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A linearity theorem for group actions on spheres with applications to homotopy representations

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Abstract. Let G be a finite group and X an equivariant $Z/|G|$-homology sphere. By Smith-theory the fixed point set X^H for a p-subgroup H is a Z/p-homology sphere of dimension $d(H)$.

THEOREM. *There exists a virtual representation $V - W \in RO(G)$ such that $d(H) = \dim V^H - \dim W^H$ holds for any subgroup H of G of prime power order.*

This applies to describing the Grothendieck group $\mathcal{D}(G)$ of dimension functions of homotopy representations for compact Lie groups G in algebraic terms.

Let G be a finite group and $T \subset \mathbf{Q}$ denote a subring which is contained in the local ring $\mathbf{Z}_{(p)}$ for any prime divisor p of the order $|G|$ of G. If X is a finitistic G-space, which is a T-homology sphere, then the fixed point set X^H is a $\mathbf{Z}_{(p)}$-homology sphere of dimension $d(H)$ for any p-subgroup H of G.

The purpose of this paper is to give a proof and an application of the following

THEOREM. *There exists a virtual representation $V - W \in RO(G)$ such that $d(H) = \dim V^H - \dim W^H$ holds for any subgroup H of G of prime power order.*

In particular d can be characterized by the combinatorial congruences in (1.1). Related linearity theorems are known for special cases: Homotopy representations of finite nilpotent groups do have stably linear dimension functions [tD1]. The theorem above is known for p-groups [Do–Ha].

Taking into account tom Dieck's analysis of group actions on homotopy spheres (compare [tD4] and [tD5]), the linearity theorem also gives sufficient conditions for the existence of group actions on homotopy spheres. An application to homotopy representations will be discussed in the second part.

Let $\mathcal{P}$ denote the open family of subgroups (i.e. $\mathcal{P}$ is a union of conjugacy classes, such that for any $K \in \mathcal{P}$ all subgroups of K are also contained in $\mathcal{P}$) of prime power order and $\mathcal{P}'$ the subfamily of cyclic ones.

(1.1) DEFINITION. Let $\mathcal{D}_{\mathcal{P}}(G)$ denote the group of functions $f : \mathcal{P} \to \mathbf{Z}$, constant on conjugacy classes, which satisfy the following relations: For every triple $H \lhd K \lhd M$ of subgroups of G with $K/H \cong \mathbf{Z}/p$ and H a p-group,

(1) $f(H) \equiv f(K) \bmod 2$, if p is odd or if $M/H \cong \mathbf{Z}/4$

(2) $f(H) \equiv f(K) \bmod 4$, if $M/H \cong$ quaternion group

(3) $f(H) + p \cdot f(M) = \Sigma f(K_i)$, if $M/H \cong \mathbf{Z}/p \times \mathbf{Z}/p$. Here of course the sum is over all K_i with $H \lhd K_i \lhd M$ and $K_i/H \cong \mathbf{Z}/p$

(4) $f(H) \cong f(K) \bmod q^{r-l}$, if $M/K \cong \mathbf{Z}/q^r$, acting on K/H with kernel of prime power order q^l.

(1.2) PROPOSITION. *If $d: \mathcal{P} \to Z$ denotes the dimension function of a finitistic G-space X which is a T-homology sphere, then $d \in \mathcal{D}_{\mathcal{P}}(G)$.*

Proof. (1) holds by Smith-theory; for (2) compare [tD1] or [Do–Ha]. (3) is the Borel formula and (4) follows from a spectral sequence argument (the proof is almost verbatim the same as the proof of [tD2; 4.1], dealing the case $q = 2$). For the reader's convenience I will recall it here. It may be assumed $H = 1$. Look at cohomology with $\mathbf{Z}/p$-coefficients of the Borel-fibration

$$(X, X^K) \to (EM \times_K X, EM \times_K X^K) \to BK.$$

By Smith theory it may be assumed $d(1) = m > n = d(K)$. Fix generators $z \in \operatorname{Coker}(H^n(X) \to H^n(X^K))$ and $u \in H^m(X)$ to obtain from the exact sequence $H^{j-1}(X^K) \xrightarrow{\delta} H^j(X, X^K) \xrightarrow{\iota} H^j(X)$ generators $y \in H^{n+1}(X, X^K)$ and $v \in H^m(X, X^K)$ such that $\delta z = y$ and $\iota v = u$. The spectral sequence of the Borel fibration has as E_2-term $E_2^{ij} = H^i(BK) \otimes H^j(X, X^K)$. The transgression $d: E_2^{0,m} \to E_2^{m-n,n+1}$ gives a relation $d(1 \otimes v) = A \otimes y$ for a unique element $0 \neq A \in H^{m-n}(BK)$.

Consider the following map of relative fibrations:

$$\begin{array}{ccc}
(X, X^K) & \xrightarrow{l_g} & (X, X^K) \\
\downarrow & & \downarrow \\
EM \times_K (X, X^K) & \xrightarrow{L_g} & EM \times_K (X, X^K) \\
\downarrow & & \downarrow \\
BK & \xrightarrow{a_g} & BK
\end{array}$$

Here l_g denotes left translation by an element $g \in M$ of order q^k. L_g is given by $(e, x) \mapsto (eg^{-1}, gx)$ and a_g is the induced map on BK; it is induced by the automorphism of K, given by conjugation by g. The induced map of spectral sequences gives a relation $A \otimes (\deg l_g) \cdot y = d(1 \otimes (\deg l_g) \cdot u) = d(L_g^*(1 \otimes u)) =$

$L_g^* d(1 \otimes u) = L_g^*(A \otimes y) = a_g^* A \otimes l_g^* y = a_g^* A \otimes (\deg l_g^K) y$. Therefore $a_g^* A = \varepsilon \cdot A$ with $\varepsilon = (\deg l_g)(\deg l_g^K)^{-1}$.

For a generator $t \in H^2(BK)$ the element A takes the form $a \cdot t^{(m-n)/2}$. The action of g on $H^2(BK)$ is by multiplication with $\gamma \in Z/l_p^*$ such that $\gamma^{q^r} = 1$ iff $(k-l)$ divides r. From $a_g^* A = \gamma^{(m-n)/2} \cdot A = \varepsilon A$ one obtains that $q^{(k-l)}$ divides $(m-n)$. ■

(1.3) THEOREM. *The dimension homomorphism $RO(G) \to \mathcal{D}_{\mathcal{P}}(G)$ is surjective for any finite group G.*

Proof. For an open subfamily $\mathcal{F}$ of $\mathcal{P}$ let $\mathcal{D}_{\mathcal{F}}(G)$ denote the functions in $\mathcal{D}_{\mathcal{P}}(G)$, restricted to $\mathcal{F}$. Note that restriction $\mathcal{D}_{\mathcal{P}}(G) \to \mathcal{D}_{\mathcal{P}'}(G)$ is injective. Hence it suffices to show that $RO(G) \to \mathcal{D}_{\mathcal{F}}(G)$ is surjective for any open subfamily $\mathcal{F}$ of $\mathcal{P}'$. I will do this inductively. Let $\mathcal{F}' \supset \mathcal{F}$ be adjacent families, i.e. $\mathcal{F}' \backslash \mathcal{F}$ is a conjugacy class (H), H being a cyclic p-group. Suppose $RO(G) \to \mathcal{D}_{\mathcal{F}}(G)$ is surjective and let $f \in \ker(\mathcal{D}_{\mathcal{P}}(G) \to \mathcal{D}_{\mathcal{F}}(G))$. I will show that for any prime q there exists a virtual representation $V_q \in \ker(RO(G) \to \mathcal{D}_{\mathcal{F}}(G))$ with $\dim V_q^H = n_q \cdot f(H)$ such that n_q divides $|G|$ and is prime to q. As the g.c.d. of the n_q is 1, a linear combination V of the V_q will satisfy $\dim V^H = f(H)$, thus showing $RO(G) \to \mathcal{D}_{\mathcal{F}'}(G)$ is surjective.

So let's fix a prime q and choose K such that K/H is a q-Sylow subgroup of NH/H.

First case. $q = p$. Note that K is a p-group. Hence by [tD1] or [Do–Ha] there exists $\tilde{V}_p \in RO(K)$ with $\dim \tilde{V}_p^L = f(L)$ for $L \leq K$. Setting $V_p = \operatorname{ind}_K^G \tilde{V}_p$, one has $\dim V_p^L = n_p \cdot f(L)$ for $L \in \mathcal{F}'$, where $n_p = |NH:K|$ is prime to p.

Second case. $q \neq p$. Note that K is a split extension by the Zassenhaus theorem. The homomorphism $\rho: K \to \operatorname{Aut} H$, induced by conjugation, has cyclic image. Let $K' < K$ be the preimage of $\{\pm 1\} < \operatorname{Aut} H$. I will construct $\tilde{U} \in RO(K')$ with $\dim \tilde{U}^L = 0$ for $L < K'$, $L \in \mathcal{F}$, and $\dim \tilde{U}^H = 2$.

Note that $\ker \rho = K''$ is nilpotent. Let W be an irreducible representation of H with $|H:\ker W| = p$ and let R denote the trivial representation. These H-representations can be viewed as K''-representations with the q-Sylow subgroup K_q'' acting trivially. In case $K'' = K'$, set $\tilde{U} = 2R - W$ (resp. $\tilde{U} = 2R - 2W$, if $p = 2$). Otherwise take $\tilde{W} = 2R - W$ and note that $\tilde{W}$ has complex structure, thus so does $\operatorname{ind}_{K''}^{K'} \tilde{W}$. But all representations of the dihedral group K'/K_2'' are of real type (here K_2'' denotes the 2-Sylow subgroup of K''). Hence there is $\tilde{U} \in RO(K')$ with $\mathbf{C} \otimes_{\mathbf{R}} \tilde{U} = \operatorname{ind}_{K''}^{K'} \tilde{W}$. Now for $U = \operatorname{ind}_{K'}^G \tilde{U}$ one has $\dim U^L = 0$ for $L \in \mathcal{F}$ and $\dim U^H = 2 \cdot |NH:K'| = 2 \cdot |NH:K| \cdot |K:K'| = 2n_q |K:K'|$. Hence the virtual representation $(f(H)/2|K:K'|) \cdot U$ can serve as V_q. ■

Note that $\mathcal{D}_{\mathcal{P}}(G)$ is a Mackey-functor, with induction map $i_H^G: \mathcal{D}_{\mathcal{P}}(H) \to \mathcal{D}_{\mathcal{P}}(G)$ defined by $i_H^G f(K) = \sum f(gKg^{-1} \cap H)$, where the sum is over double cosets $KgH \in K \backslash G/H$ (compare [tD–Pe; 9]). In the sense of Serre [Se; 12] I have just proved a "Brauer"-like statement:

(1.4) COROLLARY. *The induction map* $\bigoplus_{(H)} \mathcal{D}_{\mathcal{P}}(H) \to \mathcal{D}_{\mathcal{P}}(G)$ *is surjective, where the sum is taken over all* $\Gamma_{\mathbf{R}}$-elementary subgroups H of G.

From this view G-spheres behave like real representations.

2. Applications to homotopy representations

A *homotopy representation* of a compact Lie group G is a $G-CW$-complex X of finite orbit type such that the fixed point sets X^H are homotopy equivalent to spheres $S^{d(H)-1}$ of the same topological dimension. The function d on the set of subgroups of G is called *dimension function.* An addition law (up to G-homotopy equivalence) is defined by taking the join, with Grothendieck group $V(G)$. A thorough discussion of homotopy representations can be found in [tD–Pe].

Let $\Psi(G)$ denote the set of conjugacy classes of closed subgroups of G, topologized via the Hausdorff-metric (compare [tD6], 5.6). Let $C(G)$ denote the group of functions $f: \Psi(G) \to \mathbb{Z}$. Associating to a homotopy representation its dimension function gives a group homomorphism $d: V(G) \to C(G)$. There are a couple of partial results on the image of d. The linearity theorem applies to give a complete description of the image of d in terms of dimension functions of linear representations as well as in combinatorial terms. The following definition introduces the candidate $\mathcal{D}(G)$ for this image, which will turn out to be the right guess.

(2.1) DEFINITION. The group $\mathcal{D}(G)$ of dimension functions of G consists of all continuous functions $f: \Psi(G) \to \mathbf{Z}$ subject to the following conditions: For any finite subquotient K/H of G there exists an element $V - W \in RO(K/H)$ such that for L with $H \triangleleft L < K$,

(i) if L/H is a p-group, then $f(L) = \dim V^L - \dim W^L$

(ii) if the p-Sylow subgroup $(L/H)_p$ is normal in L/H with quotient a 2-group, then $f(L) \equiv \dim V^L - \dim W^L \bmod 2$.

Remarks. –The continuity condition corresponds to the finite orbit type condition in the definition of homotopy representations, compare [Ba; (2.4)].

– For a homotopy representation X the fixed point set X^H is a K/H-equivariant sphere. The existence of $V - W \in RO(K/H)$ satisfying (i) is a consequence of the main theorem.

- The relations in (ii) are a consequence of an Artin relation (mod 2), as given in [Do]. The Euler characteristic of X^L is determined by the dimensions $\dim X^M$ for $M < L$ and M/H a p-group, if L is as in (ii). The Lefschetz fixed point theorem gives a tight connection between this Artin relation and the orientation behaviour of group actions on homotopy spheres, compare [tD2, (1.6), (4.1) and the following remarks].
- There also is a description of $\mathcal{D}(G)$ in terms of combinatorial relations. As to (i) these are stated in (1.1). The relation corresponding to (ii) is stated in [Do].

Here are some conditions that, together, are sufficient for a function $d: \Psi(G) \to \mathbf{Z}$ to be the dimension function of a homotopy representation:

(2.2) (a) $d \in \mathcal{D}(G)$ with $d(H) \geq 0$ for $H < G$

(b) For any subgroup $H < G$ there is a unique maximal *formal isotropy group* $\hat{H} > H$ such that $d(\hat{H}) = d(H)$.

(c) The set $\mathrm{Iso}\,(d)$ of formal isotropy groups is closed under intersection, i.e. if $K, H \in \mathrm{Iso}\,(d)$, then $H \cap K \in \mathrm{Iso}\,(d)$.

(d) If $K > H$ then $d(H) \geq d(K) + \dim\,(G/\hat{K})^H - \dim W\hat{K}$

(e) There exists a G-complex A with

(i) $\mathrm{Iso}\,(d) \supset \mathrm{Iso}\,(A) \supset (S_1 \cup S_2)$, where
$S_1 = \{H \in \mathrm{Iso}\,(d) \mid d(H) \leq \dim WH + 2\}$ and
$S_2 = \{H \in \mathrm{Iso}\,(d) \mid \text{there exists } K \in \mathrm{Iso}\,(d),\ K \geq H \text{ such that } d(H) = d(K) + \dim\,(G/K)^H\}$

(ii) If $K \leq H \leq G$ and $K \in \mathrm{Iso}\,(A)$, then A^H is a WH-homotopy representation.

(2.3) THEOREM. *If the conditions* (2.2) *are satisfied, then there exists a homotopy representation X with dimension function d and* $\mathrm{Iso}\,(X) \subset \mathrm{Iso}\,(d)$.

Proof. By [Ba; 5.7] one only has to prove it for finite G. To do so, it suffices to show that for each $H < G$ there is a WH-representation sphere Y, with $\dim Y^{K/H} = d(K)$ for any p-group K/H in WH (compare [tD2; 1.7]). By [tD1; 3.6], Y need only exist stably, i.e. there is an element $V - W \in RO(WH)$ with $\dim V^K - \dim W^K = d(K)$ for any p-group K/H in WH. But this is true by the definition of $\mathcal{D}(G)$. Note that the orientation behaviour of X^H is determined by (2.1)(ii) (compare also [Ba; 5.4]). ∎

Remark. All conditions in (2.2) are necessary ones, except condition (c) (compare [Ba] and [La]). The latter paper gives an example (2.14) in which condition (c) does not hold.

Stably the picture is more pleasant: Combining with results in [tD–Pe] and [tD3], one gets the structure theorem:

(2.4) THEOREM. *There exists an exact sequence*

$$0 \to \operatorname{Pic}(A(G)) \to V(G) \xrightarrow{d} \mathscr{D}(G) \to 0.$$

Here Pic $(A(G))$ *denotes the Picard group of the Burnside ring* $A(G)$.

Proof. The kernel $v(G)$ of d was computed in [tD–Pe] for the finite group case and in [tD3] for the compact Lie case: For an appropriate G-map $f: X \to Y$ between two homotopy representations with the same dimension function the collection of degrees $(\deg f^H)$ defines an element in Pic $(A(G))$. This results in an isomorphism $v(G) \cong \operatorname{Pic}(A(G))$. By the remarks following the definition of $\mathscr{D}(G)$ the homomorphism d maps into $\mathscr{D}(G)$. Given a dimension function $d \in \mathscr{D}(G)$, all of the conditions (2.2) can be satisfied by adding the dimension function of an appropriate real representation, compare [Ba, (2.4)]. This proves surjectivity. ■

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