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1. Introduction

In this paper, we study some general properties of actions of subgroups of the mapping class group on Thurston's sphere of projective measured foliations from the point of view of dynamics. This work was strongly motivated by the work of Howard Masur on the limit sets of the two handlebody groups described in [11]. We prove that many of the properties of the limit sets of these handlebody groups are properties of limit sets for general subgroups. In particular, if the limit set is not equal to the whole sphere, we prove the existence of a nonempty open set where the group acts properly discontinuously and we describe a fundamental domain for this action.

A special case of this study has already been extensively investigated. The mapping class group of the once-punctured torus is simply $SL(2, Z)$. The Teichmüller space of a once-punctured torus is well known to be hyperbolic 2-space. Moreover, Thurston's sphere of foliations is the circle at infinity. Under these identifications, the action of the mapping class group on Thurston's sphere is simply the usual action of $SL(2, Z)$ by linear fractional transformations, (i.e. the action of $PSL(2, Z)$ in $PSL(2, C)$) [8].

Hence, one reason for making such a study is that the dynamics of the actions of subgroups of the mapping class group on Thurston's sphere ought to exhibit many of the interesting phenomena of the actions of discrete subgroups of $PSL(2, C)$ on the 2-sphere. The theorems that we prove develop this line of thought.

When one begins to think about the dynamics of the action of a subgroup of the mapping class group, it appears that there may be several interesting subsets of the sphere of projective measured foliations that could possibly play the role of a limit set. For example, one could define the limit set to be the closure of the set of accumulation points of orbits of elements of the sphere under the action of the subgroup. As other possible definitions, we could choose the closure of the set of elements which are fixed by an infinite subgroup or the closure of the set of elements which are attracting points for single elements of the group. One may also adopt a different point of view, consider the sphere as the boundary of

Teichmüller space and define the limit set to be the closure of the set of accumulation points of orbits of points in Teichmüller space. There are obviously many other possible definitions. All the sets defined above are related to one another and the choice of a definition for the limit set may be a matter of deciding which dynamical properties we want to capture.

One feature that makes this choice less obvious than in the case of subgroups of $PSL(2\,C)$ acting on S^2 is the existence of "hybrid" elements in the mapping class group, (e.g. reducible elements which have two pseudo-Anosov components). Many possible definitions of a limit set seem to yield a theory involving the ratios of the various expansion factors for the component maps. It turns out, however, that if the subgroup has "enough" pseudo-Anosov elements, then one can ignore the hybrid elements and obtain a reasonable theory in which the limit set is the closure of the fixed points of pseudo-Anosov elements. In particular, one has the fact that this set is the unique minimal closed set which is invariant under the group, which, needless to say, is a desirable feature.

The groups with sufficiently many pseudo-Anosov elements, which belong to a class of groups referred to as dynamically irreducible, are the main focus of our study. The remaining groups, including the dynamically reducible groups, are characterized in a manner following Thurston's classification of mapping classes. Among these groups, one finds the virtually abelian groups which correspond, in some sense, to the elementary subgroups of $PSL(2, C)$.

The outline of the paper is as follows. In section 2, we collect some preliminary facts about the theory of measured foliations and mapping class groups which we shall use frequently in the subsequent text.

In section 3, we give a rather broad definition of the concept of a limit set for a group action.

In section 4, we define the notion of a dynamically irreducible group (action) and give a characterization of dynamically irreducible subgroups of the mapping class group.

In section 5, we develop some basic properties of nonelementary dynamically irreducible subgroups and discuss some examples. In particular, we define a limit set for such groups which has the minimality property discussed above.

In section, 6, we prove that if the limit set for a nonelementary dynamically irreducible subgroup is not the entire sphere, then there is a region of proper-discontinuity for the subgroup which, except for a set of measure zero, is the complement of the limit set. In addition, we describe a fundamental domain for the action of the group on this region of discontinuity. The section begins with a discussion of a simple example, the cyclic group generated by a pseudo-Anosov element. Although this example is well known and well understood, it serves as motivations for the constructions in the general case. (Strictly speaking, this example is not dynamically irreducible). This section concludes with the discussion of the general case.

In section 7, the theory developed in sections 5 and 6 is extended to nonelementary dynamically reducible groups. As in the classification of elements of the mapping class group, this theory is described in terms of the limit sets of the "components" of the group. We show that outside the "join" of these limit sets, except for a set of measure zero, the group acts properly discontinuously. In particular, dynamically reducible groups always have a region of discontinuity. Again, we describe a fundamental domain for the action on this region.

Finally, in section 8, we show how the dynamics of subgroups of the mapping class group on the compactified Teichmüller space can be derived from the dynamics on the boundary.

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2. Preliminaries

Let S be a compact surface of negative Euler characteristic which is not a pair of pants. Let Γ be the mapping class group of S. Let **MF** be the space of equivalence classes of measured foliations on S. We note that in this definition, we do not consider the "zero foliation" as a point in **MF**. The set $\mathbf{R}_+$ of positive real numbers acts on **MF**, and **PMF** denotes the quotient space. We refer the reader to [1] for a general introduction to surface mapping class groups and to [5] for an exposition of Thurston's theory of measured foliations, measured foliation space and the action of Γ on that space.

Let **S** denote the set of isotopy classes of unoriented, connected, homotopically nontrivial, simple closed curves on S which are not parallel to a boundary component of S. The intersection function on $\mathbf{S} \times \mathbf{S}$, $i(,)$, is defined by the rule:

$$i(\alpha, \beta) = \min \{\text{cardinality } (a \cap b) \text{ where } a \in \alpha \text{ and } b \in \beta\}. \tag{2.1}$$

This function is clearly symmetric and Γ-invariant.

Considering **S** as a subset of **MF**, we can extend i to a symmetric, Γ-invariant, bihomogeneous function on the square of the orbit of **S** under scalar multiplication by positive reals, $R_+^* \times \mathbf{S}$. This latter set is dense in **MF** and the function extends (uniquely, of course) to a symmetric, Γ-invariant, bihomogeneous, continuous intersection function on **MF** × **MF**:

$$i: \mathbf{MF} \times \mathbf{MF} \to R_+, \tag{2.2}$$

where R_+ is the set of nonnegative reals.

The action of Γ on **MF** (or **PMF**) is a faithful action provided that S is not a closed surface of genus two, a torus with one hole, or a sphere with four holes. In each case, the kernel of the action is a cyclic subgroup of order two. (These are the only nontrivial maps of compact hyperbolic surfaces which preserve every element of **S**). We shall say that the involution is hypergeometric.

If g is an element of Γ, and g fixes every simple closed curve in **S**, then g is either the identity or the hypergeometric involution of a closed surface of genus two, a torus with one hole or a sphere with four holes. (2.3)

Hence, for most purposes, we may speak as though the action of Γ on **MF** is faithful. The same holds for **PMF**.

The elements of Γ are classified into three types, finite order, pseudo-Anosov and reducible. In discussing dynamical aspects of the action, the finite order elements play a minor role. Indeed, if we wished, we could pass to a torsion free subgroup of finite index in Γ and restrict our study to subgroups of this torsion free group [8]. In any case, the pseudo-Anosov elements and the infinite order reducible elements are of primary concern to us.

We shall say that an element L of **MF** is a *pseudo-Anosov foliation* if there is a pseudo-Anosov map, ϕ, which fixes the projective class of L, $[L]$. In this case, we see that:

$$\phi(L) = t \cdot L \quad \text{where} \quad t > 0 \quad \text{and} \quad t \neq 1. \tag{2.4}$$

We shall follow the standard convention and denote L as L_+ if $t > 1$ and L_- if $t < 1$. Similarly, we denote t as λ if $t > 1$ and λ^{-1} if $t < 1$. Hence, λ is always taken as a positive real greater than 1. We refer to L_+ as an *attracting foliation for* ϕ, and to L_- as a *repelling foliation for* ϕ. A *pseudo-Anosov pair*, $|L|$, is a pair of the form $\{L_+, L_-\}$.

Let L be a pseudo-Anosov foliation. It is a well-known consequence of the minimality of L, the unique ergodicity of L and properties of the intersection function that the set defined by the equation $i(L, .) = 0$ consists precisely of the scalar multiples of L [13].

If L is a pseudo-Anosov foliation, then $i(F, L) = 0 \Leftrightarrow [F] = [L]$. (2.5)

Let $|L|$ be a pseudo-Anosov pair, $\{L_+, L_-\}$. It follows from (2.5) that the "zero sets" of L_+ and L_- are disjoint. Hence, we can define a continuous, homogeneous, nonzero function on **MF**, $\|\ \|$, by the following rule:

$$\begin{aligned} &\|\ \|: \mathbf{MF} \to R_+ \\ &\qquad F \to \max\{i(F, L_+), i(F, L_-)\}. \end{aligned} \tag{2.6}$$

We shall refer to any continuous, homogeneous, nonzero function on **MF**, $\|\ \|$, as a *norm on* **MF**. (We do not necessarily assume that the function is defined, as in (2.5), via a pseudo-Anosov pair. This construction, however, is quite useful).

Since **PMF** is compact, all norms are equivalent. A sequence of measured foliations converges to zero or to infinity as the norms converge to zero or infinity.

Given a norm on **MF**, $\|\ \|$, we have the corresponding section from **PMF** to **MF**:

$$\begin{aligned} &s: \mathbf{PMF} \to \mathbf{MF} \\ &\qquad [F] \to F/\|F\|. \end{aligned} \tag{2.7}$$

Hence, we may regard **PMF** as a subset of **MF**, provided we observe that these sections are not Γ-equivariant. (Indeed, the action of Γ on **S** illustrates that there does not exist a Γ-equivariant section).

From the compactness of **PMF**, and the bihomogeneity of $i(,)$, we observe that i is comparable to the norm in the following sense:

There exists a positive constant, C, such that $i(F, G) \leq C.\ \|F\| \cdot \|G\|$, for each (F, G) in $\mathbf{MF} \times \mathbf{MF}$. (2.8)

Another useful construction of a norm on **MF** is obtained from the theory of pavings of **MF** ([7], [13]). One chooses finitely many train tracks which carry all measured foliations. The norm of a foliation is then defined as the sum of the weights of the foliation on the branches of a track in the collection which carries the foliation. (That this collection can be chosen in such a way to insure that this construction yields a well defined norm is not readily apparent. It is, however, a consequence of the construction of pavings.) In this version of a norm, the norms of simple closed curves are integral. Indeed, all the weights of a simple closed curve are nonnegative integers. The finiteness of the paving makes it clear that **S** is a discrete subset of **MF** which is bounded below.

There exists a positive constant, C, such that $\|\alpha\| \geq C$ for every α in $\mathbf{S}$. (2.9)

For each real number, M, $\{\alpha \in \mathbf{S} \mid \|\alpha\| \leq M\}$ is a finite set. (2.10)

From the compactness of **PMF** and these observations, one obtains the following useful fact:

For any infinite sequence of distinct simple closed curves, $\{\alpha_n\}$, there exists a subsequence, $\{\beta_n\}$, and a sequence of positive scalars, $\{r_n\}$, such that: (2.11)

(a) $r_n \cdot \beta_n \to F$ for some F in **MF**
(b) $r_n \to 0$.

Henceforth, we shall not bother to distinguish subsequences. They shall be understood in context.

A *coordinate system*, A, is a finite family of simple closed curves in $\mathbf{S}$ such that the *coordinate functions*, $\{i(\alpha, .) \mid \alpha \in \mathbf{S}\}$, parametrize **MF**. It follows from the classification of measured foliations carried over in [5] that such families exist. Of course, we may regard a coordinate system as a subset of **MF**. The Γ-invariance of $i(,)$ implies that Γ acts on coordinate systems. The next proposition shows that this action has only finite stabilizers.

PROPOSITION 2.1. *Let A be a coordinate system. The stabilizer of A in Γ, Γ_A, is a finite group.*

Proof. There is, of course, a subgroup of finite index in Γ_A, Σ, which fixes each element of A. It suffices to show that Σ is finite. Suppose, therefore, that g is an element of Σ. Then:

$$i(\alpha, g^{-1}\beta) = i(g\alpha, \beta) = i(\alpha, \beta) \quad \text{for each} \quad \beta \in \mathbf{S}, \quad \alpha \in A. \tag{1}$$

By the definition of a coordinate system, it follows that g^{-1} fixes every curve in $\mathbf{S}$, and hence, g fixes every curve in $\mathbf{S}$. By (2.3), it follows that g is either the identity or a hypergeometric involution. Hence, Σ is of order at most two. This completes the proof of proposition 2.1.

As an immediate consequence, we have:

If A and B are coordinate systems, then there are at most finitely many maps taking A to B. (2.12)

Now suppose that we are given an infinite sequence of distinct elements of Γ, $\{g_n\}$. Let A be a coordinate system. By (2.12), the collection of coordinate systems, $\{g_n(A)\}$, must be infinite. Hence, for some curve, α, in A, the collection of curves, $\{g_n(\alpha)\}$, must also be infinite (though not necessarily distinct). Therefore:

For any infinite sequence of distinct mapping classes, $\{g_n\}$, there exists a simple closed curve, α, such that $\{\|g_n(\alpha)\|\}$ is unbounded. (2.13)

The discreteness of **S** implies that the orbit in **MF** of a simple closed curve, α, under Γ is discrete. Hence, these observations are not surprising. When one considers the orbit of a pseudo-Anosov foliation in **MF**, one finds that the orbit can have accumulation points (other than 0 or ∞). Nevertheless, one can obtain similar statements, as we shall now see.

PROPOSITION 2.2 [12]. *Let* $[L]$ be a pseudo-Anosov foliation in **PMF**. *The stabilizer of* $[L]$ *in* Γ, $\Gamma_{[L]}$, *is a virtually cyclic group.*

PROPOSITION 2.3. *Let* L *be a pseudo-Anosov foliation in* **MF**. *The stabilizer of* L *in* Γ, $\Gamma_{|L|}$, *is a finite group.*

Proof. Let ϕ be a pseudo-Anosov mapping class which fixes the projective class of L, $[L]$. This mapping class generates an infinite cyclic subgroup, $gp(\phi)$, of $\Gamma_{[L]}$. Moreover, Σ is a subgroup of $\Gamma_{[L]}$ which has trivial intersection with $gp(\phi)$. The statement follows, therefore, from Proposition 2.2.

Remark. Alternatively, one can observe from [5], exposé 9, §IV, that L_Γ is a torsion subgroup of Γ and then employ the fact that Γ has a torsion free subgroup of finite index.

COROLLARY 2.4. *Let* $|L|$ *be a pseudo-Anosov pair in* **MF**. *The stabilizer of* $|L|$ *in* Γ, $\Gamma_{|L|}$ *is a finite group.*

Proof. Let $|L| = \{L_+, L_-\}$. $\Gamma_{|L|}$ has a subgroup of index at most 2 which fixes L_+ (and L_-). The statement follows from Proposition 2.3.

LEMMA 2.5. *Let* $|L|$ *be a pseudo-Anosov pair,* $\{L_+, L_-\}$ *in* **MF**. *Let* g *be an element of* Γ. *Let* t *be a positive real. If* $g(L_+) = t \cdot L_+$, *then* $g(L_-) = t^{-1} \cdot L_-$. *If* $g(L_+) = t \cdot L_-$, *then* $g(L_-) = t^{-1} \cdot L_+$.

Proof. Let ϕ be a pseudo-Anosov map fixing $[L_+]$ and $[L_-]$. Let λ be the dilitation factor for ϕ, $\lambda > 1$.

$$\phi(L_+) = \lambda \cdot L_+ \qquad \phi(L_-) = \lambda^{-1} \cdot L_-. \tag{1}$$

Suppose that $g(L_+) = t \cdot L_+$. Consider the pseudo-Anosov map $g\phi g^{-1}$, which we denote as ψ.

$$\psi(L_+) = g\phi g^{-1}(L_+) = \lambda \cdot L_+. \tag{2}$$

This implies that ϕ and ψ are both in the stabilizer of $[L_+]$. By Proposition 2.2, we conclude that ϕ and ψ must have a common power. For some nonzero integers, m and n, $\phi^m = \psi^n$. (Furthermore, by equations (1) and (2), and the fact that $\lambda > 1$, we know that $m = n$.)

Since the fixed point sets of ϕ and ψ are equal to the fixed point sets of ϕ^m and ψ^n, we conclude that ϕ and ψ have the same fixed points. On the other hand, $\mathrm{Fix}\,(\psi) = \mathrm{Fix}\,(g\phi g^{-1}) = g(\mathrm{Fix}\,(\phi))$. Hence, we conclude that g fixes $[L_-]$.

$$g(L_-) = s \cdot L_-. \tag{3}$$

To determine the value of s, we consider the intersection of L_- and L_+ which, by (2.5), we know is nonzero:

$$0 \neq i(L_+, L_-) = i(gL_-, gL_+) = s.t.i.\ (L_-, L_+). \tag{4}$$

Hence, $s \cdot t = 1$ and $gL_- = t^{-1}L_-$. A similar argument establishes the second assertion.

Remark. In the second case, we conclude that $g^2(L_+) = L_+$. Hence, the map, g, must be of finite order. This does not, however, imply that $t = 1$ (e.g. consider the maps, g, and ϕg).

COROLLARY 2.6. *Let $|L|$ be a pseudo-Anosov pair, $\{L_+, L_-\}$ in* **MF**. *Then the stabilizers of L_+ and of L_- are contained in the stabilizer of $|L|$ and the stabilizers of $[L_+]$ and of $[L_-]$ are contained in the stabilizer of $\{[L_+], [L_-]\}$.*

As an immediate consequence, we have:

If $|K|$ and $|L|$ are pseudo-Anosov pairs, then there are at most finitely many maps taking $|K|$ to $|L|$. (2.14)

We shall now establish the analog of (2.13).

LEMMA 2.7. *Let $|L|$ be a pseudo-Anosov pair, $\{L_+, L_-\}$. Let $\{g_n\}$ be an infinite sequence of distinct mapping classes. Then for some ε in $\{+, -\}$, the sequence of real numbers, $\{\|g_nL_\varepsilon\|\}$, is unbounded.*

Proof. Suppose that both sequences are bounded:

$$\|g_n L_\pm\| \leq M \quad \text{for all } n. \tag{1}$$

By (2.13), we may find a simple closed curve, α, such that $\{\|g_n^{-1}(\alpha)\|\}$ is unbounded. By (2.11), we may choose a subsequence of $\{g_n\}$ and a sequence of scalars, $\{r_n\}$, such that:

$$r_n g_n^{-1}(\alpha) \to F \quad \text{in } \mathbf{MF}, \tag{2}$$

$$r_n \to 0. \tag{3}$$

From (2.8), we conclude that:

$$i(r_n g_n^{-1}(\alpha), L_\pm) = r_n i(g_n^{-1}(\alpha), L_\pm) = r_n i(\alpha, g_n L_\pm)$$

$$i(r_n g_n^{-1}(\alpha), L_\pm) \leq r_n C \|\alpha\| \cdot \|g_n L_\pm\|. \tag{4}$$

Combining (1), (2), (3) and (4), we find that;

$$i(F, L_\pm) = 0. \tag{5}$$

But this is impossible by (2.5). Hence, one of the two sequences must be unbounded. This proves the lemma.

As should be apparent, the pseudo-Anosov foliations have some rather striking properties. In studying the dynamics of subgroups of Γ on **PMF**, therefore, it is desirable to know whether the subgroup contains any pseudo-Anosov elements. This turns out to depend primarily upon whether the group is reducible or not.

Let Σ be a subgroup of Γ. We say that Σ is *reducible* if there is a nonempty finite family of disjoint simple closed curves in **S**, A, which is preserved by Σ. We refer to A as a *reduction system* for Σ. If Σ has no reduction system, then Σ is, of course, *irreducible.* There are examples of finite irreducible groups [6]. The remaining irreducible groups are shown in [14] to contain pseudo-Anosov maps. Indeed, we have the following equivalence.

LEMMA 2.8 [14]. *Let Σ be a subgroup of Γ. Σ contains a pseudo-Anosov element if and only if Σ is an infinite irreducible group.*

We turn next to a discussion of limit sets for subgroups of Γ acting on **PMF**.

3. Limit sets

There does not seem to be a generally accepted definition of a limit set for the action of a group on a topological space. In his article on limit sets for certain handlebody groups, [11], Masur takes a limit set to be a minimal, closed invariant set. From the point of view of the dynamics of the group on the limit set, this is clearly a natural definition. However, when one is interested in the relationship of the dynamics on the limit set to that on the entire space, it seems that one might wish to adopt a broader perspective. Hence, for us, a limit set will be a less restrictive notion.

Let Σ be a group acting continuously on a space, X. A point, x, in X is a *limit point of* Σ, if there is an infinite sequence of distincts elements of Σ, $\{g_n\}$, and a point, y, in X such that x is the limit of $\{g_n(y)\}$. (We do not require that x be an accumulation point of this sequence). The *canonical limit set for* Σ acting on X, $\Lambda(\Sigma)$, is the closure of the set of limit points for Σ. A *limit set for* Σ, Λ, is a closed invariant subset of $\Lambda(\Sigma)$. (Observe that $\Lambda(\Sigma)$ is a limit set for Σ and, of course, every minimal, closed invariant set is a limit set). If Σ has a domain of discontinuity on X, Δ, (or even a wandering set), then $\Lambda(\Sigma)$ is necessarily in the complement of Δ. As observed by Thurston ([16], ch. 8), it is a remarkable occurrence when a group acts properly discontinuously on the complement of its limit set. This occurs for nonelementary subgroups of $PSL(2, C)$ acting on S^2. Except for a set of measure zero that we shall describe, we shall see that this is also true for subgroups of Γ acting on **PMF**.

4. Characterization of dynamically irreducible groups

A subgroup of Γ, Σ, is *dynamically irreducible* if it has a unique nonempty minimal closed invariant set in **PMF**. Otherwise, of course, Σ is *dynamically reducible.*

A set of pseudo-Anosov maps is *independent* if no two maps in the set have the same fixed point set in **PMF**. If the set is independent, then, by Lemma 2.5, the fixed point sets are disjoint.

Let Σ be a subgroup of Γ. Let Λ_0 be the set of pseudo-Anosov foliations for pseudo-Anosov maps in Σ. (Of course, Λ_0 may be empty). Let Λ be the closure of Λ_0. We say that Σ is *sufficiently large* if Σ has an independent pair of pseudo-Anosov maps. If Σ is sufficiently large, then Λ has a rather simple dynamical description, which we now give.

Recall that the action of a group on a topological space is said to be minimal if the orbit of every point is dense.

THEOREM 4.1. *Let Σ be sufficiently large. Λ is the unique nonempty Σ-invariant minimal closed subset of* **PMF**.

Proof. By assumption, Λ_0 and, hence, Λ are nonempty. Likewise, by definition, Λ is closed.

If L is a pseudo-Anosov foliation for a pseudo-Anosov map, ϕ, in Σ and if g is an element of Σ, then gL is a pseudo-Anosov foliation for the pseudo-Anosov element, $g\phi g^{-1}$. Therefore, Λ_0 and, hence, Λ are Σ-invariant.

To see minimality, let F and G be any two elements of Λ. We wish to prove that in every neighborhood of F, there is a Σ-translate of G.

Let U be a neighborhood of F in **PMF**. Since F is in Λ, we may choose a pseudo-Anosov foliation, F', in U which is an attracting foliation for a pseudo-Anosov map, ϕ, in Σ. (F' may be equal to F). If G is not the repelling fixed point of ϕ, then $\phi^n(G)$ converges to F'. Therefore, in this situation, for n large enough, $\phi^n(G)$ is in U.

Suppose now that G is the repelling fixed point of ϕ. Since Σ is sufficiently large, there exists a pseudo-Anosov mapping class, ψ, in Σ, whose fixed point set is disjoint from the fixed point set of ϕ. Hence, $\psi(G) \neq G$. As before, $\phi^n \circ \psi(G)$ is in U for n large enough.

This proves that the action of Σ on Λ is minimal. Moreover, it proves that any nonempty Σ-invariant closed subset of **PMF** must contain the fixed points of pseudo-Anosov elements of Σ. Therefore, any such set contains Λ. It follows that Λ is the *unique* nonempty invariant minimal closed set for Σ.

This proves the theorem.

The next statement follows immediately.

COROLLARY 4.2. *If Σ is sufficiently large, then Σ is dynamically irreducible.*

Let $|L|$ be a pseudo-Anosov pair in **MF**. Let $|[L]|$ be the corresponding pair in **PMF**. If Σ is a subgroup of the stabilizer of $|[L]|$ in Γ, $\Gamma_{|[L]|}$, we say that Σ is pseudo-Anosov *stabilizing*. If, in addition, Σ contains an element which exchanges $[L_+]$ and $[L_-]$ we say that Σ is of *symmetric type*. Otherwise, of course, Σ is of *asymmetric type*.

Suppose that Σ contains a pseudo-Anosov map, ϕ, with pseudo-Anosov pair $|L|$ in **MF**. As we have previously observed, if g is an element of Σ, then $|gL|$ is a pseudo-Anosov pair for the pseudo-Anosov element, $g\phi g^{-1}$, in Σ. Hence, Σ is either sufficiently large or pseudo-Anosov stabilizing.

LEMMA 4.3. *Let Σ be infinite and* pseudo-Anosov *stabilizing. Σ is dynamically irreducible if and only if Σ is of symmetric type.*

Proof. Under the hypothesis, $\Lambda = \{[L_+], [L_-]\}$. If Σ if not of asymmetric type, then $\{[L_+]\}$ and $\{[L_-]\}$ are distinct nonempty Γ-invariant minimal closed subsets of **PMF**. In this situation, therefore, Γ is dynamically reducible.

On the other hand, if Σ is of symmetric type, then Λ is a nonempty Γ-invariant minimal closed set. Moreover, any nonempty Γ-invariant closed set must contain either $[L_+]$ or $[L_-]$. Therefore, it must contain both. As in the sufficiently large situation, Γ is dynamically irreducible.

Now suppose that Σ does not contain a pseudo-Anosov mapping class. By Lemma 2.8, Σ is either finite or reducible. As a rule, as we shall see, reducible groups are dynamically reducible. There are, however, some exceptions which we shall now discuss.

LEMMA 4.4. *Let S be a torus with one hole or a sphere with four holes. If Σ is an infinite reducible subgroup of $\Gamma(S)$, then Σ is dynamically irreducible.*

Proof. Under the hypothesis, **PMF** is a circle, S^1. Let A be a reduction system for Σ. Then A must be a simple closed curve, α, as in figure 1 or figure 2.

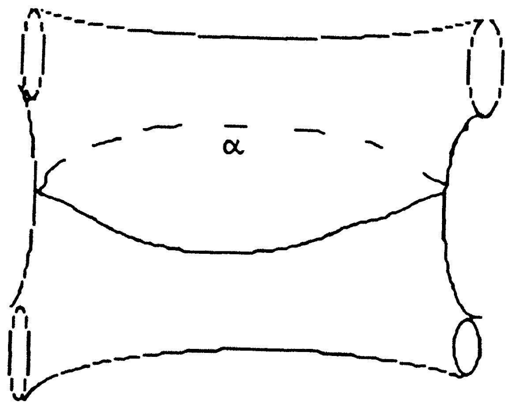

Figure 1

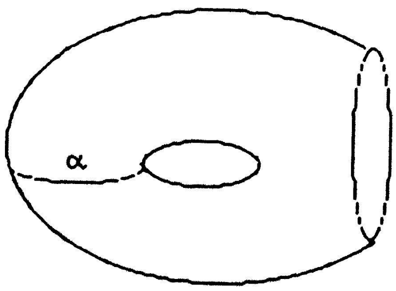

Figure 2

The stabilizer of α in Γ, Γ_α, is virtually cyclic. Since, by assumption, Σ is an infinite subgroup of Γ_α, Σ must contain a nontrivial power of D_α, D^n_α. On the other hand, D_α acts as a parabolic transformation of the circle with one fixed point. Hence, the statement follows as in the proof of Lemma 4.3.

Now we consider the "general case". If A is a finite family of disjoint simple closed curves in **S**, the *barycenter* of A, is the sum of the elements of A. In other words, the barycenter of A is the enlargement of the partial foliation whose support is a disjoint union of annuli about the components of A, foliated by leaves parallel to the components, and each with height, or total transverse measure, 1.

LEMMA 4.5. *Suppose that S is not a torus with one hole or a sphere with four holes. If Σ is a reducible subgroup of $\Gamma(S)$, then Σ is dynamically reducible.*

Proof. Suppose that there is a reduction system for Σ, A, with at least two elements. Let $\Sigma\alpha$ be the orbit under Σ of an element of A, α. Then $\Sigma\alpha$ and the barycenter of A are two distinct nonempty Σ-invariant minimal closed sets. Hence, Σ is dynamically reducible.

Suppose that Σ has no such reduction system. Then Σ must reduce along a single simple closed curve, α. Then $\{\alpha\}$ is a nonempty Σ-invariant minimal closed set. We wish to exhibit another.

The surface obtained by cutting S along α, S_α, has at most two components. By the assumption on S, at least one of these components is not a pair of pants. Hence, at least one component of S_α supports a measured foliation [5]. Let P_α be the set of projective measured foliations on S which are obtained by enlargement of measured foliations supported on components of S_α. Clearly, P_α is a nonempty closed subset of **PMF**(S). By definition, α is not in P_α. Moreover, P_α is Σ-invariant. Hence, P_α must contain a nonempty Σ-invariant minimal closed subset. Such a set provides the desired counterpart of $\{\alpha\}$. Hence, Σ is dynamically reducible.

We have discussed sufficiently large groups, groups with a single pseudo-Anosov pair, and infinite reducible groups. All that remains is the case of a finite group. But these are clearly dynamically reducible since every orbit provides a nonempty Σ-invariant minimal closed set. Hence, the following characterization of dynamically irreducible groups is an immediate consequence of the preceding lemmas and discussion.

THEOREM 4.6. *Let Σ be a subgroup of $\Gamma(S)$. Σ is dynamically irreducible if and only if it is of one of the following types:*

(1) *Σ is sufficiently large.*

(2) *Σ is infinite* pseudo-Anosov *stabilizing of symmetric type,*

(3) *Σ is infinite reducible and S is a torus with one hole or a sphere with four holes.*

On the other hand, Σ is dynamically reducible if and only if it is of one of the following types:

(4) *Σ is finite,*

(5) *Σ is pseudo-Anosov stabilizing of asymmetric type,*

(6) *Σ is infinite reducible and S is neither a torus with one hole nor a sphere with four holes.*

Remark. The types (2), (3), (4) and (5), are all virtually cyclic groups. They belong to a slightly more general class, the virtually abelian groups. These latter groups have a very simple dynamical behavior and are not of primary concern to us. We shall refer to them as *elementary groups.*

5. Limit sets for sufficiently large groups; elementary properties and some examples

In this section, Σ denotes a sufficiently large subgroup of Γ. Sufficiently large groups are rather well-behaved under standard operations on subgroups. In particular, we have the following elementary assertions.

PROPOSITION 5.1. *Let Σ_1 and Σ_2 be subgroups of Γ, such that Σ_1 is contained in Σ_2.*

(1) *If Σ_1 is sufficiently large, then Σ_2 is sufficiently large.*
(2) *If Σ_2 is sufficiently large and Σ_1 is of finite index in Σ_2, then Σ_1 is sufficiently large.*
(3) *If Σ_2 is sufficiently large and Σ_1 is infinite and normal in Σ_2, then Σ_1 is also sufficiently large.*

Proof. The first two assertions are trivial. Therefore, let Σ_2 be sufficiently large and Σ_1 be infinite and normal in Σ_2. First, we construct a pseudo-Anosov element in Σ_1.

Suppose that ϕ is a pseudo-Anosov map in Σ_2. Let $[L_-]$ and $[L_+]$ be the fixed points in **PMF** for ϕ. If Σ_1 is contained in the stabilizer of $\{[L_-], [L_+]\}$, then, by the assumption that Σ_1 is infinite and by Proposition 2.2, it follows that Σ contains a nontrivial power of ϕ.

Therefore, we may assume that there is an element, f, in Σ_1 which is not in the stabilizer of $\{[L_-], [L_+]\}$. Let ψ be the pseudo-Anosov map, $f\phi f^{-1}$. By Lemma 2.5, ϕ and ψ are independent. It follows from standard dynamical arguments (cf. [3]) that $\phi^n\psi^{-n}$ is of pseudo-Anosov type for n sufficiently large. On the other hand, $\phi^n\psi^{-n}$ is equal to $(\phi^n f\phi^{-n})f^{-1}$, which is an element of Σ_1.

In any event, therefore, we may choose a pseudo-Anosov element, ϕ, in Σ_1. Since Σ_2 is sufficiently large, there is a pseudo-Anosov map, ψ, which is independent from ϕ. Then the maps, ϕ and $\psi\phi\psi^{-1}$, are independent pseudo-Anosov maps in Σ_1. Therefore, Σ_1 is sufficiently large.

This proves Proposition 5.1.

Remark. It is easy to construct examples of nontrivial normal subgroups of sufficiently large groups which are not sufficiently large. The center of $\Gamma(S)$, where S is a closed surface of genus two, is an obvious example. There are other less obvious examples. We shall discuss such an example in the latter part of this section.

In section 4, we associated to each sufficiently large group, Σ, the following two subsets of **PMF**:

$$\Lambda_0 \text{ is set of fixed points of pseudo-Anosov mapping classes in } \Sigma, \tag{5.1}$$

$$\Lambda \text{ is the closure of } \Lambda_0. \tag{5.2}$$

We shall say that Λ is *the limit set* of Σ.

Theorem 4.1 expresses an important property of the limit set, Λ. In fact, the proof of that theorem shows that for any sufficiently large group, Σ, the associated set Λ_0 has no isolated points. (This, of course, is not the case for infinite pseudo-Anosov stabilizing groups). Therefore, Λ has no isolated points and we have the following proposition.

PROPOSITION 5.2. *Λ is a perfect set and is, therefore, uncountable.*

Here are some other properties of the limit set.

PROPOSITION 5.3. *Λ is either equal to* **PMF** *or has empty interior.*

Proof. Suppose that Λ contains an open subset, U, of **PMF**. By Theorem 4.1, for every F in Λ, there is an element, ϕ, of Σ such that $\phi(F)$ is in U. By the Σ-invariance of Λ, $\phi^{-1}(U)$ is a subset of Λ. Since $\phi^{-1}(U)$ is also an open neighborhood of F, F is in the interior of Λ. Therefore, Λ is open in **PMF**. Since Λ is also closed, it must be equal to **PMF**.

PROPOSITION 5.4. *If Σ_1 and Σ_2 are sufficiently large subgroups of Γ and Σ_1 is of finite index in Σ_2, then Σ_1 and Σ_2 have the same limit set.*

Proof. This follows easily from the fact that for any pseudo-Anosov element, ϕ, in Σ_2, there is a positive integer, n, such that ϕ^n is in Σ_1.

PROPOSITION 5.5. *If Σ_1 and Σ_2 are sufficiently large subgroups of Γ and Σ_1 is a normal subgroup of Σ_2, then Σ_1 and Σ_2 have the same limit set.*

Proof. Let Λ_1 and Λ_2 denote respectively the limit sets of Σ_1 and Σ_2.

If L is an element of **PMF** which is fixed by a pseudo-Anosov element, ϕ, in Σ_1, and if g is any mapping class in Σ_2, then $g(L)$ is a fixed point for the pseudo-Anosov mapping class, $g\phi g^{-1}$, which is in Σ_1. Hence, the set of pseudo-Anosov fixed points of Σ_1 is invariant by Σ_2. It follows that its closure, Λ_1, is invariant by Σ_2. Using Theorem 4.1, we see that Λ_1 is equal to Λ_2.

We close this section with some examples of limit sets.

EXAMPLE. 1. The mapping class group acts minimally on **PMF** (cf. [5], exposé 6, §III). Therefore, by Theorem 4.1, its limit set is equal to the whole space, **PMF**. Proposition 5.5 implies, then, that the limit set of the Torelli group is also equal to **PMF**.

EXAMPLE. 2 (*groups of Schottky type*). These are examples of groups where the limit set is a Cantor set. The construction is inspired from Klein's method for constructing free subgroups (cf. [13]). Some care should be taken if we want to insure that every element in the group is of pseudo-Anosov type. As in the case of Schottky groups, one needs, for each of the n generators, g_i, a pair of disjoint disks $\{D_i^+, D_i^-\}$, so that all of the $2n$ disks are disjoint and g_i (resp. g_i^{-1}) takes the exterior of D_i^+ (resp. D_i^-) inside D_i^- (resp. D_i^+) by a contraction (for a certain metric on **PMF**).

Both the techniques developed in [13] and [15] lead to such examples.

EXAMPLE. 3. We consider now the limit set of the group generated by the Dehn twists along two simple closed curves that fill up the surface. This group is studied in [5].

In this example, the limit set will be a Cantor set, which is naturally contained in a geometric circle of **PMF**, the term geometric referring to the natural projective piecewise-linear structure of that space.

Let a and b be two simple closed curves on S which are transverse and in minimal intersection number position such that the complement of these two curves in S is a union of cells (and annuli in the case where S has nontrivial boundary). Let n be the number of intersection points of a and b. (We assume that n is larger than two. It is easy to see that this assumption always holds for closed surfaces.)

In [5], chapter 11, there is a description of the elements of the group generated by the Dehn twists, D_a and D_b. This group is shown to be realized by a group of homeomorphisms of S which act as linear maps with respect to the singular flat metric defined on S by enlargement of the curves, a and b, to transverse measured foliations.

Let **P** be the set of projective classes of measured foliations on S which are defined by this singular flat structure as lines of constant slope. **P** is naturally parameterized by the real projective line, RP^1, since we can see a foliation by lines of constant slope on S as the image, on that surface, of the corresponding foliation of the plane by the coordinate maps associated to the "horizontal" or "vertical" atlas of S associated to a and b (see [5]). Using train track coordinates, one can easily see that there is a natural imbedding, j, of RP^1 in **PMF** as a projective piecewise-linear subvariety.

The discussion in [5] also implies that this group, Σ, is sufficiently large. The limit set of Σ is a Cantor set and is the image of the limit set in RP^1 of the group G of linear fractional transformations generated by the matrices

$$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}. \tag{5.3}$$

Σ is isomorphic to G, and an element of Σ is pseudo-Anosov iff it corresponds to a hyperbolic element in G.

Note also that if we choose a and b to be invariant under a finite order map, then the group, Σ, will have a nontrivial center; Hence, we obtain nontrivial examples of nontrivial normal subgroups of sufficiently large groups which are not sufficiently large:

EXAMPLE. 4 (*The handlebody groups*). These are the two examples studied by Howard Masur in [11]. The surface, S, is the boundary of a handlebody, and the groups considered are the subgroups of the mapping class group of S which, in one case, extend as homeomorphisms of the handlebody, and in the other case, extend and induce the identity map on the fundamental group of the handlebody. That these groups are sufficiently large can be seen from a construction of Fathi and Laudenbach [4]. Masur defines the limit sets in a different way than we do, but the two definitions coincide, as Masur shows in Theorem 1.2 of [11] that the limit sets as he defines them, are also the unique minimal invariant sets for the action of the groups. In these examples, the limit set is a proper subset of **PMF**. Unlike the previous examples, it is connected ([11], Theorem 1.2).

6. The domain of discontinuity of a sufficiently large subgroup of Γ

In this section, Σ denotes a sufficiently large subgroup of Γ, Λ_0 the set of pseudo-Anosov foliations for Σ, and Λ, its closure, the limit set.

It was proved by Masur, in his study of the limit set for the two handlebody groups [11], that these groups do not act properly discontinuously on the whole complement of their limit set, but that in each case there exists a nonempty open invariant subset on which the group acts properly discontinuously. We wish to describe a similar region for sufficiently large subgroups of Γ.

There is a set, $Z(\Lambda)$, contained in **PMF**, naturally associated to Λ, which is, in some sense, the completion of Λ with respect to the intersection function:

$$Z(\Lambda) = \{F \mid i(F, G) = 0 \text{ for some } G \text{ in } \Lambda\}. \tag{6.1}$$

Note: The intersection function is defined on **MF** and not on **PMF**. Strictly speaking, therefore, in the above definition we should refer to representatives of F and G in **MF** as having zero intersection. In the subsequent discussion, we shall frequently use this abuse of notation. The meaning, however, will be clear.

PROPOSITION 6.1. *$Z(\Lambda)$ is closed and Σ-invariant. If Λ is a proper subset of* **PMF**, *then so is $Z(\Lambda)$.*

Proof. Let F be the limit of a subsequence, $\{F_n\}$, of elements in $Z(\Lambda)$. There exists a sequence of elements, $\{G_n\}$, in Λ, such that $i(F_n, G_n)$ is zero. By compactness of Λ, the sequence, $\{G_n\}$, has a subsequence converging to a point, G, in Λ. By taking limits, $i(F, G)$ is zero and, hence, F is in $Z(\Lambda)$. Therefore, $Z(\Lambda)$ is closed.

Let F be an element of $Z(\Lambda)$ and G an element of $Z(\Lambda)$ such that $i(F, G)$ is zero. For any G in Σ, we have the identity:

$$i(gF, 'gG) = i(F, G) = 0. \tag{1}$$

Since gG is in Λ, gF is in $Z(\Lambda)$. This shows that $Z(\Lambda)$ is Σ-invariant.

Obviously, there are no uniquely ergodic foliations in $Z(\Lambda)\backslash\Lambda$. On the other hand, the set of uniquely ergodic foliations has full measure in **PMF** (see [9], [10] or [16]). It follows that $Z(\Lambda)\backslash\Lambda$ has zero measure, and therefore empty interior. Hence, if $\mathbf{PMF}\backslash\Lambda$ is nonempty, so is $\mathbf{PMF}\backslash Z(\Lambda)$.

The proof of Proposition 6.1 is now complete.

We denote the complement of $Z(\Lambda)$ in **PMF** by Δ. The rest of this section is devoted to the study of the action of Σ on Δ.

For the rest of this section, we let $|L|$ denote a pseudo-Anosov pair in **MF**, $\{L_+, L_-\}$, for some pseudo-Anosov element of Σ. We define a function on **MF**, $i(.\,, |L|)$, by the rule:

$$i(F, |L|) = \max\{i(F, L_-), i(F, L_+)\}. \tag{6.2}$$

In addition, we associate to $|L|$ the following subset of **MF**:

$$\Delta_{|L|} = \{F \in \mathbf{MF} \text{ such that for every element, } g, \text{ in } \Sigma, \quad i(F, |L|) \leq i(F, |gL|)\}. \tag{6.3}$$

Note: $\Delta_{|L|}$ is invariant by multiplication by positive reals and defines naturally a subset of **PMF**. On the other hand, $\Delta_{|L|}$ depends not on the projective classes of L_+ and L_- but on L_+ and L_-. For instance, if $|L|$ is fixed projectively by a pseudo-Anosov mapping class, ϕ, in Σ, then $\Delta_{|L|}$ is not equal to $\Delta_{|\phi L|}$. (This assertion will follow from the subsequent discussion).

LEMMA 6.2. *For any mapping class in Σ, f, we have $f(\Delta_{|L|}) = \Delta_{|fL|}$.*

Proof.

$$\begin{aligned}
&F \in f(\Delta_{|fL|}) \Leftrightarrow\\
&f^{-1}F \in f(\Delta_{|L|}) \Leftrightarrow\\
&i(f^{-1}F, |L|) \leq i(f^{-1}F, |gL|) \forall g \in \Sigma \Leftrightarrow\\
&i(F, |fL|) \leq i(F, |fgL|) \forall g \in \Sigma \Leftrightarrow\\
&i(F, |fL|) \leq i(F, |g'fL|) \text{ (where } g' = fgf^{-1} \text{ can be any element of } \Sigma) \Leftrightarrow\\
&F \in \Delta_{|fL|}.
\end{aligned}$$

This proves the lemma.

We have now the following result:

PROPOSITION 6.3. $\Lambda_0 \cap \Delta_{|L|} = \emptyset$.

Proof. Suppose, for the sake of contradiction, that there is an element, F, in $\Delta_{|L|}$ which is an attracting fixed point for the pseudo-Anosov mapping class, f, in Σ. Then:

$$i(F, |L|) \leq i(F, |gL|) \text{ for all } g \text{ in } \Sigma. \tag{1}$$

In particular, we have the inequalities:

$$i(F, |L|) \leq i(F, |f^{-n}L|) \text{ for all } n \text{ in } Z. \tag{2}$$

But we also have the identity:

$$i(F, |f^{-n}L|) = i(f^nF, |L|) = \lambda^n i(F, |L|). \tag{3}$$

Statements (2) and (3) imply that $i(F, |L|)$ is zero, which is impossible.
We have an immediate corollary.

COROLLARY 6.4. *Δ_0 is contained in* $\mathbf{PMF} \setminus \cup \{\Delta_{|gL|}, g \in \Sigma\}$.

The following propositions describe the relation of these sets to **S**.

PROPOSITION 6.5. *For every element, α, in* **S**, *the following set of numbers has a minimum:*

$$\{i(\alpha, |gL|), g \in \Sigma\}.$$

Proof. Suppose the set has no minimum. Then there exists an infinite sequence, $\{g_n\}$, in Σ such that:

$$\cdots < i(\alpha, |g_2L|) < i(\alpha, |g_1L|) < i(\alpha, |L|). \tag{1}$$

This is equivalent to the condition;

$$\cdots < i(g_2^{-1}(\alpha), |L|) < i(g_1^{-1}(\alpha), |L|) < i(\alpha, |L|). \tag{2}$$

In particular, $g_m^{-1}(\alpha)$ is not equal to $g_n^{-1}(\alpha)$ unless m is equal to n. By (2.11), we may write:

$$r_n g_n^{-1}(\alpha) \to F \quad \text{in } \mathbf{MF} \tag{3}$$

$$r_n \to 0. \tag{4}$$

It follows that:

$$
\begin{aligned}
i(r_n g_n^{-1}(\alpha), L_+) &= r_n i(g_n^{-1}(\alpha), L_+) \\
&\leq r_n i(g_n^{-1}(\alpha), |L|) \\
&\leq r_n i(\alpha, |L|).
\end{aligned} \tag{5}
$$

By taking limits, $i(F, L_+)$ is zero. By (2.5), this implies that $[F]$ is equal to $[L_+]$. By the same argument, $[F]$ is equal to $[L_-]$, which is a contradiction.

COROLLARY 6.6. **S** *is contained in* $\bigcup \{\Delta_{|gL|}, g \in \Sigma\}$.

Since some $\Delta_{|gL|}$ is non-empty, and since (by Lemma 2.6) $g^{-1}(\Delta_{|gL|}) = \Delta_{|L|}$, we conclude:

COROLLARY 6.7. *Every* $\Delta_{|L|}$ *is non-empty.*

Remark. We shall see later on that every set, $\Delta_{|L|}$, has nonempty interior if and only if Λ is not equal to **PMF**. (This will follow from Proposition 6.13).

PROPOSITION 6.8. *For any element, f, of Σ that does not fix the pair, $|L|$, in* **MF**, *the interiors of the sets, $\Delta_{|L|}$ and $\Delta_{|fL|}$ are disjoint.*

Proof. Suppose that U is a nonempty open set in **MF** contained in $\Delta_{|L|} \cap \Delta_{|fL|}$, i.e., for all $F \in U$, we have:

$$i(F, |L|) \leq i(F, |gL|) \quad \text{for every } g \text{ in } \Sigma, \tag{1}$$

$$i(F, |fL|) \leq i(F, |gfL|) \quad \text{for every } g \text{ in } \Sigma. \tag{2}$$

In particular, we have, for every F in U:

$$i(F, |fL|) \leq i(F, |fL|), \tag{3}$$

$$i(F, |L|) \leq i(F, |fL|). \tag{4}$$

This implies equality on U:

$$i(F, |L|) = i(F, |fL|) \quad \text{for all } F \text{ in } U. \tag{5}$$

Now, by definition, $i(F, |L|)$ is equal to the maximum of $i(F, L_+)$ and $i(F, L_-)$.

Since $i(.\,,.)$ is continuous, there is a smaller open set, U', on which the equality in (5) holds with $|L|$ replaced by one element of the pair $\{L_+, L_-\}$ and $|fL|$ replaced by one of $\{fL_+, fL_-\}$. On any open set, we can define a coordinate system given by intersection with finitely many foliations, so this implies that either:

$$L_- = fL_+, \quad L_+ = fL_-, \quad L_- = fL_- \quad \text{or} \quad L_+ = fL_+. \tag{6}$$

Using Lemma 2.5, we can see that this implies that f fixes the pair, $\{L_+, L_-\}$, as a pair in **MF**. This proves Proposition 6.8.

6.1. *A fundamental domain for the action of a pseudo-Anosov map on* **PMF**

The theorem that we prove next concerns the cyclic groups generated by pseudo-Anosov elements, which are dynamically reducible groups and should not, strictly speaking, appear in this section. However, this theorem illustrates the usefulness of the definition of the set, $\Delta_{|L|}$, and the study of this elementary case should give some motivation for the rest of the section.

Observe, that in this case:

$$\Lambda = Z(\Lambda) = \{L_+, L_-\}. \tag{6.4}$$

Note: From now on, we shall often consider the set, $\Delta_{|L|}$, as being a subset of **PMF**, which means that we are taking its quotient by the action of the positive reals.

THEOREM 6.9. *Let ϕ be a pseudo-Anosov element of the mapping class group and $|L|$ be an associated pseudo-Anosov pair, $\{L_+, L_-\}$. The set, $\Delta_{|L|}$, is a fundamental domain for the action of the cyclic group generated by ϕ on the space,* **PMF**. *More precisely, the following properties hold:*

(i) *The union, over $n \in Z$, of the sets, $\phi^n(\Delta_{|L|})$, is equal to* **PMF**$\setminus\{L_+, L_-\}$.
(ii) *If $n \neq m$, then $\phi^n(\Delta_{|L|})$ and $\phi^m(\Delta_{|L|})$ have disjoint interiors.*
(iii) *For any integer, n, the set, $\phi^n(\Delta_{|L|})$, contains 1 or 2 points from every orbit (except for L_+ and L_-), and the interior of this set contains at most one point from every orbit.*

The theorem will be proved using the following lemma:

LEMMA 6.10. $\Delta_{|L|} = \{F \in \mathbf{PMF}\ s.t.\ \lambda^{-1} \leq i(F, L_+)/i(F, L_-) \leq \lambda\}$.

Proof of Theorem 6.9. For every $n \in Z$, we have:

$$f(\Delta_{|L|}) = \Delta_{|fL|}. \tag{1}$$

By Lemma 6.10, we have:

$$\Delta_{|f^n L|} = \{F \in \mathbf{PMF} \text{ s.t. } \lambda^{-1} \leq i(F, \lambda^n L_+)/i(F, \lambda^{-n} L_-) \leq \lambda\}$$
$$= \{F \in \mathbf{PMF} \text{ s.t. } \lambda^{-(2n+1)} \leq i(F, L_+)/i(F, L_-) \leq \lambda^{2n-1}\}. \tag{2}$$

The theorem follows easily from this characterization and Proposition 6.8.

Proof of Lemma 6.10. Consider the following two subsets of the sphere, **PMF**:

$$D_1 = \{F \in \mathbf{PMF} \text{ s.t. } i(F, L_+) \leq i(F, L_-)\} \tag{1}$$

$$D_2 = \{F \in \mathbf{PMF} \text{ s.t. } i(F, L_-) \leq i(F, L_+)\}. \tag{2}$$

The lemma can be proved separately for points in D_1 and points in D_2. Suppose F is an element of D_1. Then F is in $\Delta_{|L|}$ if and only if:

$$i(F, L_-) \leq \max\{\lambda^n i(F, L_+), \lambda^{-n} i(F, L_-)\} \text{ for every } n \in Z. \tag{3}$$

This is equivalent to:

$$i(F, L_-) \leq \lambda^n i(F, L_+) \tag{4}$$

or

$$i(F, L_-) \leq \lambda^{-n} i(F, L_-) \text{ for every } n \in Z.$$

These two inequalities are easily seen to be equivalent to the statement that:

$$i(F, L_-) \leq \lambda^n i(F, L_+) \text{ for every } n \geq 1. \tag{5}$$

That is:

$$i(F, L_-) \leq \lambda i(F, L_+). \tag{6}$$

It follows that for every F in D_1 we have:

$$F \in \Delta_{|L|} \Leftrightarrow \lambda^{-1} \leq i(F, L_+)/i(F, L_-) \leq \lambda. \tag{7}$$

The same reasoning applies for elements F in D_2.
This proves Lemma 6.10.

6.2. Proper discontinuity on Δ

Theorem 6.16, that we shall prove below, shows that in the most general case, the group, Σ, acts properly discontinuously on Δ. Furthermore, Propositions 6.13 and 6.15 show that $\Delta_{|L|} \cap \Delta$ defines a fundamental region for the action of Σ on Δ.

LEMMA 6.11. *Let F be an element of Δ, and $\{g_n\}$ be an infinite sequence of distinct mapping classes in Σ. Then the sequence of numbers, $\{i(F, |g_nL|)\}$, is unbounded.*

Proof. By Lemma 2.7, we know that at least one of the two sequences, $\{\|g_nL_+\|\}$ and $\{\|g_nL_-\|\}$, is unbounded.

Suppose $\{\|g_nL_+\|\}$ is unbounded (the other case is similar). Then there exists a sequence, $\{r_n\}$, of positive real numbers, such that:

$$r_ng_nL_+ \to G \quad \text{in } \mathbf{MF} \tag{1}$$

$$r_n \to 0. \tag{2}$$

The projective class of G is in Λ.
We may write:

$$r_n^{-1}i(F, r_ng_nL_+) = i(F, g_nL_+) \leq i(F, |g_nL|). \tag{3}$$

If the last term was bounded independently of n, then $i(F, r_ng_nL_+)$ would have to converge to zero. This would imply that $i(F, G)$ is zero, which is a contradiction.

PROPOSITION 6.12. *Suppose that $\Delta_{|L|}$ contains a point in Δ. Then the set of mapping classes, $\{g \mid \Delta_{|L|}$ is contained in $\Delta_{|gL|}\}$, is finite.*

Proof. Suppose that $\{g_n\}$ is an infinite sequence of distinct mapping classes with $\Delta_{|L|}$ contained in $\Delta_{|g_nL|}$, and let F be an element of $\Delta \cap \Delta_{|L|}$. Then, for every n, we have:

$$i(F, |L|) \leq i(F, |g_nL|) \leq i(F, |L|). \tag{1}$$

This implies that:

$$i(F, |L|) = i(F, |g_nL|). \tag{2}$$

But this contradicts Lemma 6.11.

PROPOSITION 6.13. *For every F in Δ, there exists an element, g, in Σ such that F is in $\Delta_{|gL|}$.*

Proof. Suppose, on the contrary, that F is not in $\Delta_{|gL|}$ for any g in Σ. Then there exists an infinite sequence, $\{g_n\}$, of distinct mapping classes such that:

$$\cdots < i(F, |g_2L|) < i(F, |g_1L|) < i(F, |L|). \tag{1}$$

Therefore, the sequence, $\{i(F, |g_nL|)\}$, is bounded, and this contradicts Lemma 6.11.

PROPOSITION 6.14. *Let K be a compact set in Δ. Then the set of mapping classes, $\{g \in \Sigma \mid K \cap \Delta_{|gL|} \neq \emptyset\}$, is finite.*

Proof. Suppose there exists an infinite sequence, $\{g_n\}$, of distinct mapping classes such that $K \cap \Delta_{|g_nL|}$ is nonempty for every n.

Let F_n be an element of $K \cap \Delta_{|g_nL|}$.

$$i(F_n, |g_nL|) \leq i(F_n, |L|). \tag{1}$$

By Lemma 2.7, we know that one of the sequences, $\{\|g_nL_+\|\}$ and $\{\|g_nL_-\|\}$, is unbounded. Suppose the first one is unbounded. (The argument works for both cases). Taking subsequences, we can write:

$$r_ng_nL_+ \to G \quad \text{in } \mathbf{MF} \tag{2}$$

$$r_n \to 0 \tag{3}$$

$$F_n \to F \quad \text{in } \mathbf{MF}. \tag{4}$$

It follows that:

$$\begin{aligned} i(F, G) &= \lim i(F_n, r_ng_nL_+) \\ &\leq \lim r_n i(F_n, |g_nL|) \\ &\leq \lim r_n i(F_n, |L|) \\ &= 0. \end{aligned} \tag{5}$$

This contradicts the fact that G is in Λ and F is in Δ.

PROPOSITION 6.15. *Let F be a point in Δ. Then the Σ-orbit of F intersects $\Delta_{|L|}$ only finitely many times.*

Proof. Suppose that $g_n(F)$ is in $\Delta_{|L|}$ for infinitely many g_n. Then:

$$i(g_n(F), |L|) \leq i(g_n(F), |g_nL|) = i(F, |L|). \tag{1}$$

Therefore, $i(F, |g_nL|)$ is bounded independently of n.
This contradicts Lemma 6.11.

THEOREM 6.16. *Σ acts properly discontinuously on Δ.*

Proof. We show that for every compact subset, K, in Δ, there are only finitely many mapping classes, g, in Σ such that the intersection, $g(K) \cap K$, is nonempty.

By Proposition 6.14, there are only finitely many mapping classes, g, for which the sets $\Delta_{|gL|}$ have nonempty intersection with K. Let $D_1, \ldots, D_N$ be these sets.

Suppose now that there is an infinite sequence, $\{f_n\}$, of distinct mapping classes such that for every n, the intersection, $K \cap f_n(K)$, is nonempty. Then $f_n^{-1}(K) \cap K$ is nonempty for every n.

Let F_n be an element of $f_n^{-1}(K) \cap K$. By Proposition 6.13, there is a mapping class, g_n, in Σ, such that F_n is in $\Delta_{|g_nL|}$. Since F_n is in K, g_n must be among the finitely many maps described above. By assumption, $f_n(F_n)$ is also in K. On the other hand:

$$f_n(F_n) \in f_n(\Delta_{|g_nL|}) = \Delta_{|f_ng_nL|}. \tag{1}$$

Hence, $\Delta_{|f_ng_nL|}$ must be one of the sets $D_1, \ldots, D_N$.

By taking a subsequence, we find an infinite sequence of distinct mapping classes, $\{f_ng_n\}$, with all the sets, $\Delta_{|f_ng_nL|}$, being equal. This contradicts Proposition 6.12.

7. The action of an infinite reducible group

In this section, we consider the action on **PMF** of an infinite reducible group, Σ. The class of infinite reducible groups includes all the nonelementary dynamically reducible groups as well as all the infinite elementary groups except for the pseudo-Anosov stabilizing groups. In extending the results of the previous section to this class of groups, therefore, we shall have described the dynamics of all groups except the finite groups and the pseudo-Anosov stabilizing groups. The dynamics of these latter groups, is, on the other hand, particularly simple.

Given a reduction system for Σ, A, we may consider the action of Σ on the set of components of the surface obtained by splitting S along A, S_A. The kernel of this action, Σ_A, by definition, preserves each component of S_A. The restrictions of Σ_A to these components will be called the *components of* Σ_A. (Σ_A, of course, may not be the direct product of its components).

It is easy to see that if A is a maximal reduction system for Σ, then the components of Σ_A are irreducible groups. By Lemma 2.8, it follows that each component is either finite or contains a pseudo-Anosov mapping class. We shall refer to these latter components as *pseudo-Anosov components of* Σ_A. The corresponding component of S_A will be called a *pseudo-Anosov component of* S. It is easy to see that the pseudo-Anosov components of S are well-defined independently of the choice of maximal reduction system.

Let B be the set of boundary components of pseudo-Anosov components of S. Of course, B may be empty, but it is a natural reduction system for Σ. Indeed, it must be contained in any maximal reduction system for Σ (cf. [2]).

Let S_t be the surface obtained from S by excising the pseudo-Anosov components of S. It follows from standard arguments that the restriction of Σ_B to S_t, Σ_t, is a virtually abelian group. In fact, it is a finite extension of a (possibly trivial) subgroup of $D(T)$, where T is a reduction system for Σ_B and $D(T)$ is the free abelian group generated by Dehn twists about the components of T.

Let T be a minimal reduction system for Σ_B. T is then a canonical reduction system for Σ_B. The *essential reduction system for* Σ, A_Σ, is the union of B and T. It is a canonical reduction system for Σ, since it is the unique minimal reduction system for Σ such that each component of the associated stabilizer is either pseudo-Anosov or finite.

From the assumption that Σ is infinite, it follows that A_Σ is nonempty.

7.1. *Limit sets for infinite reducible groups*

Let Σ be an infinite reducible group. Let A be the essential reduction system for Σ, $B \cup T$, as described above.

Let S_B be decomposed as follows:

$$S_B = S_1 \cup \cdots \cup S_N \cup S_t \tag{7.1}$$

The restriction of Σ_B to S_i, Σ_i, is pseudo-Anosov for $i = 1, \ldots, N$. (7.2)

(Of course, if B is empty, N is zero).

We associate the following sets to Σ:

Λ_0^i is the set of projective measured foliations in **PMF** which are obtained by enlargement of fixed points of pseudo-Anosov elements in Σ_i, where $i = 1, \ldots, N$. (7.3)

Λ^i is the closure of Λ_0^i. (7.4)

Λ is the union of $\Lambda^1, \ldots, \Lambda^N$ and A. (7.5)

We shall refer to Λ as *the limit set for* Σ. It is a limit set for Σ in the sense of section 3. On the other hand, Σ may not act minimally on Λ (cf. Lemma 4.3). If B is nonempty, for instance, then B and the union of $\Lambda_1, \ldots, \Lambda_N$ form two disjoint closed invariant sets.

The *components of* Λ are the nonempty Σ-invariant minimal closed subsets of Λ.

PROPOSITION 7.1. *There are finitely many components of* Λ. *The set,* Λ, *is the union of its components.*

Proof. It is clear from the definition of Λ that Λ is also the limit set for Σ_A. Certainly, the Σ_A components of Λ are contained in the Σ components of Λ. Furthermore, (cf. Lemma 4.3), the Σ components of Λ are the Σ orbits of the Σ_A components of Λ. Hence, we can assume that Σ is equal to Σ_A.

Under this assumption, we can completely describe the components of Λ. If Σ_i is pseudo-Anosov stabilizing of asymmetric type then Λ^i is a pseudo-Anosov pair which splits into two components of Λ. Otherwise, by Lemma 4.3 and Theorem 4.1, Λ^i is a component of Λ. Finally, A splits up into the orbits of A under Σ, each of which is a component of Λ. The proposition follows immediately.

Remark. It is not true, in general, that Σ has finitely many nonempty invariant minimal closed sets. If, for example, Σ is a cyclic group generated by a map with two pseudo-Anosov components of the same expansion factor, then every point on the join of the corresponding attracting fixed points is such a set. (This join is a segment, and, hence, uncountable). It appears, however, that the failure of a finiteness statement is due solely to this "join construction".

7.2. *The zero set and the domain of discontinuity*

As for sufficiently large groups, we define the zero set of Λ, $Z(\Lambda)$, as a completion with respect to the intersection function:

$Z(\Lambda)$ is the set of projective classes of measured foliations which have zero intersection with a foliation in Λ. (7.6)

Since Λ is a proper subset of **PMF**, the proof of Proposition 6.1 establishes the following proposition.

PROPOSITION 7.2. *$Z(\Lambda)$ is a closed Σ-invariant proper subset of* **PMF**.

Remark. As in [13], we can also take the join of the sets $\Lambda_1, \ldots, \Lambda_N$ and the components of A. This set need not be a limit set in the sense of section 3. It is, on the other hand, a subset of $Z(\Lambda)$.

We define Δ to be the complement of $Z(\Lambda)$ in **PMF**. As in section 6, we shall prove that Δ is a domain of discontinuity for Σ.

A *complete system for* Σ, $|C|$, is a subset of **MF** which is the union of the following subsets of **MF**:

$|L^i| = \{L^i_+, L^i_-\}$ a pseudo-Anosov pair for an element of Σ^i,
$i = 1, \ldots, N$. (7.7)

α^*, a curve in **S** which has nontrivial intersection with α, associated to every component α of A. (7.8)

We define a function on **MF**, $i(.\,, |C|)$, by the rule:

$i(F, |C|) = \max\{i(F, L^i_-), i(F, L^i_+), i(F, \alpha^*)$ where $i = 1, \ldots, N$, and $\alpha \in A\}$. (7.9)

Again, we associate to $|C|$ the following subset of **MF**:

$\Delta_{|C|} = \{F \in$ **MF** such that for every element g in Σ,
$i(F, |C|) \le i(F, |gC|)\}$. (7.10)

We discuss generalizations of the basic properties of the sets, $\Delta_{|L|}$, of the previous section. We shall not provide proofs where the arguments carry over directly.

LEMMA 7.3. *For any mapping class, f, in Σ, we have $f(\Delta_{|C|}) = \Delta_{|fC|}$.*

PROPOSITION 7.4. *$\Lambda^i_0 \cap \Delta_{|C|} = \varnothing$ for $i = 1, \ldots, N$.*

COROLLARY 7.5. *Λ_0^i is contained in* $\boldsymbol{PMF} \setminus \cup \{\Delta_{|gC|}, g \in \Sigma\}$.

Remark. We shall see that every set, $\Delta_{|C|}$, has nonempty interior. (This will follow from Proposition 7.8).

We shall now discuss the proper discontinuity of Σ on Δ.

7.3. *Proper discontinuity on Δ*

As in section 6.2, we shall now prove that Σ acts properly discontinuously on Δ and $\Delta_{|C|}$ provides a fundamental region for the action of Σ on Δ.
Index the components of S_A:

$$S_A = S_1 \cup \cdots \cup S_P \quad \text{where } N \le P. \tag{7.11}$$

By restricting Σ_A to S_A, we obtain a reduction homomorphism, ρ, whose kernel K is a subgroup of $D(A)$:

$$1 \to K \to \Sigma_A \to \Pi_{i=1,\ldots,N}\Gamma(S_i) \tag{7.12}$$

K is contained in $D(A)$. (7.13)

Moreover, by restriction to S_i, we have projections onto the i-th component:

$$\pi_i : \Sigma_A \to \Gamma(S_i) \quad i = 1, \ldots, P. \tag{7.14}$$

We define $\|C\|$ by the rule:

$$\|C\| = \max\{\|F\|, F \in C\}. \tag{7.15}$$

LEMMA 7.6. *Let $\{g_n\}$ be an infinite sequence of distinct mapping classes in Σ. Then the sequence of numbers, $\{\|g_nC\|\}$, is unbounded.*

Proof. The stable subgroup of Σ, Σ_A, is of finite index in Σ. Hence, there are finitely many left cosets of Σ_A in Σ. Thus, we may assume that there is an element, h, in Σ, such that the sequence, $\{g_n\}$, is contained in the left coset, $h\Sigma_A$.

$$g_n = hh_n, \quad h_n \in \Sigma_A. \tag{1}$$

It follows that:

$$\|g_nC\| \geq M \,.\, \|h_nC\| \quad \text{for some positive constant, } M. \tag{2}$$

Hence, we can assume that $\{g_n\}$ is contained in Σ_A.

Suppose that $\{\|g_nC\|\}$ is bounded. By applying Lemma 2.7 to the pseudo-Anosov components of S, we find that the corresponding sequences of restrictions are finite:

$$\{\pi_i g_n\} \text{ is finite } 1 \leq i \leq N. \tag{3}$$

Hence, by a coset argument as before, we may assume that $\{g_n\}$ acts trivially on the pseudo-Anosov components of S.

Since the restrictions to the remaining components of S_A are finite, we may assume, in addition, that $\{g_n\}$ is contained in the kernel, K. Thus, we may write g_n as a product of powers of Dehn twists about the components of A.

$$A = \{\alpha_1, \ldots, \alpha_M\} \tag{4}$$

$$D_i = \text{the Dehn twist about } \alpha_i \tag{5}$$

$$g_n = D_1^{p(1,n)} \cdots D_M^{p(K,n)}. \tag{6}$$

Since $\{g_n\}$ is infinite, we may assume that:

$$p(1, n) \rightarrow \pm\infty. \tag{7}$$

Using a norm constructed from a paving of **MF** which is compatible with A, as in [13], we easily see that:

$$\|g_n \alpha_1^*\| \rightarrow \infty. \tag{8}$$

Hence, $\{\|g_nC\|\}$ is unbounded.

The following lemma is proved in the same manner as Lemma 6.11 by using Lemma 7.6 instead of Lemma 2.7.

LEMMA 7.7. *Let F be an element of Δ, and $\{g_n\}$ be an infinite sequence of distinct mapping classes in Σ. Then the sequence of numbers, $\{i(F, |g_nC|)\}$, is unbounded.*

PROPOSITION 7.8. *For every F in Δ, there exists an element, g, in Σ such that F is in the set, $\Delta_{|gC|}$.*

PROPOSITION 7.9. *The following set of mapping classes is finite*: $\{g \in \Sigma \mid \Delta_{|C|}$ *is contained in* $\Delta_{|gL|}\}$.

Proof. Since Δ is nonempty, the proof follows through as for Proposition 6.13.

PROPOSITION 7.10. *Let K be a compact set in Δ. Then the following set of mapping classes is finite*: $\{g \in \Sigma \mid K \cap \Delta_{|gC|} \neq \emptyset\}$.

PROPOSITION 7.12. *Let F be a point in Δ. Then the Σ-orbit of F intersects $\Delta_{|C|}$ only finitely many times.*

THEOREM 7.17. *Σ acts properly discontinuously on Δ.*

8. A Remark on the dynamics on Teichmüller space

In this section, we consider the space, **PMF**, as Thurston's boundary at infinity of **T**, the Teichmüller space of the surface, (see [5], exposé 8). We recall that the topology on the union $\mathbf{T} \cup \mathbf{PMF}$ is defined by first imbedding the two spaces **T** and **MF** into the space $\mathbf{R}_+^{\mathbf{S}}$ of functions on the set **S** of nontrivial homotopy classes of simple closed curves, *via* the functions $\alpha \to l(y, \alpha)$ if $y \in \mathbf{T}$ (where $l(\,,)$ denotes the length of the closed geodesic in the class α), and $\alpha \to i(y, \alpha)$ if $y \in \mathbf{MF}$. The topology is then the induced topology on the projectivized space $\mathbf{PR}_+^{\mathbf{S}}$.

It is well-known that the mapping class group, Γ, acts properly discontinuously on **T**. Therefore, a limit point for Γ acting on $\mathbf{T} \cup \mathbf{PMF}$ is necessarily in **PMF**. We shall not go into the details of the dynamics of the action of subgroups of Γ on $\mathbf{T} \cup \mathbf{PMF}$, but we wish to point out how one can relate such a study to the study we have made of the dynamics of the action of Γ on **PMF**.

Let Σ be a subgroup of the mapping class group acting on **PMF**, and let $\Lambda(\Sigma)$ be the canonical limit set for this action, as defined in section 3.

Let $Z\Lambda(\Sigma)$ be the subset of **PMF** defined as:

$$\{F \mid i(F, x) = 0 \text{ for some point } x \text{ in } \Lambda(\Sigma)\}. \tag{8.1}$$

As described earlier, the set $Z\Lambda(\Sigma) \backslash \Lambda(\Sigma)$ has zero measure in **PMF**. We prove the following result on the dynamics of the group action on the compactified space:

PROPOSITION 8.1. *Let $[F]$ in* **PMF** *be a limit point (in the sense of section 3) for the group action of Σ on* $\mathbf{T} \cup \mathbf{PMF}$. *Then $[F]$ is in $Z\Lambda(\Sigma)$.*

Proof. Suppose that $\{g_n\}$ is an infinite sequence of distinct mapping classes in Σ and y is a point in $\mathbf{T} \cup \mathbf{PMF}$ such that $g_n(y)$ converges to $[F]$ in the topology of the functional space $\mathbf{PR}_+^{\mathbf{S}}$ (cf. [5]).

If y is in $\mathbf{PMF}$, the proposition is clear, since it follows immediately from sections 6 and 7 that Σ acts properly discontinuously on $\mathbf{PMF} \backslash Z\Lambda(\Sigma)$.

So we can assume that y is an element of $\mathbf{T}$.

By [5], exposé 8, Corollaire II.3, we can write:

$$r_n g_n(y) \to F \quad \text{in the space } \mathbf{R}_+^{\mathbf{S}}, \tag{1}$$

$$r_n \to 0. \tag{2}$$

We now use the "projection" $q : \mathbf{T} \to \mathbf{MF}$, described in [5]. We shall follow the notation of that section.

By [5], exposé 8, there is a neighborhood of F in $\mathbf{T} \cup \mathbf{PMF}$ of the form $V \cup W$, where V is a subset of $\mathbf{T}$ and W a subset of $\mathbf{PMF}$, on which there is a projection

$$q : V \to \mathbf{MF}$$

satisfying

$$i(q(v), \alpha) \leq I(v, \alpha) \quad \forall \alpha \in \mathbf{S} \quad \text{and} \quad v \in V \tag{3}$$

$$r_n q_n \to F \quad \text{in} \quad \mathbf{MF} \Leftrightarrow r_n y_n \to F \quad \text{in } \mathbf{R}_+^{\mathbf{S}} \tag{4}$$

where we can assume $y_n \in V$ and $r_n \to 0$.

Since the elements, $\{g_n\}$, are all distinct, we may choose, by (2.13), an element, α, of $\mathbf{S}$ such that the sequence, $\{g_n(\alpha)\}$, is unbounded. Taking a subsequence as usual, we can write:

$$s_n g_n(\alpha) \to G \quad \text{in } \mathbf{MF}, \tag{5}$$

$$s_n \to 0. \tag{6}$$

Note that G, being the limit of a sequence, $\{g_n(\alpha)\}$, in $\mathbf{PMF}$, is in $\Lambda(\Sigma)$.

Let $y_n = g_n(y)$ and $\alpha_n = g_n(\alpha)$. We have the following identity:

$$I(y_n, \alpha_n) = I(y, \alpha). \tag{7}$$

By using (3) and (7), we have the inequality:

$$i(r_n q(y_n), s_n \alpha_n) \leq r_n s_n I(y_n, \alpha_n) = r_n s_n I(y, \alpha). \tag{8}$$

By taking limits and using (4), we conclude that $i(F, G)$ is zero. Therefore, F is in $Z\Lambda(\Sigma)$.

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