

Zeitschrift: Commentarii Mathematici Helvetici
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 64 (1989)

Artikel: A linearity theorem for group actions on spheres with applications to homotopy representations.
Autor: Bauer, Stefan
DOI: <https://doi.org/10.5169/seals-48938>

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A linearity theorem for group actions on spheres with applications to homotopy representations

STEFAN BAUER

Abstract. Let G be a finite group and X an equivariant $\mathbb{Z}/|G|$ -homology sphere. By Smith-theory the fixed point set X^H for a p -subgroup H is a \mathbb{Z}/p -homology sphere of dimension $d(H)$.

THEOREM. *There exists a virtual representation $V - W \in RO(G)$ such that $d(H) = \dim V^H - \dim W^H$ holds for any subgroup H of G of prime power order.*

This applies to describing the Grothendieck group $\mathcal{D}(G)$ of dimension functions of homotopy representations for compact Lie groups G in algebraic terms.

Let G be a finite group and $T \subset \mathbb{Q}$ denote a subring which is contained in the local ring $\mathbb{Z}_{(p)}$ for any prime divisor p of the order $|G|$ of G . If X is a finitistic G -space, which is a T -homology sphere, then the fixed point set X^H is a $\mathbb{Z}_{(p)}$ -homology sphere of dimension $d(H)$ for any p -subgroup H of G .

The purpose of this paper is to give a proof and an application of the following

THEOREM. *There exists a virtual representation $V - W \in RO(G)$ such that $d(H) = \dim V^H - \dim W^H$ holds for any subgroup H of G of prime power order.*

In particular d can be characterized by the combinatorial congruences in (1.1). Related linearity theorems are known for special cases: Homotopy representations of finite nilpotent groups do have stably linear dimension functions [tD1]. The theorem above is known for p -groups [Do-Ha].

Taking into account tom Dieck's analysis of group actions on homotopy spheres (compare [tD4] and [tD5]), the linearity theorem also gives sufficient conditions for the existence of group actions on homotopy spheres. An application to homotopy representations will be discussed in the second part.

Let \mathcal{P} denote the open family of subgroups (i.e. \mathcal{P} is a union of conjugacy classes, such that for any $K \in \mathcal{P}$ all subgroups of K are also contained in \mathcal{P}) of prime power order and \mathcal{P}' the subfamily of cyclic ones.

(1.1) **DEFINITION.** Let $\mathcal{D}_{\mathcal{P}}(G)$ denote the group of functions $f: \mathcal{P} \rightarrow \mathbb{Z}$,

constant on conjugacy classes, which satisfy the following relations: For every triple $H \triangleleft K \triangleleft M$ of subgroups of G with $K/H \cong \mathbb{Z}/p$ and H a p -group,

- (1) $f(H) \equiv f(K) \pmod{2}$, if p is odd or if $M/H \cong \mathbb{Z}/4$
- (2) $f(H) \equiv f(K) \pmod{4}$, if $M/H \cong$ quaternion group
- (3) $f(H) + p \cdot f(M) = \sum f(K_i)$, if $M/H \cong \mathbb{Z}/p \times \mathbb{Z}/p$. Here of course the sum is over all K_i with $H \triangleleft K_i \triangleleft M$ and $K_i/H \cong \mathbb{Z}/p$
- (4) $f(H) \equiv f(K) \pmod{q^{r-l}}$, if $M/K \cong \mathbb{Z}/q^r$, acting on K/H with kernel of prime power order q^l .

(1.2) PROPOSITION. If $d: \mathcal{P} \rightarrow \mathbb{Z}$ denotes the dimension function of a finitistic G -space X which is a T -homology sphere, then $d \in \mathcal{D}_\varphi(G)$.

Proof. (1) holds by Smith-theory; for (2) compare [tD1] or [Do-Ha]. (3) is the Borel formula and (4) follows from a spectral sequence argument (the proof is almost verbatim the same as the proof of [tD2; 4.1], dealing the case $q = 2$). For the reader's convenience I will recall it here. It may be assumed $H = 1$. Look at cohomology with \mathbb{Z}/p -coefficients of the Borel-fibration

$$(X, X^K) \rightarrow (EM \times_K X, EM \times_K X^K) \rightarrow BK.$$

By Smith theory it may be assumed $d(1) = m > n = d(K)$. Fix generators $z \in \text{Coker}(H^n(X) \rightarrow H^n(X^K))$ and $u \in H^m(X)$ to obtain from the exact sequence $H^{j-1}(X^K) \xrightarrow{\delta} H^j(X, X^K) \xrightarrow{i} H^j(X)$ generators $y \in H^{n+1}(X, X^K)$ and $v \in H^m(X, X^K)$ such that $\delta z = y$ and $\iota v = u$. The spectral sequence of the Borel fibration has as E_2 -term $E_2^{ij} = H^i(BK) \otimes H^j(X, X^K)$. The transgression $d: E_2^{0,m} \rightarrow E_2^{m-n, n+1}$ gives a relation $d(1 \otimes \underset{\downarrow \psi}{u}) = A \otimes y$ for a unique element $0 \neq A \in H^{m-n}(BK)$.

Consider the following map of relative fibrations:

$$\begin{array}{ccc} (X, X^K) & \xrightarrow{l_g} & (X, X^K) \\ \downarrow & & \downarrow \\ EM \times_K (X, X^K) & \xrightarrow{L_g} & EM \times_K (X, X^K) \\ \downarrow & & \downarrow \\ BK & \xrightarrow{a_g} & BK \end{array}$$

Here l_g denotes left translation by an element $g \in M$ of order q^k . L_g is given by $(e, x) \mapsto (eg^{-1}, gx)$ and a_g is the induced map on BK ; it is induced by the automorphism of K , given by conjugation by g . The induced map of spectral sequences gives a relation $A \otimes (\deg l_g) \cdot y = d(1 \otimes (\deg l_g) \cdot u) = d(L_g^*(1 \otimes u)) =$

$L_g^* d(1 \otimes u) = L_g^*(A \otimes y) = a_g^* A \otimes l_g^* y = a_g^* A \otimes (\deg l_g^K) y$. Therefore $a_g^* A = \varepsilon \cdot A$ with $\varepsilon = (\deg l_g)(\deg l_g^K)^{-1}$.

For a generator $t \in H^2(BK)$ the element A takes the form $a \cdot t^{(m-n)/2}$. The action of g on $H^2(BK)$ is by multiplication with $\gamma \in \mathbb{Z}/p^*$ such that $\gamma^{q'} = 1$ iff $(k-l)$ divides r . From $a_g^* A = \gamma^{(m-n)/2} \cdot A = \varepsilon A$ one obtains that $q^{(k-l)}$ divides $(m-n)$. ■

(1.3) THEOREM. *The dimension homomorphism $RO(G) \rightarrow \mathcal{D}_p(G)$ is surjective for any finite group G .*

Proof. For an open subfamily \mathcal{F} of \mathcal{P} let $\mathcal{D}_{\mathcal{F}}(G)$ denote the functions in $\mathcal{D}_p(G)$, restricted to \mathcal{F} . Note that restriction $\mathcal{D}_p(G) \rightarrow \mathcal{D}_{\mathcal{F}}(G)$ is injective. Hence it suffices to show that $RO(G) \rightarrow \mathcal{D}_{\mathcal{F}}(G)$ is surjective for any open subfamily \mathcal{F} of \mathcal{P} . I will do this inductively. Let $\mathcal{F}' \supset \mathcal{F}$ be adjacent families, i.e. $\mathcal{F}' \setminus \mathcal{F}$ is a conjugacy class (H) , H being a cyclic p -group. Suppose $RO(G) \rightarrow \mathcal{D}_{\mathcal{F}}(G)$ is surjective and let $f \in \ker(\mathcal{D}_p(G) \rightarrow \mathcal{D}_{\mathcal{F}}(G))$. I will show that for any prime q there exists a virtual representation $V_q \in \ker(RO(G) \rightarrow \mathcal{D}_{\mathcal{F}}(G))$ with $\dim V_q^H = n_q \cdot f(H)$ such that n_q divides $|G|$ and is prime to q . As the g.c.d. of the n_q is 1, a linear combination V of the V_q will satisfy $\dim V^H = f(H)$, thus showing $RO(G) \rightarrow \mathcal{D}_{\mathcal{F}}(G)$ is surjective.

So let's fix a prime q and choose K such that K/H is a q -Sylow subgroup of NH/H .

First case. $q = p$. Note that K is a p -group. Hence by [tD1] or [Do-Ha] there exists $\tilde{V}_p \in RO(K)$ with $\dim \tilde{V}_p^L = f(L)$ for $L \leq K$. Setting $V_p = \text{ind}_K^G \tilde{V}_p$, one has $\dim V_p^L = n_p \cdot f(L)$ for $L \in \mathcal{F}'$, where $n_p = |NH:K|$ is prime to p .

Second case. $q \neq p$. Note that K is a split extension by the Zassenhaus theorem. The homomorphism $\rho: K \rightarrow \text{Aut } H$, induced by conjugation, has cyclic image. Let $K' < K$ be the preimage of $\{\pm 1\} < \text{Aut } H$. I will construct $\tilde{U} \in RO(K')$ with $\dim \tilde{U}^L = 0$ for $L < K'$, $L \in \mathcal{F}$, and $\dim \tilde{U}^H = 2$.

Note that $\ker \rho = K''$ is nilpotent. Let W be an irreducible representation of H with $|H: \ker W| = p$ and let R denote the trivial representation. These H -representations can be viewed as K'' -representations with the q -Sylow subgroup K''_q acting trivially. In case $K'' = K'$, set $\tilde{U} = 2R - W$ (resp. $\tilde{U} = 2R - 2W$, if $p = 2$). Otherwise take $\tilde{W} = 2R - W$ and note that \tilde{W} has complex structure, thus so does $\text{ind}_{K''}^{K'} \tilde{W}$. But all representations of the dihedral group K'/K''_2 are of real type (here K''_2 denotes the 2-Sylow subgroup of K''). Hence there is $\tilde{U} \in RO(K')$ with $\mathbb{C} \otimes_{\mathbb{R}} \tilde{U} = \text{ind}_{K''}^{K'} \tilde{W}$. Now for $U = \text{ind}_K^{K'} \tilde{U}$ one has $\dim U^L = 0$ for $L \in \mathcal{F}$ and $\dim U^H = 2 \cdot |NH:K'| = 2 \cdot |NH:K| \cdot |K:K'| = 2n_q |K:K'|$. Hence the virtual representation $(f(H)/2 |K:K'|) \cdot U$ can serve as V_q . ■

Note that $\mathcal{D}_\varphi(G)$ is a Mackey-functor, with induction map $i_H^G: \mathcal{D}_\varphi(H) \rightarrow \mathcal{D}_\varphi(G)$ defined by $i_H^G f(K) = \sum f(gKg^{-1} \cap H)$, where the sum is over double cosets $KgH \in K \backslash G/H$ (compare [tD–Pe; 9]). In the sense of Serre [Se; 12] I have just proved a “Brauer”-like statement:

(1.4) COROLLARY. *The induction map $\bigoplus_{(H)} \mathcal{D}_\varphi(H) \rightarrow \mathcal{D}_\varphi(G)$ is surjective, where the sum is taken over all $\Gamma_{\mathbf{R}}$ -elementary subgroups H of G .*

From this view G -spheres behave like real representations.

2. Applications to homotopy representations

A *homotopy representation* of a compact Lie group G is a G – CW -complex X of finite orbit type such that the fixed point sets X^H are homotopy equivalent to spheres $S^{d(H)-1}$ of the same topological dimension. The function d on the set of subgroups of G is called *dimension function*. An addition law (up to G -homotopy equivalence) is defined by taking the join, with Grothendieck group $V(G)$. A thorough discussion of homotopy representations can be found in [tD–Pe].

Let $\Psi(G)$ denote the set of conjugacy classes of closed subgroups of G , topologized via the Hausdorff-metric (compare [tD6], 5.6). Let $C(G)$ denote the group of functions $f: \Psi(G) \rightarrow \mathbb{Z}$. Associating to a homotopy representation its dimension function gives a group homomorphism $d: V(G) \rightarrow C(G)$. There are a couple of partial results on the image of d . The linearity theorem applies to give a complete description of the image of d in terms of dimension functions of linear representations as well as in combinatorial terms. The following definition introduces the candidate $\mathcal{D}(G)$ for this image, which will turn out to be the right guess.

(2.1) DEFINITION. The group $\mathcal{D}(G)$ of dimension functions of G consists of all continuous functions $f: \Psi(G) \rightarrow \mathbb{Z}$ subject to the following conditions: For any finite subquotient K/H of G there exists an element $V - W \in RO(K/H)$ such that for L with $H \triangleleft L < K$,

- (i) if L/H is a p -group, then $f(L) = \dim V^L - \dim W^L$
- (ii) if the p -Sylow subgroup $(L/H)_p$ is normal in L/H with quotient a 2-group, then $f(L) \equiv \dim V^L - \dim W^L \pmod{2}$.

Remarks. –The continuity condition corresponds to the finite orbit type condition in the definition of homotopy representations, compare [Ba; (2.4)].

– For a homotopy representation X the fixed point set X^H is a K/H -equivariant sphere. The existence of $V - W \in RO(K/H)$ satisfying (i) is a consequence of the main theorem.

- The relations in (ii) are a consequence of an Artin relation (mod 2), as given in [Do]. The Euler characteristic of X^L is determined by the dimensions $\dim X^M$ for $M < L$ and M/H a p -group, if L is as in (ii). The Lefschetz fixed point theorem gives a tight connection between this Artin relation and the orientation behaviour of group actions on homotopy spheres, compare [tD2, (1.6), (4.1) and the following remarks].
- There also is a description of $\mathcal{D}(G)$ in terms of combinatorial relations. As to (i) these are stated in (1.1). The relation corresponding to (ii) is stated in [Do].

Here are some conditions that, together, are sufficient for a function $d: \Psi(G) \rightarrow \mathbf{Z}$ to be the dimension function of a homotopy representation:

- (2.2) (a) $d \in \mathcal{D}(G)$ with $d(H) \geq 0$ for $H < G$
- (b) For any subgroup $H < G$ there is a unique maximal *formal isotropy group* $\hat{H} > H$ such that $d(\hat{H}) = d(H)$.
- (c) The set $\text{Iso}(d)$ of formal isotropy groups is closed under intersection, i.e. if $K, H \in \text{Iso}(d)$, then $H \cap K \in \text{Iso}(d)$.
- (d) If $K > H$ then $d(H) \geq d(K) + \dim(G/\hat{K})^H - \dim W\hat{K}$
- (e) There exists a G -complex A with
- (i) $\text{Iso}(d) \supset \text{Iso}(A) \supset (S_1 \cup S_2)$, where
 $S_1 = \{H \in \text{Iso}(d) \mid d(H) \leq \dim WH + 2\}$ and
 $S_2 = \{H \in \text{Iso}(d) \mid \text{there exists } K \in \text{Iso}(d), K \geq H \text{ such that } d(H) = d(K) + \dim(G/K)^H\}$
- (ii) If $K \leq H \leq G$ and $K \in \text{Iso}(A)$, then A^H is a WH -homotopy representation.

(2.3) THEOREM. *If the conditions (2.2) are satisfied, then there exists a homotopy representation X with dimension function d and $\text{Iso}(X) \subset \text{Iso}(d)$.*

Proof. By [Ba; 5.7] one only has to prove it for finite G . To do so, it suffices to show that for each $H < G$ there is a WH -representation sphere Y , with $\dim Y^{K/H} = d(K)$ for any p -group K/H in WH (compare [tD2; 1.7]). By [tD1; 3.6], Y need only exist stably, i.e. there is an element $V - W \in RO(WH)$ with $\dim V^K - \dim W^K = d(K)$ for any p -group K/H in WH . But this is true by the definition of $\mathcal{D}(G)$. Note that the orientation behaviour of X^H is determined by (2.1)(ii) (compare also [Ba; 5.4]). ■

Remark. All conditions in (2.2) are necessary ones, except condition (c) (compare [Ba] and [La]). The latter paper gives an example (2.14) in which condition (c) does not hold.

Stably the picture is more pleasant: Combining with results in [tD–Pe] and [tD3], one gets the structure theorem:

(2.4) THEOREM. *There exists an exact sequence*

$$0 \rightarrow \text{Pic}(A(G)) \rightarrow V(G) \xrightarrow{d} \mathcal{D}(G) \rightarrow 0.$$

Here $\text{Pic}(A(G))$ denotes the Picard group of the Burnside ring $A(G)$.

Proof. The kernel $v(G)$ of d was computed in [tD–Pe] for the finite group case and in [tD3] for the compact Lie case: For an appropriate G -map $f: X \rightarrow Y$ between two homotopy representations with the same dimension function the collection of degrees $(\deg f^H)$ defines an element in $\text{Pic}(A(G))$. This results in an isomorphism $v(G) \cong \text{Pic}(A(G))$. By the remarks following the definition of $\mathcal{D}(G)$ the homomorphism d maps into $\mathcal{D}(G)$. Given a dimension function $d \in \mathcal{D}(G)$, all of the conditions (2.2) can be satisfied by adding the dimension function of an appropriate real representation, compare [Ba, (2.4)]. This proves surjectivity. ■

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SONDERFORSCHUNGSBEREICH 170: "Geometrie und Analysis" Mathematisches Institut, Bunsenstr. 3–5, D-3400 Göttingen

University of Chicago, Department of Mathematics, 5734 University Avenue, Chicago, IL 60637

Received March 10, 1987.