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A proof of the Chern–Lashof conjecture in dimensions greater than five

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Summary. Chern and Lashof ([1], [2]) conjectured that if a smooth manifold M^m has an immersion into $\mathbf{R}^w$, then the best possible lower bound for its total absolute curvature is the Morse number $\mu(M)$. We give a proof of this when $m > 5$. Under the same dimension restriction, our methods allow us to show that $\mu(M)$ is still the best possible lower bound among immersions within a fixed regular homotopy class except in the case $w = m + 1 =$ even, for which the best lower bound is $\max\{\mu(M), 2|d|\}$, where d the degree of the Gauss map.

1. Introduction

Let $i: M^m \to \mathbf{R}^w$ be a smooth immersion of a smooth closed manifold in Euclidean space. Then the *total absolute curvature* $\tau(i)$ of the immersion is defined (cf. §2). Roughly speaking, $\tau(i)$ is a measure of the total amount of bending of the tangent plane of M^m in $\mathbf{R}^w$ as the point of tangency moves over M. The total absolute curvature depends on the immersion, and can be altered even by small changes of the immersion within its regular homotopy class.

The Morse number $\mu(M)$ is a topological invariant of M, and consequently does not depend on the immersion. For smooth manifolds of dimension greater than five it is the number of cells in the smallest C.W. complex of the same simple homotopy type as M (cf. [11]). In 1957 Chern and Lashof [2] proved that $\tau(i)$ is bounded below by the Morse number $\mu(M)$. In the same paper, they conjecture that $\mu(M)$ is the *best possible* lower bound for $\tau(i)$ as the immersion $i: M^m \to \mathbf{R}^w$ varies over all possibilities. We give:

THEOREM A. *The Chern–Lashof conjecture holds for smooth immersions of smooth closed manifolds of dimension greater than five.*

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Before stating our second result we recall some facts about the normal degree of a codimension one immersion $i: M^m \to \mathbf{R}^{m+1}$ of an oriented smooth manifold M (cf. Hopf [3], Milnor [5]). The normal degree is defined as follows. Let $g: M^m \to S^m$ be the Gauss map associating to each point $x \in M$ its oriented unit normal vector. The degree $d \in \mathbf{Z}$ of this map is the normal degree of the immersion, and is clearly an invariant of the regular homotopy class of i. The following theorem is due to Hopf, Milnor and Smale (and depends upon Adam's theorem that only the spheres of dimensions 1, 3 and 7 are parallelizeable).

NORMAL DEGREE THEOREM ([3], [5], [12]). *Let $i: M^m \to \mathbf{R}^{m+1}$ be a smooth immersion of a smooth closed oriented manifold with normal degree d. Then*:

i) *If m is even then $d = \chi(M)/2$.*

ii) *If m is odd but not* 1, 3 *or* 7 *then*:

 a) *If M is parallelizeable then d is even.*

 b) *If M is not parallelizeable then d is odd.*

 Moreover, subject to these restrictions, every possible integer arises as the normal degree of some immersion $M^m \to \mathbf{R}^{m+1}$.

iii) *If $m =$ 1, 3, or 7 then M is parallelizeable and d can be any integer.*

In 1961 Kuiper [4] explicitly raised the question of computing the best possible lower bound for τ of immersions of M^2 in $\mathbb{R}^3$ varying over all possibilities *in a prescribed regular homotopy class*. An answer to this question has been given recently by Pinkall [9]. Here is the answer to the analogue of Kuiper's question for manifolds of dimension greater than five.

THEOREM B. *Let $i: M^m \to \mathbf{R}^w$ be a smooth immersion of a smooth closed manifold of dimension greater than five. Let $\tau[i]$ denote the infimum of $\tau(j)$ as j varies over all immersions in the regular homotopy class* [i] *of i.*

i) *If $m + 1 < w$, of if $m + 1 = w =$ odd then $\tau[i] = \mu(M)$.*

ii) *If $m + 1 = w =$ even then M is orientable and* $\inf \tau[i] = \max\{\mu(M), 2|d|\}$, *where d is the normal degree of i.*

Note that for $m > 5$ Theorem B provides (except in the case $m + 1 = w =$ even) a stronger form of the Chern–Lashof conjecture showing that $\mu(M)$ is in fact the best possible lower bound for τ within each regular homotopy class. In the exceptional case $m + 1 = w =$ even it shows that there are counterexamples to this stronger form of the Chern–Lashof conjecture unless $|d| \leq \mu(M)/2$. The examples (due to Milnor) provided by part ii) of the Normal Degree Theorem show that this inequality need not hold. However the Normal Degree Theorem ii) also says that we can always find another immersion (in a different regular homotopy class) of the same manifold for which $|d| = 1$ or 0. Since $\mu(M) \geq 2$ it follows that for this new immersion $\max\{\mu(M), 2|d|\} = \mu(M)$. Thus Theorem B implies Theorem A.

Our paper is organized as follows. In §2 we recall the definition of total absolute curvature and the Kuiper–Wilson reduction of the computation of its best lower bound to the immersed Morse number γ. We also mention a few known results for small m. In §3 we state the fundamental handle manipulation results based on work of Perron and Rourke, and in §4 we use these to show that in codimension greater than one there is no obstruction to $\gamma = \mu(M)$. In §5 we study the codimension one case more carefully, describing several techniques for manipulating the handles. In §6 we study the local degree of the Gauss mapping and derive the inequality $\tau(M) \geq \max\{\mu(M), 2|d|\}$ for the odd codimension one case. In §7 we combine the result of §5 and §6 to finish the proof of the codimension one case. Finally in §8 we give examples of codimension one immersions of odd dimensional spheres in Euclidean space which are "tight within their regular homotopy class," having arbitrary odd normal degree, and raise a question about their uniqueness.

Throughout this paper $C_*(M)$ refers to the chain complex of the universal cover of M as a right $\mathbf{Z}\pi_1(M)$ module.

2. The Kuiper–Wilson reduction to Morse theory

Let $g: S^q(M) \to S^{w-1}$ be the Gauss map from the unit normal sphere bundle of M to the unit sphere in $\mathbf{R}^w$ arising from the immersion $i: M \to \mathbf{R}^w$. The total absolute curvature of i is defined to be:

$$\tau(i) = \int_{S^q(M)} |g^*\omega|,$$

where ω is the rotation invariant volume form on S^{w-1} of unit total volume (cf. [1]).

A change of variables to the sphere S^{w-1} yields the formula:

$$\tau(i) = \int_{S^{w-1}} \nu(e)\omega$$

where $\nu(e) = \#\{x \in M \mid g(x) = e\}$. If e is a regular value of g (which holds for almost all $e \in S^{w-1}$, then $h_e \equiv \langle e, \cdot \rangle \mid M$ is a Morse function and $v(e) = C(h_e) \equiv$ the number of critical points of h_e. Hence:

$$\tau(i) = \int_{S^{w-1}} C(h_e)\omega,$$

so that $\tau(i) \geq \mu(M)$ for all immersions (cf. [2] for all this).

Now we define the *immersed Morse number* $\gamma = \gamma[i]$ to be $\inf C(x_1 \circ j \mid M)$ where j varies over all immersions in the regular homotopy class $[i]$ of i. It follows immediately that $\tau[i] \geq \gamma[i] \geq \mu(M)$.

Here is the Kuiper–Wilson observation.

THEOREM 2.1 ([4], [16]). *Let $i: M^m \to \mathbf{R}^w$ be a smooth immersion of a smooth closed manifold in Euclidean space. Then $\tau[i] = \gamma[i]$.*

For a sketch of the proof see Sharpe [11]. The importance of this result for us is that it reduces the problem of computing the best lower bound for τ to "the Morse theory of immersions."

We mention some results on Theorems A and B for small m. Transversality arguments show that every immersion $M^m \to \mathbf{R}^w$ is regularly homotopic to an immersion $M^m \to \mathbf{R}^{2m}$ and hence to an immersion $M^m \to \mathbf{R}^{2m+1}$ with last coordinate a Morse function on M with $\mu(M)$ critical points (cf. Kuiper–Wilson op. cit.). Thus both A and B are true for $w > 2m$ and the interesting cases occur for $w \leq 2m$, which we now assume. When $m = 1$ Theorem A is obvious, and Theorem B follows from Whitney's classification of immersions of S^1 in $\mathbf{R}^2$ by their normal degree [15]. For $m = 2$ and $w = 3$, Theorem A follows from the classification of 2-manifolds, and examples of immersions of these in $\mathbf{R}^3$ given over the years. Pinkall's work cited above is Theorem B in this case. I know of no evidence that Theorems A and B do not generalize to all dimensions.

3. Manipulating immersed handles

In the "abstract (i.e. non-ambient) Morse theory" of handle rearrangement and cancellation (as described in Milnor [7] p. 37 & 45 for example) the intention is to alter (under certain circumstances) a given Morse function $f: M \to \mathbf{R}$ so as to obtain specific effects on its critical point structure (the reordering of two adjacent critical values or the removal of a complementary pair of critical points.

In the "embedded Morse theory" of Rourke [10] and Perron [8] (codimension $\geq$ two) and Sharpe [11] (codimension one) the aim is to do exactly

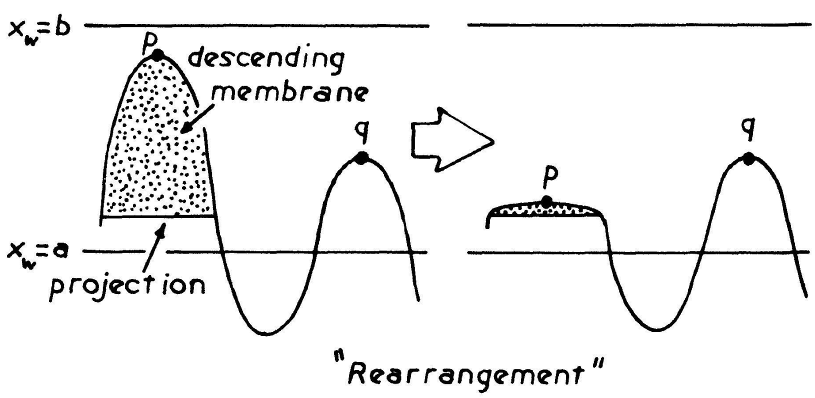

Fig. 3.1. "Rearrangment"

the same things, except that the Morse function is in fact a composite $f = x_w \circ i : M \to \mathbf{R}$, (where $i : M \to \mathbf{R}^w$ is an embedding and x_w is the last coordinate function), and we are only allowed to alter f ambiently i.e. by varying i through an isotopy. Figures 3.1 and 3.2 indicate Perron's processes for ambient rearrangement and cancellation.

In this section we adapt the techniques for rearrangement and cancellation of embedded handle theory to the (less restrictive) case in which i is an immersion and again we are only allowed to alter f ambiently, which here means varying i through a regular homotopy. It turns out that the hypotheses which work in case of "abstract Morse theory" almost always work in "immersed Morse theory." The exception occurs in the codimension one case for the cancellation theorem and can already be seen in the case of immersions of the circle in the plane as shown in the Figures 3.3 and 3.4. In both cases the critical points p and q form an abstractly cancelling pair. In the case of Figure 3.3 there is no way to cancel the critical points p and q *ambiently* since such a cancellation would alter the normal

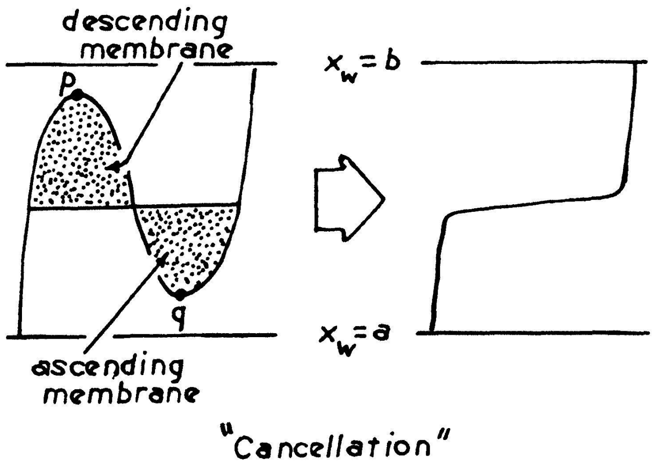

Fig. 3.2. "Cancellation"

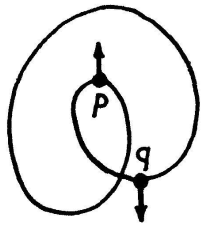

Fig. 3.3

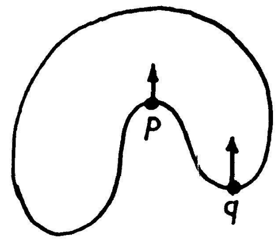

Fig. 3.4

degree of the immersion. Such a pair of abstractly cancelling critical points we call an "oppositely sensed" or "obstinate" handle pair. On the other hand Figure 3.4 shows a pair of "similarly sensed", or "removeable" critical points which can obviously be removed ambiently.

Let us assume that f has exactly two critical points p and q in the range $M_{[a,b]}=f^{-1}([a, b])$, and that both lie in the interior of $M_{[a,b]}$. Let $\alpha=\operatorname{index}_p(f)$ and $\beta=\operatorname{index}_q(f)$, and assume that $f(p)\geq f(q)$. A choice of a gradient-like vector field ξ_0 for f on $M_{[a,b]}$ (with $\xi_0(f)=-1$ off a small neighborhood of p and q) gives rise to a handlebody structure on $M_{[a,b]}$. For the moment we are interested in the handle h^α of dimension α corresponding to the upper critical point p.

By putting i in general position along the core of h^α (by means of a small regular homotopy of i), the image of the core in $\mathbf{R}^w$ will have transverse self-intersection of dimension $\delta=2\alpha-w\leq 2\alpha-(m+1)$. Let us assume until further notice that $2\alpha\leq m+1$ so that $\delta\leq 0$. Thus the image of the core has at most a finite number of point self-intersections. Since $\mathbf{R}^w$ is simply connected (and $m>5$), these intersection points can be "piped down" (cf. [14]) below the level of q, as in the Figure 3.5. After increasing a to a' if necessary we may assume that $i\mid$(core of h^α) is an embedding and thus, after possibly choosing smaller normal discs, $i\mid h^\alpha$ is an embedding.

We now indicate the changes needed in Perron's basic lemmas in order that they apply to the immersion case. We use the ascending and descending membranes as in Perron ([8], def 2.3) except that our membranes are allowed to have interior intersection with M. The effect of this is that whenever we push M across a membrane we may introduce new self-intersections on M. However Perron's lemma 2.4 ([8]) for rearranging handles now holds, with regular homotopy replacing isotopy, even in codimension one. In Perron's lemma 2.9 ([8]) for ambient cancellation we may weaken the condition that the interiors of the projections of the two membranes be disjoint by assuming that this is true only for some collar neighborhoods of their boundaries. As above, the effect of

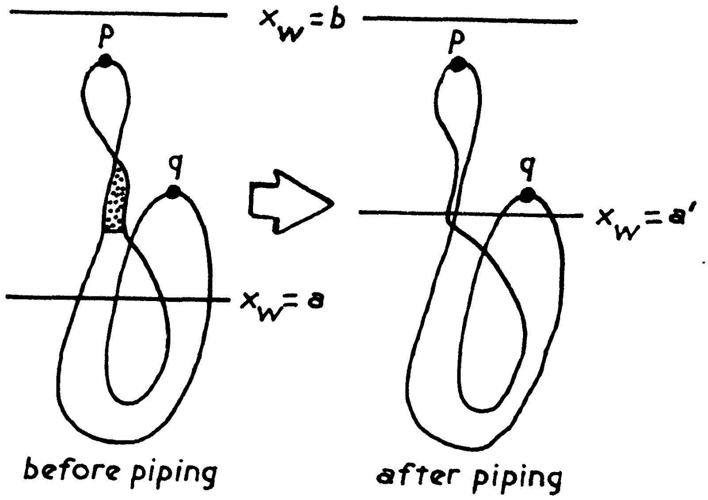

Fig. 3.5

this is that during the cancellation procedure we may introduce new self-intersections on M, so the cancellation process involves a regular homotopy rather than an isotopy.

Let us consider the meaning of the condition that there exist collars for the projections which are disjoint. This condition follows by general position except in codimension one. If it fails in codimension one it means that there is a sequence of points common to the two collars which converges to a point common to their boundaries. But condition 1 of Perrons lemma 2.9 ([8]) tells us there is just one such point, and it follows that the two collars lie (locally) on the same side of M at this point. Thus, in the codimension one case, to cancel a pair of critical points we need only assume that, in addition to being an abstractly cancelling pair, the critical points are "removeable," by which we mean that the upward normal directions (i.e. the positive x_w direction) at p and q give the same orientation to the normal line bundle of M in $\mathbf{R}^w$ if we compare them along the (unique homotopy class of) path T joining p to q in M along the core of p and the co-core of q. In this case we say the corresponding handles have the same "sense."

Note that if M is oriented then the use of T is unnecessary. In fact in this case we can give each critical point a canonical sense of "up" or "down" corresponding as its oriented normal direction points up or down. If M is non-orientable we can do this non-canonically by choosing base point paths to the critical points. Then a critical point is "up" or "down" according as upward normal orientation at the base point carried along to the critical point by means of its base point path is up or down. In this case T gives an element $\lambda \in \pi_1(M)$ which is orientation preserving if and only if the corresponding handles have the same sense as in the paragraph above.

Now let us consider the modifications to these arguments in the case that the inequality $2\alpha \leq m+1$ fails. In this case we use the Morse function $-f=(-x_w)\circ i$ whose handles are the dual of those of f. The effect is to interchange the roles of p and q in the rearrangement and cancellation theorems. Writing $\alpha^* = m-\alpha$ etc. we have $2\alpha^* = 2m-2\alpha < 2m-(m+1) = m-1$. Thus in the case $\beta \geq \alpha$ (rearrangement theorem) we get $2\beta^* \leq 2\alpha^* < m-1 < m+1$, and in the case $\alpha = \beta+1$ (cancellation theorem) we get $2\beta^* = 2\alpha^* + 2 < m+1$. In both cases we recover the inequality required for proceeding as above. Clearly the regular homotopy we obtain which rearranges or cancels the critical points for $-f$ will do the same for f.

Thus we can state the main result of this section as follows:

THEOREM 3.6. *Let $i: M^m \to \mathbf{R}^w$ be a smooth immersion of a smooth closed manifold such that $f = x_w \circ i: M^m \to \mathbf{R}$ is a Morse function with exactly two critical points p and q (of index α and β respectively) in $M_{[a,b]} = f^{-1}([a,b])$, with $a < f(q) \leq f(p) < b$. Then:*

i) (*Immersed Rearrangement*) *If $\alpha \leq \beta$ then there is a regular homotopy of i with support in $M_{[a,b]}$ which moves it to a new position i' so that $f' = x_w \circ i'$ has the same critical points as f (with the same indices) but $f'(p) < f'(q)$.*

ii) (*Immersed Cancellation*) *Suppose that $\alpha = \beta + 1$ and that the descending sphere of p meets the ascending sphere of q transversely in a single point in some level $f^{-1}(c)$ (where $f(q) < c < f(p)$). If $w = m+1$ we assume in addition that p and q are similarly sensed. Then there is a regular homotopy of i with support in $M_{[a,b]}$ to a new position i' so that $f' = x_w \circ i'$ has the no critical points in $M_{[a,b]}$.*

4. Codimension greater than one

Consider the case of an immersion $i: M^m \to \mathbf{R}^w$ where $m+1 < w$. After a small regular homotopy of i (a small rotation of $\mathbf{R}^w$ will do) we may assume that $f = x_w \circ i: M^m \to \mathbf{R}$ is a Morse function. It gives rise to a handlebody structure on M with free chain complex (over $\Lambda = \mathbf{Z}\pi_1(M)$):

$$0 \longrightarrow C_m \xrightarrow{\partial_m} C_{m-1} \xrightarrow{\partial_{m-1}} \cdots \longrightarrow C_1 \xrightarrow{\partial_1} C_0 \longrightarrow 0$$

Now there is a sequence of $\pi_1(M)$-simple base changes, stabilizations, and destabilizations which will convert C_* into a chain complex with total rank $\mu(M)$. When $m > 5$, just as in the proof of the s-cobordism theorem, each of the simple base changes can be realized by variation of the gradient-like vector field ξ_0 for f combined with the immersed handle rearrangement 3.6(i). The destabilizations can all be realized by immersed cancellation 3.6(ii), and the stabilizations can be realized trivially (cf. Perron [8] lemma 2.10). This proves Theorem B when the codimension >1.

5. Moving obstinate handles in codimension one

We can try to do the same thing as in §4 to remove handles in the codimension one case, except here we can only stabilize and destabilize with (abstractly cancelling) pairs of critical points of the same sense. In this section we describe the following three techniques:

1) "trading up" (or "trading down") obstinate handle pairs.
2) "exchange of sense" of two oppositely sensed handles of the same dimension.
3) "reversal of sense" of a handle at the expense of introducing or removing an obstinate handle pair.

First we study "trading up". Given a oppositely sensed pair of handles (h^{k+1}, h^k), we first insert a trivial similarly sensed pair (h_1^{k+2}, h_1^{k+1}). By adding h^{k+1} to h_1^{k+1} along *some* path we get an *abstractly* cancelling pair (h_1^{k+1}, h^k). We can insure that this pair is similarly sensed as follows. If M is orientable, then choose the similarly sensed pair (h_1^{k+2}, h_1^{k+1}) to have the same sense as h^k. If M is non-orientable, then we can *choose* the path along which the handle addition takes place so that (h_1^{k+1}, h^k) is similarly sensed. Thus we can cancel the similarly sensed pair (h_1^{k+1}, h^k), and we are left with a oppositely sensed pair (h_1^{k+2}, h^{k+1}) in one dimension higher. We leave the special case $k=0$ to the reader. Figure 5.1 is a diagram of this process. Similarly we can also trade handles down.

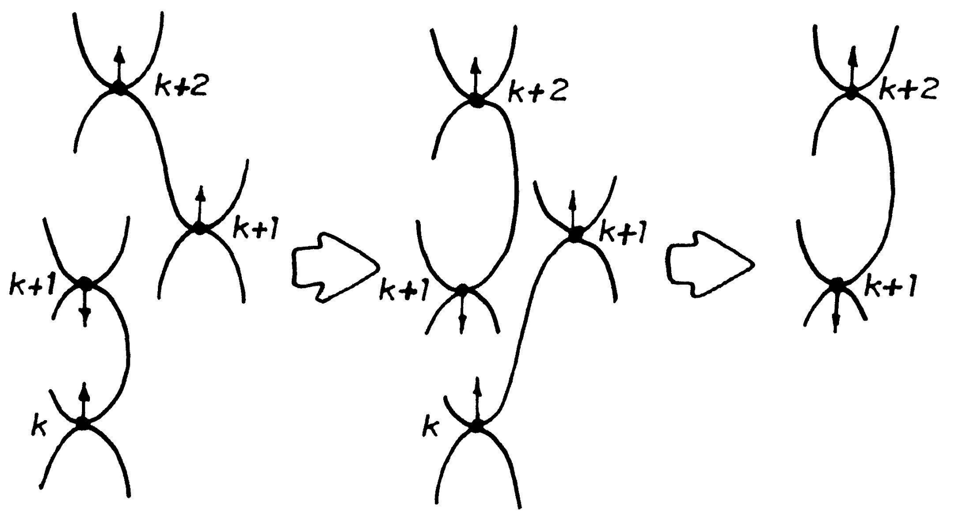

Fig. 5.1

Next we study "exchange of sense." Assume first that M is orientable. Consider two oppositely sensed handles of dimension k, say h_1^k (up) and h_2^k (down) with $e_1, e_2 \in C_k(M)$ the corresponding basis elements. Now consider the familiar sequence of simple base change corresponding to handle addition (and change of orientation and base point paths at the last stage) all of which can be realized ambiently (cf. 3.6):

$$\begin{aligned}
&e_1^k, e_2^k\\
&\to e_1^k + e_2^k\lambda, e_2^k \quad \text{where } \lambda \in \pi_1(M)\\
&\to e_1^k + e_2^k\lambda, e_2^k - (e_1^k + e_2^k\lambda)\lambda^{-1} = e_1^k + e_2^k\lambda, -e_1^k\lambda^{-1}\\
&\to e_1^k + e_2^k\lambda + (-e_1^k\lambda^{-1})\lambda, -e_1^k\lambda^{-1} = e_2^k\lambda, -e_1^k\lambda^{-1}\\
&\to e_2^k, e_1^k
\end{aligned}$$

Of course the senses of the handles are not changed by the realizations of these operations; that is, the sense of the new h_2^k is up and the sense of the new h_1^k is down. After interchanging the *names* of the new handles as well, we see that the total effect of these operations is to return the same chain complex while exchanging the senses of two of the handles.

In the non-orientable case we choose the path along which the handle addition takes place so that λ is orientation preserving, and the argument above still applies.

Finally we study "reversal of sense." This is a simple application of "exchange." Given a k handle (up), introduce a removeable pair with dimensions k and $k+1$ (both down) and exchange the senses of the two k handles to get one k handle (down) plus an obstinate pair with dimensions k (up) and $k+1$ (down). On the other hand, given a k handle (up) together with an obstinate pair consisting of a k handle (down) and a $k+1$ handle (up), then we can exchange the senses of the k handles to get a k handle (down) together with a removeable pair and this latter can be cancelled.

It is now easy to see that given a codimension one immersion $i: M^m \to \mathbf{R}^{m+1}$, we can combine these three techniques with 3.6 to get a regular homotopy to a new position $i': M^m \to \mathbf{R}^{m+1}$, so that $f' = x_{m+1} \circ i'$ has $\mu(M)$ critical points individually sensed as we choose, together with $2v$ other critical points coming from v obstinate pairs in a single pair of dimensions k and $k+1$, which we can select at will. However the number v will of course vary with the choices of sense that we make.

In the orientable case the v obstinate pairs may all be assumed to have the same sense (i.e. that the k handles are all up or all down) for if not we can exchange the senses of some of the k handles of these pairs to make some of them

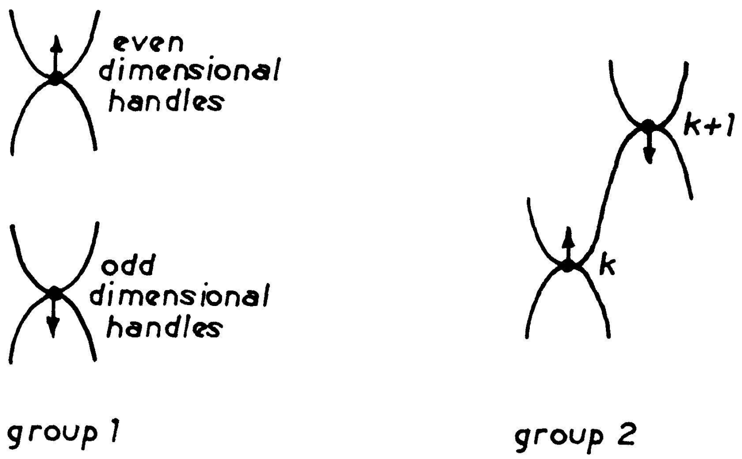

Fig. 5.2

removeable, and then get rid of them. Moreover we can assume all the k handles are up, if we are willing to perhaps change k. For they are either all up or all down, and if the latter then trade them all up one dimension.

In the non-orientable case we have even more freedom to remove obstinate pairs. Given any two obstinate pairs of k and $k+1$ handles we can exchange senses of the k handles along a path for which the corresponding element in $\pi_1(M)$ is non-orientable to get two removeable pairs and then get rid of them. Thus in the non-orientable case we may assume that $\nu = 0$ or 1. Thus we have:

PROPOSITION 5.2. *Every smooth immersion $i: M^m \to \mathbf{R}^{m+1}$ with $m > 5$ is regularly homotopic to an immersion i' for which the handles of $f' = x_{m+1} \circ i'$ fall (non-canonically) into two groups as in Figure 5.2 consisting of:*

1) *$\mu(M)$ "necessary" handles such that those of even dimension are up and those of odd dimension are down (let μ_k be the number of handles of dimension k of this type), and*
2) *ν obstinate handle pairs in dimensions $k+1$ and k with the k handles all up. If M is non-orientable, then $\nu = 1$ or 0.*

6. Local degree

We will need the following simple lemma to compute local degrees in the case when M is orientable:

LEMMA 6.1. *Let $i: M^m \to \mathbf{R}^{m+1}$ be an immersion of an orientable manifold with Gauss map $g: M^m \to S^m$. Assume that $f = x_{m+1} \circ i$ has a nondegenerate critical point of index k at $p \in M$, and that $g(p) = e_{m+1}$. Then g has local degree $(-1)^{m-k}$ at p. If $g(p) = -e_{m+1}$, then g has local degree $(-1)^k$ at p.*

Proof. We can choose an oriented coordinate system $(x, y) \in \mathbf{R}^k \times \mathbf{R}^{m-k}$ on M near p so that p has coordinates $(0, 0)$, and $f = c - x^2 + y^2$. Moreover (x, y) extends over a neighborhood of $i(p)$ in $\mathbf{R}^{m+1}$ so that (x, y, x_{m+1}) is an oriented local coordinate system on $\mathbf{R}^{m+1}$, in which M is given by the equation $x^2 - y^2 + x_{m+1} = 0$. Assume that this local coordinate system (x, y, x_{m+1}) is the restriction of the standard global coordinate system on $\mathbf{R}^{m+1}$ (we shall leave the general case to the reader). Thus g is given there by $g(x, y) = (2x, -2y, 1)$, and hence has local degree $(-1)^{m-k}$. In the other case where $g(p) = -e_{m+1}$ a similar argument gives the local degree of g at p as $(-1)^k$.

COROLLARY 6.2. *The number of critical points of f is at least* $2\,|d|$.

Proof. Let $C_+(f)$ (respectively $C_-(f)$) be the number of critical points of f with the upward (respectively downward) sense. Since the local degree at each critical point is ± 1, we have:

$$C_+(f) \geq |\Sigma \text{ local degree of } g \text{ at } c| = |d|,$$

where the summation is over critical points c of f with the upward sense. Similarly $C_-(f) \geq |d|$, so $C(f) = C_+(f) + C_-(f) \geq 2\,|d|$.

Remark 6.3. In the case of m even, the Normal Degree Theorem gives $|d| = |\chi/2|$ and hence the corollary gives $C(f) \geq |\chi|$. Since we already have $C(f) \geq \mu(M) \geqslant |\chi|$ (cf. §2) this is not new. In the case of m odd however, the Normal Degree Theorem says that $|d|$ is unbounded, so the inequality $C(f) \geq 2\,|d|$ gives a nontrivial restriction on $C(f)$ supplementary to the condition $C(f) \geq \mu(M)$. Thus we have $C(f) \geq \max\,\{\mu(M), 2\,|d|\}$.

7. The case of codimension one

We begin by eliminating the non-orientable odd dimensional manifolds.

LEMMA 7.1. *No smooth closed non-orientable odd dimensional manifold immerses with codimension one in Euclidean space.*

Proof. An immersion $f: M^m \to \mathbf{R}^{m+1}$ yields a normal line bundle ν which in turn gives a map $g: M^m \to \mathbf{P}^m$ such that $\nu = g^*(\eta)$ and $\tau_M = g^*(\tau_{\mathbf{P}^m})$, where η is the canonical bundle over $\mathbf{P}^m$. Thus $\tau_M \oplus \varepsilon = g^*(\tau_{\mathbf{P}^m} \oplus \varepsilon) = g^*(\eta^{m+1}) = \nu^{m+1}$. Thus $\omega_1(M) = (m+1)\omega_1(\nu) = 0$ since m is odd. Hence M is orientable.

For the rest of this section we assume that the handles have been arranged according to the description in Proposition 5.2.

Consider the case of a non-orientable, and hence even dimensional, manifold. There is at most one oppositely sensed pair. If such a pair exists we merely trade it down until the dimensions are 1 and 0, and then do a "reversal of sense" between the 0 handle with one of the minima along an appropriate (orientation reversing if necessary) path. This changes the oppositely sensed pair to a similarly sensed one, and we remove it, leaving exactly $\mu(M)$ handles.

Now consider the case of an orientable even dimensional manifold. According to the Normal Degree Theorem the degree of the Gauss map is $\chi/2$; calculating by Lemma 6.1 with both the up and the down handles respectively, we get:

$$\chi/2 = (-1)^k v + \sum \mu_{2k} = -\{(-1)^k v + \sum \mu_{2k-1}\}.$$

It follows that $(-1)^k 2v + \sum \mu_k = 0$, and hence k is odd. In particular:

$$\chi/2 = -v + \sum \mu_{2k} = v - \sum \mu_{2k-1}$$

If $\chi/2 \geq 0$ then $v \leq \sum \mu_{2k}$ while if $\chi/2 \leq 0$ then $v \leq \sum \mu_{2k-1}$. In the first case we can use the "reversal of sense" to remove all the obstinate pairs (after trading them up or down as necessary) by using them to change the senses of some of the even dimensional handles. In the other case we remove them by changing the senses of some of the odd dimensional handles.

Finally we consider the case of an oriented odd dimensional manifold. Let us orient M so that the normal degree $d \geq 0$. Calculating by the Lemma 6.1 with both the up and the down handles gives:

$$d = (-1)^{k+1} v - \sum \mu_{2k} = (-1)^{k+1} v - \sum \mu_{2k-1}$$

Thus $2d + \mu = (-1)^{k+1} 2v$ and hence k is odd. thus we can reduce v by changing the senses of $\min\{\mu, v\}$ handles in the first group by the "reversal of sense" trick. This leaves $\mu + 2(v - \min\{\mu, v\}) = \max\{\mu, 2|d|\}$ handles. But by the final remark of §6 this is a lower bound for the number of critical points, so it is the best possible lower bound. This finishes the proof of Theorem B.

8. Concluding remarks

What we might call the "geometric question" in the subject of total absolute curvature of immersions is the following. Given an immersion $i: M^m \to \mathbf{R}^w$ whose total absolute curvature achieves the minimum within its regular homotopy class, what can one say about M itself and the immersion i? The hypothesis can be framed as follows. For $m > 5$ an immersion $i: M \to \mathbf{R}^w$ has minimal absolute curvature within its regular homotopy class iff for almost all $e \in S^{w-1}$, the function $x \mapsto i(x) \cdot e$ is a Morse function with exactly $\gamma[i]$ critical points. Such an immersion may be called "tight within its regular homotopy class". We note that in the case when $\gamma[i] = \mu(M)$ this notion of tightness appears to be slightly more general than the usual one which requires also that $\mu(M) = \sum \dim H_k(M, K)$ for some field K. In any case, the usual notion of tightness is very strong, many manifolds having no tight immersion in any Euclidean space (c.f. Thorbergson [13] for highly connected even dimensional examples of this).

Let $i: S^m \to \mathbf{R}^w$ be an immersion which is tight within its regular homotopy class. Assume first that $m > 5$ and we are *not* in the exceptional case $m + 1 = w =$ even. Then Theorem B shows that $\gamma[i] = \mu(S^m) = 2$. Thus the immersion is tight, and by the work of Chern and Lashof [1], $i(S^m)$ is the boundary of a convex ball in some $m + 1$ dimensional affine subspace of $\mathbf{R}^w$. In particular i is regularly homotopic either to the standard embedding or the antipodal embedding. Now let us consider the exceptional case $m + 1 = w =$ even. In this case there are immersions $i: S^m \to \mathbf{R}^{m+1}$ which are tight within their regular homotopy class with any odd normal degree d as indicated in Figure 8.1. This picture immerses S^1 with normal degree $d = 2k - 1$ in $\mathbf{R}^2$ as an immersed submanifold Y. The immersed hypersurface of revolution X in $\mathbf{R}^{m+1}$ obtained by revolving Y about the y axis is:

$$X = \{(x, y, z) \in \mathbf{R} \times \mathbf{R} \times \mathbf{R}^{m-1} \mid (\sqrt{\{x^2 + z^2\}}, y) \in Y\}.$$

By symmetry, the critical point structure of the function h_e on X does not change

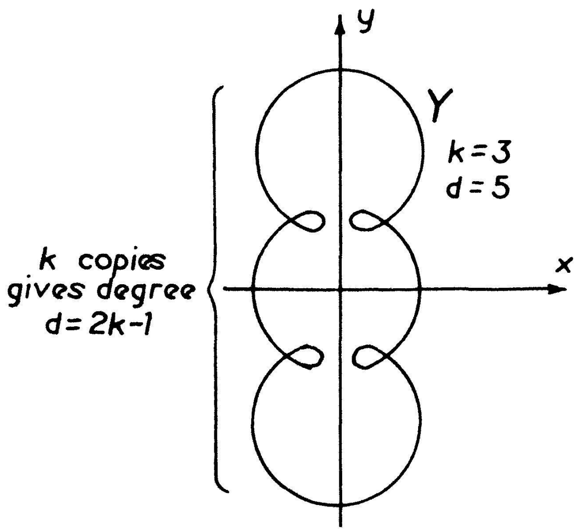

Fig. 8.1

if we rotate e about the y axis into the original plane, where (for e not vertical) it clearly has exactly $2d$ critical points. Using Lemma 6.1 it is easily checked that X has normal degree d. It follows that X is tightly immersed within its regular homotopy class. Is it true that for m odd any smooth immersion $S^m \to \mathbf{R}^{m+1}$ of odd degree d which is tightly immersed within its regular homotopy class is regularly homotopic to this one?

REFERENCES

[1] S. Chern & R. K. Lashof, *On the total curvature of immersed manifolds*, I. Am. J. Math. *79* (1957) 306–318.

[2] S. Chern & R. K. Lashof, *On the total curvature of immersed manifolds*, II. Mich. J. Math. *5* (1958) 5–12.

[3] H. Hopf, *Vectorfelder in n-dimensionalen Mannigfaltkeiten*, Math. Ann. *96* (1926/27) 225–250.

[4] N. H. Kuiper, *Convex Immersions of Closed Surfaces in* E^3, Comm. Math. Helv. *35* (1961) 85–92.

[5] J. W. Milnor, *On the immersions of n-manifolds in* $(n+1)$*-space*, Comm. Math. Helv. *30* (1955) 275–284.

[6] J. W. Milnor, *Morse Theory*, Ann. of Math. Studies, no. 51, Princeton University Press, Princeton N.J. 1963.

[7] J. W. Milnor, *Lectures on the h-Cobordism Theorem*, Notes by L. Seibenmann and J. Sondow. Mathematical Notes. Princeton University Press, Princeton, N.J., 1965.

[8] B. Perron, *Pseudo-Isotopies de Plongement en codimension* 2, Bull. Soc. Math. France, *103* (1975) 289–339.

[9] U. Pinkall, *Tight Surfaces and Regular Homotopy*, Topology *25* (1986) 475–482.

[10] C. Rourke, *Embedded Handle Theory*. Topology of Manifolds, Edited by C. Cantrell and C. Edwards. University of Georgia (1969).

[11] R. W. Sharpe, *Total Absolute Curvature and Embedded Morse Numbers*, to appear in The Journal of Differential Geometry.

[12] S. Smale, *The classification of immersions of spheres in Euclidean space*, Annals of Math. *69* (1959) 327–344.

[13] G. Thorbergson, *Tight immersions of highly connected manifolds*, Comment. Math. Helvetici *61* (1986) 102–121.

[14] C. T. C. Wall, *Surgery on Compact Manifolds*, Academic Press 1970.

[15] H. Whitney, *On regular closed curves in the plane*, Composito Math. *4* (1937) 276–284.

[16] Wilson, *The total absolute curvature of immersed manifolds*, Journal L.M.S. *40* (1965) 362–366.

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