \in \mathcal{H}$ is defined by $\Phi(x_i) = g_i x_i g_i^{-1}$ for some $g_i \in \bar{F}$. If $g_i \in \rho'(\pi_1(S^{n+2} - L'))$ then we can replace μ_i' by conjugates so that now $\theta(\bar{L}) = \theta(\bar{L}')$. If $g_i \notin \rho'(\pi_1(S^{n+2} - L'))$ we apply Lemma 1, as above, to replace $\bar{L}'$ by a concordant link so that $g_i \in \rho'(\pi_1(S^{n+2} - L'))$ and the result follows.

To complete the classification of links up to ω-concordance we have the following theorem:

THEOREM 2. *For any $\alpha \in \pi_{n+2}(K_\infty)$, there exists a based link $\bar{L}$ such that $\theta(\bar{L}) = \alpha$.*

Proof. Choose $f: S^{n+2} \to K_\infty$ representing α and make f transverse regular at the midpoints of the 2-cells in K_∞. This yields a collection of framed submanifolds $L = \{L_1, \ldots, L_n\}$ of S^{n+2} and a homomorphism $\rho: \pi_1(S^{n+2} - L) \to \bar{F}$ so that, for any meridian μ_i of any component of L_i, $\rho(\mu_i)$ is conjugate to x_i. Suppose each L_i is connected. Then, by Lemma 7, L is a link. If we choose meridian $\{\mu_i\}$, then $\rho(\mu_i) = g_i x_i g_i^{-1}$, for some $g_i \in \bar{F}$. If $g_i \in \rho(\pi_1(S^{n+2} - L))$ then we can change $\{\mu_i\}$ so that $\rho(\mu_i) = x_i$. The resulting based link $\bar{L}$ will then satisfy $\theta(\bar{L}) = \alpha$. But if $g_i \notin \rho(\pi_1(S^{n+2} - L))$ we can apply Lemma 1 to replace $\bar{L}$ by a concordant link so that $g_i \in \rho(\pi_1(S^{n+2} - L))$ and we are done.

We must deal with the situation where some L_i is disconnected. We construct a framed cobordism $V = \{v_1, \ldots, v_m\}$ from L to $L' = \{L_1', \ldots, L_m'\}$ with fewer components than L and an extension of ρ to $\bar{\rho}: \pi_1((I \times S^{n+2}) - V) \to \bar{F}$ so that, for any meridian $\bar{\mu}_i$ of any component of V_i, $\bar{\rho}(\bar{\mu}_i)$ is conjugate to x_i. By Lemma 7, we can perform the construction in the definition of θ to extend f on $0 \times S^{n+2}$ to a map $\bar{f}: I \times S^{n+2} \to K_\infty$ so that $\bar{f} \mid 1 \times S^{n+2}$ yields L' and $\rho' = \bar{\rho} \circ i_{1*}$, where $i_1: S^{n+2} - L' \to (I \times S^{n+2}) - V$ is the inclusion.

Choose any meridians v, v' of different components of L_i. Then $\rho(v') = g\rho(\omega)g^{-1}$, for some $g \in \bar{F}$. If $g \in \rho(\pi_1(S^{n+2} - L))$ we can apply Lemma 2 to obtain V. Otherwise we apply Lemma 1 to replace L by a concordant L'', which is based with respect to a collection of meridians including v and v', and homomorphisms $\rho'': \pi_1(S^{n+2} - L'') \to \bar{F}$ such that $g \in \rho''(\pi_1(S^{n+2} - L''))$. We can now apply Lemma 2 to L'' to obtain the cobordism V from L'' to L'.

6. We establish a relationship between the algebraic closure $\bar{F}$ of the free group F and the fundamental group of the Vogel localization EW of $W = K(F, 1)$ used by LeDimet [L].

THEOREM 3. $\bar{F} \approx \pi_1(EW)/\pi_1(EW)_\omega$.

Proof. We recall two properties of the localization functor E. Suppose X is a finite CW-complex.

(i) EX is the inductive limit of finite subcomplexes

$$X = X_0 \subset X_1 \subset \cdots \subset X_n \subset X_{n+1} \subset \cdots$$

such that X_n/X is contractible for every n.

(ii) EX is "local" i.e. for a finite CW-complex pair (K, L) such that K/L is contractible, any map $L \to EX$ has a unique (up to homotopy) extension $K \to EX$.

Now consider the homomorphism $\eta: F \to \pi_1(EW)$ induced by the inclusion of (i), and an isomorphism $\pi_1(W) \approx F$, and the inclusion $i: F \to \bar{F}$. We define a homomorphism $\theta: \pi_1(EW) \to \bar{F}$ such that $\mu \circ \eta = i$ and show that θ is onto and $\operatorname{Ker} \theta = \pi_1(EW)_\omega$.

To define θ we construct homomorphism $\theta_n: \pi_1(W_n) \to \bar{F}$ which are consistent with the inclusions $W_n \subseteq W_{n+1}$ and take θ to be the limit. Suppose θ_n is already defined and let $g_1, \ldots, g_p$ be a set of generators of $\pi_1(W_{n+1})$. By (i) $\pi_1(W_{n+1})$ is normally generated by the image of the inclusion $j: \pi_1(W_n) \to \pi_1(W_{n+1})$. Thus there are words $w_i(y_1, \ldots, y_k, z_1, \ldots, z_p)$ such that $w_i(1, \ldots, 1, z_1, \ldots, z_p) = 1$ $(i = 1, \ldots, p)$ and $g_i = w_i(jh_1, \ldots, jh_k, g_1, \ldots, g_p)$ for some $h_i \in \pi_1(W_n)$. Now let

$$G = \frac{\pi_1(W_n) * F(z_1, \ldots, z_p)}{\langle z_i^{-1} w_i(h_1, \ldots, h_k, z_1, \ldots, z_p) \rangle}$$

Then j factors $\pi_1(W_k) \to G \xrightarrow{j'} \pi_1(W_{n+1})$ in an obvious manner, where $j'(z_i) = g_i$. By the definition of algebraically closed, θ_n extends to a unique homomorphism $\theta'_n: G \to \bar{F}$. Since $\bar{F}$ is residually nilpotent, $\theta'_n(G_\omega) = 1$. On the other hand we can apply Stallings theorem [S] to conclude that j' induces an isomorphism

$G/G_\omega \overset{\approx}{\rightarrow} \pi_1(W_{n+1})/\pi_1(W_{n+1})_\omega$ ($H_2(\pi_1(W_{n+1}))=0$ because $H_2(W_{n+1})=0$, and j' is onto).

We show that θ is onto. Any element of $\bar{F}$ is part of a collection $g_1,\ldots,g_k \in \bar{F}$ satisfying equations $g_i = w_i(x_1,\ldots,x_m,g_1,\ldots,g_k)$ $(i=1,\ldots,k)$ where $w_i(x_1,\ldots,x_m,y,\ldots,y_k)$ are words satisfying $w_i(1,\ldots,1,y_1,\ldots,y_k)=1$ ($\{x_i\}$ are a basis of F). Let K be a complex obtained by adjoining 1-cells $e_1,\ldots,e_k$ to W, and then 2-cells $E_1,\ldots,E_k$ via attaching maps representing $y_i^{-1}w_i(x_1,\ldots,x_m,y_1,\ldots,y_k)-x_i$ is represented by the i-th 1-cell of W and y_j by e_j. Now K/W is contractible and so the inclusion $W\rightarrow EW$ extends to a unique map $f:K\rightarrow EW$. Let $\bar{g}_i=f_*(y_i)$ and so we have equations $\bar{g}_i=w_i(x_1,\ldots,x_m,\bar{g}_1,\ldots,\bar{g}_k)$ in $\pi_1(EW)$. It follows that $\theta(\bar{g}_i)=g_i$ by the uniqueness of solutions in $\bar{F}$.

To see that $\operatorname{Ker}\theta=\pi_1(EW)_\omega$ note that η induces an isomorphism $\hat{\eta}:\hat{F}\overset{\approx}{\rightarrow}\widehat{\pi_1(EW)}$ of the nilpotent completions, by Stallings Theorem. Since θ is uniquely determined, we see that it coincides with the composition $\pi_1(EW)\rightarrow \widehat{\pi_1(EW)} \xrightarrow{\hat{\eta}^{-1}} \hat{F}$. Since $\hat{\eta}^{-1}$ is an isomorphism and $\operatorname{Ker}\{A\rightarrow\hat{A}\}$ is A_ω for any group A, the result follows.

7. We now turn to the subject of E-links and homology boundary links. We recall some definitions (see [C], [Sm]).

DEFINITIONS. 1) A link L of multiplicity m is a *homology boundary link* if the fundamental group of its complement admits an epimorphism onto the free group of rank m.

2) A group E is a *(finite) E-group* if it is the fundamental group of a (finite) 2-complex K with $H_2(K)=0$ and $H_1(K)$ torsion-free; *rank* $E=\operatorname{rank} H_1(E)$ in the finite case.

3) A link L of multiplicity m is a *(finite) E-link* if the fundamental group π of its complement admits a homomorphism $\phi:\pi\rightarrow E$ where E is a (finite) E-group of rank m, $\phi(\pi)$ normally generates E and ϕ kills longitudes.

It is shown in [C] that any internal band sum of a boundary link is a sublink of a homology boundary link, as well as a finite E-link. In fact, it can be seen that the class of (finite) E-links is closed under the operation of internal band sum. It is also shown in [C] that any E-link or sublink of a homology boundary link has vanishing Orr invariant. We prove the analogous result for θ.

PROPOSITION 5. *If $\bar{L}$ is any based link whose associated unbased link L is either a finite E-link or a sublink of a homology boundary link, then $\theta(\bar{L})=0$.*

Proof. We only deal with the case of an E-link since it will be shown below that any sublink of a homology boundary link is a finite E-link. Using the notation of the Definition, let $\bar{\mu}_i = \phi(\mu_i)$ where $\{\mu_i\}$ are the meridians for $\bar{L}$. Now since $H_2(E) = 0$ and rank $E = m$, the homomorphism $F \to E$ defined by $x_i \mapsto \mu_i$ extends to an isomorphism $\hat{F} \approx \hat{E}$ of the nilpotent completions of the lower central series. Let $\rho: E \to \hat{F}$ be defined by the inverse of this isomorphism. Thus $\rho(\bar{\mu}_i) = x_i$ and, since $\{\bar{\mu}_i\}$ normally generate E and E is finitely generated, $\rho(E) \subseteq \bar{F}$. By Lemma 8, $\rho \circ \phi$ coincides with the homomorphism used to define $\theta(\bar{L})$.

Now the argument proceeds as in [C]. Using ϕ one can define an element α of $\pi_{n+2}(K_E)$, where $K_E = K(\tilde{E}, 1)$ with 2-cells attached by maps representing $\bar{\mu}_i$. ($\tilde{E} = E/N$, where N is the maximal perfect subgroup of E). Then $\theta(\bar{L})$ is the image of α in $\pi_{n+2}(K_\infty)$ via a map $K_E \to K_\infty$ defined by ρ. But K_E is contractible and so $\alpha = 0$.

PROPOSITION 6. *If L is a sublink of a homology boundary link then L is a finite E-link. In fact, ϕ is an epimorphism.*

Proof. Suppose L is a sublink of a homology boundary link $\tilde{L}$ with $(m+k)$-components. Then the fundamental group $\tilde{\pi}$ of the complement of $\tilde{L}$ admits an epimorphism $\tilde{\phi}: \tilde{\pi} \to \tilde{F}$, where $\tilde{F}$ is the free group of rank $m+k$. Let $\mu_1, \ldots, \mu_k$ be meridians of the components of $\tilde{L}$ deleted to obtain L and let $r_i = \tilde{\phi}(\mu_i)$. There is an obvious isomorphism of $\tilde{\pi}/\langle \mu_1, \ldots, \mu_k \rangle$ with the fundamental group π of the complement of L, inducing an epimorphism $\phi: \pi \to \tilde{F}/\langle r_1, \ldots, r_k \rangle$. It is clear that E is a finite E-group. That ϕ kills longitudes will follow from the fact that $\tilde{\phi}$ kills longitudes. To see this note that $\tilde{F}$ is normally generated by the elements $r_1, \ldots, r_{m+k}$ which are images, under $\tilde{\phi}$, of meridians of $\tilde{L}$. Thus $r_i \notin [\tilde{F}, \tilde{F}]$. Now if $a \in \tilde{F}$ is the image of a longitude, then $[a, r_i] = 1$ for some r_i, and, therefore, $a = r_i^t$ for some $t \geq 0$. Since $a \in [\tilde{F}, \tilde{F}]$, we conclude $a = 1$.

8. We will now deal only with *one-dimensional* links. To understand the relationship between finite E-links and sublinks of homology boundary links, we need to bring in one more link invariant.

Let L be a (one-dimensional) link with group π and let $\lambda \subseteq \pi$ be the normal closure of the longitudes of L. Set $G = \pi/\lambda$. We define an element $\alpha_L \in H_3(G)$ as follows. Let M be the oriented manifold obtained by doing 0-framed surgery on S^3 along L. Then $\pi_1(M) = G$ and so the associated map $M \to K(G, 1)$ carries the fundamental class of M to α_L.

THEOREM 4. *L is a sublink of a homology boundary link if and only if L is a finite E-link where ϕ also satisfies*:

(i) *ϕ is an epimorphism*

(ii) *$\phi_*(\alpha_L) = 0$ (ϕ induces a homomorphism $G \to E$)*

COROLLARY. *If L is a finite E-link such that $\phi_*(\alpha_L) = 0$, then L is concordant to a sublink of a homology boundary link.*

Remarks. (a) If the Whitehead conjecture is true, then $H_3(E) = 0$ and so (ii) is vacuous.

(b) It is an interesting question whether the converse of the Corollary is true.

Proof of Theorem 4. In view of Proposition 6, we only have to check (ii). Let $\tilde{M}$ be the oriented manifold obtained by doing 0-framed surgery on S^3 along $\tilde{L}$. The natural map $\tilde{\pi} \to \pi$ induces $\psi: \tilde{G} \to H$, where $\tilde{G} = \tilde{\pi}/\tilde{\lambda}$ ($\tilde{\lambda}$ is the normal closure of the longitudes of $\tilde{L}$) and H is a quotient of G by the normal closure of the extra components of $\tilde{L}$. Then ϕ induces $\phi': H \to E$. There is an obvious cobordism V between M and $\tilde{M}$, using the handles added to M along the extra components of $\tilde{L}$, and the maps $M \to K(G, 1) \to K(H, 1)$ and $\tilde{M} \to K(\tilde{G}, 1) \to K(H, 1)$ extend to a map $V \to K(H, 1)$. Thus $\alpha_{\tilde{L}}$ and α_L map to the same element of H. Applying ϕ' we deduce $\phi_*(\alpha_L) = \tilde{\psi}_* \circ \tilde{\phi}_*(\alpha_{\tilde{L}})$, where $\tilde{\psi}: \tilde{F} \to E$ is the quotient map. But $\tilde{\phi}_*(\alpha_{\tilde{L}}) \in H_3(\tilde{F}) = 0$.

Now suppose L is a finite E-link and let K be a finite complex with fundamental group E and $H_2(K) = 0$. The first step is to construct a map $f: X \to K$, where X is the complement of a tubular neighborhood of L such that $f_*: \pi_1(X) = \pi \to E = \pi_1(K)$ coincides with ϕ. In fact we construct a map $f': M \to K$ and define $f = f' \mid X$. The existence of f' is shown, by the following lemma, to be equivalent to (ii).

LEMMA 9. *Let M be a closed connected oriented 3-manifold, K a finite 2-complex and $\phi: \pi_1(M) \to \pi_1(K)$ an epimorphism. Then there exists a map $f: M \to K$ such that $f_*: \pi_1(M) \to \pi_1(K)$ agrees with ϕ if and only if $\phi_*(\alpha) = 0$ where $\alpha \in H_3(\pi_1(M))$ is defined by $[M]$.*

Proof. It is easy to see that the existence of f implies $\alpha = 0$ since $H_3(K) = 0$. For the converse, we begin by choosing a map $f_0: M_0 \to K$ inducing ϕ on π_1, where $M_0 = M -$ disk. The obstruction to extending f_0 over M is an element $\beta \in \pi_2(K)$ - up to the action of $\pi_1(K)$. Now, an examination of the spectral sequence of the universal covering of K results in a short exact sequence:

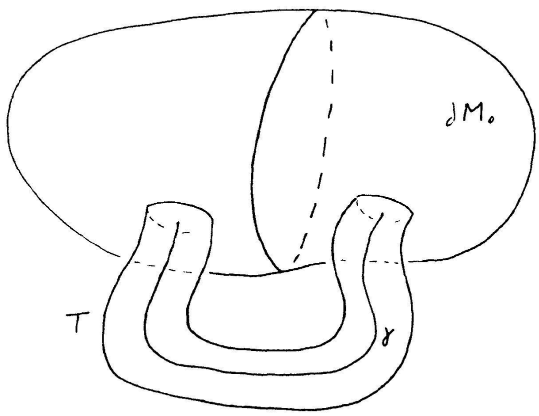
$$0 \to H_3(\pi_1(K)) \to \pi_2(K) \otimes_{\mathbb{Z}\pi} \mathbb{Z} \to H_2(K)$$

The class of β in $\pi_2(K) \otimes_{\mathbb{Z}\pi} \mathbb{Z}$ is surely the image of α under the first homomorphism of this sequence but we only need to show it is the image of some multiple of α. To see this consider the corresponding spectral sequence for M and M_0 resulting in a commutative diagram with exact rows:

$$\begin{array}{ccccccc}
 & 0 \longrightarrow & H_3(\pi_1(K)) & \longrightarrow & \pi_2(K) \otimes_{\mathbb{Z}\pi} \mathbb{Z} & \longrightarrow & H_2(K) \\
 & & \uparrow \phi_* & & \uparrow f_{0*} & & \uparrow f_{0*} \\
 & & H_3(\pi_1(M)) & \longrightarrow & \pi_2(M_0) \otimes_{\mathbb{Z}\pi} \mathbb{Z} & \xrightarrow{\rho_2} & H_2(M_0) \\
 & & \| & & \downarrow \rho_1 & & \downarrow \\
H_3(M) & \longrightarrow & H_3(\pi_1(M)) & \longrightarrow & \pi_2(M) \otimes_{\mathbb{Z}\pi} \mathbb{Z} & \longrightarrow & H_2(M)
\end{array}$$

It suffices to observe that β comes from a class in $\pi_2(M_0) \otimes_{\mathbb{Z}\pi} \mathbb{Z}$ which lies in kernel ρ_1 and in kernel ρ_2.

To prove the lemma we now show that it is possible to change f_0 so that β changes by an arbitrary element of the form $(g-1)\xi$, where $g \in \pi_1(K)$, $\xi \in \pi_2(K)$. Since ϕ is onto we can choose an arc γ in M_0 which begins and ends transversely on ∂M_0 but maps via f_0 to a closed path representing g. A tubular neighbourhood T of γ is diffeomorphic to $I \times D^2$. If we assume $f_0 \mid T$ is constant on the fibers $t \times D^2$, we can define $f_1 \mid T$ so that $f_1 \mid 0 \times D^2$ represents ξ, $f_1 \mid 1 \times D^2$ represents $g \cdot \xi$ and $f_1 \mid I \times S^1 = f_0 \mid I \times S^1$. If we let $f_1 \mid M - T = f_0 \mid M - T$, we obtain the required mapping.



For each 2-cell of K choose an interior point x_i. We may assume each x_i is a regular value and set $L_i' = f^{-1}(x_i)$. If $L' = L \cup \bigcup_i L_i'$ and $\tilde{F} = \pi_1(K')$, where K' is the 1-skeleton of K, then f induces a homomorphism $\phi' : \pi_1(S^3 - L') \to \tilde{F}$ which forms part of a commutative diagram:

$$\begin{array}{ccc}
\pi_1(S^3 - L') & \xrightarrow{\phi'} & \tilde{F} \\
\downarrow & & \downarrow \phi \\
\pi_1(S^3 - L) & \longrightarrow & E
\end{array}$$

We would like ϕ' to be onto and we apply Lemma 1 to arrange this. Since ϕ is onto, $\tilde{F}$ is generated by Image ϕ' and Kernel $\{\tilde{F} \to E\}$. But the latter group is normally generated by the attaching maps of the 2-cells of K and these are image of meridians of the $\{L_i'\}$. In Lemma 1 let $M = S^3 - L$ and $L_0 = \bigcup_i L_i'$. We conclude that there is a concordance V from L' to L'', which is a product on L, such that ϕ' extends over $\pi_1(S^3 - V)$ yielding an epimorphism $\phi'': \pi_1(S^3 - L'') \to \tilde{F}$. We still have $L \subseteq L''$. If we knew each L_i'' consisted of a single component, then L'' would be a homology boundary link and and the proof would be complete.

Note that we may assume each L_i'' is non-empty by adjoining trivial components if necessary. Now any two components of L_i'' have meridians which, by construction, map to conjugates in $\tilde{F}$ of the attaching map of the 2-cell containing x_i. Thus we may apply Lemma 2 to connect these components. By repeating this procedure we eventually have replaced L'' by $\tilde{L}$ so that $\tilde{L}_i$ is connected, for each i. This completes the proof.

Proof of Corollary. This follows immediately from Theorem 4 and Lemma 1, which asserts that L is concordant to a link L' such that ϕ extends over the complement of the concordance making L' an E-link with ϕ' which is onto. We also need to note that α_L is a concordance invariant in the following sense. If L is concordant to L' via a concordance V and G, G', H are the fundamental groups of the complements of L, L' and V, respectively, modulo the normal closures of the longitudes, then $i_*(\alpha_L) = i'_*(\alpha_{L'})$ where $i: G \to M$ and $i': G' \to M$ are the obvious homomorphisms. This follows from an easy construction of a cobordism W between M and M', where M, M' are obtained by surgery on S^3 along L, L', using V, so that G, G' and H are the fundamental groups of M, M' and W.

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Received June 10, 1987