Zeitschrift: Commentarii Mathematici Helvetici

Herausgeber: Schweizerische Mathematische Gesellschaft

Band: 64 (1989)

Artikel: Rigidity of convex domains in manifolds with nonnegative Ricci and

sectional curvature.

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DOI: https://doi.org/10.5169/seals-48939

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Rigidity of convex domains in manifolds with nonnegative Ricci and sectional curvature

VIKTOR SCHROEDER and MARTIN STRAKE

1. Introduction

This paper is motivated by rigidity results of Gromov [BGS, §5] which were generalized in [SZ]. One of these results is the following rigidity theorem for convex domains in manifolds of nonnegative sectional curvature $K \ge 0$ [SZ, Theorem 5]:

Let X be a complete manifold with $K \ge 0$, B a compact strictly convex region in X and U a neighborhood of ∂B . If the metric in $U \setminus B$ is locally symmetric of rank ≥ 3 , then the metric is also locally symmetric in B.

A similar rigidity result cannot be expected in the category of manifolds with nonnegative Ricci-curvature $Ric \ge 0$ since a symmetric space of non-compact type has positive Ricci-curvature and a small local modification of the metric is possible within this category.

If however the metric in $U \setminus B$ is assumed to be flat, then the above result implies that the metric is flat in B and one can generalize this to the case $Ric \ge 0$:

THEOREM 1. Let M be a compact Riemannian manifold with convex boundary and nonnegative Ricci-curvature. Assume that the sectional curvature is identically zero in some neighborhood U of ∂M and that one of the following conditions holds:

- a) ∂M is simply connected
- b) dim ∂M is even and ∂M is strictly convex in some point $p \in \partial M$ Then M is flat.

We remark here that the proof of Theorem 1 is quite different from the proofs in [SZ] where the rigidity part of the Rauch comparison theorems is used in an essential way. This tool can obviously not work for $Ric \ge 0$. Instead we use more global arguments. An easy argument shows that M can be isometrically embedded into a manifold N such that $N \setminus M$ is the complement of a compact set in euclidean space. The Bishop-Gromov inequality then implies that N (and hence also M) is flat. If one uses instead the solution of the positive mass

conjecture, then the argument shows that Theorem 1 holds also for nonnegative scalar curvature.

Thus the condition that the metric is flat in a whole neighborhood of ∂M is very strong. One might expect that, for $Ric \ge 0$, it suffices to assume that the sectional curvature vanishes only on the boundary. We can prove this in the special case of a metric ball:

THEOREM 2. Let M be a Riemannian manifold of dimension $n \ge 3$ and let $B = B_r(p_0)$ a convex metric ball embedded by the exponential map \exp_{p_0} with boundary $H = \partial B$. Assume that the Ricci-curvature is nonnegative on B and that

- a) $K(\sigma) = 0$ for all 2-planes with footpoint on H which are tangent to H, if n is odd.
- b) H is strictly convex and $K(\sigma) = 0$ for all 2-planes with footpoint on H, if n is even.

Then B is flat.

In the proof of this result we use ideas from [GW]. We finally prove the rigidity of a product $M = M_1 \times M_2$ with noncompact factors and $K \ge 0$ under a compact modification of the metric which preserves $K \ge 0$.

THEOREM 3. Let M_1 , M_2 be complete noncompact Riemannian manifolds with sectional curvature $K \ge 0$. Let $\Omega \subset M := M_1 \times M_2$ be the complement of a compact subset. If $\phi: \Omega \to \bar{M}$ is an isometric embedding, where \bar{M} is a complete manifold with $K \ge 0$ and dim $\bar{M} = \dim M$ then ϕ extends in a unique way to an isometry $\bar{\phi}: M \to \bar{M}$.

This result was stated (without proof) by Gromov [BGS, p. 75] but we think that the proof is not at all trivial. Note that one cannot expect such a result for $\text{Ric} \ge 0$: If M_1 , M_2 are noncompact with K > 0, then the products has Ric > 0 and one can deform the metric locally. The examples of [SY] show that Ric > 0 allows even surgery constructions starting from products. However there is a rigidity result for $\text{Ric} \ge 0$ if M contains a line, i.e. splits as $M' \times \mathbb{R}$ by the Cheeger-Gromoll splitting theorem [CG]. Let \bar{M} be a manifold which coincides with M outside of a compact set. It is not difficult to show that also \bar{M} contains a line and splits as $\bar{M}' \times \mathbb{R}$. From this one concludes that M is isometric to \bar{M} .

In section 4 we give an example of a manifold $M = M_1 \times M_2$ with compact factor M_1 and a manifold \bar{M} which is isometric to M outside of compact sets but which is not diffeomorphic to M.

We would like to thank J. Eschenburg, Min-Oo, M. Müter and W. Ziller for helpful discussions.

2. Rigidity for nonnegative Ricci-curvature

The proof of Theorem 1 is based on the following observation:

LEMMA 1. Let M^n be a compact Riemannian manifold with convex boundary and assume that M is flat in some neighborhood U of ∂M . Then there exists an isometric embedding $f: M \to N^n$, where N is a complete open manifold which is flat outside of f(M). If in addition ∂M is simply connected then $N \setminus f(M)$ is isometric to $\mathbb{R}^n \setminus C$, where C is a compact subset of \mathbb{R}^n .

Remark. If the Ricci-curvature is nonnegative on M and M is not flat then N has only one end. This is easily seen by the splitting theorem of Cheeger-Gromoll, comp. [CG].

Proof of Lemma 1. For $\varepsilon > 0$ let $U_{\varepsilon} := \{p \in M \mid \operatorname{dist}(p, \partial M) \leq \varepsilon\}$. Then for ε small enough U_{ε} is a subset of U and can be identified with $\partial M \times [-\varepsilon, 0]$, where (p, t) corresponds to $\exp t\eta_p$ and η_p denotes the outer normal field along ∂M . Consider the universal covering $S \to \partial M$ and the group Γ of decktransformations. Then $U_{\varepsilon} \cong \partial M \times [-\varepsilon, 0]$ is diffeomorphic to $(S \times [-\varepsilon, 0])/\Gamma$, where Γ operates trivially on the second factor. The product $S \times [-\varepsilon, 0]$ carries a flat metric induced from the metric on $U_{\varepsilon} \cong \partial M \times [-\varepsilon, 0]$. As $S \times [-\varepsilon, 0]$ is simply connected, there is an isometric immersion $D_0: S \times [-\varepsilon, 0] \to \mathbb{R}^n$ (developing map, comp. [Th]). Define $\xi \stackrel{\text{def}}{=} (D_0)_* \partial / \partial t$, then ξ is the outer unit normal vector field along D_0 . As the immersion D_0 is convex, we can extend D_0 to an immersion $D: S \times [-\varepsilon, \infty)$ by

$$D(p, t) = D_0(p, 0) + t\xi(p, 0)$$

and the pull back metric on $S \times [-\varepsilon, \infty)$ is flat and agrees on $S \times [-\varepsilon, 0]$ with the given metric. Clearly Γ operates isometrically and U_{ε} can be considered as a subset of $N_0 := (S \times [-\varepsilon, \infty)/\Gamma)$. Under this identification M is a subset of $N := (M \setminus U_{\varepsilon}) \cup N_0$.

Now assume that ∂M is simply connected. Then $U_{\varepsilon} \cong \partial M \times [-\varepsilon, 0]$ is also simply connected and we can consider the isometric immersion $D_0: \partial M \times [-\varepsilon, 0] \to \mathbb{R}^n$. As ∂M is compact and convex and since dim $\partial M > 1$, D_0 is an embedding by the theorem of Sackstedter [S]. If $B \subset \mathbb{R}^n$ denotes the bounded components of $\mathbb{R}^n \setminus D_0(\partial M \times \{-\varepsilon\})$ then we can define $N := M \cup_{D_0} (\mathbb{R}^n \setminus B)$. \square

Proof of Theorem 1. a) By Lemma 1 we may assume that M is a subset of the manifold N, where $N \setminus M$ is isometric to $\mathbb{R}^n \setminus C$. As C is compact the limit

 $\lim\inf_{t\to\infty}v_p(t)/t^n$ is equal to $\liminf_{t\to\infty}v_0(t)/t^n$, where $v_p(t)$ resp. $v_0(t)$ denotes the volume of a ball of radius t with center p in N resp. center 0 in the euclidean space \mathbb{R}^n . Now the condition $\mathrm{Ric}\geq 0$ on N implies that N is isometric to \mathbb{R}^n by the rigidity part of the Bishop-Gromov inequality [G].

b) As ∂M is strictly convex in some point $p \in \partial M$ (i.e. the Weingarten-map with respect to the outer unit normal is strictly positive definte at p) and as M is flat in some neighborhood of ∂M we may assume without loss of generality that ∂M is strictly convex everywhere. (This can be shown by iterating a standard convolution process for the distance function ρ of the boundary ∂M . This method leads to a strictly convex C^{∞} -function $\bar{\rho}$ which is arbitrarily close to ρ , comp. [ES]. Note that by the remark above, we can assume that ∂M has only one component.) Consider the orientation covering $\bar{M} \to M$. Then \bar{M} satisfies the same conditions as M and in particular the intrinsic curvature of $\partial \bar{M}$ is strictly positive by the Gauss-equation. Furthermore $\partial \bar{M}$ is orientable and evendimensional. Therefore $\partial \bar{M}$ is simply connected by the Lemma of Synge [CE]. Thus a) implies that \bar{M} (and therefore M) is flat. \square

Proof of Theorem. 2. The proof is subdivided into two steps. Let L resp. L_0 be the Weingarten map of H resp. $S_r(0)$ with respect to the outer unit normal vector, where $S_r(0)$ denotes the standard euclidean sphere of radius r.

(i) First we will show that $Ric \ge 0$ on B implies

$$A \stackrel{\text{def}}{=} \int_{H} \det L \, dV \le \int_{S_{r}(0)} \det L_{0} \, dV_{0} = \text{vol}(S_{1}(0)) \tag{1}$$

Furthermore $A = \text{vol}(S_1(0))$ is only possible if $B_r(p)$ is isometric to $B_r(0_p) \subset T_p M = \mathbb{R}^n$.

As Ric ≥ 0 on B the Gromov-Bishop inequality [G] gives (compare B with the euclidean ball $B_r(0)$):

$$\operatorname{vol}(H) \le \operatorname{vol}(S_r(0)) \tag{2}$$

The equality holds if and only if B is isometric to $B_r(0)$. A similar comparison argument shows:

trace
$$(L) \leq \operatorname{trace}(L_0)$$

The arithmetic-geometric mean inequality gives

$$0 \le \det(L)^{1/m} \le \frac{1}{m} \operatorname{trace}(L) \le \frac{1}{m} \operatorname{trace}(L_0) = \det(L_0)^{1/m} = r^{-1}$$

and therefore

$$\int_{H} \det L \, dV \le \int_{H} r^{-m} \, dV = r \, \text{vol} \, (S_r(0)) = \text{vol} \, (S_1(0))$$

where equality holds iff B is isometric to an euclidean ball, compare (2).

(ii) Now we want to show that condition a) resp. b) of Theorem 3 implies:

$$\int_{H} \det L \, dV = \operatorname{vol} \left(S_1(0) \right)$$

Then by (1) we have $K \equiv 0$ on B.

 α) Assume that n is odd. As $K(\sigma) = 0$ for all 2-planes σ tangent to H the Gauss equation implies det L = G, where G is the Gauss-Bonnet integrand of the even dimensional orientable hypersurface H. Therefore:

$$\int_{H} \det L \, dV = \int_{H} G \, dV = \frac{\chi(H)}{2} \operatorname{vol}(S_{1}(0)) = \operatorname{vol}(S_{1}(0))$$

 β) Assume that n is even. As H is simply connected (dim $H \ge 2$) the curvature condition $K(\sigma) = 0$ for all 2-planes with footpoint in H implies the existence of a parellel orthonormal trivialisation E_1, \ldots, E_n of the bundle $TM|_H$. Let N denote the outer unit normal field of H. Define a Gauss-map $\phi: H \to S_1(0)$ by

$$\phi(p) = \sum_{k=1}^{n} \langle N(p), E_k \rangle e_k$$

where e_1, \ldots, e_n denotes the standard orthonormal basis of \mathbb{R}^n . Then

$$\phi_* x = \sum_{k=1}^n \langle Lx, E_k \rangle e_k$$

and

$$\phi^* dV_0 = (\det L) dV$$

Therefore

$$\int_{H} \det L \, dV = \deg (\phi) \int_{S_{1}(0)} dV_{0} = \deg (\phi) \, \text{vol} \, (S_{1}(0))$$

As L is positive definite the differential ϕ_* is nonsingular and therefore ϕ is a local diffeomorphism and hence a covering map. $S_1(0)$ is simply connected hence

 ϕ is an (orientation-preserving) diffeomorphism and therefore $\deg(\phi) = +1$, which completes the proof. \Box

Remark. 1) If n is even, $n \ge 3$ and H is convex (but not necessarily strictly convex) then $\deg(\phi) = 0$ implies that the tangent bundle $TH \cong TS_1(0)$ is trivial and therefore $\dim H = n - 1 \in \{3, 7\}$ (comp. [GW, Lemma 9]). Hence Theorem 2 part b) remains true if H is only convex and $n \ge 3$, $n \ne 4$, 8.

2) In the case that $\dim M = 3$ and that the sectional curvature K is nonnegative, one can prove a version for arbitrary convex sets (comp. [SS] Theorem 2):

Let M be a compact Riemannian manifold of dimension 3 and with nonnegative sectional curvature. Assume that the boundary ∂M is strictly convex and that $K(\sigma) = 0$ for all 2-planes σ which are tangent to ∂M . Then M is flat.

3. Rigidity of products

For the proof of Theorem 3 we recall some facts from the structure theory of a complete open manifold M with nonnegative sectional curvature (see [CG], [CE] ch. 8):

If C is a compact totally convex subset in M with nonempty boundary ∂C , then also the sets

$$C^{t} = \{ p \in C \mid d(p, \partial C) \ge t \}$$

are totally convex. Let $C^{\max} = C^a$ where $a = \sup\{t \ge 0 \mid C^t \ne \emptyset\}$. Then $\dim C^{\max} < \dim C$. By the basic construction of [CG] there exists an exhaustion of M by compact totally convex subsets C_t , $t \ge 0$ such that $C_t = C_{t+s}^s$ and $C_0 = C_t^{\max}$ for all t, s > 0. In particular $\dim C_0 < \dim C_t = \dim M$ for all t > 0. If $C(1) \stackrel{\text{def}}{=} C_0$ has nontrivial boundary, then let $C(2) \stackrel{\text{def}}{=} C(1)^{\max}$. We obtain a sequence $C_0 = C(1) \supset \cdots \supset C(k) = \Sigma$, where k is the smallest integer such that C(k) is wouthout boundary. $\Sigma = C(k)$ is called a soul of M.

In the theorem we investigate a product $M = M_1 \times M_2$. For the factors M_i , i = 1, 2, we have the exhaustions $C_{i,i}$ and the chain $C_i(1) \supset \cdots \supset C_i(k_i) = \Sigma_i$, where Σ_i is the soul of M_i .

We also recall the following construction of Sharafudtinov [Sh], see also [Y]: Let C be a compact totally convex subset in M with nonempty boundary ∂C . Then there exists a strong deformation retract $\psi_t: C \to C'$ which is distance nonincreasing. Thus there exists also a contraction map $\psi_t: C_t \to C(1)$ and finally a contraction $\psi: C \to \Sigma$.

For the proof of the theorem the following notation is useful: Let $D \subset M$ and $\bar{D} \subset \bar{M}$ be subsets. We say that $\phi(D)$ and \bar{D} coincide outside of a compact set and we write $\phi(D) \stackrel{c}{=} \bar{D}$, if there are compact sets $K \subset M$ and $\bar{K} \subset \bar{M}$ such that $\phi|_{D \setminus K} : D \setminus K \to \bar{M}$ is an isometry from $D \setminus K$ onto $\bar{D} \setminus \bar{K}$. Note that we can use this notation even when D is not completely contained in Ω .

We prove first that $\phi(M) \stackrel{c}{=} \bar{M}$, i.e. that $Q \stackrel{\text{def}}{=} \bar{M} \setminus \phi(\Omega)$ is compact. Therefore we can assume that $\Omega = M \setminus C_a$ for a suitable a > 0. Since C_a is totally convex and dim $M = \dim \bar{M}$ also Q is totally convex because every geodesic which enters $\phi(\Omega)$ cannot leave $\phi(\Omega)$. If Q is noncompact then there exists a sequence $q_i \in Q$ with $d(q_i, \partial Q) \to \infty$. Furthermore there are $p_i \in \phi(\Omega)$ with $d(p_i, \partial Q) \to \infty$. Then a sequence of minimizing geodesics from q_i to p_i has an accumulation line which intersects ∂Q . By Toponogov's splitting theorem \bar{M} splits as $\bar{M}' \times \mathbb{R}$. We can assume that $(x, 0) \in \partial Q$ for a point $x \in \bar{M}'$ and $(x, t) \in \phi(\Omega)$ for t > 0 and $(x, t) \in Q$ for $t \leq 0$.

Let y be a point in \bar{M}' . For $t_0 > 0$ large enough, $(y, t_0) \in \phi(\Omega)$ and $(y, -t_0) \in Q$. Thus the line $\{y\} \times \mathbb{R}$ intersects ∂Q . Since ∂Q is compact, the distance d(x, y) is universally bounded and \bar{M}' is compact. But this is impossible since M is a product of two noncompact factors. The contradiction shows that Q is compact and $\phi(M) \stackrel{c}{=} \bar{M}$.

For the rest of the proof we will assume (without loss of generality) that Ω is the complement of $C_{1,a} \times C_{2,a}$ in $M = M_1 \times M_2$ for a suitable positive constant a.

We consider the cylinder $Z := C_{1,a} \times M_2$ in M. Let $\bar{Z} \stackrel{\text{def}}{=} \bar{M} \setminus \phi(M \setminus Z)$. We claim that \bar{Z} is a totally convex subset of \bar{M} . Note that the complement of Z in M is isometric to the complement of \bar{Z} in \bar{M} . Since Z is totally convex, every geodesic leaving Z cannot return. Thus the same is true for \bar{Z} and hence \bar{Z} is also totally convex.

We claim that $\bar{Z}^{\max} = \bar{Z}^a$ and $\bar{Z}^{\max} \stackrel{c}{=} \phi(Z^a) = \phi(C_1(1) \times M_2)$. Since $\bar{M} \stackrel{c}{=} \phi(M)$ it is clear that $\bar{Z} \stackrel{c}{=} \phi(Z)$ and $\bar{Z}^t \stackrel{c}{=} \phi(Z^t)$. It follows that dim $\bar{Z}^a < \dim \bar{Z}$ and hence $\bar{Z}^{\max} = \bar{Z}^a$ and $\bar{Z}^a \stackrel{c}{=} \phi(Z^a)$. Thus we have proved that $\bar{Z}(1) \stackrel{c}{=} \phi(Z(1))$. In the same way we obtain $\bar{Z}(2) \stackrel{c}{=} \phi(Z(2))$ and finally $\bar{Z}(k_1) \stackrel{c}{=} \phi(Z(k_1)) = \phi(\Sigma_1 \times M_2)$. For the proof of Theorem 3 the following result is essential

LEMMA 2. $S \stackrel{\text{def}}{=} \bar{Z}(k_1)$ is complete without boundary and isometric to the product $\Sigma_1 \times M_2$.

Proof of Lemma 2. The proof consists of three steps:

- 1. We show that S is complete without boundary.
- 2. Through every point $x \in S$ there exists a totally geodesic submanifold isometric to M_2 .

- 3. We show that if $M_2(x)$ and $M_2(y)$ are two of these submanifolds of S, then there exists a totally geodesic and isometric immersion $G:[0, r] \times M_2 \rightarrow S$ such that $G(0, M_2) = M_2(x)$ and $G(r, M_2) = M_2(y)$. From this fact we derive the product structure.
- 1. Let us assume to the contrary that $\partial S \neq \emptyset$. Then ∂S lies in a compact set since S coincides with $\phi(\Sigma_1 \times M_2)$ outside of a compact set. For t sufficiently large, the set $S^t = \{p \in S \mid d(p, \partial S) \geq t\}$ is contained in the set where S coincides with the product $\phi(\Sigma_1 \times M_2)$ and we can define the projection $\pi: S^t \to M_2$. Let $\psi: \bar{Z} \to S$ and $\psi_t: S \to S^t$ be Sharafudtinov retractions. It is easy to check that the construction of the maps ψ , ψ_t (compare [Y]) also works in our context where \bar{Z} is not compact. Note that outside of a compact set ψ coincides with the product map $\psi^1 \times id$, where $\psi^1: C_{1,a} \to \Sigma_1$ is a Sharafudtinov retraction in M_1 . Choose $x_1 \in \partial C_{1,a}$ and let $i: M_2 \to \{x_1\} \times M_2$ be an isometric embedding of M_2 into ∂Z . Then $\alpha = \pi \circ \psi_t \circ \psi \circ \phi \circ i$ is a map from M_2 onto a proper subset of M_2 which coincides with the identity outside of a compact set. Such a map is impossible for topological reasons.

It follows that $S = \bar{Z}(k_1)$ is the soul of the cylinder \bar{Z} and $S \stackrel{c}{=} \phi(\Sigma_1 \times M^2)$.

2. We prove that though every point $x \in S$ there exists a totally geodesic submanifold isometric to M_2 .

Consider a point $\phi(x_1, x_2) \in S$, where $x_1 \in \Sigma_1$ and $x_2 \in M_2$, i.e. a point outside of the compact set. Let $\gamma: [0, \infty) \to M_1$ be a unit speed ray with $\gamma(0) = x_1$. It follows from the basic construction in [CG] that $\gamma(t) \in \partial C_{1,t}$ for $t \ge 0$. We consider the geodesic $\bar{\gamma}(s) = \phi(\gamma(s), x_2)$ in \bar{M} . Since $\bar{Z}^{\max} = \bar{Z}^a$ it follows that $d(\phi(x_1, x_2), \partial \bar{Z}) \ge a$. Since $\phi(\gamma(a), x_2) \in \partial \bar{Z}$, this geodesic is minimizing up to $\partial \bar{Z}$ and since the constant a can be choosen arbitrarily large, $\bar{\gamma}$ is a ray in \bar{M} . Let $\bar{c}: \mathbb{R} \to S$ be a geodesic in S with $\bar{c}(0) = \varphi(x_1, x_2)$. Let W(t) be the parallel vectorfield along $\bar{c}(t)$ with $W(0) = \bar{\gamma}$. It follows from [CG] Theorem 1.10 that

$$H(s, t) \stackrel{\text{def}}{=} \exp_{\bar{c}(t)} sW(t) \tag{3}$$

is a totally geodesic isometric immersion of the flat halfplane $[0, \infty) \times \mathbb{R}$ into \overline{M} . Let $c: \mathbb{R} \to M_2$ be a geodesic with $c(0) = x_2$ and let $\overline{c}: \mathbb{R} \to S$ be the geodesic such that $\overline{c}(t) = \phi(x_1, c(t))$ for |t| small, then one checks easily that

$$H(s, t) \stackrel{c}{=} \phi(\gamma(s), c(t)) \tag{4}$$

For b>0 we consider the manifold $\gamma(b)\times M_2\subseteq M$. For b sufficiently large, $\gamma(b)\times M_2$ is completely contained in Ω . Let $Y\stackrel{\text{def}}{=}\phi(\gamma(b)\times M_2)\subseteq \bar{M}$. Note that

 $(-\dot{\gamma}(b), 0)$ defines a globally parallel vectorfield V on Y. By construction we obtain for x_2 outside of a compact subset of M_2 that

$$\exp bV(\phi(\gamma(b), x_2) = \phi(\gamma(0), x_2) \in S$$

We claim that the map $\theta(y) = \exp_y bV(y)$ is a totally geodesic isometric embedding of Y into S. Let therefore $c: \mathbb{R} \to M_2$ be any geodesic of M_2 which does not stay in a compact subset. We obtain the flat halfspace H(s, t) as in (4) which contains the geodesic $t \mapsto \phi(\gamma(b), c(t))$ in Y. It follows that the map θ is an isometry along the geodesic $\phi(\gamma(b), c(t))$. By the structure theory of M_2 it is clear that only a zero-set of geodesics stays in a compact set. Thus θ is an isometry. More generally, it follows from Rauch's comparison theorem that the map

$$D:[0, b] \times Y \rightarrow \bar{M}$$

 $(s, y) \mapsto \exp_{v} sV(y)$

is a totally geodesic isometric embedding. Since $D(b, M_2)$ is contained in S outside of a compact set and S is totally geodesic, it follows that $D(b, M_2) \subseteq S$.

Because $S \stackrel{c}{=} \phi(\Sigma_1 \times M_2)$ there exists a compact set K_2 in M_2 such that S is isometric to $\Sigma_1 \times \Omega_2$ outside of a compact set, where Ω_2 is the complement of K_2 . We just have proved, that every fiber $\{x_1\} \times \Omega_2$ is a subset of a complete totally geodesic submanifold isometric to M_2 . We denote this submanifold with $M_2(x_1)$. Let x be an arbitrary point in S, then consider a ray $c:[0,\infty) \to S$ starting in x. This ray is finally contained in $\Sigma_1 \times \Omega_2$ and since Σ_1 is compact, it is contained in a fiber $\{x_1\} \times \Omega_2$. Thus $x \in M_2(x_1)$ and every point of S is contained in $M_2(x_1)$ for a suitable x_1 .

3. Let $x_1, y_1 \in \Sigma_1$ and $\alpha: [0, r] \to \Sigma_1$ a minimal geodesic between them where $r = d(x_1, y_1)$. We claim: There exists a totally geodesic and isometric embedding $G: [0, r] \times M_2 \to S$ such that $G(0, M_2) = M_2(x_1)$ and $G(r, M_2) = M_2(y_1)$.

Before we prove this claim, we show that this implies S isometric to $\Sigma_1 \times M_2$. First the above claim shows that the manifolds $M_2(x_1)$ define a foliation of S and hence also an integrable distribution. If c is any geodesic in S, then c is contained in the image of an isometric embedding G as above. It follows that the distribution is invariant under parallel translation and hence S is a product by the de Rham splitting theorem. Since $S = \phi(\Sigma_1 \times M_2)$ it is clear that S is isometric to $\Sigma_1 \times M_2$.

To prove the claim, we consider $M_2(x_1) \stackrel{c}{=} \phi(\{x_1\} \times M_2)$, $M_2(y_1) \stackrel{c}{=} \phi(\{y_1\} \times M_2)$ and canonical isometries $\phi_x : M_2 \rightarrow M_2(x_1)$, $\phi_y : M_2 \rightarrow M_2(y_1)$. We first assume that

the distance $r = d(x_1, y_1)$ is small enough, such that for every $z \in M_2$ there exists a unique minimal geodesic from $\phi_x(z)$ to $\phi_y(z)$. Since S is a product outside of a compact set this is possible for small $r \ge 0$. Let $\pi: M_2(x_1) \to M_2(y_1)$ be the projection which maps $\phi_x(z)$ onto $\phi_y(z)$. Let $c: \mathbb{R} \to M_2$ be a geodesic which does not stay in a compact set and let c_x and c_y be the geodesics in $M_2(x_1)$ and $M_2(y_1)$ such that $c_x(t) \stackrel{c}{=} \phi(\{x_1\} \times c(t))$ and $c_y(t) \stackrel{c}{=} \phi(\{y_1\} \times c(t))$. We can assume that $c(0) \in \Omega_2$, i.e. near to 0, $c_x(t)$ and $c_y(t)$ bound a flat totally geodesic strip.

We want to show that $c_x[0, \infty)$ and $c_y[0, \infty)$ bound a totally geodesic flat strip. The set of all t such that $c_x[0, t]$ and $c_y[0, t]$ bound a flat strip isometric to $[0, t] \times [0, r]$ is clearly closed. To prove that the set is open we assume that $c_x[0, t_0]$ and $c_y[0, t_0]$ bound a flat strip and let $t_1 \ge t_0$ with $t_1 - t_0$ small. It follows from Rauch's comparison theorem [CE, pg. 29], that $r_1 \stackrel{\text{def}}{=} d(c_x(t_1), c_y(t_1)) \le r$ and that equality implies that also $c_x[0, t_1]$ and $c_y[0, t_1]$ bound flat strip. Thus it remains to show that $r_1 \ge r$.

Therefore choose a ray $\gamma:[0,\infty)\to M_1$ with $\gamma(0)=x_1\in\Sigma_1$ and consider the ray $\bar{\gamma}(s)=\phi(\gamma(s),\,c(0))$ in \bar{M} . In S we have the piecewise geodesic formed by the three pieces $c_x[0,\,t_1],\;\;\beta[0,\,r_1],\;\;c_y[0,\,t_1],\;\;$ where $\beta:[0,\,r_1]\to S$ is the minimal geodesic from $c_x(t_1)$ to $c_y(t_1)$. Let $w\stackrel{\text{def}}{=}\bar{\gamma}(0)$ and W be the parallel vectorfield along the piecewise geodesic, i.e we parellel translate w from $c_x(0)$ along c_x to $c_x(t_1)$, from there along β to $c_y(t_1)$ and then back along c_y to $c_y(0)$.

As in (3) we thus obtain three totally geodesic immersions

$$F^{1}(s, t) = \exp_{c_{x}(t)} sW(c_{x}(t))$$

$$F^{2}(s, t) = \exp_{\beta(t)} sW(\beta(t))$$

$$F^{3}(s, t) = \exp_{c_{y}(t)} sW(c_{y}(t))$$

where F^1 and F^2 is defined on $[0, \infty) \times [0, t_1]$ and F^3 on $[0, \infty) \times [0, r_1]$.

By (4) $F^1(s, t) \stackrel{c}{=} \phi(\gamma(s), c(t))$ and in the same way $F^3(s, t) \stackrel{c}{=} \phi(\gamma^*(s), c(t))$, where γ^* is the M_1 component of the ray $\phi^{-1} \circ \bar{\gamma}^*$ where $\bar{\gamma}^*(s) = F^3(s, 0)$.

Choose b > 0 sufficiently large such that $F^{i}(b, t) \in \phi(\Omega)$ for all i and t. Then

$$r_1 = d(c_x(t_1), c_y(t_2))$$

$$= d(F^2(0, 0), F^2(0, r_1))$$

$$= d(F^2(b, 0), F^2(b, r_1))$$

$$= d(\phi(\gamma(b), c(t_1)), \phi(\gamma^*(b), c(t_1)))$$

where b is arbitrary. For b sufficiently large

$$d(\phi(\gamma(b), c(t_1)), \phi(\gamma^*(b), c(t_1))) = d(\gamma(b), \gamma^*(b))$$

Now γ and γ^* are rays in M_1 with $\gamma(b)$, $\gamma^*(b) \in \partial C_{1,b}$ for all b. It is then a consequence of the first variation formula, that $d(\gamma(t), \gamma^*(t))$ is monotone increasing. Thus

$$d(\gamma(b), \gamma^*(b)) \ge d(\gamma(0), \gamma^*(0)) = r$$

It follows that $c_x[0, \infty)$ and $c_y[0, \infty)$ bound a flat strip and with the same argument $c_x(\mathbb{R})$ and $c_y(\mathbb{R})$ bound a flat strip. Since the geodesics which leave every compact set are dense, this argument shows that $d(\pi(z), z) = r$ for all $z \in M_2(x_1)$. In particular $M_2(x_1)$ and $M_2(y_1)$ have no common points. Since by assumption for every point $z \in M_2(x_1)$ there is a unique minimal geodesic to the corresponding point in $M_2(y_1)$, there exists a unit vectorfield W on $M_2(x_1)$ such that $\pi(z) = \exp_z rW(z)$. The flat strip argument from above shows that along every geodesic \bar{c} in $M_2(x_1)$ which does not stay in a compact subset W is a parallel normal vectorfield. It follows from the denseness of these geodesics that W is a parallel normal unit vectorfield.

Since $\pi(z) = \exp_z rW(z)$ is an isometry, it follows from Rauch's theorem that the map

$$[0, r] \times M_2(x_1) \rightarrow S, \qquad (s, z) \mapsto \exp_z sW(z)$$

is a totally geodesic isometric immersion. Since it is an embedding outside of a compact set one checks easily that it is an embedding.

We have assumed that r is sufficiently small. In the general case let $x_1, y_1 \in \Sigma_1$ be arbitrary and α a minimal geodesic joining them. Let $\bar{\alpha}$ be the minimal geodesic $\bar{\alpha}(s) = \phi(\alpha(s), x_2)$ between $\phi(x_1, x_2)$ and $\phi(y_1, x_2)$ where $x_2 \in \Omega_2$. The above argument shows that $\bar{\alpha}(0)$ extends to a globally parallel vectorfield on $M_2(x_1)$. One checks easily that

$$(s, z) \mapsto \exp_z sW(z)$$

is an isometric embedding also in this case. Thus we have proved the lemma. \Box

We are now able to complete the proof of Theorem 3. Let $\bar{c}: \mathbb{R} \to \bar{M}$ be any geodesic with $\bar{c}(0) \in \bar{M} \setminus \bar{Z}$. We claim that there exists a totally geodesic isometric immersion $G: \mathbb{R} \times M_2 \to \bar{M}$ such that \bar{c} is contained in the image of G.

Since $\bar{c}(0) \in \phi(\Omega)$ there exists a point $x_1 \in M_1$ such that $\bar{c}(0) \in Y \stackrel{\text{def}}{=} \phi(\{x_1\} \times M_2)$. We can assume that $\dot{c}(0)$ is not tangent to Y. Let w' be the normal component of \dot{c} and $w \stackrel{\text{def}}{=} w'/||w'||$. Then w extends to a globally parallel unit

normal vectorfield on Y. We consider the map

$$G: \mathbb{R} \times Y \to \overline{M}$$

 $G(s, y) \stackrel{\text{def}}{=} \exp_y sW(y)$

By Rauchs theorem, the map $G_s = G(s, .)$ from Y to \bar{M} is distance nonincreasing for small $s \ge 0$ and the rigidity part of this theorem states that if G_s is isometric for $s \ge 0$, then $G|_{\{0,s\} \times Y}$ is an isometric immersion.

Thus we have to show that G_s is an isometry. Let therefore $i: M_2 \to \{x_1\} \times M_2$ the embedding, $\pi: S \to M_2$ the distance nonincreasing projection onto the M_2 -factor of $S \cong \Sigma_1 \times M_2$, let $\psi: \bar{Z} \to S$ be the Sharafudtinov-retraction as in the proof of Lemma 3.

We can assume that $G_s(Y) \subset \bar{Z}$ since G_s is clearly an isometry as long as the image lies in $\bar{M} \setminus \bar{Z}$. Then we have the distance nonincreasing map $\pi \circ \psi \circ G_s \circ \phi \circ i : M_2 \to M_2$ which is the identity outside of a compact set. Such a map has to be an isometry (compare Lemma 1, 2 in [Sh]). It follows that G_s is an isometry.

Since the set of geodesics which leave \bar{Z} is dense, one checks easily that through every point of \bar{M} there is a totally geodesic submanifold isometric to M_2 and that the distribution defined by the tangent spaces of these manifolds is invariant under parallel translation (compare the proof of the splitting $S = \Sigma_1 \times M_2$ in the proof of Lemma 2). It follows from the de Rham decomposition that \bar{M} splits a factor M_2 and since $\bar{M} \stackrel{c}{=} \phi(M_1 \times M_2)$ it is clear that \bar{M} is isometric to $M_1 \times M_2$. Obviously ϕ extends in a unique way to an isometry $\bar{\phi}: M \to \bar{M}$. \square

4. Flexibility of products with nonnegative curvature

Let $M = M_1 \times M_2$ be an open product manifold with sectional curvature $K \ge 0$ where the factor M_1 is compact. We ask how flexible is this product with respect to modifications of the metric within compact sets which preserve $K \ge 0$.

If M_2 has K > 0 (or at least K > 0 at one point), then one can deforme the metric on M_2 in a compact set. In this case the soul of M is isometric to $M_1 \times \{p\}$ and the factor M_1 survives in the new metric.

Consider now a manifold M_2 which is diffeomorphic to \mathbb{R}^{k+1} and $M_2 \setminus C_2$ is isometric to $(S^k, g_E) \times [0, \infty)$ for a compact subset C_2 of M_2 , where g_E is the standard metric on the sphere. It is easy to construct rotational symmetric metrics

of this type. Choose $M_1 = (S^k, g_E)$ then $M = M_1 \times M_2$ is isometric to $S^k \times S^k \times [0, \infty)$ outside of a compact set C where C is isometric to $S^k \times C_2$. Note that we can glue $S^k \times C_2$ in different ways onto the boundary of $S^k \times S^k \times [0, \infty)$ and thus one cannot see from the structure of $M \setminus C$ which S^k factor survives in a manifold M which is isometric to M outside of a compact set.

One can even not see the topological structure of the manifold by looking only to the complement of a compact set. Consider therefore $M_2^* = (S^3, g_1) \times (\mathbb{R}^2, g_2)/S^1$, where we choose some left-invariant metric g_1 on S^3 and a rotational symmetric metric g_2 on \mathbb{R}^2 . S^1 operates diagonally on the product, where it rotates the Hopf-circles on S^3 and acts by rotations on (\mathbb{R}^2, g_2) .

We choose g_2 such that (\mathbb{R}^2, g_2) is isometric to $S_a^1 \times [0, \infty)$, outside of a compact set, where S_a^1 is a circle of radius a. Then, outside of a compact set, M_2^* is isometric to $(S^3, g_3) \times [0, \infty)$, where g_3 is also a left-invariant metric on S^3 . If we choose g_1 suitable then M_2^* is isometric to $(S^3, g_E) \times [0, \infty)$ outside of a compact set. Let $M_1 = (S^3, g_E)$. Then the product $M = M_1 \times M_2$ (for k = 3) is isometric to $\bar{M} = M_1 \times M_2^*$ outside of compact sets, but M and \bar{M} have different topology. In particular their souls are not isometric, sos!

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Received August 3, 1987