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A generalized Hopf formula for higher homology groups

RALPH STÖHR

1. Introduction

Let G be a group, $1 \to R \to F \to G \to 1$ a free presentation of G. One of the pioneering results on homology of groups was *Hopf's formula* (H. Hopf [9]): $H_2G = R \cap F'/[R, F]$, providing a purely group theoretical interpretation of the second homology group of G. In this paper we obtain a generalized Hopf formula expressing the even-dimensional homology groups $H_{2c}G$ in terms of free presentations of G. Specifically, let $1 \to R_i \to F_i \to G \to 1$ $(i = 1, 2, \ldots, c)$ be free presentations of G and let $F = F_1 * F_2 * \cdots * F_c$ be the free product of the F_i. We identify the F_i and R_i with their canonical images in F. In Section 3 we state our main result (Theorem 1), the isomorphism

$$H_{2c}G = ([R_1, R_2, \ldots, R_c] \cap N)\gamma_{c+1}R/[R_1, R_2, \ldots, R_c, F]\gamma_{c+1}R,$$

where R and N are canonically defined normal subgroups of the free group F. Under the natural assumption that $[R_1, \ldots, R_c]$ is R_1 in case $c = 1$, this formula holds for all positive integers c and for $c = 1$ it coincides with the classical Hopf formula. Concerning the odd-dimensional homology groups, we prove in Section 7 that, under the additional assumption that G is finite, $H_{2c-1}G$ is isomorphic to a certain factor of the center of $F/\gamma_{c+1}R$ (Theorem 2).

An alternative interpretation of higher homology groups has been given by B. Conrad [4], whose approach generalizes the notion of the multiplier of a group, which was introduced by I. Schur [15] and later recognised as an interpretation of H_2G.

Concerning cohomology we mention, that a number of authors have given interpretations of the higher cohomology groups $H^k(G, A)$ extending the classical work of Eilenberg and Mac Lane [5] (see Holt [8], Huebschmann [10], further

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references can be found in Mac Lane's Historical Note [11]). The basic idea in all this work is to combine Whitehead's notion of crossed modules with Yoneda's interpretation of Ext as classes of long exact sequences. An entirely different description of $H^k(G, A)$ has been given by K. W. Gruenberg [6]. Gruenberg's approach involves free presentations, free products and commutator subgroups. Some basic ideas of Gruenberg's construction have been adapted in the present paper.

The arrangement of this paper is as follows. Notations and some preliminary notions will be introduced in Section 2. Our main result, the generalized Hopf formula, will be stated in Section 3. The proof is given in Section 6 by exploiting the preliminary discussion in Sections 4 and 5. Finally, in Section 7, we prove the above mentioned result on odd-dimensional homology of finite groups.

2. Preliminaries

Let H be a group, $a_1, \ldots, a_c, b$ elements of H and $S_1, \ldots, S_c$ subgroups of H. As usual we define

$$a^b = b^{-1}ab, \qquad [a_1, a_2] = a_1^{-1}a_2^{-1}a_1a_2,$$

$$[a_1, \ldots, a_c] = [[a_1, \ldots, a_{c-1}], a_c]$$

and, for $c \geqslant 2$,

$$[S_1, \ldots, S_c] = gp\{[a_1, \ldots, a_c]; a_i \in S_i (i = 1, \ldots, c)\}.$$

In case $c = 1$ it will be convenient to assume that $[S_1, \ldots, S_c]$ is simply S_1. The lower central series $H = \gamma_1 H \supseteq H' = \gamma_2 H \supseteq \gamma_3 H \supseteq \cdots$ of H is defined inductively by $\gamma_{i+1}H = [\gamma_i H, H] (i \geqslant 1)$, i.e.

$$\gamma_i H = [H, \ldots, H] \text{ (i times, $i \geqslant 1$).}$$

Let F be a free group with free basis X. The quotient $F/\gamma_{c+1}F$ is the free nilpotent group of class c with free generators $\tilde{x} = x\gamma_{c+1}F$ ($x \in X$). If G is any group, F^G denotes the free group with free basis $Y = \{a_{x,g}; x \in X, g \in G\}$. The action of G on Y defined by $a_{x,g}^h = a_{x,gh}$ ($h \in G$) induces an action of G on the free nilpotent group $F^G/\gamma_{c+1}F^G$. The semidirect product of $F^G/\gamma_{c+1}F^G$ with G via the induced action is the $\mathfrak{N}_c$-verbal wreath product of $F/\gamma_{c+1}F$ by G and is written for short as $F/\gamma_{c+1}F \operatorname{wr}_{\mathfrak{N}_c} G$ ($\mathfrak{N}_c$ is the standard notation for the variety of all nilpotent of class at most c groups, see [14]). The normal subgroup $F^G/\gamma_{c+1}F^G$ is termed the base group of the wreath product.

Suppose

$$1 \longrightarrow R \longrightarrow F \xrightarrow{\pi} G \longrightarrow 1 \tag{1}$$

is an exact sequence of groups, i.e. a free presentation of G. We record here a special case of Shmel'kin's general embedding theorem.

LEMMA 1 (A. L. Shmel'kin [16]). *The mapping* $x\gamma_{c+1}R \to x\pi \cdot \tilde{a}_{x,1}$ $(x \in X)$ *extends to an embedding of* $F/\gamma_{c+1}R$ *into* $F/\gamma_{c+1}F \, wr_{\mathfrak{N}_c} \, G$.

An important feature of the Shmel'kin embedding is that the subgroup $R/\gamma_{c+1}R$ is mapped isomorphically into the base group $F^G/\gamma_{c+1}F^G$.

In case $c = 1$ the above verbal wreath product is simply the ordinary restricted direct wreath product $F/F' \operatorname{wr} G$ (see [14]) and the embedding of Lemma 1 is the Magnus embedding $F/R' \to F/F' \operatorname{wr} G$ (W. Magnus [12]). The base group $F^G_{ab} = F^G/(F^G)'$ of the direct wreath product becomes a G-module via conjugation in $F/F' \operatorname{wr} G$. It is easily seen that F^G_{ab} is a free G-module with free generators $a_{x,1}(F^G)'$ $(x \in X)$. The free abelian group $R_{ab} = R/R'$ carries, by conjugation in F, the structure of a G-module, which is usually called the relation module of G associated with the free presentation (1). The restriction of the Magnus embedding to R_{ab} induces an embedding $\mu : R_{ab} \to F^G_{ab}$. Let IG denote the augmentation ideal of the integral group ring $\mathbb{Z}G$ and let σ denote the surjection $F^G_{ab} \to IG$ defined by $a_{x,1}(F^G)' \to 1 - x\pi$. By a theorem of Blackburn [2],

$$0 \longrightarrow R_{ab} \xrightarrow{\mu} F^G_{ab} \xrightarrow{\sigma} IG \longrightarrow 0 \tag{2}$$

is an exact sequence of G-modules. In another context the exact sequence (2) appeared already in Gruenberg's paper [17]. It is customarily referred to as the relation sequence (stemming from the free presentation (1)) and the embedding μ is called the Magnus embedding for modules (see also [3], p. 43).

Well-known facts and standard notations concerning cohomology of groups will be used without citing special references; these however can easily be found in K. Brown's book [3]. In particular, if $A_1, \dots, A_n$ are G-modules, the tensor product $A_1 \otimes \cdots \otimes A_n$ (over $\mathbb{Z}$) will always be regarded as a G-module with diagonal action. Also, G-modules (including abelian subquotients of (multiplicative) groups with induced G-action) will be written additively. Concerning commutator calculus, which will be used in Sections 4 and 5, we refer to [13].

3. Theorem 1

For the rest of this paper, let G be a group, $c \geqslant 1$ a fixed positive integer,

$$1 \longrightarrow R_i \longrightarrow F_i \xrightarrow{\pi_i} G \longrightarrow 1 \qquad (i = 1, 2, \ldots, c) \tag{3}$$

free presentations of G and let X_i denote a set of free generators for F_i. The free product

$$F = F_1 * F_2 * \cdots * F_c$$

is itself a free group with free basis $X = X_1 \cup \cdots \cup X_c$. We will identify the F_i and R_i with their canonical images in F. Let $\tilde{F}$ denote the free nilpotent group $F/\gamma_{c+1}F$ and consider the verbal wreath product $\tilde{F}\,\mathrm{wr}_{\mathfrak{N}_c}\,G$. The base group of the latter is the free nilpotent group $\tilde{F}^G = F^G/\gamma_{c+1}F^G$ with free generators $\tilde{a}_{x,g} = a_{x,g}\gamma_{c+1}F^G$ $(x \in X, g \in G)$. The mapping

$$x \longrightarrow x\pi_i \cdot \tilde{a}_{x,1} \qquad (x \in X_i, i = 1, \ldots, c) \tag{4}$$

extends to a homomorphism

$$\alpha : F \to \tilde{F}\,\mathrm{wr}_{\mathfrak{N}_c}\,G.$$

The kernel of this homomorphism can be described as follows. Let π denote the homomorphism form F onto G defined by $x\pi = x\pi_i$ $(x \in X_i, i = 1, \ldots, c)$ and put $R = \ker \pi$. Then

$$1 \longrightarrow R \longrightarrow F \xrightarrow{\pi} G \longrightarrow 1$$

is a free presentation of G. By construction, there is a commutative triangle

$$\begin{tikzcd} F \arrow[r, "\alpha"] \arrow[dr] & \tilde{F}\mathrm{wr}_{\mathfrak{N}_c} G \\ & F/\gamma_{c+1}R \arrow[u] \end{tikzcd}$$

where the vertical homomorphism is the Shmel'kin embedding for $F/\gamma_{c+1}R$ (see Lemma 1) and the diagonal is the natural homomorphism from F onto its quotient $F/\gamma_{c+1}R$. It follows that the kernel of α coincides with $\gamma_{c+1}R$.

Now consider the lower central series of the base group of $\tilde{F} \operatorname{wr}_{\mathfrak{N}_c} G$. The last non-trivial term is $\gamma_c\tilde{F}^G = \gamma_c F^G/\gamma_{c+1}F^G$. This is an abelian normal subgroup of $\tilde{F} \operatorname{wr}_{\mathfrak{N}_c} G$ and so is the commutator $[\gamma_c\tilde{F}^G, G]$. The mapping (4) defines a homomorphism

$$\beta: F \to \tilde{F} \operatorname{wr}_{\mathfrak{N}_c} G/[\gamma_c\tilde{F}^G, G].$$

We denote the kernel of this homomorphism by N. Now we are able to state our main result expressing $H_{2c}G$ as a certain factor of the free group F.

THEOREM 1. *There is an isomorphism*

$$H_{2c}G = ([R_1, \dots, R_c] \cap N)\gamma_{c+1}R/[R_1, \dots, R_c, F]\gamma_{c+1}R.$$

The proof of the theorem will be given in Section 6. We conclude this section with an examination of the case $c = 1$. In this case one has $F = F_1$, $R = R_1$ and the homomorphism β maps F into

$$F_{ab} \operatorname{wr} G/[F_{ab}^G, G] \cong F_{ab} \times G$$

(direct product). Consequently, we have $N = F' \cap R$ and the formula of Theorem 1 gives

$$H_2G = (R \cap F')R'/[R, F]R' = R \cap F'/[R, F],$$

the classical Hopf formula.

4. The verbal wreath product $\tilde{F} \operatorname{wr}_{\mathfrak{N}_c} G$

In this section we examine the verbal wreath product $\tilde{F} \operatorname{wr}_{\mathfrak{N}_c} G$. Let $\tilde{F}_i$ and $\tilde{F}_i^G$ $(i = 1, \dots, c)$ denote the quotients $F_i/\gamma_{c+1}F_i$ and $F_i^G/\gamma_{c+1}F_i^G$, respectively. Obviously, the free nilpotent groups $\tilde{F}_i^G$ are canonically embedded in the base group $\tilde{F}^G$ and the verbal wreath products $\tilde{F}_i \operatorname{wr}_{\mathfrak{N}_c} G$ are canonically embedded in $\tilde{F} \operatorname{wr}_{\mathfrak{N}_c} G$ in the obvious way (in fact, $\tilde{F}^G$ is the $\mathfrak{N}_c$-coproduct of the $\tilde{F}_i^G$, see [14], Chapter 1, §8). Now consider $\gamma_c\tilde{F}^G$, the c-th term of the lower central series of $\tilde{F}^G$. This is a free abelian group and the G-action on $\tilde{F}^G$ induces on $\gamma_c\tilde{F}^G$ the structure of a G-module. Obviously, $[\tilde{F}_1^G, \tilde{F}_2^G, \dots, \tilde{F}_c^G]$ is a submodule of $\gamma_c\tilde{F}^G$.

LEMMA 2. *The submodule* $[\tilde{F}_1^G, \dots, \tilde{F}_c^G]$ *is a direct summand of* $\gamma_c\tilde{F}^G$ *and* *the mapping*

$$[\tilde{a}_{x_1,g_1}, \ldots, \tilde{a}_{x_c,g_c}] \to a_{x_1,g_1}(F_1^G)' \otimes \cdots \otimes a_{x_c,g_c}(F_c^G)',$$

where $x_i \in X_i$, $g_i \in G$ $(i = 1, \ldots, c)$ *extends to an isomorphism*

$$\psi : [\tilde{F}_1^G, \ldots, \tilde{F}_c^G] \xrightarrow{\sim} (F_1^G)_{ab} \otimes \cdots \otimes (F_c^G)_{ab}.$$

Proof. Assume the free generators $\tilde{a}_{x,g}$ $(x \in X, g \in G)$ of $\tilde{F}^G$ totally ordered so that for all $x_i \in X_i$, $g_i \in G$ $(i = 1, \ldots, c)$

$$a_{x_2,g_2} < a_{x_1,g_1} < a_{x_3,g_3} < \cdots < a_{x_c,g_c}.$$

The free abelian group $\gamma_c \tilde{F}^G$ has a free $\mathbb{Z}$-basis consisting of all basic commutators of weight c defined over the $\tilde{a}_{x,g}$ (= basic commutatiors of weight 1) with respect to the above introduced ordering. Then $[F_1^G, \ldots, F_c^G]$ is, as a $\mathbb{Z}$-module, freely generated by all left-normed basic commutators $[\tilde{a}_{x_1,g_1}, \ldots, \tilde{a}_{x_c,g_c}]$ $(x_i \in X_i, g_i \in G, i = 1, \ldots, c)$ and $\gamma_c \tilde{F}^G$ decomposes (as a $\mathbb{Z}$ – module) into the direct sum

$$\gamma_c \tilde{F}^G = [F_1^G, \ldots, F_c^G] \oplus A, \tag{5}$$

where A is (freely) generated by all remaining basic commutators of weight c.

In fact, (5) is a direct decomposition of $\gamma_c \tilde{F}^G$ as a G-module and, in particular, $[\tilde{F}_1^G, \ldots, \tilde{F}_c^G]$ is a direct summand of $\gamma_c \tilde{F}^G$, as desired. To verify this, it suffices to show that A is a G-submodule. For, let $v = v(\tilde{a}_{x_1,g_1}, \ldots, \tilde{a}_{x_c,g_c})$ be a basic commutator of weight c involving the free generators $\tilde{a}_{x_i,g_i}, \ldots, \tilde{a}_{x_c,g_c}$, where $x_i \in X_{k(i)}$, $1 \le k(i) \le c$, $g_i \in G$ $(i = 1, \ldots, c)$ and suppose that v is one of the generators of A. We have to show that, for any $g \in G$,

$$v^g = v(\tilde{a}_{x_1,g_1 g}, \ldots, \tilde{a}_{x_c,g_c g})$$

is also in A.

Case 1. v is not left-normed. Then there is obviously no left-normed basic commutator in the unique $\mathbb{Z}$-linear combination of basic commutators expressing v^g. Hence, $v^g \in A$.

Case 2. v is left normed and $k(i) = k(j)$ for some i,j $(1 \le i < j \le c)$. Then any basic commutator occurring in the unique $\mathbb{Z}$-linear combination expressing v^g involves the free generators $\tilde{a}_{x_i,g_i g}$ and $\tilde{a}_{x_j,g_j g}$ with $k(i) = k(j)$. Hence, $v^g \in A$.

Case 3. v is left-normed and $k(i) \neq k(j)$ for all i,j $(i \leq i < j \leq c)$. Then we have $v = [\tilde{a}_{x_i,g_i}, \ldots, \tilde{a}_{x_c,g_c}]$, where $x_i \in X_{k(i)}$ and $(k(1), k(2), \ldots, k(c)) \neq (1, 2, \ldots, c)$. But in this case $v^g = [\tilde{a}_{x_i,g_1g}, \ldots, \tilde{a}_{x_c,g_cg}]$ is itself a left normed basic commutator with $(k(1), \ldots, k(c)) \neq (1, \ldots, c)$ and, consequently, $v^G \in A$.

Now it remains to check that the mapping defined in the lemma extends to the desired isomorphism. But this is clear, since the mapping is obviously compatible with the corresponding G-actions and provides a one-one correspondence of free $\mathbb{Z}$-bases of $[\tilde{F}_1^G, \ldots, \tilde{F}_c^G]$ and $(F_1^G)_{ab} \otimes \cdots \otimes (F_c^G)_{ab}$. $\square$

5. The quotient $[R_1, \ldots, R_c]\gamma_{c+1}R/\gamma_{c+1}R$

Now we return to F. In this section we examine the quotient

$$[R_1, \ldots, R_c]\gamma_{c+1}R/\gamma_{c+1}R. \tag{6}$$

Being a subgroup of the free abelian group $\gamma_c R/\gamma_{c+1}R$, (6) is itself a free abelian group. Conjugation in F induces on $\gamma_c R/\gamma_{c+1}R$ the structure of an F-module. Since R acts trivially, $\gamma_c R/\gamma_{c+1}R$ may be regarded as a G-module by defining

$$(m\gamma_{c+1}R) \cdot g = m^y \gamma_{c+1}R,$$

where $m \in \gamma_c R$, $g \in G$ and $y \in F$ with $y\pi = g$. For $r_i \in R_i$ $(i = 1, \ldots, c)$ one has obviously

$$\begin{aligned}([r_1, \ldots, r_c]\gamma_{c+1}R) \cdot g &= [r_1, \ldots, r_c]^y \gamma_{c+1}R \\ &= [r_1^y, \ldots, r_c^y]\gamma_{c+1}R \\ &= [r_1^{y_1}, \ldots, r_c^{y_c}]\gamma_{c+1}R,\end{aligned}$$

where $y_i \in F_i$ and $y_i\pi = y\pi = g$. Hence, (6) is a submodule of $\gamma_c R/\gamma_{c+1}R$.

Let $\alpha_i (1 \leq i \leq c)$ denote the restriction of the homomorphism α to the free factor F_i of F. Then we have a commutative diagram

$$\begin{tikzcd} F_i \arrow[r, "\alpha_i"] \arrow[dr] & \tilde{F}_i \operatorname{wr}_{\mathfrak{N}_c} G \arrow[r, hook] & \tilde{F} \operatorname{wr}_{\mathfrak{N}_c} G, \\ & F_i/\gamma_{c+1}R_i \arrow[u] & \end{tikzcd}$$

where the vertical homomorphism is the Shmel'kin embedding for $F_i/\gamma_{c+1}R_i$.

Consequently, α maps the free factor F_i onto the image of this Shmel'kin embedding. In particular, the subgroup R_i is mapped into $\tilde{F}_i^G \subseteq \tilde{F}^G$ and α_i induces a monomorphism $\mu_i : (R_i)_{ab} \to (F_i^G)_{ab}$ (if $r \in R_i$, then $(rR_i')\mu_i = r\alpha_i(F_i^G)')$, which is just the Magnus embedding for modules stemming from the free presentation (3).

Let $r_i \in R_i$ $(i = 1, \ldots, c)$ and consider the commutator $[r_1, \ldots, r_c] \in [R_1, \ldots, R_c]$. The homomorphism α maps $[r_1, \ldots, r_c]$ into $[\tilde{F}_1^G, \ldots, \tilde{F}_c^G]$. Hence, the isomorphism ψ from Lemma 2 can be applied to $[r_1, \ldots r_c]\alpha$. The result is

$$\begin{aligned}[r_1, \ldots, r_c]\alpha\psi &= [r_1\alpha_1, \ldots, r_c\alpha_c]\psi \\ &= r_1\alpha_1(F_1^G)' \otimes \cdots \otimes r_c\alpha_c(F_c^G)' \\ &= r_1R_1'\mu_1 \otimes \cdots \otimes r_cR_c'\mu_c \\ &= (r_1R_1' \otimes \cdots \otimes r_cR_c')\mu_1 \otimes \cdots \otimes \mu_c.\end{aligned}$$

It follows, that the image of $[R_1, \ldots, R_c]$ under the homomorphism α is isomorphic to the image of the canonical homomorphism

$$\mu_1 \otimes \cdots \otimes \mu_c : (R1)_{ab} \otimes \cdots \otimes (R_c)_{ab} \to (F_1^G)_{ab} \otimes \cdots \otimes (F_c^G)_{ab},$$

which is the tensor product of the Magnus embeddings μ_i $(i = 1, \ldots, c)$. Hence, we have proved the following.

LEMMA 3. *The mapping*

$$[r_1, \ldots, r_c]\gamma_{c+1}R \to r_1R_1' \otimes \cdots \otimes r_cR_c'$$

($r_i \in R_i$, $i = 1, \ldots, c$) *extends to an isomorphism*

$$\varphi : [R_1, \ldots, R_c]\gamma_{c+1}R/\gamma_{c+1}R \xrightarrow{\sim} (R_1)_{ab} \otimes \cdots \otimes (R_c)_{ab}. \quad \square$$

Moreover, we have seen that there is a commutative square

$$\begin{array}{ccc} [R_1, \ldots, R_c]\gamma_{c+1}R/\gamma_{c+1}R & \longrightarrow & [\tilde{F}_1^G, \ldots, \tilde{F}_c^G] \\ \varphi \downarrow \wr & & \downarrow \wr \psi \\ (R_1)_{ab} \otimes \cdots \otimes (R_c)_{ab} & \xrightarrow{\mu_1 \otimes \cdots \otimes \mu_c} & (F_1^G)_{ab} \otimes \cdots \otimes (F_c^G)_{ab} \end{array} \tag{7}$$

where the upper horizontal homomorphism is induced by the restriction of α to $[R_1, \ldots, R_c]$.

6. Proof of Theorem 1

Now we proceed to the proof of Theorem 1. The homomorphism β (see Section 3) maps the subgroup $[R_1, \ldots, R_c] \subseteq F$ into

$$[\bar{F}_1^G, \ldots, \bar{F}_c^G][\gamma_c \bar{F}^G, G]/[\gamma_c \bar{F}^G, G] \subseteq \gamma_c \bar{F}^G/[\gamma_c \bar{F}^G, G].$$

Clearly, $\gamma_{c+1}R$ and $[R_1, \ldots, R_c, F]$ are in the kernel of β. Hence, the restriction of β to $[R_1, \ldots, R_c]$ induces a homomorphism

$$\begin{aligned}\beta^* : &[R_1, \ldots, R_c]\gamma_{c+1}R/[R_1, \ldots, R_c, F]\gamma_{c+1}R\\ &\to [\bar{F}_1^G, \ldots, \bar{F}_c^G][\gamma_c \bar{F}^G, G]/[\gamma_c \bar{F}^G, G].\end{aligned}$$

The kernel of this homomorphism is

$$\ker \beta^* = ([R_1, \ldots, R_c] \cap N)\gamma_{c+1}R/[R_1, \ldots, R_c, F]\gamma_{c+1}R.$$

Consequently, the theorem will be proved once we show that $\ker \beta^*$ is isomorphic to $H_{2c}G$.

There are canonical isomorphisms

$$\begin{aligned}&[R_1, \ldots, R_c]\gamma_{c+1}R/[R_1, \ldots, R_c, F]\gamma_{c+1}R\\ &\quad \cong ([R_1, \ldots, R_c]\gamma_{c+1}R/\gamma_{c+1}R) \otimes_G \mathbb{Z},\\ &\quad \gamma_c \bar{F}^G/[\gamma_c \bar{F}^G, G] \cong \gamma_c \bar{F}^G \otimes_G \mathbb{Z},\end{aligned} \tag{8}$$

where $\mathbb{Z}$ (the ring of integers) is considered as a trivial G-module. Since $[\bar{F}_1^G, \ldots, \bar{F}_c^G]$ is a direct summand of $\gamma_c \bar{F}^G$, the latter implies that there is an isomorphism

$$[\bar{F}_1^G, \ldots, \bar{F}_c^G][\gamma_c \bar{F}^G, G]/[\gamma_c \bar{F}^G, G] \cong [\bar{F}_1^G, \ldots, \bar{F}_c^G] \otimes_G \mathbb{Z}. \tag{9}$$

Using (8) and (9), β^* can be rewritten as

$$\beta^* : ([R_1, \ldots, R_c]\gamma_{c+1}R/\gamma_{c+1}R) \otimes_G \mathbb{Z} \to [\bar{F}_1^G, \ldots, \bar{F}_c^G] \otimes_G \mathbb{Z}$$

and this is, in view of the commutative square (7), equivalent to the canonical homomorphism

$$(\mu_1 \otimes \cdots \otimes \mu_c) \otimes 1 : ((R_1)_{ab} \otimes \cdots \otimes (R_c)_{ab}) \otimes_G \mathbb{Z} \\ \to ((F_1^G)_{ab} \otimes \cdots \otimes (F_c^G)_{ab}) \otimes_G \mathbb{Z}.$$

It remains to show that the kernel of this homomorphism is $H_{2c}G$. In fact, this is an easy consequence of the results in K. W. Gruenberg's paper [17]. However, for completeness we give a formal argument.

The commutative triangle

$$\begin{tikzcd}[column sep=small]
((R_1)_{ab} \otimes \cdots \otimes (R_c)_{ab}) \otimes_G Z \arrow[rr] \arrow[dr, "(\mu_1 \otimes 1 \otimes \cdots \otimes 1) \otimes 1"'] & & ((F_1^G)_{ab} \otimes (F_2^G)_{ab} \otimes \cdots \otimes (F_c^G)_{ab}) \otimes_G Z \\
& ((F_1^G)_{ab} \otimes (R_2)_{ab} \otimes \cdots \otimes (R_c)_{ab}) \otimes_G Z \arrow[ur, "(1 \otimes \mu_2 \otimes \cdots \otimes \mu_c) \otimes 1"'] &
\end{tikzcd}$$

defines a decomposition of $(\mu_1 \otimes \cdots \otimes \mu_c) \otimes 1$. We claim that $(1 \otimes \mu_2 \otimes \cdots \otimes \mu_c) \otimes 1$ is injective. Indeed, put

$$P = (F_2^G)_{ab} \otimes \cdots \otimes (F_c^G)_{ab} / ((R_2)_{ab} \otimes \cdots \otimes (R_c)_{ab}) \mu_2 \otimes \cdots \otimes \mu_c$$

and consider the short exact sequence

$$0 \to (F_1^G)_{ab} \otimes (R_2)_{ab} \otimes \cdots \otimes (R_c)_{ab} \\ \to (F_1^G)_{ab} \otimes (F_2^G)_{ab} \otimes \cdots \otimes (F_c^G)_{ab} \to (F_1^G)_{ab} \otimes P \to 0.$$

Since $(F_1^G)_{ab} \otimes P$ is a free G-module, this sequence remains exact after tensoring (over G) with $\mathbb{Z}$. Hence, $(1 \otimes \mu_2 \otimes \cdots \otimes \mu_c) \otimes 1$ is injective and we can state that

$$\ker (\mu_1 \otimes \cdots \otimes \mu_c) \otimes 1 = \ker (\mu_1 \otimes 1 \otimes \cdots \otimes 1) \otimes 1.$$

Now consider the short exact sequence

$$0 \to (R_1)_{ab} \otimes (R_2)_{ab} \otimes \cdots \otimes (R_c)_{ab} \\ \to (F_1^G)_{ab} \otimes (R_2)_{ab} \otimes \cdots \otimes (R_c)_{ab} \\ \to IG \otimes (R_2)_{ab} \otimes \cdots \otimes (R_c)_{ab} \to 0, \tag{10}$$

obtained by tensoring the relation sequence for G associated with (3) ($i = 1$) with $(R_2)_{ab} \otimes \cdots \otimes (R_c)_{ab}$, and note that $(F_1^G)_{ab} \otimes (R_2)_{ab} \otimes \cdots \otimes (R_c)_{ab}$ is a free

G-module. By tensoring (10) with $\mathbb{Z}$ (over G) we get

$$\ker(\mu_1 \otimes 1 \otimes \cdots \otimes 1) \otimes 1 = \mathrm{Tor}_1^G(IG \otimes (R_2)_{ab} \otimes \cdots \otimes (R_c)_{ab}, \mathbb{Z}).$$

By using the well-known reduction identities

$$\mathrm{Tor}_k^G(IG \otimes B, \mathbb{Z}) = \mathrm{Tor}_{k+1}^G(B, \mathbb{Z})$$
$$\mathrm{Tor}_k^G((R_i)_{ab} \otimes B, \mathbb{Z}) = \mathrm{Tor}_{k+2}^G(B, \mathbb{Z})$$

(dimension shifting) one gets

$$\mathrm{Tor}_1^G(IG \otimes (R_2)_{ab} \otimes \cdots \otimes (R_c)_{ab}, \mathbb{Z}) = \mathrm{Tor}_{2c}^G(\mathbb{Z}, \mathbb{Z}) = H_{2c}G$$

and this completes the proof of the theorem. □

We conclude this section with the following

Remark. A simple analysis of our proof shows that Theorem 1 can be generalized as follows. Let $w(x_1, \ldots, x_c)$ be a basic commutatur of weight c with independent entries $x_1, \ldots, x_c$. Then there is an isomorphism

$$H_{2c}G = (w(R_1, \ldots, R_c) \cap N)\gamma_{c+1}R/[w(R_1, \ldots, R_c), F]\gamma_{c+1}R,$$

where

$$w(R_1, \ldots, R_c) = gp\{w(g_1, \ldots, g_c), g_i \in R_i \ (i = 1, \ldots, c)\}.$$

7. Theorem 2

Let $Z(F/\gamma_{c+1}R)$ denote the center of the quotient $F/\gamma_{c+1}R$. It is well-known that $Z(F/\gamma_{c+1}R)$ coincides with $(\gamma_c R/\gamma_{c+1}R)^G$, the group of fixed points on $\gamma_c R/\gamma_{c+1}R$, and that this group is non-trivial if and only if G is a finite group (see [1] for the case $c = 1$, [7] for the general case). Assume now that G is finite and let τ denote the trace map, i.e. the G-module homomorphism defined by

$$m\tau = m\Big(\sum_{g \in G} g\Big), \qquad m \in \gamma_c R/\gamma_{c+1}R.$$

Note that $(\gamma_c R/\gamma_{c+1}R)\tau \subseteq (\gamma_c R/\gamma_{c+1}R)^G$.

THEOREM 2. *Let G be a finite group. Then there is an isomorphism*

$$H_{2c-1}G \cong (Z(F/\gamma_{c+1}R) \cap [R_1, \ldots, R_c]\gamma_{c+1}R/\gamma_{c+1}R)/ \\ ([R_1, \ldots, R_c]\gamma_{c+1}R/\gamma_{c+1}R)\tau.$$

Proof. In view of

$$Z(F/\gamma_{c+1}R) \cap [R_1, \ldots, R_c]\gamma_{c+1}R/\gamma_{c+1}R = ([R_1, \ldots, R_c]\gamma_{c+1}R/\gamma_{c+1}R)^G$$

and Lemma 3, the group on the right hand side of the formula in Theorem 2 is isomorphic to

$$((R_1)_{ab} \otimes \cdots \otimes (R_c)_{ab})^G/((R_1)_{ab} \otimes \cdots \otimes (R_c)_{ab})\tau.$$

This quotient is, by definition, the Tate cohomology group $\hat{H}^0(G, (R_1)_{ab} \otimes \cdots \otimes (R_c)_{ab})$. By using repeatedly the isomorphism

$$\hat{H}^k(G, (R_i)_{ab} \otimes B) \cong \hat{H}^{k-2}(G, B)$$

(dimension shifting), we get

$$\hat{H}^0(G, (R_1)_{ab} \otimes \cdots \otimes (R_c)_{ab}) \cong \hat{H}^{-2c}(G, \mathbb{Z}) = H_{2c-1}G$$

and this completes the proof of the theorem. □

Added in proof. Alternative generalizations of Hopf's formula have been obtained independently in recent papers by A. Rodicio [18] and R. Brown and G. Ellis [19]. Finally, I would like to thank K. W. Gruenberg and the referee for helpful comments on this paper.

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Akademie der Wissenschaften der DDR
Karl-Weierstraß-Institut für Mathematik
Mohrenstr. 39
Berlin
DDR - 1086
German Democratic Republic

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