Comment. Math. Helvetici 64 (1989) 173–186 0010-2571/89/020173-14$1.50 + 0.20/0

Rigidity of convex domains in manifolds with nonnegative Ricci and sectional curvature

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1. Introduction

This paper is motivated by rigidity results of Gromov [BGS, §5] which were generalized in [SZ]. One of these results is the following rigidity theorem for convex domains in manifolds of nonnegative sectional curvature $K \geq 0$ [SZ, Theorem 5]:

Let X be a complete manifold with $K \geq 0$, B a compact strictly convex region in X and U a neighborhood of ∂B. If the metric in $U \setminus B$ is locally symmetric of rank ≥ 3, then the metric is also locally symmetric in B.

A similar rigidity result cannot be expected in the category of manifolds with nonnegative Ricci-curvature $\mathrm{Ric} \geq 0$ since a symmetric space of non-compact type has positive Ricci-curvature and a small local modification of the metric is possible within this category.

If however the metric in $U \setminus B$ is assumed to be flat, then the above result implies that the metric is flat in B and one can generalize this to the case $\mathrm{Ric} \geq 0$:

THEOREM 1. *Let M be a compact Riemannian manifold with convex boundary and nonnegative Ricci-curvature. Assume that the sectional curvature is identically zero in some neighborhood U of ∂M and that one of the following conditions holds:*

a) *∂M is simply connected*

b) dim *∂M is even and ∂M is strictly convex in some point $p \in \partial M$*

Then M is flat.

We remark here that the proof of Theorem 1 is quite different from the proofs in [SZ] where the rigidity part of the Rauch comparison theorems is used in an essential way. This tool can obviously not work for $\mathrm{Ric} \geq 0$. Instead we use more global arguments. An easy argument shows that M can be isometrically embedded into a manifold N such that $N \setminus M$ is the complement of a compact set in euclidean space. The Bishop–Gromov inequality then implies that N (and hence also M) is flat. If one uses instead the solution of the positive mass conjecture, then the argument shows that Theorem 1 holds also for nonnegative scalar curvature.

Thus the condition that the metric is flat in a whole neighborhood of ∂M is very strong. One might expect that, for $\mathrm{Ric} \geq 0$, it suffices to assume that the sectional curvature vanishes only on the boundary. We can prove this in the special case of a metric ball:

THEOREM 2. *Let M be a Riemannian manifold of dimension $n \geq 3$ and let $B = B_r(p_0)$ a convex metric ball embedded by the exponential map $\exp_{p_0}$ with boundary $H = \partial B$. Assume that the Ricci-curvature is nonnegative on B and that*

a) *$K(\sigma) = 0$ for all 2-planes with footpoint on H which are tangent to H, if n is odd.*

b) *H is strictly convex and $K(\sigma) = 0$ for all 2-planes with footpoint on H, if n is even.*

Then B is flat.

In the proof of this result we use ideas from [GW]. We finally prove the rigidity of a product $M = M_1 \times M_2$ with noncompact factors and $K \geq 0$ under a compact modification of the metric which preserves $K \geq 0$.

THEOREM 3. *Let M_1, M_2 be complete noncompact Riemannian manifolds with sectional curvature $K \geq 0$. Let $\Omega \subset M := M_1 \times M_2$ be the complement of a compact subset. If $\phi : \Omega \to \bar{M}$ is an isometric embedding, where $\bar{M}$ is a complete manifold with $K \geq 0$ and $\dim \bar{M} = \dim M$ then ϕ extends in a unique way to an isometry $\bar{\phi} : M \to \bar{M}$.*

This result was stated (without proof) by Gromov [BGS, p. 75] but we think that the proof is not at all trivial. Note that one cannot expect such a result for $\mathrm{Ric} \geq 0$: If M_1, M_2 are noncompact with $K > 0$, then the products has $\mathrm{Ric} > 0$ and one can deform the metric locally. The examples of [SY] show that $\mathrm{Ric} > 0$ allows even surgery constructions starting from products. However there is a rigidity result for $\mathrm{Ric} \geq 0$ if M contains a line, i.e. splits as $M' \times \mathbb{R}$ by the Cheeger–Gromoll splitting theorem [CG]. Let $\bar{M}$ be a manifold which coincides with M outside of a compact set. It is not difficult to show that also $\bar{M}$ contains a line and splits as $\bar{M}' \times \mathbb{R}$. From this one concludes that M is isometric to $\bar{M}$.

In section 4 we give an example of a manifold $M = M_1 \times M_2$ with compact factor M_1 and a manifold $\bar{M}$ which is isometric to M outside of compact sets but which is not diffeomorphic to M.

We would like to thank J. Eschenburg, Min-Oo, M. Müter and W. Ziller for helpful discussions.

2. Rigidity for nonnegative Ricci-curvature

The proof of Theorem 1 is based on the following observation:

LEMMA 1. *Let M^n be a compact Riemannian manifold with convex boundary and assume that M is flat in some neighborhood U of ∂M. Then there exists an isometric embedding $f: M \to N^n$, where N is a complete open manifold which is flat outside of $f(M)$. If in addition ∂M is simply connected then $N \setminus f(M)$ is isometric to $\mathbb{R}^n \setminus C$, where C is a compact subset of $\mathbb{R}^n$.*

Remark. If the Ricci-curvature is nonnegative on M and M is not flat then N has only one end. This is easily seen by the splitting theorem of Cheeger–Gromoll, comp. [CG].

Proof of Lemma 1. For $\varepsilon > 0$ let $U_\varepsilon := \{p \in M \mid \operatorname{dist}(p, \partial M) \leq \varepsilon\}$. Then for ε small enough U_ε is a subset of U and can be identified with $\partial M \times [-\varepsilon, 0]$, where (p, t) corresponds to $\exp t\eta_p$ and η_p denotes the outer normal field along ∂M. Consider the universal covering $S \to \partial M$ and the group Γ of decktransformations. Then $U_\varepsilon \cong \partial M \times [-\varepsilon, 0]$ is diffeomorphic to $(S \times [-\varepsilon, 0])/\Gamma$, where Γ operates trivially on the second factor. The product $S \times [-\varepsilon, 0]$ carries a flat metric induced from the metric on $U_\varepsilon \cong \partial M \times [-\varepsilon, 0]$. As $S \times [-\varepsilon, 0]$ is simply connected, there is an isometric immersion $D_0: S \times [-\varepsilon, 0] \to \mathbb{R}^n$ (developing map, comp. [Th]). Define $\xi \overset{\text{def}}{=} (D_0)_* \partial/\partial t$, then ξ is the outer unit normal vector field along D_0. As the immersion D_0 is convex, we can extend D_0 to an immersion $D: S \times [-\varepsilon, \infty)$ by

$$D(p, t) = D_0(p, 0) + t\xi(p, 0)$$

and the pull back metric on $S \times [-\varepsilon, \infty)$ is flat and agrees on $S \times [-\varepsilon, 0]$ with the given metric. Clearly Γ operates isometrically and U_ε can be considered as a subset of $N_0 := (S \times [-\varepsilon, \infty)/\Gamma)$. Under this identification M is a subset of $N := (M \setminus U_\varepsilon) \cup N_0$.

Now assume that ∂M is simply connected. Then $U_\varepsilon \cong \partial M \times [-\varepsilon, 0]$ is also simply connected and we can consider the isometric immersion $D_0: \partial M \times [-\varepsilon, 0] \to \mathbb{R}^n$. As ∂M is compact and convex and since $\dim \partial M > 1$, D_0 is an embedding by the theorem of Sackstedter [S]. If $B \subset \mathbb{R}^n$ denotes the bounded components of $\mathbb{R}^n \setminus D_0(\partial M \times \{-\varepsilon\})$ then we can define $N := M \cup_{D_0} (\mathbb{R}^n \setminus B)$. □

Proof of Theorem 1. a) By Lemma 1 we may assume that M is a subset of the manifold N, where $N \setminus M$ is isometric to $\mathbb{R}^n \setminus C$. As C is compact the limit

$\liminf_{t\to\infty} v_p(t)/t^n$ is equal to $\liminf_{t\to\infty} v_0(t)/t^n$, where $v_p(t)$ resp. $v_0(t)$ denotes the volume of a ball of radius t with center p in N resp. center 0 in the euclidean space $\mathbb{R}^n$. Now the condition $\mathrm{Ric} \geq 0$ on N implies that N is isometric to $\mathbb{R}^n$ by the rigidity part of the Bishop–Gromov inequality [G].

b) As ∂M is strictly convex in some point $p \in \partial M$ (i.e. the Weingarten-map with respect to the outer unit normal is strictly positive definte at p) and as M is flat in some neighborhood of ∂M we may assume without loss of generality that ∂M is strictly convex everywhere. (This can be shown by iterating a standard convolution process for the distance function ρ of the boundary ∂M. This method leads to a strictly convex C^∞-function $\bar{\rho}$ which is arbitrarily close to ρ, comp. [ES]. Note that by the remark above, we can assume that ∂M has only one component.) Consider the orientation covering $\bar{M} \to M$. Then $\bar{M}$ satisfies the same conditions as M and in particular the intrinsic curvature of $\partial \bar{M}$ is strictly positive by the Gauss-equation. Furthermore $\partial \bar{M}$ is orientable and even-dimensional. Therefore $\partial \bar{M}$ is simply connected by the Lemma of Synge [CE]. Thus a) implies that $\bar{M}$ (and therefore M) is flat. □

Proof of Theorem. 2. The proof is subdivided into two steps. Let L resp. L_0 be the Weingarten map of H resp. $S_r(0)$ with respect to the outer unit normal vector, where $S_r(0)$ denotes the standard euclidean sphere of radius r.

(i) First we will show that $\mathrm{Ric} \geq 0$ on B implies

$$A \overset{\mathrm{def}}{=} \int_H \det L \, dV \leq \int_{S_r(0)} \det L_0 \, dV_0 = \mathrm{vol}\,(S_1(0)) \tag{1}$$

Furthermore $A = \mathrm{vol}\,(S_1(0))$ is only possible if $B_r(p)$ is isometric to $B_r(0_p) \subset T_pM = \mathbb{R}^n$.

As $\mathrm{Ric} \geq 0$ on B the Gromov–Bishop inequality [G] gives (compare B with the euclidean ball $B_r(0)$):

$$\mathrm{vol}\,(H) \leq \mathrm{vol}\,(S_r(0)) \tag{2}$$

The equality holds if and only if B is isometric to $B_r(0)$.

A similar comparison argument shows:

$$\mathrm{trace}\,(L) \leq \mathrm{trace}\,(L_0)$$

The arithmetic-geometric mean inequality gives

$$0 \leq \det\,(L)^{1/m} \leq \frac{1}{m}\,\mathrm{trace}\,(L) \leq \frac{1}{m}\,\mathrm{trace}\,(L_0) = \det\,(L_0)^{1/m} = r^{-1}$$

and therefore

$$\int_H \det L\, dV \leq \int_H r^{-m}\, dV = r \operatorname{vol}(S_r(0)) = \operatorname{vol}(S_1(0))$$

where equality holds iff B is isometric to an euclidean ball, compare (2).

(ii) Now we want to show that condition a) resp. b) of Theorem 3 implies:

$$\int_H \det L\, dV = \operatorname{vol}(S_1(0))$$

Then by (1) we have $K \equiv 0$ on B.

α) Assume that n is odd. As $K(\sigma) = 0$ for all 2-planes σ tangent to H the Gauss equation implies $\det L = G$, where G is the Gauss–Bonnet integrand of the even dimensional orientable hypersurface H. Therefore:

$$\int_H \det L\, dV = \int_H G\, dV = \frac{\chi(H)}{2} \operatorname{vol}(S_1(0)) = \operatorname{vol}(S_1(0))$$

β) Assume that n is even. As H is simply connected ($\dim H \geq 2$) the curvature condition $K(\sigma) = 0$ for all 2-planes with footpoint in H implies the existence of a parellel orthonormal trivialisation $E_1, \ldots, E_n$ of the bundle $TM|_H$. Let N denote the outer unit normal field of H. Define a Gauss-map $\phi : H \to S_1(0)$ by

$$\phi(p) = \sum_{k=1}^{n} \langle N(p), E_k \rangle e_k$$

where $e_1, \ldots, e_n$ denotes the standard orthonormal basis of $\mathbb{R}^n$. Then

$$\phi_* x = \sum_{k=1}^{n} \langle Lx, E_k \rangle e_k$$

and

$$\phi^* dV_0 = (\det L)\, dV$$

Therefore

$$\int_H \det L\, dV = \deg(\phi) \int_{S_1(0)} dV_0 = \deg(\phi) \operatorname{vol}(S_1(0))$$

As L is positive definite the differential ϕ_* is nonsingular and therefore ϕ is a local diffeomorphism and hence a covering map. $S_1(0)$ is simply connected hence

ϕ is an (orientation-preserving) diffeomorphism and therefore $\deg(\phi) = +1$, which completes the proof. $\square$

Remark. 1) If n is even, $n \geq 3$ and H is convex (but not necessarily strictly convex) then $\deg(\phi) = 0$ implies that the tangent bundle $TH \cong TS_1(0)$ is trivial and therefore $\dim H = n - 1 \in \{3, 7\}$ (comp. [GW, Lemma 9]). Hence Theorem 2 part b) remains true if H is only convex and $n \geq 3$, $n \neq 4, 8$.

2) In the case that $\dim M = 3$ and that the sectional curvature K is nonnegative, one can prove a version for arbitrary convex sets (comp. [SS] Theorem 2):

Let M be a compact Riemannian manifold of dimension 3 and with nonnegative sectional curvature. Assume that the boundary ∂M is strictly convex and that $K(\sigma) = 0$ for all 2-planes σ which are tangent to ∂M. Then M is flat.

3. Rigidity of products

For the proof of Theorem 3 we recall some facts from the structure theory of a complete open manifold M with nonnegative sectional curvature (see [CG], [CE] ch. 8):

If C is a compact totally convex subset in M with nonempty boundary ∂C, then also the sets

$$C^t = \{p \in C \mid d(p, \partial C) \geq t\}$$

are totally convex. Let $C^{\max} = C^a$ where $a = \sup\{t \geq 0 \mid C^t \neq \emptyset\}$. Then $\dim C^{\max} < \dim C$. By the basic construction of [CG] there exists an exhaustion of M by compact totally convex subsets C_t, $t \geq 0$ such that $C_t = C^s_{t+s}$ and $C_0 = C^{\max}_t$ for all $t, s > 0$. In particular $\dim C_0 < \dim C_t = \dim M$ for all $t > 0$. If $C(1) \overset{\text{def}}{=} C_0$ has nontrivial boundary, then let $C(2) \overset{\text{def}}{=} C(1)^{\max}$. We obtain a sequence $C_0 = C(1) \supset \cdots \supset C(k) = \Sigma$, where k is the smallest integer such that $C(k)$ is wouthout boundary. $\Sigma = C(k)$ is called a soul of M.

In the theorem we investigate a product $M = M_1 \times M_2$. For the factors M_i, $i = 1, 2$, we have the exhaustions $C_{i,t}$ and the chain $C_i(1) \supset \cdots \supset C_i(k_i) = \Sigma_i$, where Σ_i is the soul of M_i.

We also recall the following construction of Sharafudtinov [Sh], see also [Y]: Let C be a compact totally convex subset in M with nonempty boundary ∂C. Then there exists a strong deformation retract $\psi_t : C \to C^t$ which is distance nonincreasing. Thus there exists also a contraction map $\psi_t : C_t \to C(1)$ and finally a contraction $\psi : C \to \Sigma$.

For the proof of the theorem the following notation is useful: Let $D \subset M$ and $\bar{D} \subset \bar{M}$ be subsets. We say that $\phi(D)$ and $\bar{D}$ coincide outside of a compact set and we write $\phi(D) \stackrel{c}{=} \bar{D}$, if there are compact sets $K \subset M$ and $\bar{K} \subset \bar{M}$ such that $\phi|_{D \setminus K}: D \setminus K \to \bar{M}$ is an isometry from $D \setminus K$ onto $\bar{D} \setminus \bar{K}$. Note that we can use this notation even when D is not completely contained in Ω.

We prove first that $\phi(M) \stackrel{c}{=} \bar{M}$, i.e. that $Q \stackrel{\text{def}}{=} \bar{M} \setminus \phi(\Omega)$ is compact. Therefore we can assume that $\Omega = M \setminus C_a$ for a suitable $a > 0$. Since C_a is totally convex and $\dim M = \dim \bar{M}$ also Q is totally convex because every geodesic which enters $\phi(\Omega)$ cannot leave $\phi(\Omega)$. If Q is noncompact then there exists a sequence $q_i \in Q$ with $d(q_i, \partial Q) \to \infty$. Furthermore there are $p_i \in \phi(\Omega)$ with $d(p_i, \partial Q) \to \infty$. Then a sequence of minimizing geodesics from q_i to p_i has an accumulation line which intersects ∂Q. By Toponogov's splitting theorem $\bar{M}$ splits as $\bar{M}' \times \mathbb{R}$. We can assume that $(x, 0) \in \partial Q$ for a point $x \in \bar{M}'$ and $(x, t) \in \phi(\Omega)$ for $t > 0$ and $(x, t) \in Q$ for $t \leqslant 0$.

Let y be a point in $\bar{M}'$. For $t_0 > 0$ large enough, $(y, t_0) \in \phi(\Omega)$ and $(y, -t_0) \in Q$. Thus the line $\{y\} \times \mathbb{R}$ intersects ∂Q. Since ∂Q is compact, the distance $d(x, y)$ is universally bounded and $\bar{M}'$ is compact. But this is impossible since M is a product of two noncompact factors. The contradiction shows that Q is compact and $\phi(M) \stackrel{c}{=} \bar{M}$.

For the rest of the proof we will assume (without loss of generality) that Ω is the complement of $C_{1,a} \times C_{2,a}$ in $M = M_1 \times M_2$ for a suitable positive constant a. We consider the cylinder $Z := C_{1,a} \times M_2$ in M. Let $\bar{Z} \stackrel{\text{def}}{=} \bar{M} \setminus \phi(M \setminus Z)$. We claim that $\bar{Z}$ is a totally convex subset of $\bar{M}$. Note that the complement of Z in M is isometric to the complement of $\bar{Z}$ in $\bar{M}$. Since Z is totally convex, every geodesic leaving Z cannot return. Thus the same is true for $\bar{Z}$ and hence $\bar{Z}$ is also totally convex.

We claim that $\bar{Z}^{\max} = \bar{Z}^a$ and $\bar{Z}^{\max} \stackrel{c}{=} \phi(Z^a) = \phi(C_1(1) \times M_2)$. Since $\bar{M} \stackrel{c}{=} \phi(M)$ it is clear that $\bar{Z} \stackrel{c}{=} \phi(Z)$ and $\bar{Z}' \stackrel{c}{=} \phi(Z')$. It follows that $\dim \bar{Z}^a <$ *dim* $\bar{Z}$ and hence $\bar{Z}^{\max} = \bar{Z}^a$ and $\bar{Z}^a \stackrel{c}{=} \phi(Z^a)$. Thus we have proved that $\bar{Z}(1) \stackrel{c}{=} \phi(Z(1))$. In the same way we obtain $\bar{Z}(2) \stackrel{c}{=} \phi(Z(2))$ and finally $\bar{Z}(k_1) \stackrel{c}{=} \phi(Z(k_1)) = \phi(\Sigma_1 \times M_2)$. For the proof of Theorem 3 the following result is essential

LEMMA 2. $S \stackrel{\text{def}}{=} \bar{Z}(k_1)$ *is complete without boundary and isometric to the product* $\Sigma_1 \times M_2$.

Proof of Lemma 2. The proof consists of three steps:

1. We show that S is complete without boundary.
2. Through every point $x \in S$ there exists a totally geodesic submanifold isometric to M_2.

3. We show that if $M_2(x)$ and $M_2(y)$ are two of these submanifolds of S, then there exists a totally geodesic and isometric immersion $G:[0, r]\times M_2 \to S$ such that $G(0, M_2)=M_2(x)$ and $G(r, M_2)=M_2(y)$. From this fact we derive the product structure.

1. Let us assume to the contrary that $\partial S \neq \emptyset$. Then ∂S lies in a compact set since S coincides with $\phi(\Sigma_1 \times M_2)$ outside of a compact set. For t sufficiently large, the set $S^t=\{p\in S \mid d(p, \partial S)\geq t\}$ is contained in the set where S coincides with the product $\phi(\Sigma_1\times M_2)$ and we can define the projection $\pi: S^t \to M_2$. Let $\psi:\bar{Z}\to S$ and $\psi_t: S\to S^t$ be Sharafudtinov retractions. It is easy to check that the construction of the maps ψ, ψ_t (compare [Y]) also works in our context where $\bar{Z}$ is not compact. Note that outside of a compact set ψ coincides with the product map $\psi^1\times id$, where $\psi^1: C_{1,a}\to \Sigma_1$ is a Sharafudtinov retraction in M_1. Choose $x_1\in \partial C_{1,a}$ and let $i: M_2\to \{x_1\}\times M_2$ be an isometric embedding of M_2 into ∂Z. Then $\alpha=\pi\circ\psi_t\circ\psi\circ\phi\circ i$ is a map from M_2 onto a proper subset of M_2 which coincides with the identity outside of a compact set. Such a map is impossible for topological reasons.

It follows that $S=\bar{Z}(k_1)$ is the soul of the cylinder $\bar{Z}$ and $S \stackrel{c}{=} \phi(\Sigma_1\times M^2)$.

2. We prove that though every point $x\in S$ there exists a totally geodesic submanifold isometric to M_2.

Consider a point $\phi(x_1, x_2)\in S$, where $x_1\in\Sigma_1$ and $x_2\in M_2$, i.e. a point outside of the compact set. Let $\gamma:[0,\infty)\to M_1$ be a unit speed ray with $\gamma(0)=x_1$. It follows from the basic construction in [CG] that $\gamma(t)\in\partial C_{1,t}$ for $t\geq 0$. We consider the geodesic $\bar{\gamma}(s)=\phi(\gamma(s), x_2)$ in $\bar{M}$. Since $\bar{Z}^{\max}=\bar{Z}^a$ it follows that $d(\phi(x_1, x_2), \partial\bar{Z})\geq a$. Since $\phi(\gamma(a), x_2)\in\partial\bar{Z}$, this geodesic is minimizing up to $\partial\bar{Z}$ and since the constant a can be choosen arbitrarily large, $\bar{\gamma}$ is a ray in $\bar{M}$. Let $\bar{c}:\mathbb{R}\to S$ be a geodesic in S with $\bar{c}(0)=\varphi(x_1, x_2)$. Let $W(t)$ be the parallel vectorfield along $\bar{c}(t)$ with $W(0)=\dot{\bar{\gamma}}$. It follows from [CG] Theorem 1.10 that

$$H(s, t) \stackrel{\text{def}}{=} \exp_{\bar{c}(t)} sW(t) \tag{3}$$

is a totally geodesic isometric immersion of the flat halfplane $[0,\infty)\times\mathbb{R}$ into $\bar{M}$. Let $c:\mathbb{R}\to M_2$ be a geodesic with $c(0)=x_2$ and let $\bar{c}:\mathbb{R}\to S$ be the geodesic such that $\bar{c}(t)=\phi(x_1, c(t))$ for $|t|$ small, then one checks easily that

$$H(s, t) \stackrel{c}{=} \phi(\gamma(s), c(t)) \tag{4}$$

For $b>0$ we consider the manifold $\gamma(b)\times M_2\subseteq M$. For b sufficiently large, $\gamma(b)\times M_2$ is completely contained in Ω. Let $Y\stackrel{\text{def}}{=}\phi(\gamma(b)\times M_2)\subseteq\bar{M}$. Note that

$(-\dot{\gamma}(b), 0)$ defines a globally parallel vectorfield V on Y. By construction we obtain for x_2 outside of a compact subset of M_2 that

$$\exp bV(\phi(\gamma(b), x_2) = \phi(\gamma(0), x_2) \in S$$

We claim that the map $\theta(y) = \exp_y bV(y)$ is a totally geodesic isometric embedding of Y into S. Let therefore $c: \mathbb{R} \to M_2$ be any geodesic of M_2 which does not stay in a compact subset. We obtain the flat halfspace $H(s, t)$ as in (4) which contains the geodesic $t \mapsto \phi(\gamma(b), c(t))$ in Y. It follows that the map θ is an isometry along the geodesic $\phi(\gamma(b), c(t))$. By the structure theory of M_2 it is clear that only a zero-set of geodesics stays in a compact set. Thus θ is an isometry. More generally, it follows from Rauch's comparison theorem that the map

$$\begin{aligned} D: [0, b] \times Y &\to \tilde{M} \\ (s, y) &\mapsto \exp_y sV(y) \end{aligned}$$

is a totally geodesic isometric embedding. Since $D(b, M_2)$ is contained in S outside of a compact set and S is totally geodesic, it follows that $D(b, M_2) \subseteq S$.

Because $S \stackrel{c}{=} \phi(\Sigma_1 \times M_2)$ there exists a compact set K_2 in M_2 such that S is isometric to $\Sigma_1 \times \Omega_2$ outside of a compact set, where Ω_2 is the complement of K_2. We just have proved, that every fiber $\{x_1\} \times \Omega_2$ is a subset of a complete totally geodesic submanifold isometric to M_2. We denote this submanifold with $M_2(x_1)$. Let x be an arbitrary point in S, then consider a ray $c: [0, \infty) \to S$ starting in x. This ray is finally contained in $\Sigma_1 \times \Omega_2$ and since Σ_1 is compact, it is contained in a fiber $\{x_1\} \times \Omega_2$. Thus $x \in M_2(x_1)$ and every point of S is contained in $M_2(x_1)$ for a suitable x_1.

3. Let $x_1, y_1 \in \Sigma_1$ and $\alpha: [0, r] \to \Sigma_1$ a minimal geodesic between them where $r = d(x_1, y_1)$. We claim: There exists a totally geodesic and isometric embedding $G: [0, r] \times M_2 \to S$ such that $G(0, M_2) = M_2(x_1)$ and $G(r, M_2) = M_2(y_1)$.

Before we prove this claim, we show that this implies S isometric to $\Sigma_1 \times M_2$. First the above claim shows that the manifolds $M_2(x_1)$ define a foliation of S and hence also an integrable distribution. If c is any geodesic in S, then c is contained in the image of an isometric embedding G as above. It follows that the distribution is invariant under parallel translation and hence S is a product by the de Rham splitting theorem. Since $S \stackrel{c}{=} \phi(\Sigma_1 \times M_2)$ it is clear that S is isometric to $\Sigma_1 \times M_2$.

To prove the claim, we consider $M_2(x_1) \stackrel{c}{=} \phi(\{x_1\} \times M_2)$, $M_2(y_1) \stackrel{c}{=} \phi(\{y_1\} \times M_2)$ and canonical isometries $\phi_x: M_2 \to M_2(x_1)$, $\phi_y: M_2 \to M_2(y_1)$. We first assume that the distance $r = d(x_1, y_1)$ is small enough, such that for every $z \in M_2$ there exists a unique minimal geodesic from $\phi_x(z)$ to $\phi_y(z)$. Since S is a product outside of a compact set this is possible for small $r \geq 0$. Let $\pi : M_2(x_1) \to M_2(y_1)$ be the projection which maps $\phi_x(z)$ onto $\phi_y(z)$. Let $c : \mathbb{R} \to M_2$ be a geodesic which does not stay in a compact set and let c_x and c_y be the geodesics in $M_2(x_1)$ and $M_2(y_1)$ such that $c_x(t) \stackrel{c}{=} \phi(\{x_1\} \times c(t))$ and $c_y(t) \stackrel{c}{=} \phi(\{y_1\} \times c(t))$. We can assume that $c(0) \in \Omega_2$, i.e. near to 0, $c_x(t)$ and $c_y(t)$ bound a flat totally geodesic strip.

We want to show that $c_x[0, \infty)$ and $c_y[0, \infty)$ bound a totally geodesic flat strip. The set of all t such that $c_x[0, t]$ and $c_y[0, t]$ bound a flat strip isometric to $[0, t] \times [0, r]$ is clearly closed. To prove that the set is open we assume that $c_x[0, t_0]$ and $c_y[0, t_0]$ bound a flat strip and let $t_1 \geq t_0$ with $t_1 - t_0$ small. It follows from Rauch's comparison theorem [CE, pg. 29], that $r_1 \stackrel{\text{def}}{=} d(c_x(t_1), c_y(t_1)) \leq r$ and that equality implies that also $c_x[0, t_1]$ and $c_y[0, t_1]$ bound flat strip. Thus it remains to show that $r_1 \geq r$.

Therefore choose a ray $\gamma : [0, \infty) \to M_1$ with $\gamma(0) = x_1 \in \Sigma_1$ and consider the ray $\bar{\gamma}(s) = \phi(\gamma(s), c(0))$ in $\bar{M}$. In S we have the piecewise geodesic formed by the three pieces $c_x[0, t_1]$, $\beta[0, r_1]$, $c_y[0, t_1]$, where $\beta : [0, r_1] \to S$ is the minimal geodesic from $c_x(t_1)$ to $c_y(t_1)$. Let $w \stackrel{\text{def}}{=} \dot{\bar{\gamma}}(0)$ and W be the parallel vectorfield along the piecewise geodesic, i.e we parellel translate w from $c_x(0)$ along c_x to $c_x(t_1)$, from there along β to $c_y(t_1)$ and then back along c_y to $c_y(0)$.

As in (3) we thus obtain three totally geodesic immersions

$$F^1(s, t) = \exp_{c_x(t)} sW(c_x(t))$$
$$F^2(s, t) = \exp_{\beta(t)} sW(\beta(t))$$
$$F^3(s, t) = \exp_{c_y(t)} sW(c_y(t))$$

where F^1 and F^2 is defined on $[0, \infty) \times [0, t_1]$ and F^3 on $[0, \infty) \times [0, r_1]$.

By (4) $F^1(s, t) \stackrel{c}{=} \phi(\gamma(s), c(t))$ and in the same way $F^3(s, t) \stackrel{c}{=} \phi(\gamma^*(s), c(t))$, where γ^* is the M_1 component of the ray $\phi^{-1} \circ \bar{\gamma}^*$ where $\bar{\gamma}^*(s) = F^3(s, 0)$.

Choose $b > 0$ sufficiently large such that $F^i(b, t) \in \phi(\Omega)$ for all i and t. Then

$$\begin{aligned} r_1 &= d(c_x(t_1), c_y(t_2)) \\ &= d(F^2(0, 0), F^2(0, r_1) \\ &= d(F^2(b, 0), F^2(b, r_1)) \\ &= d(\phi(\gamma(b), c(t_1)), \phi(\gamma^*(b), c(t_1))) \end{aligned}$$

where b is arbitrary. For b sufficiently large

$$d(\phi(\gamma(b), c(t_1)), \phi(\gamma^*(b), c(t_1))) = d(\gamma(b), \gamma^*(b))$$

Now γ and γ^* are rays in M_1 with $\gamma(b), \gamma^*(b) \in \partial C_{1,b}$ for all b. It is then a consequence of the first variation formula, that $d(\gamma(t), \gamma^*(t))$ is monotone increasing. Thus

$$d(\gamma(b), \gamma^*(b)) \geq d(\gamma(0), \gamma^*(0)) = r$$

It follows that $c_x[0, \infty)$ and $c_y[0, \infty)$ bound a flat strip and with the same argument $c_x(\mathbb{R})$ and $c_y(\mathbb{R})$ bound a flat strip. Since the geodesics which leave every compact set are dense, this argument shows that $d(\pi(z), z) = r$ for all $z \in M_2(x_1)$. In particular $M_2(x_1)$ and $M_2(y_1)$ have no common points. Since by assumption for every point $z \in M_2(x_1)$ there is a unique minimal geodesic to the corresponding point in $M_2(y_1)$, there exists a unit vectorfield W on $M_2(x_1)$ such that $\pi(z) = \exp_z rW(z)$. The flat strip argument from above shows that along every geodesic $\bar{c}$ in $M_2(x_1)$ which does not stay in a compact subset W is a parallel normal vectorfield. It follows from the denseness of these geodesics that W is a parallel normal unit vectorfield.

Since $\pi(z) = \exp_z rW(z)$ is an isometry, it follows from Rauch's theorem that the map

$$[0, r] \times M_2(x_1) \to S, \qquad (s, z) \mapsto \exp_z sW(z)$$

is a totally geodesic isometric immersion. Since it is an embedding outside of a compact set one checks easily that it is an embedding.

We have assumed that r is sufficiently small. In the general case let $x_1, y_1 \in \Sigma_1$ be arbitrary and α a minimal geodesic joining them. Let $\bar{\alpha}$ be the minimal geodesic $\bar{\alpha}(s) = \phi(\alpha(s), x_2)$ between $\phi(x_1, x_2)$ and $\phi(y_1, x_2)$ where $x_2 \in \Omega_2$. The above argument shows that $\dot{\bar{\alpha}}(0)$ extends to a globally parallel vectorfield on $M_2(x_1)$. One checks easily that

$$(s, z) \mapsto \exp_z sW(z)$$

is an isometric embedding also in this case. Thus we have proved the lemma. □

We are now able to complete the proof of Theorem 3. Let $\bar{c}: \mathbb{R} \to \bar{M}$ be any geodesic with $\bar{c}(0) \in \bar{M} \setminus \bar{Z}$. We claim that there exists a totally geodesic isometric immersion $G: \mathbb{R} \times M_2 \to \bar{M}$ such that $\bar{c}$ is contained in the image of G.

Since $\bar{c}(0) \in \phi(\Omega)$ there exists a point $x_1 \in M_1$ such that $\bar{c}(0) \in Y \overset{\text{def}}{=} \phi(\{x_1\} \times M_2)$. We can assume that $\dot{\bar{c}}(0)$ is not tangent to Y. Let w' be the normal component of $\dot{\bar{c}}$ and $w \overset{\text{def}}{=} w'/\|w'\|$. Then w extends to a globally parallel unit normal vectorfield on Y. We consider the map

$$G: \mathbb{R} \times Y \to \bar{M}$$

$$G(s, y) \stackrel{\text{def}}{=} \exp_y sW(y)$$

By Rauchs theorem, the map $G_s = G(s, .)$ from Y to $\bar{M}$ is distance nonincreasing for small $s \geq 0$ and the rigidity part of this theorem states that if G_s is isometric for $s \geq 0$, then $G|_{[0,s] \times Y}$ is an isometric immersion.

Thus we have to show that G_s is an isometry. Let therefore $i: M_2 \to \{x_1\} \times M_2$ the embedding, $\pi: S \to M_2$ the distance nonincreasing projection onto the M_2-factor of $S \cong \Sigma_1 \times M_2$, let $\psi: \bar{Z} \to S$ be the Sharafudtinov-retraction as in the proof of Lemma 3.

We can assume that $G_s(Y) \subset \bar{Z}$ since G_s is clearly an isometry as long as the image lies in $\bar{M} \backslash \bar{Z}$. Then we have the distance nonincreasing map $\pi \circ \psi \circ G_s \circ \phi \circ i: M_2 \to M_2$ which is the identity outside of a compact set. Such a map has to be an isometry (compare Lemma 1, 2 in [Sh]). It follows that G_s is an isometry.

Since the set of geodesics which leave $\bar{Z}$ is dense, one checks easily that through every point of $\bar{M}$ there is a totally geodesic submanifold isometric to M_2 and that the distribution defined by the tangent spaces of these manifolds is invariant under parallel translation (compare the proof of the splitting $S = \Sigma_1 \times M_2$ in the proof of Lemma 2). It follows from the de Rham decomposition that $\bar{M}$ splits a factor M_2 and since $\bar{M} \stackrel{c}{=} \phi(M_1 \times M_2)$ it is clear that $\bar{M}$ is isometric to $M_1 \times M_2$. Obviously ϕ extends in a unique way to an isometry $\bar{\phi}: M \to \bar{M}$. □

4. Flexibility of products with nonnegative curvature

Let $M = M_1 \times M_2$ be an open product manifold with sectional curvature $K \geq 0$ where the factor M_1 is compact. We ask how flexible is this product with respect to modifications of the metric within compact sets which preserve $K \geq 0$.

If M_2 has $K > 0$ (or at least $K > 0$ at one point), then one can deforme the metric on M_2 in a compact set. In this case the soul of M is isometric to $M_1 \times \{p\}$ and the factor M_1 survives in the new metric.

Consider now a manifold M_2 which is diffeomorphic to $\mathbb{R}^{k+1}$ and $M_2 \backslash C_2$ is isometric to $(S^k, g_E) \times [0, \infty)$ for a compact subset C_2 of M_2, where g_E is the standard metric on the sphere. It is easy to construct rotational symmetric metrics of this type. Choose $M_1 = (S^k, g_E)$ then $M = M_1 \times M_2$ is isometric to $S^k \times S^k \times [0, \infty)$ outside of a compact set C where C is isometric to $S^k \times C_2$. Note that we can glue $S^k \times C_2$ in different ways onto the boundary of $S^k \times S^k \times [0, \infty)$ and thus one cannot see from the structure of $M \setminus C$ which S^k factor survives in a manifold $\bar{M}$ which is isometric to M outside of a compact set.

One can even not see the topological structure of the manifold by looking only to the complement of a compact set. Consider therefore $M_2^* = (S^3, g_1) \times (\mathbb{R}^2, g_2)/S^1$, where we choose some left-invariant metric g_1 on S^3 and a rotational symmetric metric g_2 on $\mathbb{R}^2$. S^1 operates diagonally on the product, where it rotates the Hopf-circles on S^3 and acts by rotations on $(\mathbb{R}^2, g_2)$.

We choose g_2 such that $(\mathbb{R}^2, g_2)$ is isometric to $S_a^1 \times [0, \infty)$, outside of a compact set, where S_a^1 is a circle of radius a. Then, outside of a compact set, M_2^* is isometric to $(S^3, g_3) \times [0, \infty)$, where g_3 is also a left-invariant metric on S^3. If we choose g_1 suitable then M_2^* is isometric to $(S^3, g_E) \times [0, \infty)$ outside of a compact set. Let $M_1 = (S^3, g_E)$. Then the product $M = M_1 \times M_2$ (for $k = 3$) is isometric to $\bar{M} = M_1 \times M_2^*$ outside of compact sets, but M and $\bar{M}$ have different topology. In particular their souls are not isometric, sos!

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Received August 3, 1987