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A new proof of the compactness theorem for integral currents

BRIAN WHITE

One of the most fundamental results in the geometric measure theory of currents is the compactness theorem, which states that for each $k < \infty$, the set of integer-multiplicity rectifiable currents (henceforth simply called "rectifiable currents") with mass and boundary mass $\leq k$ is compact with respect to the weak topology. The compactness theorem is an immediate consequence of the Banach–Alaoglu theorem and the closure theorem, which states that this set is closed with respect to the weak topology in the space of all currents. The original proof of the closure theorem by Federer and Fleming [FF] requires the structure theorem for sets of finite Hausdorff measure, the proof of which is very difficult. Bruce Solomon [SB] and later F. J. Almgren, Jr. [A] succeeded in giving proofs of the closure theorem that do not use the structure theorem. However, their proofs rely on various facts about multivalued functions. Those facts, although elementary in comparison with the structure theorem, nevertheless require a somewhat lengthy development. In this paper we give a straightforward proof of the closure theorem that uses neither the structure theorem nor multivalued functions. Up to a point the proof is essentially the one given by Solomon (which in turn uses ideas from the original proof by Federer and Fleming), but where that proof uses multivalued functions, the one here substitutes an argument using the constancy, theorem, the Poincare inequality, and the fact that a set of finite n-dimensional Hausdorff measure has upper n-dimensional density ≤ 1 almost everywhere.

I would like to thank Bruce Solomon for suggesting a number of changes in the manuscript.

1. Preliminaries

For the most part, we follow the terminology and notion of Leon Simon's book [S], which differ slightly from those in Federer's treatise [F]. In particular,

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here rectifiable currents need not have compact support (and so would be called locally rectifiable currents in [F]). Also, $B(x, r)$ denotes the *open* ball of radius r centered at x, $\mathbf{M}_W(T)$ denotes the mass of T in the open set W, and $\rightharpoonup$ denotes weak convergence (either of currents or of Radon measures). We recall the following facts from [S]:

1. *Currents with Locally Finite Mass*

If μ is a Radon measure on $U \subset R^{n+k}$ and $\vec{v}$ is a μ-measurable and locally μ-integrable n-vectorfield on U, then $\mu \wedge \vec{v} \in \mathcal{D}_n(U)$ is the n-dimensional current with locally finite mass defined by

$$(\mu \wedge \vec{v})\omega = \int_U \langle \omega(x), \vec{v}(x) \rangle \, d\mu(x)$$

Conversely, if $T \in \mathcal{D}_n(U)$ is an n-dimensional current with locally finite mass, then $T = \|T\| \wedge \vec{T}$ where $\|T\|$ is the Radon measure such that

$$\|T\| W = \mathbf{M}_W(T) = \mathbf{M}(T \llcorner W)$$

for every open $W \subset U$ and $\vec{T}$ is a unit n-vectorfield (i.e., $|\vec{T}x| \equiv 1$). (In [S], the measure $\|T\|$ is usually written μ_T.)

2. *Slicing*

If $T \in \mathcal{D}_n(U)$ and $\mathbf{M}_W(T) + \mathbf{M}_W(\partial T) < \infty$ for every open $W \Subset U$ and if $f : U \to R$ is lipschitz, then the slice of T by f at r is defined to be

$$\langle T, f, r \rangle = \partial(T \llcorner \{f < r\}) - (\partial T) \llcorner \{f < r\}$$

(In this paper we only slice T with $\partial T = 0$, so that the second term is 0.) The two facts we need about slicing are first, that if T is rectifiable, then so is $\langle T, f, r \rangle$ for $\mathcal{L}^1$-almost every r [S, 28], and second:

SLICING LEMMA. *Suppose $f : U \to R$ is lipschitz and $T_i \in \mathcal{D}_n(U)$ is a sequence of currents such that $T_i \rightharpoonup T$ and*

$$\sup_i \, (\mathbf{M}_W(T_i) + \mathbf{M}_W(\partial T_i)) < \infty$$

for every $W \Subset U$. Then for $\mathcal{L}^1$-almost every r, there is a subsequence i' such that

$$\langle T_{i'}, f, r \rangle \rightharpoonup \langle T, f, r \rangle \tag{1}$$

and

$$\sup_i \left(\mathbf{M}_W(\langle T_{i'}, f, r\rangle) + \mathbf{M}_W(\partial\langle T_{i'}, f, r\rangle)\right) < \infty \qquad (W \Subset U)$$

Furthermore, if for some $W_0 \Subset U$,

$$\lim \left(\mathbf{M}_{W_0}(T_i) + \mathbf{M}_{W_0}(\partial T_i)\right) = 0$$

then the subsequence may be chosen so that

$$\lim \left(\mathbf{M}_{W_0}(\langle T_{i'}, f, r\rangle) + \mathbf{M}_{W_0}(\partial\langle T_{i'}, f, r\rangle)\right) = 0$$

Proof. First pass to a subsequence i' such that $\|T_{i'}\| + \|\partial T_{i'}\|$ converges to a Radon measure μ. Then it is not hard to show that (1) holds except for the countably many r such that $\mu\{f = r\} \neq 0$.

The other conclusions follow (after passing to further subsequences) from the facts that

$$\int_a^{*b} \mathbf{M}_W(\{T_i, f, r\rangle)\, dr \leq \mathrm{Lip}\,(f)\mathbf{M}_W(T_i \llcorner \{a < r < b\})$$

and

$$\partial\langle T_i, f, r\rangle = -\langle \partial T_i, f, r\rangle$$

See [S, 28.10].

3. *Densities*

If μ is a Borel regular measure on $U \subset R^{n+k}$, $M \subset R^{n+k}$, and $x \in U$, then we define the upper density

$$\Theta^{n*}(\mu, M, x) = \limsup_{r \to 0} \frac{\mu(M \cap B(x, r))}{\alpha(n) r^n}$$

where $\alpha(n)$ is the volume of the unit n-ball. We use $\Theta^{n*}(\mu, x)$ as an abbreviation of $\Theta^{n*}(\mu, R^{n+k}, x)$. The lower densities Θ^n_* are defined in the same way with "inf" instead of "sup."

The two basic facts about densities we will use are that

$$\mu(W) \geq \delta \mathscr{H}^n\{x \in W : \Theta^{n*}(\mu, x) \geq \delta\}$$

if W is open and that

$$\Theta^{n*}(\mathscr{H}^n, M, x) \leq 1 \quad \text{for } \mathscr{H}^n\text{-almost every } x$$

if $\mathscr{H}^n(M) < \infty$. See chapter 1.3 of [S].

4. Lebesgue Points

THEOREM. *If μ is a Radon measure on R^{n+k} and f is a locally μ-integrable function on R^{n+k}, then for μ-almost every x,*

$$\lim_{r \to 0} \frac{1}{\mu(B(x, r))} \int_{B(x,r)} |f(y) - f(x)| \, d\mu(y) = 0$$

This follows fairly easily from the weaker result in which one integrates $f(y) - f(x)$ instead of $|f(y) - f(x)|$. The weaker result is part of theorem 4.7 in [S]. See [F, 2.9.9] for the stronger theorem.

5. The Boundary Rectifiability Theorem

THEOREM. *If $T \in D_n(U)$ is a rectifiable current such that $\mathbf{M}_W(\partial T) < \infty$ for every $W \Subset U$, then ∂T is also rectifiable.*

This is an easy consequence [S, 30.8] of the deformation theorem and the $(n-1)$-dimensional case of the closure theorem. Thus we can use this theorem in a proof by induction of the closure theorem. (Note that the 0 and 1 dimensional cases of the boundary rectifiability theorem are trivial.)

2. Two lemmas

The proof of the closure theorem will be reduced to showing that a boundaryless current with rectifiable slices must be rectifiable. Part (2) of the following lemma plays an important role.

2.1. LOWER DENSITY LEMMA

(1) *If $T \in \mathscr{D}_n(U)$ and $\mathbf{M}_W(T) + \mathbf{M}_W(\partial T) < \infty$ for every $W \Subset U$, then for $\|T\|$-almost every x,*

$$\lim_{r \to 0} \frac{\lambda(x, r)}{\|T\| B(x, r)} = 1$$

where

$$\lambda(x, r) = \inf \{\mathrm{M}(S) : \partial S = \partial(T \llcorner B(x, r)),\ S \in \mathscr{D}_n(U)\}$$

(2) *If in addition* $\partial T = 0$ *and if* $\partial(T \llcorner B(x, r))$ *is rectifiable for every* x *and almost every* r, *then there is a* $\delta > 0$ *such that*

$$\Theta^n_*(\|T\|, x) > \delta$$

for $\|T\|$*-almost every* $x \in U$.

Remark. This lemma and its proof are essentially due to W. Fleming [FW], who used them in proving the closure theorem for rectifiable flat chains mod p. The version here is taken from [SB].

Proof. Clearly $\lambda(x, r) \leq \|T\| B(x, r)$, so if (1) is false then there is an $\varepsilon > 0$ and a set X such that $\|T\| X > 0$ and such that for each $x \in X$, there exist arbitrarily small balls $B(x, r)$ with

$$\lambda(x, r) < (1 - \varepsilon)\mathrm{M}(T \llcorner B(x, r))$$

Note that X can be chosen so that $X \subset W$ for some open subset $W \Subset U$. By the Besicovitch covering theorem, for every $\rho > 0$ there exists a disjoint collection $B_1, B_2, \ldots \subset W$ of such balls covering $\|T\|$-almost all of X and having radii $\leq \rho$. Let S_i be a current such that

$$\partial S_i = \partial(T \llcorner B_i)$$

and

$$\mathrm{M}(S_i) < (1 - \varepsilon)\mathrm{M}(T \llcorner B_i)$$

Let $T_\rho = T - \sum T \llcorner B_i + \sum S_i$. Then for every n-form ω,

$$(T - T_\rho)\omega = \sum (T \llcorner B_i - S_i)\omega$$

$$= \sum (\partial(q_i \times (T \llcorner B_i - S_i)))\omega$$

(where q_i is the center of B_i)

$$= \sum (q_i \times (T \llcorner B_i - S_i))\, d\omega$$

$$\leq \sum \mathbf{M}(q_i \times (T \llcorner B_i - S_i)) \sup |d\omega|$$

$$\leq \sum \rho \mathbf{M}(T \llcorner B_i - S_i) \sup |d\omega|$$

$$\leq \sum 2\rho \mathbf{M}(T \llcorner B_i) \sup |d\omega|$$

$$\leq \rho 2 M_W(T) \sup |d\omega|$$

so $T_\rho \rightharpoonup T$ as $\rho \to 0$. Thus by the lower semicontinuity of mass

$$\mathbf{M}_W(T) \leq \liminf_{\rho \to 0} \mathbf{M}_W(T_\rho).$$

But

$$\mathbf{M}_W(T_\rho) \leq \mathbf{M}_W\Big(T - \sum T \llcorner B_i\Big) + \sum \mathbf{M}_W(S_i)$$

$$\leq \mathbf{M}_W\Big(T - \sum T \llcorner B_i\Big) + (1 - \varepsilon) \sum \mathbf{M}_W(T \llcorner B_i)$$

$$\leq \mathbf{M}_W(T) - \varepsilon \sum \mathbf{M}_W(T \llcorner B_i)$$

$$\leq \mathbf{M}_W(T) - \varepsilon \, \|T\| X$$

which is a contradiction. This proves (1).

To prove (2), let x be a point at which (1) holds and let $f(r) = \mathbf{M}(T \llcorner B(x, r))$. Then for all sufficiently small r (say $r < R$),

$$f(r) < 2\lambda(x, r).$$

For almost every r,

$$\mathbf{M}(\partial(T \llcorner B(x, r))) \leq f'(r)$$

(by slicing theory applied to the function $\operatorname{dist}(x, \cdot)$ [S, 28.9]), and since $\partial(T \llcorner B(x, r))$ is rectifiable, the isoperimetric inequality [S, 30.1] implies that

$$\lambda(x, r)^{(n-1)/n} \leq c(f'(r))$$

for almost every r. Thus

$$(2f(r))^{(n-1)/n} \leq c(f'(r))$$

or

$$\frac{d}{dr}(f(r)^{1/n}) \geq c'$$

so (since f is non-decreasing)

$$f(r) \geq (c')^{1/n} r^n$$

for $r < R$. ■

By taking a limit of dilations of a current at a point one gets currents that are described by the following lemma.

2.2. CONSTANT VECTORFIELD LEMMA. *Let* $T = \|T\| \wedge \vec{T} \in \mathscr{D}_n(R^{n+k})$ *be a current of locally finite mass such that* $\partial T = 0$ *and such that* $\vec{T}x = \vec{\omega} \in \Lambda_n(R^{n+k})$ *for every* x. *Let* V *be the subspace of* R^{n+k} *of vectors* v *such that* T *is invariant under translations in the direction of* v. *(That is,* $v \in V$ *if and only if* T *is invariant under translations by* rv *for every real number* r.*) Then* $\vec{w} \in \Lambda_n(V)$.

Proof. By making an orthogonal change of coordinates we can assume that V has a basis consisting of a sub-collection of the standard basis vectors $e_1, \ldots, e_{n+k}$. It then suffices to show, for instance, that if $e_1 \notin V$, then $\vec{w}$ (when expressed in terms of the usual basis $\{e_{i_1} \wedge \cdots \wedge e_{i_n}\}$) has no $e_1 \wedge e_{i_2} \wedge \cdots \wedge e_{i_n}$ components. By mollification we can also assume that $\|T\| = \mathscr{L}^{n+k} \llcorner \theta$ for a smooth function θ. Now if $f \in C^\infty(R^{n+k})$ has compact support, then

$$\begin{aligned} 0 &= (\partial T)(f\, dx_2\, dx_3 \cdots dx_n) \\ &= T(d(f\, dx_2\, dx_3 \cdots dx_n)) \\ &= T\left(\sum \frac{\partial f}{\partial x_i}\, dx_i\, dx_2 \cdots dx_n\right)\Big) \\ &= \int \sum a_i \theta(x) \frac{\partial f}{\partial x_i}\, d\mathscr{L}^n x \end{aligned}$$

(where a_i is the $e_i \wedge e_2 \wedge \cdots \wedge e_n$ component of $\vec{w}$)

$$= -\int \sum a_i \frac{\partial \theta}{\partial x_i} f(x)\, d\mathcal{L}^n x$$

Since this holds for all f with compact support,

$$\sum a_i \frac{\partial \theta}{\partial x_i} \equiv 0$$

so T is invariant under translations in the direction $\sum a_i e_i$. Thus $a_1 = 0$. The same argument works for the other n-vectors of the form $e_1 \wedge e_{i_2} \wedge \cdots \wedge e_{i_n}$. ■

3. The closure theorem

THEOREM. *Let T_i be a sequence of n-dimensional rectifiable currents in $U \subset R^{n+k}$ with*

$$\sup_i \left(\mathbf{M}_W(T_i) + \mathbf{M}_W(\partial T_i)\right) < \infty \tag{1}$$

for every $W \Subset U$. If T_i converges weakly to a current T, then T is also rectifiable.

Proof. The proof is by induction, so we will assume that the result is true for $(n-1)$-dimensional currents. (The 0-dimensional case is trivial).

First observe that the general case of the theorem follows from the special case when $U = R^{n+k}$ and

$$\sup_i \left(\mathbf{M}(T_i) + \mathbf{M}(\partial T_i)\right) < \infty \tag{2}$$

For if $B(a, R) \subset U$, then the slicing Lemma 1.3 (with $W = B(a, R/2)$) implies that there is an $r \in (0, R/2)$ and a subsequence i' such that

$$T_{i'} \llcorner B(a, r) \to T \llcorner B(a, r)$$

and

$$\sup_{i'} \mathbf{M}(\langle T_{i'}, \operatorname{dist}(a, \cdot), r \rangle) < \infty$$

i.e.,

$$\sup_{i'} (\mathbf{M}(\partial(T_{i'} \llcorner B(a, r)) - (\partial T_{i'}) \llcorner B(a, r)) < \infty$$

Adding this to (1) with $W = B(a, r)$ gives

$$\sup_{i'} (\mathbf{M}(T_i \llcorner B(a, r)) + \mathbf{M}(\partial(T_{i'} \llcorner B(a, r))) < \infty$$

Thus the special case of the theorem implies that $T \llcorner B(a, r)$ is rectifiable, and thus (since a is arbitrary) that T is rectifiable.

Hence from now on we will assume (2) and that $U = R^{n+k}$.

Next we claim that the general case follows from the case when $\partial T = 0$. For from the $(n-1)$-dimensional case of the theorem (via the boundary rectifiability Theorem 1.5), we have that ∂T_i is rectifiable for each i. Since $\partial T_i \rightharpoonup \partial T$, the induction hypothesis also implies that ∂T is rectifiable. By the isoperimetric theorem there is a rectifiable current R of finite mass with $\partial R = \partial T$. Since $T_i - R \rightharpoonup T - R$ and $\partial(T - R) = 0$, the special case of the theorem implies that $T - R$ is rectifiable and hence that T is rectifiable.

Thus we will assume that $\partial T = 0$.

If f is any lipschitz function on R^{n+k}, then for almost every r, the slice $\langle T_i, f, r \rangle$ is rectifiable. By the slicing lemma, for almost every r there is a subsequence i' such that

$$\sup_{i} (\mathbf{M}(\langle T_{i'}, f, r \rangle + \mathbf{M}(\partial \langle T_{i'}, f, r \rangle)) < \infty$$

and

$$\langle T_{i'}, f, r \rangle \rightharpoonup \langle T, f, r \rangle$$

Thus (by the $(n-1)$-dimensional case of the closure theorem) $\langle T, f, r \rangle$ is rectifiable.

So far we have that $\partial T = 0$, $\mathbf{M}(T) < \infty$, and that for every lipschitz function f, the slice $\langle T, f, r \rangle = \partial(T \llcorner \{f < r\})$ is rectifiably for almost every r. From these three facts alone we will prove that T is rectifiable.

By the lower density Lemma 2.1 there is a $\delta < 0$ such that

$$\Theta^n_*(\|T\|, x) \geq \delta$$

for $\|T\|$-almost every x. Thus if we let

$$M = \{x \in R^{n+k} : \Theta^n_*(\|T\|, x) \geq \delta\}$$

then

$$\|T\|(R^{n+k} \setminus M) = 0 \tag{3}$$

Also the basic density properties (1.3) of Radon measures imply that

$$\mathscr{H}^n(M) \leq \delta^{-1} \|T\|(M) < \infty$$

Finally, we recall [S, 26.29] that (since $\mathrm{M}(T) + \mathrm{M}(\partial T) < \infty$) if $\mathscr{H}^n(E) = 0$ then $\|T\|(E) = 0$. Thus by (3), $\|T\|$ is absolutely continuous with respect to $\mathscr{H}^n \llcorner M$. Letting θ be the Radon–Nykodym derivative we have

$$T\omega = \int_M \langle \omega(x), \vec{T}x \rangle \theta(x)\, d\mathscr{H}^n x$$

or

$$T\omega = \int_M \langle \omega(x), \vec{\tau}x \rangle\, d\mathscr{H}^n x$$

where $\vec{\tau}x = \theta(x)\vec{T}x$. We must show that M is n-rectifiable and that for $\mathscr{H}^n$-almost every $x \in M$, $\vec{T}x$ is a simple unit n-vector associated with $\mathrm{Tan}_\alpha M$ and $\theta(x)$ is an integer.

For $\mathscr{H}^n$-almost every a in M we have

$$\Theta^{n*}(\mathscr{H}^n, M, a) \leq 1 \tag{4}$$

$$\lim_{r \to 0} \frac{1}{\mathscr{H}^n(M \cap B(a, r))} \int_{M \cap B(a,r)} |\vec{\tau}x - \vec{\tau}a|\, d\mathscr{H}^n x = 0 \tag{5}$$

by Section 1.3 and Section 1.4 (applied to the measure $\mathscr{H}^n \llcorner \mathrm{M}$). Fix any such point $a \in M$. Note that by (5) and the definition of M,

$$\Theta^n_*(\mathscr{H}^n, M, a) = |\vec{\tau}a|^{-1}\, \Theta^n_*(\|T\|, a) > 0 \tag{6}$$

Let Λ be a sequence of positive numbers converging to 0. Define $\eta_{\lambda,a} : R^{n+k} \to$

R^{n+k} by $\eta_{\lambda,a}(x) = \lambda^{-1}(x-a)$. Then for every $R < \infty$,

$$\begin{aligned}
&\limsup_{\lambda \in \Lambda} (\mathscr{H}^n \llcorner \eta_{\lambda,a} M) B(0, R) \\
&\quad = \limsup_{\lambda \in \Lambda} \lambda^{-n} \mathscr{H}^n(M \cap B(a, \lambda R)) \\
&\quad \leq \alpha(n) R^n \Theta^{n*}(\mathscr{H}^n, M, a) \\
&\quad < \infty
\end{aligned}$$

(where $\alpha(n)$ is the volume of the unit n-ball). Thus [S, 4.4] there exists a subsequence Λ' and a Radon measure μ such that

$$\mathscr{H}^n \llcorner \eta_{\lambda,a} M \rightharpoonup \mu \quad (\lambda \in \Lambda') \tag{7}$$

Let $\vec{w} = \vec{\tau} a$. It follows immediately that

$$(\mathscr{H}^n \llcorner \eta_{\lambda,a} M) \wedge \vec{w} \rightharpoonup \mu \wedge \vec{w} \quad (\lambda \in \Lambda')$$

We claim also that

$$T_\lambda \rightharpoonup \mu \wedge \vec{w} \quad (\lambda \in \Lambda') \tag{8}$$

and that

$$\|T_\lambda\| \rightharpoonup |\vec{w}| \mu \quad (\lambda \in \Lambda') \tag{9}$$

where $T_\lambda = \eta_{\lambda,a\#} T$. For note that for every $R < \infty$,

$$\begin{aligned}
\mathbf{M}_{B(0,R)}(T_\lambda - (\mathscr{H}^n \llcorner \eta_{\lambda,a} M) \wedge \vec{w}) &= \lambda^{-n} \mathbf{M}_{B(a,\lambda R)}(T - (\mathscr{H}^n \llcorner M) \wedge \vec{w}) \\
&= \lambda^{-n} \int_{M \cap B(a,\lambda R)} |\vec{\tau} x - \vec{w}| \, d\mathscr{H}^n x \\
&\to 0
\end{aligned}$$

by (4) and (5). Thus we have (8) and (9).

Since for every lipschitz f, almost all the slices $\langle T_\lambda, f, r \rangle$ are rectifiable, it follows from (8), (9), the slicing lemma, and the $(n-1)$-dimensional case of the closure theorem that almost all the slices $\langle \mu \wedge \vec{w}, f, r \rangle$ are rectifiable. Thus we have the lower density bound

$$\Theta^n_*(\mu, x) \geq \delta / |\vec{w}| > 0 \tag{10}$$

for μ-almost every x. We also have from (8) that $\partial(\mu \wedge \vec{w}) = 0$. Thus by the constant vectorfield Lemma 2.2, μ is translation invariant in at least n directions. The lower density bound (10) implies that it is translation invariant in at most n directions (because, just as with M, the set of x where (10) holds has locally finite $\mathcal{H}^n$ measure). Thus (again by the constant vectorfield lemma), $\vec{w}$ is a simple n-vector, and there is a collection $P_1, \ldots, P_p$ of n-planes parallel to the n-dimensional subspace determined by $\vec{w}$ such that

$$\mu = \sum_{j=1}^{p} \alpha_j (\mathcal{H}^n \llcorner P_j)$$

Note that

$$\sum_{j=1}^{p} \alpha_j \leq \Theta^{n*}(\mathcal{H}^n, M, a) \leq 1$$

by (4) and (7). By the lower density bound (10), each α_j is $\geq \delta/|\vec{w}|$, so p must be finite.

We claim that $p = 1$. For suppose not. Let

$$3\varepsilon < \min \{\operatorname{dist}(P_i, P_j) : P_i \neq P_j\}$$

Fix a j and let $f(x) = \operatorname{dist}(x, P_j)$. We suppose without loss of generality that P_j is parallel to $R^n \times \{0\} \subset R^{n+k}$. Let $U = B(0, R) \subset R^n$ and $W = U \times R^k \subset R^{n+k}$. Now

$$\begin{aligned} \lim_{\lambda \in \Lambda'} \mathbf{M}_W(T_\lambda \llcorner \{\varepsilon < f < 2\varepsilon\}) &\leq |\vec{w}| \mu(\bar{W} \cap \{\varepsilon \leq f \leq 2\varepsilon\}) \quad \text{(by (9))} \\ &= 0 \end{aligned}$$

so by the slicing lemma, there exists a further subsequence Λ'' and a $t \in (\varepsilon, 2\varepsilon)$ such that

$$\begin{aligned} T_\lambda \llcorner \{f < t\} &\to (\mu \wedge \vec{w}) \llcorner \{f < t\} \qquad (\lambda \in \Lambda'') \\ &= \alpha_j (\mathcal{H}^n \llcorner P_j) \wedge \vec{w} \\ &= \alpha_j |\vec{w}| [P_j] \end{aligned}$$

and such that

$$\mathbf{M}_W(\partial(T_\lambda \llcorner \{f < t\})) \to 0 \qquad (\lambda \in \Lambda'')$$

Let $T'_\lambda = \Pi(T_\lambda \llcorner \{f < t\})$ where $\Pi: R^{n+k} \to R^k$ is the orthogonal projection. According to the Poincare inequality [S, 6.4] (see also [S, 26.28]), there exist $\beta_\lambda \in R$ such that

$$\mathbf{M}_U(T'_\lambda - \beta_\lambda[U]) \leq c\mathbf{M}_U(\partial T'_\lambda)$$

As $\lambda \to 0$, the right hand side tends to 0, hence $\beta_\lambda \to \beta = \alpha_j |\vec{w}|$ and

$$\mathbf{M}_U(T'_\lambda - \beta[U]) \to 0$$

But

$$\beta \mathscr{L}^n(U \setminus \Pi(\eta_{\lambda,a} M \cap \{f < t\})) \leq \mathbf{M}_U(T'_\lambda - \beta[U])$$

so

$$\lim \mathscr{L}^n(U \setminus \Pi(\eta_{\lambda,a} M \cap \{f < t\})) = 0$$

Thus

$$\begin{aligned}
\mathscr{L}^n(U) &= \lim_{\lambda \in \Lambda''} \mathscr{L}^n(\Pi(\eta_{\lambda,a} M \cap \{f < t\}) \cap U) \\
&\leq \liminf_{\lambda \in \Lambda''} \mathscr{H}^n((\eta_{\lambda,a} M \cap \{f < t\}) \cap W) \\
&\leq \mu(\{f \leq t\} \cap \bar{W}) \\
&= \alpha_j \mathscr{H}^n(U)
\end{aligned}$$

Thus each α_j is ≥ 1. By (4), (6), and (7)

$$0 < \Theta^n_*(\mathscr{H}^n, M, a) \leq \sum \alpha_j \leq \Theta^{n*}(\mathscr{H}^n, M, a) \leq 1$$

Hence $p = 1$ and $\alpha_1 = 1$.

Furthermore, P_1 must pass through the origin since for every $r > 0$,

$$\begin{aligned}
\mu(B(0, 2r)) &\geq \liminf (\mathscr{H}^n \llcorner \eta_{\lambda,a} M) B(0, r) \\
&= \liminf \lambda^{-n} \mathscr{H}^n(M \cap B(a, \lambda r)) \\
&\geq \alpha(n) r^n \Theta^n_*(\|T\|, a) \\
&> 0
\end{aligned}$$

by (6).

Finally, note that the plane P_1 is determined by $\vec{w} = \vec{\tau}a$ and therefore does not depend on the particular sequence Λ'' of λ's. We have shown that the set M has an approximate tangent plane at $\mathcal{H}^n$-almost every $a \in M$. Hence M is n-rectifiable [S, 11.6].

We also showed that $\vec{\tau}a$ is a simple n-vector associated with the $\operatorname{Tan}_a M$. Thus it remains only to show that $\theta(a) = |\vec{\tau}a| = |\vec{w}|$ is an integer. But this follows easily from the facts that $\mu = \mathcal{H}^n \llcorner P_1$ and that $\mu \wedge \vec{w}$ has (integer multiplicity) rectifiable slices. ■

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