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# The vanishing of Steenrod squares on $H$-spaces

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## §0. Introduction

The structure of the mod 2 cohomology of an $H$-space as a module over the Steenrod algebra is the topic of this paper. We assume throughout that the mod 2 homology is an associative ring. Under these hypotheses we prove the following theorem:

THEOREM A. *Let $X$ be a simply connected finite $H$-space. Then for $r \geq 0$, $k > 0$*

$$\sigma^* QH^{2^r+2^{r+1}k-1} \subseteq Sq^{2^r} H^*(\Omega X)$$

*and*

$$Sq^{2^r} QH^{2^r+2^{r+1}k-1}(X) = 0.$$

In the primitively generated case, Theorem A was proved by Thomas [10] by using techniques which involve the projective plane. In the primitively generated case it is not necessary to suspend to the loop space; the primary information is true in $H^*(X)$.

Since the appearance of Thomas' theorems for primitively generated $H$-spaces, topologists have wondered what the appropriate generalizations should be for $H$-spaces that do not admit primitive mod 2 Hopf structures. Using Thomas' results, one is able to show that the Lie groups $E_6$, $E_7$ and $E_8$ could not admit primitively generated mod 2 cohomology rings since they all have nine dimensional generators that do not lie in the image of $Sq^2$. It is shown in the proof here that in fact the *suspension* of a nine dimensional generator must lie in the image of $Sq^2$. Thus, Thomas' results hold when one *suspends* to the loop space of a finite $H$-space. One cannot always desuspend back because there are transpotence elements which are not suspensions.

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The proof is based on two inductive arguments. In this paper we assume the following inductive hypothesis $\mathcal{P}(r)$: For $r' < r$

$$QH^{2^{r'}+2^{r'+1}l-1}(X) = Sq^{2^{r'}l}QH^{2^{r'}+2^{r'}l-1}(X)$$

and

$$Sq^{2^{r'}}QH^{2^{r'}+2^{r'+1}l-1}(X) = 0.$$

We prove that $\mathcal{P}(r)$ implies

$$Sq^{2^r}QH^{2^r+2^{r+1}k-1}(X) = 0,$$

This paper can therefore be considered the sister paper to [7]. The inductive hypothesis of this paper is $\mathcal{P}(r)$. The end result of this paper is $\mathcal{S}(r)$, and $\mathcal{S}(r)$ is the inductive hypothesis of the paper [7].

$$\mathcal{S}(r): \mathcal{P}(r) \quad \text{and} \quad Sq^{2^r}QH^{2^r+2^{r+1}k-1}(X) = 0.$$

In [7], we show that $\mathcal{S}(r)$ implies $\mathcal{P}(r+1)$. The results of these two papers together yield $\mathcal{S}(r)$ and $\mathcal{P}(r)$ for all $r$. This paper proves the inductive hypothesis for [7] and [7] proves the inductive hypothesis for this paper.

Theorem A shows many Steenrod operations take indecomposables to decomposables. Browder [1] first showed that the Bockstein maps even indecomposables to decomposables. This implied a one connected finite $H$-space is two connected. It was then shown using techniques similar to these here that a three connected finite $H$-space is 6 connected and a seven connected finite $H$-space is fourteen connected [4, 6]. In a subsequent paper we will prove a finite $H$-space has first nonvanishing homotopy group in degrees one, three, seven, or fifteen [7].

The proof of $\mathcal{P}(r)$ depends on a thorough understanding of the interplay between the cohomology of the $H$-space and the cohomology of the loop space. Use of the Eilenberg Moore spectral sequence, together with the fact that the loop space has no homology two torsion allows us to conclude that the module of cohomology primitives consists of transpotence and suspension elements only. We then apply a theorem of Lin and Williams [9] which restricts the indeterminacy of a secondary cohomology operation in the cohomology of the loop space. This restriction on the indeterminacy has the effect of limiting our consideration only to the action of the Steenrod algebra on suspension and transpotence elements. Information about suspension elements can be pulled back to the original $H$-space, since the suspension map is a monomorphism on odd dimensional indecomposables. The action of the Steenrod algebra on transpotence elements is shown to be dependent on the degree of the Steenrod operation and the degree of the Steenrod operation; in particular, for the operations we will be considering, they will often map transpotence elements into suspension elements.

Throughout the paper we will use the following conventions. The symbol $X$ will be reserved for a one connected $H$-space with the following properties:

(1) $H_*(X; \mathbb{Z}_2)$ is an associative ring.
(2) For $r' \leq r$ the module

$$\sum_{l=1}^{\infty} QH^{2^{r'}+2^{r'+1}l-1}(X; \mathbb{Z}_2)$$

is a finite dimensional vector space.
(3) $QH^{\text{even}}(X; \mathbb{Z}_2) = 0$.

These hypotheses hold for all finite one connected $H$-spaces that satisfy condition 1, and all known finite $H$-spaces admit an $H$-structure that satisfies condition 1.

We will use the following simplified notation

$$H^* = H^*(X; \mathbb{Z}_2) \qquad H_* = H_*(X; \mathbb{Z}_2)$$
$$Q^* = QH^*(X; \mathbb{Z}_2)$$

If there are no coefficients, it will be understood that the coefficients are $\mathbb{Z}_2$. In section 3 it will be useful to let

$$P^* = PH^*(\Omega X).$$

In chapter one a universal example is constructed. Roughly speaking, a Postnikov space $E$ is constructed which has an element $v \in H^*(E)$ with $\bar{\Delta} v = u \otimes u$. The construction is designed to reflect information obtained from inductive hypothesis $\mathcal{P}(r)$. The reader is encouraged to read the proof of [6] which is an excellent model to motivate the more general construction given here in chaper one.

In chapter two we show that it may not in general be possible to lift our $H$-space $X$ to $E$. However, a result of Lin and Williams [9] allows us to map $\Omega X$ to $\Omega E$ by an $H$-map $\tilde{f}$. The element $\sigma^* v \in PH^*(\Omega E)$ has nonzero $c$-obstruction $c(\sigma^* v) = \sigma^* u \otimes \sigma^* u$. If $x \in Q^{2^r+2^{r+1}k-1}$ and $\tilde{f}^*(\sigma^* u) = \sigma^* x$, then

$$c(\tilde{f}(\sigma^* v)) = \sigma^* x \otimes \sigma^* x + c(\tilde{f})^*(\sigma^* v).$$

It is shown that $c(\tilde{f})^*(\sigma^* v)$ must "cancel" $\sigma^* x \otimes \sigma^* x$. In our situation $c(\tilde{f})^*$ will involve primary operations.

In chapter three, we show that our judicious choice of Steenrod factorizations forces $\sigma^* x$ to lie in the image of $Sq^{2^r}$. This completes the proof of Theorem A given the hypotheses of $\mathscr{P}(r)$. Note that $\mathscr{P}(1)$ was proved in [5]. Therefore by the process described above, we obtain $\mathscr{S}(1)$. $\mathscr{S}(1)$ implies $\mathscr{P}(2)$ which implies $\mathscr{S}(2)$ and so on.

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## §1. Construction of the Universal Example

Consider the following hypothesis $\mathscr{P}(r)$: For $r' < r$

$$Q^{2^{r'}+2^{r'+1}l-1} = Sq^{2^{r'}l} Q^{2^{r'}+2^{r'}l-1} \quad \text{for all} \quad l > 0$$

and

$$Sq^{2^{r'}} Q^{2^{r'}+2^{r'+1}l-1} = 0.$$

THEOREM 1.1.(r). *Assuming $\mathscr{P}(r)$, then for $r > 1$,*

$$\sigma^* Q^{2^r+2^{r+1}k-1} \subseteq Sq^{2^r} PH^{2^{r+1}k-2}(\Omega X)$$

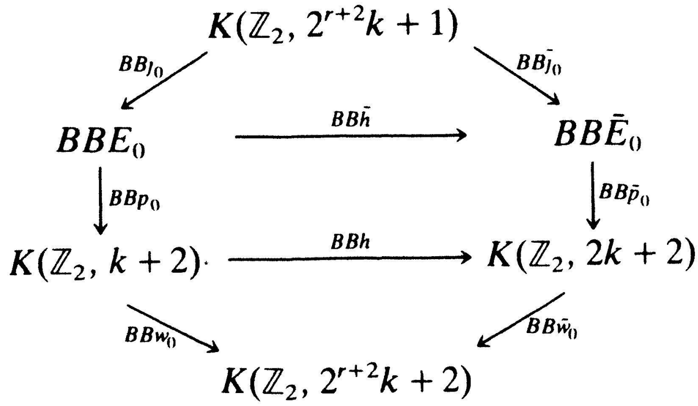
*and*

$$Sq^{2^r} Q^{2^r+2^{r+1}k-1} = 0.$$

The proof of Theorem 1.1(r) will occupy the next couple of chapters. In chapter 1 we build the universal example. In chapter 2 we prove there is a lifting. In chapter 3 the $c$-invariant is used to prove Theorem 1.1(r).

The proof is modeled on the proof that $\sigma^* Q^{8k+3} \subseteq \operatorname{im} Sq^4$. The reader is encouraged to become familiar with the techniques of that proof [6].

We now build the second stage of the universal example $E$. Consider the following commutative diagram of maps and spaces for $r \geq 1$.



Here $BBE_0$, $BB\bar{E}_0$ are the fibres of $BBw_0$, $BB\bar{w}_0$ respectively.

$$BBw_0^*(i_{2^{r+2}k+2}) = Sq^{2^{r+1}k}Sq^{2^rk}\cdots Sq^{2k}Sq^k i_{k+2}$$
$$BB\bar{w}_0^*(i_{2^{r+2}k+2}) = Sq^{2^{r+1}k}Sq^{2^rk}\cdots Sq^{2k} i_{2k+2}$$
$$BBh^*(i_{2k+2}) = Sq^k i_{k+2}.$$

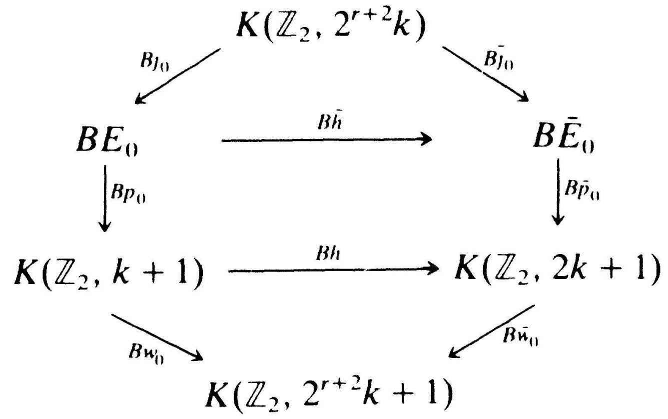
By induction on $r$, it is easy to show that for $2 \le i \le r$

$$Sq^{2^i}[Sq^{2^{r+1}k}Sq^{2^rk}\cdots Sq^{2k}]i_{2k+2} = 0. \tag{1.1}$$

Also

$$Sq^{2^r}Sq^1[Sq^{2^{r+1}k}\cdots Sq^{2k}]i_{2k+2} = 0 \tag{1.2}$$
$$Sq^2[Sq^{2^{r+1}k}\cdots Sq^{2k}]i_{2k+2} = (Sq^{2^rk}\cdots Sq^{2k} i_{2k+2})^2. \tag{1.3}$$

Looping the above diagram, we get




Equations (1.1) and (1.2) imply there are suspension elements $B\bar{v}_i \in H^*(B\bar{E}_0)$ for $2 \le i \le r$ and for $i = 0$ where $B\bar{j}_0^*(B\bar{v}_i) = Sq^{2^i} i_{2^{r+2}k}$ for $2 \le i \le r$ and

$$B\bar{j}_0^*(B\bar{v}_0) = Sq^{2^r}Sq^1 i_{2^{r+2}k},$$

Equation (1.3) implies there is an element $B\bar{v}_1 \in H^*(B\bar{E}_0)$ with $B\bar{j}_0^*(B\bar{v}_1) = Sq^2 i_{2^{r+2}k}$ and

$$\bar{\Delta} B\bar{v}_1 = Sq^{2^rk}\cdots Sq^{2k} i_{2k+1} \otimes Sq^{2^rk}\cdots Sq^{2k} i_{2k+1}. \tag{1.4}$$

Now since $r \ge 1$, $\Omega\bar{E}_0 \simeq K(\mathbb{Z}_2, 2k-1) \times K(\mathbb{Z}_2, 2^{r+2}k-2)$ as $H$-spaces. Letting

$\bar{v}_i = \sigma^*(B\bar{v}_i)$, then for $1 \le i \le r$

$$\sigma^*(\bar{v}_i) = \alpha_i i_{2k-1} \otimes 1 + 1 \otimes Sq^{2^i} i_{2^r+2k-2}$$
$$\sigma^*(v_0) = \alpha_0 i_{2k-1} \otimes 1 + 1 \otimes Sq^{2^r} Sq^1 i_{2^r+2k-2}$$

where $\alpha_i \in \mathscr{A}(2)$.

If the $\alpha_i$ are nontrivial, we can change $B\bar{v}_i$ by $B\bar{p}_0^*(\alpha_i i_{2k+1})$ so that

$$\sigma^*(\bar{v}_i) = 1 \otimes Sq^{2^i} i_{2^r+2k-2}$$

for $1 \le i \le r$

$$\sigma^*(\bar{v}_0) = 1 \otimes Sq^{2^r} Sq^1 i_{2^r+2k-2}.$$

Then since

$$Sq^{2^r} Sq^{2^r} = \sum_{i=1}^{r-1} Sq^{2^{r+1}-2^i} Sq^{2^i} + Sq^{2^r-1} Sq^{2^r} Sq^1$$

it follows that

$$\sum_{i=1}^{r-1} Sq^{2^{r+1}-2^i} \bar{v}_i + Sq^{2^r-1} \bar{v}_0 \in \ker \sigma^* \cap PH^*(\bar{E}_0).$$

But $\sigma^* : QH^{\mathrm{odd}}(\bar{E}_0) \to PH^{\mathrm{even}}(\Omega\bar{E}_0)$ is monic. Hence

$$\sum_{i=1}^{r-1} Sq^{2^{r+1}-2^i} \bar{v}_i + Sq^{2^r-1} \bar{v}_0 = 0.$$

Therefore

$$\sigma^* \left[ \sum_{i=1}^{r-1} Sq^{2^{r+1}-2^i} B\bar{v}_i + Sq^{2^r-1} B\bar{v}_0 \right] = 0.$$

Since $\sigma^* : QH^{4l}(B\bar{E}_0) \to PH^{4l-1}(\bar{E}_0)$ is also monic equation (1.4) implies

$$\sum_{i=1}^{r-1} Sq^{2^{r+1}-2^i} B\bar{v}_i + Sq^{2^r-1} B\bar{v}_0$$

is primitive decomposable and in $\ker B\bar{j}_0^*$.

It follows that there is a primary operation $\alpha$ of odd degree such that

$$\sum_{i=1}^{r-1} Sq^{2^{r+1}-2^i} B\bar{v}_i + Sq^{2^r-1} B\bar{v}_0 = [B\bar{p}_0^*(\alpha i_{2k+1})]^2. \tag{1.5}$$

Now define $Bv_i = (B\bar{h})^*(B\bar{v}_i)$. Then equation (1.5) implies

$$\sum_{i=1}^{r-1} Sq^{2^{r+1}-2^i} Bv_i + Sq^{2^r-1} Bv_0 = [B\bar{p}_0^*(\alpha Sq^k i_{k+1})]^2. \tag{1.6}$$

$\alpha Sq^k i_{k+1}$ is a cup product square, therefore the right hand side of (1.6) is at least a fourth power.

We have the following theorem.

THEOREM 1.2. *There exist elements* $\sigma^*(Bv_i) = v_i \in H^*(E_0)$ *which satisfy*

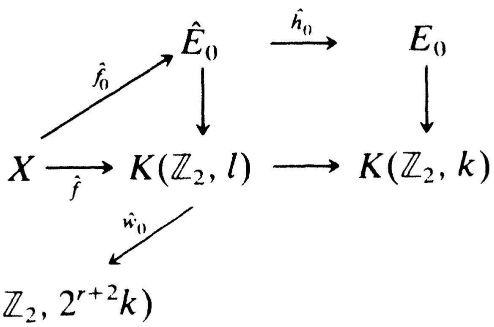
(1) $\sum_{i=1}^{r-1} Sq^{2^{r+1}-2^i} Bv_i + Sq^{2^r-1} Bv_0$ *is four fold decomposable,*

(2) $\sum_{i=1}^{r-1} Sq^{2^{r+1}-2^i} v_i + Sq^{2^r-1} v_0 = 0.$

*Further, suppose* $k = 2^i l$ *where* $l$ *is odd and there is a generator* $x_l \in H^l(X)$ *with height* $2^{r+2+i}$. *Then there is a map* $\hat{f}_0 : X \to E_0$ *with* $\sigma^* \hat{f}_0^*(v_r) = Sq^{2^r} \varphi_{2^{r+2+i}}(x_l)$.

*Proof.* (1) and (2) follow from (1.6).

Note that $\varphi_{2^{r+2+i}}(x_l)$ is realized by a map into the following two stage system $\hat{E}_0$.




$\hat{w}_0^*(i_{2^{r+2}k}) = (i_l)^{2^{r+2+i}}$. Define $\hat{f}^*(i_l) = x_l$. We have

$$\Omega \hat{E}_0 \simeq K(\mathbb{Z}_2, l-1) \times K(\mathbb{Z}_2, 2^{r+2}k - 2)$$

and

$$\Omega \hat{f}_0^*(i_{2^{r+2}k-2}) = \varphi_{2^{r+2+i}}(x_l).$$

But there exist maps $\hat{h}_0: \hat{E}_0 \to E_0$ and

$$\Omega\hat{h}_0^*(1 \otimes i_{2^{r+2}k+2}) = i_{2^{r+2}k-2} \in H^*(\Omega\hat{E}_0).$$

Let $\bar{f}_0 = \hat{h}_0\hat{f}_0$. It follows that

$$\begin{aligned} \sigma^*\bar{f}_0^*(v_r) &= \Omega\bar{f}_0^*(1 \otimes Sq^{2^r} i_{2^{r+2}k-2}) \\ &= Sq^{2^r}\varphi_{2^{r+2+l}}(x_l) \quad \text{Q.E.D.} \end{aligned}$$

The universal example $E$ will be a loop space with an element $v \in H^*(E)$ with $\bar{\Delta}v = u \otimes u$. If $E$ is an $H$-space, let $P_2E$ denote the projective plane of $E$. Recall there is an exact triangle with the property that if $i(y) = u$ then $\lambda(u \otimes u) = y^2$, [2].

$$\begin{array}{ccc} H^*(P_2E) & \xrightarrow{\;i\;} & IH^*(E) \\ & {\scriptstyle\lambda}\nwarrow \quad \swarrow{\scriptstyle\bar{\Delta}} & \\ & IH^*(E) \otimes IH^*(E) & \end{array}$$

We will show there is such an element $y \in H^*(P_2E)$. The following proposition is a consequence of the Adem relations and calculations of binomial coefficients.

PROPOSITION 1.3.
(a) *Let* $\theta_r = Sq^{2^{r+1}k} + Sq^{2^{r+1}k-2^{r-1}}Sq^{2^{r-1}}$. *Then*

$$Sq^{2^r+2^{r+1}k} = Sq^{2^r}\theta_r + \sum_{i=0}^{r-2} (Sq^{2^{r+1}k}Sq^{2^r-2^i} + Sq^{2^{r+1}k+2^r-2^i})Sq^{2^i}$$

(b) *Let* $i \le r-2$. *Then on elements of degree* $2^i + 2^{r-1} + 2^r k$

$$Sq^{2^{r+1}k+2^r-2^i}Sq^{2^r k+2^{r-1}} = \sum_{l\le i-1} Sq^{2^{r+1}k+2^r-2^l}Sq^{2^r k+2^{r-1}-2^i+2^l}$$

$$Sq^{2^r-2^i}Sq^{2^{r-1}+2^r k} = \sum_{l\le i-1} Sq^{2^r-2^l+2^r k}Sq^{2^{r-1}-2^i+2^l}.$$

(c) *Let* $i \le r-2$. *On elements of degree* $2^i + 2^{r+1}k$

$$Sq^{2^{r+1}-2^i}Sq^{2^{r+1}k} = \sum_{l\le i-1} Sq^{2^{r+1}k+2^r-2^l}Sq^{2^r-2^i+2^l}.$$

*Proof.* (b) and (c) follow by direct application of the Adem relations and computing binomial coefficients. For (a) note that

$$Sq^{2^r+2^{r+1}k} = \sum_{i=0}^{r-2} Sq^{2^{r+1}k+2^r-2^i} Sq^{2^i} + Sq^{2^r} Sq^{2^{r+1}k} + Sq^{2^{r-1}+2^{r+1}k} Sq^{2^{r-1}}$$

and

$$Sq^{2^{r-1}+2^{r+1}k} = Sq^{2^{r+1}k} Sq^{2^{r-1}} + Sq^{2^r} Sq^{2^{r+1}k-2^{r-1}}$$

$$Sq^{2^{r-1}} Sq^{2^{r-1}} = \sum_{i=0}^{r-2} Sq^{2^r-2^i} Sq^{2^i}. \quad \text{Q.E.D.}$$

We now define spaces $BK$, $BK_0$ and a map $Bw: BK \to BK_0$. The fibre of $Bw$ will be $BE$ and $E$ will be the desired universal example. Let

$$BK = BE_0 \times K(\mathbb{Z}_2, 2^r + 2^{r+1}k) \times \prod_{i=1}^{r-2} K(\mathbb{Z}_2, 2^i + 2^{r-1} + 2^r k) \times \prod_{i=1}^{r-1} K(\mathbb{Z}_2, 2^i + 2^{r+1}k)$$

$$BK_0 = K(\mathbb{Z}_2, 2^r + 2^{r+2}k) \times \prod_{i=0}^{r-2} K(\mathbb{Z}_2, 2^i + 2^r + 2^{r+1}k)$$
$$\times \prod_{i=1}^{r-1} K(\mathbb{Z}_2, 2^i + 2^{r+2}k) \times K(\mathbb{Z}_2, 1 + 2^r + 2^{r+2}k)$$
$$\times \prod_{i \le r-2} \prod_{l \le i-1} K(\mathbb{Z}_2, 2^r k + 2^r + 2^l)$$
$$\times \prod_{i \le r-2} \prod_{l \le i-1} K(\mathbb{Z}_2, 2^{r+1}k + 2^r + 2^l)$$
$$\times \prod_{i \le r-1} \prod_{l \le i-1} K(\mathbb{Z}_2, 2^{r+1}k + 2^r + 2^l)$$

Define $Bw: BK \to BK_0$ by

$$(Bw)^*(i_{2^r+2^{r+2}k}) = \theta_r i_{2^r+2^{r+1}k} - Bv_r$$
$$(Bw)^*(i_{2^i+2^r+2^{r+1}k}) = Sq^{2^i} i_{2^r+2^{r+1}k} - Sq^{2^{r-1}+2^r k} i_{2^i+2^{r-1}+2^r k}$$
$$(Bw)^*(i_{2^i+2^{r+2}k}) = Bv_i - Sq^{2^{r+1}k} i_{2^i+2^{r+1}k} \qquad 1 \le i \le r-1$$

$$(Bw)^*(i_{1+2^r+2^{r+2}k}) = Bv_0$$

$$(Bw)^*(i_{2^ik+2^r+2^l}) = Sq^{2^{r-1}-2^i+2^l} i_{2^i+2^{r-1}+2^rk}$$

$$(Bw)^*(i_{2^{r+1}k+2^r+2^l}) = Sq^{2^rk+2^{r-1}+2^l-2^i} i_{+2^i+2^{r-1}+2^rk}$$

$$(Bw)^*(i'_{2^{r+1}k+2^r+2^l}) = Sq^{2^r-2^i+2^l} i_{2^i+2^{r+1}k}\,.$$

We have a diagram

$$\begin{array}{ccc} K_0 & & \\ \downarrow{\scriptstyle Bj} & & \\ BE & & \\ \downarrow{\scriptstyle Bp} & & \\ BK & \xrightarrow[Bw]{} & BK_0 \end{array}$$

Consider the element $z \in H^*(BK_0)$ defined by

$$\begin{aligned} z = {} & Sq^{2^r} i_{2^r+2^{r+2}k} + \sum_{i=0}^{r-2} (Sq^{2^{r+1}k} Sq^{2^r-2^i} + Sq^{2^{r+1}k+2^r-2^i}) i_{2^i+2^r+2^{r+1}k} \\ & + \sum_{i=1}^{r-1} Sq^{2^{r+1}-2^i} i_{2^i+2^{r+2}k} + Sq^{2^r-1} i_{1+2^r+2^{r+2}k} \\ & + Sq^{2^{r+1}k} \sum_{l \le i-1} Sq^{2^r+2^l+2^rk} i_{2^rk+2^r+2^l} \\ & + \sum_{l \le i-1} Sq^{2^{r+1}k+2^r-2^l} (i_{2^{r+1}k+2^r+2^l} + i'_{2^{r+1}k+2^r+2^l}) \end{aligned} \tag{1.7}$$

LEMMA 1.4. *Modulo three fold decomposables,*

$$(Bw)^*(z) = (i_{2^r+2^{r+2}k})^2.$$

*Proof.* It is a lengthy but simple calculation. We have, using the formulas for $(Bw)^*$,

$$(Bw)^*[Sq^{2^r} i_{2^r+2^{r+2}k}] = Sq^{2^r} \theta_r i_{2^r+2^{r+1}k} - Sq^{2^r} Bv_r$$

$$(Bw)^*\left[\sum_{i=0}^{r-2} (Sq^{2^{r+1}k} Sq^{2^r-2^i} + Sq^{2^{r+1}k+2^r-2^i}) i_{2^i+2^r+2^{r+1}k}\right]$$

$$= \sum_{i=0}^{r-2} (Sq^{2^{r+1}k} Sq^{2^r-2^i} + Sq^{2^{r+1}k+2^r-2^i}) Sq^{2^i} i_{2^r+2^{r+1}k}$$

$$- \sum_{i=0}^{r-2} (Sq^{2^{r+1}k} Sq^{2^r-2^i} + Sq^{2^{r+1}k+2^r-2^i}) Sq^{2^{r-1}+2^r k} i_{2^i+2^{r-1}+2^r k}.$$

By Proposition 1.3(a) the sum of these two terms is

$$(i_{2^r+2^{r+1}k})^2 - Sq^{2^r} Bv_r - \sum_{i=0}^{r-2} (Sq^{2^{r+1}k} Sq^{2^r-2^i} + Sq^{2^{r+1}k+2^r-2^i}) Sq^{2^{r-1}+2^r k} i_{2^i+2^{r-1}+2^r k}.$$

By Proposition 1.3(b) the last term in the above sum is cancelled by

$$(Bw)^*\left[\sum_{l\le i-1} Sq^{2^{r+1}k+2^r-2^l}(i_{2^{r+1}k+2^r+2^l}) + Sq^{2^{r+1}k} \sum_{l\le i-1} Sq^{2^r-2^l+2^r k} i_{2^r k+2^r+2^l}\right].$$

Now consider

$$(Bw)^*\left[\sum_{i=1}^{r-1} Sq^{2^{r+1}-2^i} i_{2^i+2^{r+2}k} + Sq^{2^r-1} i_{1+2^r+2^{r+2}k}\right]$$

$$= \sum_{i=1}^{r-1} Sq^{2^{r+1}-2^i}(Bv_i - Sq^{2^{r+1}k} i_{2^i+2^{r+1}k}) + Sq^{2^r-1} Bv_0$$

$$= Sq^{2^r} Bv_r + \text{four fold decomposables}$$

$$+ \sum_{i=1}^{r-1} Sq^{2^{r+1}-2^i} Sq^{2^{r+1}k} i_{2^i+2^{r+1}k} \text{ by Theorem 1.2.}$$

By Proposition 1.3(c) the last term in the above sum is cancelled by

$$(Bw)^*\left[\sum_{l\le i-1} Sq^{2^{r+1}k+2^r-2^l} i'_{2^{r+1}k+2^r+2^l}\right].$$

Therefore $(Bw)^*(z) = (i_{2^r+2^{r+1}k})^2$ + four fold decomposables. Q.E.D.

Looping $Bw$ we have a diagram

$$\begin{array}{ccc} \Omega K_0 & & \\ \downarrow{\scriptstyle j} & & \\ E & & \\ \downarrow{\scriptstyle p} & & \\ K & \xrightarrow[w]{} & K_0 \end{array}$$

PROPOSITION 1.5. *There exists an element* $v \in H^*(E)$ *with* $\bar{\Delta} v = u \otimes u$ *and* $j^*(v) = (\sigma^*)^2(z)$, $u = p^*(i_{2^r+2^{r+1}k-1})$. *Therefore* $\sigma^*(v) \in H^*(\Omega E)$ *has*

$$c(\sigma^* v) = \sigma^* u \otimes \sigma^* u.$$

*Proof.* By Lemma 1.4 $(Bp)^*(i_{2^r+2^{r+1}k}) \in H^*(BE)$ has the property that $(Bp)^*(i_{2^r+2^{r+1}k})^2$ is four fold decomposable. Since $P_2E \subseteq BE$, let $y$ be the image of $(Bp)^*(i_{2^r+2^{r+1}k})$ in $H^*(P_2E)$. Then since $H^*(P_2E)$ has no 3-fold products, $y^2 = 0$. Further $H^*(P_2E) \to H^*(E)$ maps $y$ to a primitive $u = p^*(i_{2^r+2^{r+1}k-1})$. It follows that there exists a $v \in H^*(E)$ with the desired properties. Q.E.D.

## §2. Construction of a lifting

As the reader may have surmised, the space $E$ was constructed to facilitate the lifting of a map $X \xrightarrow{f} K$ to $E$. Such a lifting, however, does not obviously exist because we do not know that $\theta_r x = \bar{f}_0^*(v_r)$. At best, we know that $\theta_r x - \bar{f}_0^*(v_r)$ is odd decomposable. Since $Q^{\text{even}} = 0$, we can conclude $\theta_r x - \bar{f}_0^*(v_r)$ is three fold decomposable.

Our strategy will be to prove that there is a map $f : X \to K$ such that $\operatorname{im}(wf)^*$ is three fold decomposable. Recall that $H^*(P_2\Omega X)$ has all three fold products trivial. Therefore, there is a commutative diagram

$$\begin{array}{ccccccc} & & & & E & & \\ & & & \nearrow & \downarrow & & \\ P_2\Omega X & \longrightarrow & X & \xrightarrow[f]{} & K & \xrightarrow[w]{} & K_0 \end{array}$$

This will be the goal of this chapter.

It will be convenient to recall the definition of $w$

$$w^*(i_{2^r+2^{r+2}k-1}) = \theta_r i_{2^r+2^{r+1}k-1} - v_r \tag{2.1}$$

$$w^*(i_{2^i+2^r+2^{r+1}k-1}) = Sq^{2^i} i_{2^r+2^{r+1}k-1} - Sq^{2^{r-1}+2^r k} i_{2^i+2^{r-1}+2^r k-1} \tag{2.2}$$

$$w^*(i_{2^i+2^{r+2}k-1}) = v_i - Sq^{2^{r+1}k} i_{2^i+2^{r+1}k-1} \quad \text{for} \quad 1 \le i \le r-1 \tag{2.3}$$

$$w^*(i_{2^r+2^{r+2}k}) = v_0 \tag{2.4}$$

$$w^*(i_{2^r k+2^r+2^l-1}) = Sq^{2^{r-1}-2^i+2^l} i_{2^i+2^{r-1}+2^r k-1} \tag{2.5}$$

$$w^*(i_{2^{r+1}k+2^r+2^l-1}) = Sq^{2^r k+2^{r-1}+2^l-2^i} i_{2^i+2^{r-1}+2^r k-1} \tag{2.6}$$

$$w^*(i'_{2^{r+1}k+2^r+2^l-1}) = Sq^{2^r-2^i+2^l} i_{2^i+2^{r+1}k-1}. \tag{2.7}$$

Let $\bar{x} \in Q^{2^r+2^{r+1}k-1}$ have representative $x$. By downward induction on $k$, we may assume that for $k' > k$

$$\sigma^* Q^{2^r+2^{r+1}k'-1} \subseteq Sq^{2^r} PH^{2^{r+1}k'-2}(\Omega X).$$

Then $\sigma^*(\theta_r x) = Sq^{2^r} y$ for some $y \in PH^{2^{r+2}k-2}(\Omega X)$. If $y$ is a transpotence element, Theorem 1.2 implies there is a lift $\bar{f}_0 : X \to E_0$ such that $\sigma^*[\bar{f}_0^*(v_r)] = Sq^{2^r} y$. If $y$ is a suspension element, then choose $\bar{f}_0 : X \to K(\mathbb{Z}_2, 2^{r+2}k - 1) \xrightarrow{j_0} E_0$. In either case

$$\sigma^*(\theta_r x) = Sq^{2^r} y = \sigma^*[\bar{f}_0^*(v_r)].$$

But $\sigma^* : Q^{\text{odd}} \to PH^{\text{even}}(\Omega X)$ is monic. Therefore $\theta_r x - \bar{f}_0^*(v_r)$ is odd decomposable and therefore three fold decomposable. This defines $f : X \to K$ into the factors $K(\mathbb{Z}_2, 2^r + 2^{r+1}k - 1) x E_0$.

Now $\mathscr{P}(r)$ implies

$$Sq^{2^r} x = Sq^{2^{r-1}+2^r k} x_{2^r+2^{r-1}+2^r k-1} + \text{three fold decomposables.}$$

Hence, let

$$f^*(i_{2^r+2^{r-1}+2^r k-1}) = x_{2^r+2^{r-1}+2^r k-1}.$$

Similarly,

$$\bar{f}_0^*(v_0) = Sq^{2^{r+1}k} x_{2^r+2^{r+1}k-1} + \text{three fold decomposables.}$$

So let

$$f^*(i_{2^r+2^{r+1}k-1}) = x_{2^r+2^{r+1}k-1}.$$

Since $Q^{\text{even}} = 0$, $\bar{f}_0^*(v_0)$ is decomposable. In the construction for $E_0$, $\bar{E}_0$, $\bar{f}_0^*(v_0) = \bar{f}_0^*(\bar{h}^*(\bar{v}_0))$. Applying the Cartan formula [4] to $\bar{\Delta} \bar{f}_0^*(\bar{h}^*(\bar{v}_0))$ yields

$$\bar{\Delta} \bar{f}_0^*(\bar{h}^*(\bar{v}_0)) \in \xi H^* \otimes H^* + H^* \otimes \xi H^* + \operatorname{im} Sq^{2^r-1}$$
$$\{f_0^*(\bar{h}^*(\bar{v}_0))\} \in P^{2^r+2^{r+2}k}(H^* // \xi H^*) = 0.$$

Therefore $f_0^*(v_0)$ is three fold decomposable.

To complete the proof that $\operatorname{im}(wf)^*$ is three fold decomposable we need the following lemma.

LEMMA 2.1. *Let* $i<r$, $l>0$. *Then if* $2^i$ *appears in the dyadic expansion of* $n$ *then*

$$Sq^n Q^{2^i+2^{i+1}l-1}=0.$$

*Proof.* Write

$$n=2^{i_1}+2^{i_2}+\cdots+2^{i_{s-1}}+2^i+2^{i+1}m$$

for $m\geq 0$, $0\leq i_1<i_2<\cdots<i_{s-1}$.

Then modulo doubletons of degree $\leq 2^i$

$$Sq^n=Sq^{2^{i_1}}Sq^{2^{i_2}}\cdots Sq^{2^{i_{s-1}}}Sq^{2^i}Sq^{2^{i+1}m}.$$

A simple argument shows all doubletons of degree less than or equal to $2^i$ annihilate $Q^{2^i+2^{i+1}l-1}$. Now $Sq^{2^i}Sq^{2^{i+1}m}Q^{2^i+2^{i+1}l-1}=0$ by $\mathcal{P}(r)$. Hence $Sq^nQ^{2^i+2^{i+1}l-1}=0$. Q.E.D.

Lemma 2.1 implies

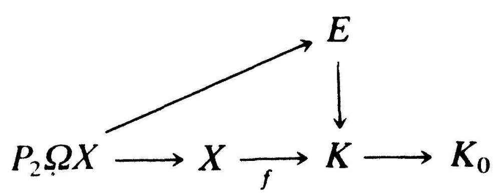
$$Sq^{2^{r+1}-2^i+2^l}x_{2^i+2^{r-1}+2^rk-1}, \qquad Sq^{2^r+2^{r-1}+2^l-2^i}x_{2^i+2^{r+1}+2^rk-1}$$

and $Sq^{2^r-2^i+2^l}x_{2^i+2^{r+1}k-1}$ are three fold decomposable. We have shown

THEOREM 2.2. *There is a map* $f:X\to K$ *such that* $\operatorname{im}(wf)^*$ *is three fold decomposable.*

## §3. Analysis of indeterminacy

By Theorem 2.2, there is a lifting




Looping the entire diagram and using the fact that $\Omega X$ is an $H$-retract of $\Omega P_2\Omega X$

we have a diagram

$$\begin{array}{ccccc} & & \Omega^2 K_0 & & \\ & & \big\downarrow{\scriptstyle \Omega j} & & \\ & & \Omega E & & \\ & \overset{\tilde{f}}{\nearrow} & \big\downarrow{\scriptstyle \Omega p} & & \\ \Omega K & \xrightarrow[\Omega f]{} & \Omega K & \longrightarrow & \Omega K_0 \end{array}$$

where $\tilde{f}$ is an $H$-lifting.

Since $\Omega E_0 \simeq K(\mathbb{Z}_2, k-1) \times K(\mathbb{Z}_2, 2^{r+2}k-2)$ and $H^*(\Omega X)$ is even dimensional, $\Omega f$ has no $c$-obstruction.

It will be useful to use abbreviated notation here to identify certain submodules of $H^*(\Omega X)$.

DEFINITION 3.1. Let $P$ denote the submodule $PH^*(\Omega X)$. Let $F_2 \subseteq H^*(\Omega X)$ be the submodule generated by primitives, two fold products of suspensions and divided squares of suspensions. Let $D \subseteq H^*(\Omega X)$ be the submodule of decomposables of $H^*(\Omega X)$.

A theorem of Lin and Williams [9] shows if $\{a_i\}$ are the Steenrod operations appearing in the expansion for $z$, equation (1.7), then

$$\sigma^* x \otimes \sigma^* x \in \sum a_i(P \otimes F_2 + F_2 \otimes P). \tag{3.1}$$

Since $H^*(\Omega X)$ is bicommutative and even dimensional we have

Primitives and indecomposables in degrees congruent to two mod four are isomorphic. (3.2)

If $\gamma_2(u)$ is a divided square of a primitive $u$ and $Sq^l u = 0$, then $Sq^{2l}\gamma_2(u)$ is a primitive plus a decomposable. (3.3)

By [5],

$P^{4l} = \xi P^{2l}$, hence $Sq^2$ maps primitives to decomposables. (3.4)

Let $A = P^{2^r+2^{r+1}k-2}$, $B = P^{2^r+2^r k-2}$.

LEMMA 3.1. (a) *If* $\deg a_i \equiv 0 \bmod 4$, *then*

$$a_i(P \otimes F_2 + F_2 \otimes P) \cap (A \otimes A) \subseteq a_i(P \otimes P)$$

(b) *If* $\deg a_i \equiv 2 \bmod 4$, *then*

$$a_i(P \otimes F_2 + F_2 \otimes P) \cap (A \otimes A)$$

*is a sum of terms of the form*

$$\alpha_1 P \otimes \beta_1 F_2 + \beta_2 F_2 \otimes \alpha_1 P$$

*where* $\alpha_i$, $\beta_i \in \mathcal{A}(2)$ *and* $\deg \alpha_i \equiv 0 \bmod 4$ *and* $\deg \beta_i \equiv 2 \bmod 4$.

*Proof.* By (3.2) $A$ consists of indecomposables. Now if $\alpha \in \mathcal{A}(2)$, $x \in P$ and $\alpha x \in A$, then by (3.4) $\deg \alpha \equiv 0 \bmod 4$ and $\deg x \equiv 2 \bmod 4$. Therefore, if $a_i(P \otimes F_2) \cap (A \otimes A)$ is nontrivial it consists of sums of terms of the form $\alpha x \otimes \beta y$ where degree $\beta \equiv 0 \bmod 4$ in case (a). Then $y \in F_2$ and $\deg y \equiv 2 \bmod 4$. Since there are no divided squares in this degree, $y$ must be a primitive. Hence

$$a_i(P \otimes F_2) \cap (A \otimes A) \subseteq a_i(P \otimes P).$$

Similarly

$$a_i(F_2 \otimes P) \cap (A \otimes A) \subseteq a_i(P \otimes P).$$

This proves (a).

Now in case (b) if $\deg a_i \equiv 2 \bmod 4$ then

$$a_i(P \otimes F_2) \cap (A \otimes A)$$

consists of terms of the form $\alpha_1 x \otimes \beta_1 y$ where $\deg \alpha_1 \equiv 0 \bmod 4$ and $\deg \beta_1 \equiv 2 \bmod 4$. Similarly $a_i(F_2 \otimes P) \cap (A \otimes A)$ is spanned by terms of the form $\beta_2 z \otimes \alpha_2 w$ where $\deg \alpha_2 \equiv 0 \bmod 4$ and $\deg \beta_2 \equiv 2 \bmod 4$. This proves the lemma. Q.E.D.

Now equation (1.7) implies the $a_i$ here the following form:

$$Sq^{2^r} \tag{3.5}$$

$$Sq^{2^{r+1}k}Sq^{2^r-2^i} + Sq^{2^{r+1}k+2^r-2^i} \qquad i \leq r-2 \tag{3.6}$$

$$Sq^{2^{r+1}-2^i} + Sq^{2^r-1} \qquad i \le r-1 \tag{3.7}$$

$$Sq^{2^{r+1}k} Sq^{2^r-2^l+2^rk} \qquad l \le r-2 \tag{3.8}$$

$$Sq^{2^{r+1}k+2^r-2^l} \qquad l \le r-2 \tag{3.9}$$

It will be useful to use the associativity of $H_*$ at certain points in the argument although one suspects that there are proofs where the associativity of $H_*$ is not needed. Let $R = \{x \in H^* \mid \bar{\Delta}x \in \xi H^* \otimes H^*\}$. Then recall the exact sequence [8]

$$0 \to \xi H^* \to R \to Q^* \to 0.$$

*It follows that if* $r_1, r_2 \in R^{\text{odd}}$ and $r_1 - r_2$ is decomposable, then in fact $r_1 = r_2$. (3.10)

We will use this repeatedly.

LEMMA 3.2. *Let $\gamma \in \mathscr{A}(2)$ have degree $2^j + 2^{j+1}m \quad m \ge 0$. Then if $j < i$*

$$\gamma PH^{2^i+2^{i+1}l-2}(\Omega X) \subseteq \operatorname{im} \sigma^*.$$

*Proof.* Let $y \in PH^{2^i+2^{i+1}l-2}(\Omega X)$. We may assume $y$ is not a suspension. Then $y = \varphi_{2^i}(x_{2l+1})$ for some generator $x_{2l+1}$ of height $2^i$. Then $y$ comes from a map into the bundle $\bar{E}$ which is the fibre of $\bar{w}: K(\mathbb{Z}_2, 2l+1) \to K(\mathbb{Z}_2, 2^{i+1}l + 2^i)$

$$\bar{w}^*(i_{2^{i+1}l+2^i}) = (i_{2l+1})^{2^i}.$$

But then $\gamma i_{2^{i+1}l+2} \in \ker \bar{w}^*$ so there is a $\bar{v} \in H^*(\bar{E})$ with

$$\bar{j}^*(\bar{v}) = \gamma(i_{2^{i+1}l+2^i-1})$$

$$\begin{array}{ccccc}
 & & K(\mathbb{Z}_2, 2^{i+1}l + 2^i - 1) & & \\
 & & \downarrow & & \\
 & \overset{\bar{f}}{\nearrow} & \bar{E} & & \\
 & & \downarrow & & \\
\Omega & \longrightarrow & K(\mathbb{Z}_2, 2l+1) & \xrightarrow[\bar{w}]{} & K(\mathbb{Z}_2, 2^{i+1}l + 2^i)
\end{array}$$

If $\bar{f}^*p^*(i_{2l+1}) = x_{2l+1}$ then $\bar{v}$ may be chosen so that $\sigma^*\bar{f}^*(\bar{v}) = \gamma y$. Q.E.D.

LEMMA 3.3. *Let $\delta$ be a doubleton of degrees $2^i$ for $i < r$. Then for $l > 0$*

$$\sigma^* Q^{2^r+2^{r+1}l-1} \cap \delta PH^*(\Omega X) = 0.$$

*Proof.* Let $\delta = \alpha_0 Sq^{2^i}\alpha_1 Sq^{2^i}\alpha_2$ where $\deg \alpha_j \equiv 0 \bmod 2^{i+1}$. Then if $z \in PH^*(\Omega X)$ and

$$\delta z \in \sigma^* Q^{2^r+2^{r+1}l-1},$$

degree $\alpha_2 z = 2^{i+1}m - 2$ for some $m > 0$. We will show $Sq^{2^i}\alpha_2 z$ is a suspension. If $\alpha_2 z$ is not a suspension, then $\alpha_2 z = \varphi_{2^j}(x_1)$ for some $j \geq i+1$. But then Lemma 3.1 implies $Sq^{2^i}\alpha_2 z$ is a suspension. We have then

$$\alpha_1 Sq^{2^i}\alpha_2 z \in \sigma^* Q^{2^i+2^{i+1}n-1} \quad \text{for some} \quad n > 0.$$

By $\mathcal{P}(r)$, $Sq^{2^i}\alpha_2 Sq^{2^i}\alpha_2 z = 0$. Hence $\delta z = 0$. Q.E.D.

PROPOSITION 3.4. $\sigma^* Q^{2^r+2^{r+1}k-1} \cap Sq^{2^{r-1}} PH^{2^{r-1}+2^r k-2}(\Omega X) = 0$.

*Proof.* Let $\bar{x} \in Q^{2^r+2^{r+1}k-1}$ and suppose $\sigma^* x = Sq^{2^{r-1}} y$ for some primitive $y$. By $\mathcal{P}(r)$ $y$ cannot be a suspension. So $y \in \varphi_{2^{r-1}}(x_{4k+1})$. We may assume $x_{4k+1} \in R$ since $H^*$ has a Borel decomposition with generators in $R$ [5].

Therefore $\mathcal{P}(r)$ implies $x_{4k+1} = Sq^{2k} x_{2k+1}$ for some $x_{2k+1} \in R$. Hence

$$0 = (x_{4k+1})^{2^{r-1}} = Sq^{2^r k}(x_{2k+1}^{2^{r-1}}).$$

Applying $Sq^{2^{r-1}}$, we get

$$0 = Sq^{2^{r-1}} Sq^{2^r k}(x_{2k+1}^{2^{r-1}}) = (x_{2k+1})^{2^r}.$$

So $x_{2k+1}$ has height either $2^{r-1}$ or $2^r$. We will show that either case is impossible.

*Case* 1. $x_{2k+1}$ has height $2^r$. Consider the following commutative diagram

$$\begin{array}{ccc}
E_1 & \longrightarrow & E_2 \\
\downarrow & & \downarrow \\
K(\mathbb{Z}_2, 2k+1) & \xrightarrow{Sq^{2k}} & K(\mathbb{Z}_2, 4k+1) \\
{\scriptstyle w_1}\downarrow & & \downarrow{\scriptstyle w_2} \\
K(\mathbb{Z}_2, 2^{r+1}k + 2^r) & \xrightarrow[Sq^{2^{r-1}}]{} & K(\mathbb{Z}_2, 2^{r+1}k + 2^{r-1})
\end{array}$$

where

$$w_1^*(i_{2^{r+1}k+2^r}) = (i_{2k+1})^{2^r}$$
$$w_2^*(i_{2^{r+1}k+2^{r-1}}) = (i_{4k+1})^{2^{r-1}}.$$

This proves $\varphi_{2^r}(x_{2k+1}) = Sq^{2^{r-1}}\varphi_{2^{r-1}}(x_{4k+1}) = \sigma^*x$. But a transpotence is not a suspension so we have a contradiction.

*Case* 2. $x_{2k+1}$ has height $2^{r-1}$. Consider the diagram

$$\begin{array}{ccc}
E_1 & \longrightarrow & E_2 \\
\downarrow & & \downarrow \\
K(\mathbb{Z}_2, 2k+1) & \xrightarrow{Sq^{2k}} & K(\mathbb{Z}_2, 4k+1) \\
{\scriptstyle w_1}\downarrow & & \downarrow{\scriptstyle w_2} \\
K(\mathbb{Z}_2, 2^r k + 2^{r-1}) & \xrightarrow[Sq^{2^r k}]{} & K(\mathbb{Z}_2, 2^{r+1}k + 2^{r-1})
\end{array}$$

$$w_1^*(i_{2^r k+2^{r-1}}) = (i_{2k+1})^{2^{r-1}}$$
$$w_2^*(i_{2^{r+1}k+2^{r-1}}) = (i_{4k+1})^{2^{r-1}}.$$

Then this implies

$$\varphi_{2^{r-1}}(x_{4k+1}) = Sq^{2^r k}\varphi_{2^{r-1}}(x_{2k+1})$$

But then

$$\sigma^*x = Sq^{2^{r-1}}\varphi_{2^{r-1}}(x_{4k+1}) = Sq^{2^{r-1}}Sq^{2^r k}\varphi_{2^{r-1}}(x_{2k+1})$$
$$= \delta\varphi_{2^{r-1}}(x_{2k+1})$$

where $\delta$ is a doubleton of degree $\leq 2^{2-2}$.

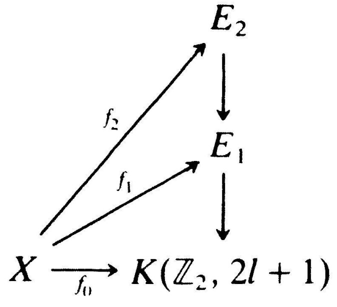
By Lemma 3.3 this implies $\sigma^*x = 0$ which is a contradiction since $\sigma^*: Q^{\text{odd}} \to PH^{\text{even}}(\Omega X)$ is monic. Q.E.D.

LEMMA 3.5. *Suppose* $Sq^m x_{2l+1} = 0$ *and* $x_{2l+1}$ *is a generator of height* $2^i$. *Then*

$$Sq^{2^i m}\varphi_{2^i}(x_{2l+1})$$

*is a suspension.*

*Proof.* Let $w_1: K(\mathbb{Z}_2, 2l+1) \to K(\mathbb{Z}_2, 2l+m+1)$ be defined by $w_1^*(i_{2l+m+1}) = Sq^m i_{2l+1}$. Let $E_1$ be the fibre of $w_1$. Define $w_2: E_1 \to K(\mathbb{Z}_2, 2^{i+1}l + 2^i)$ by $w_2^*(i_{2^{i+1}l+2^i}) = p_1^*(i_{2l+1})^{2^i}$. Let $E_2$ be fibre $w_2$. Then there is a lift




and $\Omega E_2 \simeq \Omega E_1 \times K(\mathbb{Z}_2, 2^{i+1}l + 2^i - 2)$

$$\Omega \tilde{f}_2^*[1 \otimes Sq^{2^i m} i_{2^{i+1}l+2^i-2}] = \sigma^* \tilde{f}_2^*(v_2) = Sq^{2^i m} \varphi_{2^i}(x_{2l+1})$$

where $j_2^*(v_2) = Sq^{2^i m} i_{2^{i+1}l+2^i-1}$. Q.E.D.

PROPOSITION 3.6. *Let $n = 2^j + 2^{j_1} + 2^{j_1+1}m$ for some $m \geq 0$*

$$j < j_1 \leq i \leq r, \qquad l \geq 0.$$

*Then $PH^{2^i+2^{i+1}l-2}(\Omega X) \cap Sq^n PH^*(\Omega X) = 0$.*

*Proof.* $Sq^n = Sq^{2^j} Sq^{2^{j_1}} Sq^{2^{j_1+1}m} + \delta$ where $\delta$ is a doubleton of degree less than $2^i$. By Lemma 3.3, if the intersection is nonempty, there is a primitive $y$ of degree

$$2^i - 2^j - 2^{j_1} + 2^{i+1}l - 2 \quad \text{with} \quad Sq^{2^j} Sq^{2^{j_1}} y \neq 0.$$

*Claim*: $Sq^{2^{j_1}} y$ is a suspension.

If $y$ is not a suspension, then $y = \varphi_{2^j}(z)$ where $\deg z = 2^{j_1-j} + 2^{j_1-j+1}u - 1$ where $u > 0$. We may assume $z \in R$, so $Sq^{2^{j_1-j}} z = 0$ by $\mathcal{P}(r)$. Hence by Lemma 3.5, $Sq^{2^{j_1}} y$ is a suspension. Now the claim implies $0 \neq Sq^{2^j} Sq^{2^{j_1}} y \in Sq^{2^j} \sigma^* Q^{2^i - 2^j + 2^{i+1}l - 1}$. But $\mathcal{P}(r)$ implies $Sq^{2^j} Q^{2^i - 2^j + 2^{i+1}l - 1} = 0$. Q.E.D.

Suppose $u$ has dyadic expansion

$$2^{i_1} + 2^{i_2} + \cdots 2^{i_s} \qquad i_1 < i_2 < \cdots < i_s.$$

We define the *dyadic length of $u$* to be $s$.

Now consider the indeterminacy described in equations (3.5) to (3.9).

LEMMA 3.7.

$$Sq^{2^r}(PH^*(\Omega X)\otimes PH^*(\Omega X))\cap(A\otimes A)$$
$$\subseteq Sq^{2^r}PH^*(\Omega X)\otimes PH^*(\Omega X)+PH^*(\Omega X)\otimes Sq^{2^r}PH^*(\Omega X).$$

*Proof.* By Proposition 3.3 $A\cap Sq^{2^{r-1}}PH^*(\Omega X)=0$. If

$$(Sq^{\alpha}\otimes Sq^{2^r-\alpha})(PH^*(\Omega X)\otimes PH^*(\Omega X))\cap(A\otimes A)$$

is nontrivial, then since $\alpha\neq 2^{r-1}$, by Proposition 3.6, $\alpha$ cannot have dyadic length greater than one. Hence $\alpha$ is either zero or $2^r$. Q.E.D.

LEMMA 3.8.

$$Sq^{2^{r+1}k}(PH^*(\Omega X)\otimes PH^*(\Omega X))\cap(A\otimes A)$$
$$\subseteq Sq^{2^rk}PH^*(\Omega X)\otimes Sq^{2^rk}PH^*(\Omega X).$$

*Proof.* If

$$[Sq^{2^rk+\alpha}PH^*(\Omega X)\otimes Sq^{2^rk-\alpha}PH^*(\Omega X)]\cap(A\otimes A)$$

is nontrivial, then for degree reasons $0\le\alpha<2^r$. But if $\alpha\neq 0$, either $2^rk+\alpha$ or $2^rk-\alpha$ has the form $2^j+2^{j_1}+2^{j_1+1}m$ for $j<j_1\le r$. By Proposition 3.6, $\alpha=0$. Q.E.D.

PROPOSITION 3.9.

$$(B\otimes B)\cap Sq^{2^r-2^i}(PH^*(\Omega X)\otimes PH^*(\Omega X))$$
$$\subseteq Sq^{2^{r-1}}PH^*(\Omega X)\otimes Sq^{2^{r-2}}PH^*(\Omega X)$$
$$+Sq^{2^{r-2}}PH^*(\Omega X)\otimes Sq^{2^{r-1}}PH^*(\Omega X)$$

*if $i=r-2$. If $i<r-2$ the intersection is trivial.*

*Proof.* If $[Sq^{u_1}PH^*(\Omega X)\otimes Sq^{u_2}PH^*(\Omega X)]\cap(B\otimes B)$ is nontrivial and $u_1+u_2=2^r-2^i$, then by Proposition 3.6, both $u_1$ and $u_2$ must have dyadic length 1. Q.E.D.

PROPOSITION 3.10.

$$(B \otimes B) \cap Sq^{2^r-2^l+2^rk}(PH^*(\Omega X) \otimes PH^*(\Omega X))$$
$$\subseteq Sq^{2^{r-1}}PH^*(\Omega X) \otimes Sq^{2^{r-2}}PH^*(\Omega X)$$
$$+ Sq^{2^{r-2}}PH^*(\Omega X) \otimes Sq^{2^{r-1}}PH^*(\Omega X)$$

*if* $l = r-2$. *If* $l < r-2$ *the intersection is trivial.*

*Proof.* Again as in Proposition 3.9, if $u_1 + u_2 = 2^r - 2^l + 2^rk$, by Proposition 3.6,

$$u_1 = \alpha_1 + 2^r l_1$$
$$u_2 = \alpha_2 + 2^r l_2$$

where $\alpha_1 + \alpha_2 = 2^r - 2^l$. Hence $\alpha_1$ must be either $Sq^{2^{r-1}}$ or $Sq^{2^{r-2}}$ and $\alpha_2$ is the other. $Sq^{u_i} = Sq^{\alpha_i} Sq^{2^r l_i} + \delta$ where $\delta$ is a doubleton of degree less than $2^r$. Q.E.D.

THEOREM 3.11.

(a) $(A \otimes A) \cap (Sq^{2^{r+1}k} Sq^{2^r-2^l})(PH^*(\Omega X) \otimes PH^*(\Omega X))$
$\subseteq Sq^{2^r}PH^*(\Omega X) \otimes PH^*(\Omega X) + PH^*(\Omega X) \otimes Sq^{2^r}PH^*(\Omega X)$.

(b) $(A \otimes A) \cap (Sq^{2^{r+1}k} Sq^{2^r-2^l+2^rk})(PH^*(\Omega X) \otimes PH^*(\Omega X))$
$\subseteq Sq^{2^r}PH^*(\Omega X) \otimes PH^*(\Omega X) + PH^*(\Omega X) \otimes Sq^{2^r}PH^*(\Omega X)$.

*Proof.* By Propositions 3.9 and 3.10 elements of the intersection belong to

$$Sq^{2^rk} Sq^{2^{r-1}} PH^*(\Omega X) \otimes PH^*(\Omega X) + \text{twist}.$$

But

$$Sq^{2^{r-1}} Sq^{2^rk} = Sq^{2^rk} Sq^{2^{r-1}} + [Sq^{2^{r-1}}, Sq^{2^r}] Sq^{2^r(k-1)}$$

plus a doubleton of degree less than $2^r$. Hence by Proposition 3.4

$$Sq^{2^rk} Sq^{2^{r-1}} PH^*(\Omega X) \cap A \subseteq Sq^{2^r} PH^*(\Omega X). \qquad \text{Q.E.D.}$$

PROPOSITION 3.12.

(a) *For* $i \leq r-2$, $(A \otimes A) \cap Sq^{2^{r+1}k+2^r-2^i}(PH^*(\Omega X) \otimes PH^*(\Omega X)) = 0.$

(b) *For* $i \leq r-1$, $(A \otimes A) \cap Sq^{2^{r+1}-2^i}(PH^*(\Omega X) \otimes PH^*(\Omega X)) = 0.$

*Proof.* Again Propositions 3.6 and 3.4 imply the intersections are trivial unless $i = r-2$ for (a) and $i = r-1$ for (b). Then $Sq^{2^{r+1}k+2^r-2^i}$ must split as $Sq^{2^rk+2^{r-1}} \otimes Sq^{2^rk+2^{r-2}}$ + twist. But Proposition 3.4, and Lemma 3.3 imply this is trivial.

In (b) it must split as

$$Sq^{2^r} \otimes Sq^{2^{r-1}}.$$

Again Proposition 3.4 implies the intersection is trivial. Q.E.D.

COROLLARY 3.13. *If* $\sigma^*x \otimes \sigma^*x$ *lies in the indeterminacy of degrees congruent to zero* mod 4 *in equations* (3.5) *to* (3.9), *then* $\sigma^*x \in \operatorname{im} Sq^{r^2}$.

*Proof.* By Lemma 3.7 if $\sigma^*x \otimes \sigma^*x \in \operatorname{im} Sq^{2^r}(P \otimes P)$ we have $\sigma^*x \in \operatorname{im} Sq^{2^r}$. By Theorem 3.11(a) if $\sigma^*x \otimes \sigma^*x \in Sq^{2^{r+1}k}Sq^{2^r-2^i}(P \otimes P)$ then $\sigma^*x \in \operatorname{Im} Sq^{2^r}$. By Proposition 3.12,

$$\sigma^*x \otimes \sigma^*x \notin Sq^{2^{r+1}k+2^r-2^i}(P \otimes P)$$

and

$$\sigma^*x \otimes \sigma^*x \notin Sq^{2^{r+1}-2^i}(P \otimes P).$$

By Theorem 3.11(b) if $\sigma^*x \otimes \sigma^*x \in Sq^{2^{r+1}k}Sq^{2^r-2^i+2^rk}(P \otimes P)$ then $\sigma^*x \in \operatorname{im} Sq^{2^r}$. Equation (3.9) is the same indeterminacy as in equation (3.6). Hence all the indeterminacy of degree congruent to zero mod four yields $\sigma^*x \in \operatorname{im} Sq^{2^r}$. Q.E.D.

It remains to check the indeterminacy of degrees congruent to two mod four. From equations (3.5) to (3.9) such indeterminacy has the form

$$Sq^{2^{r+1}k}Sq^{2^r-2} + Sq^{2^{r+1}k+2^r-2}. \qquad (3.12)$$

$$Sq^{2^{r+1}-2}. \qquad (3.13)$$

$$Sq^{2^{r+1}k}Sq^{2^r-2+2^rk}. \qquad (3.14)$$

For excess reasons,

$$\sigma^*x \otimes \sigma^*x \notin (Sq^{2^{r+1}k+2^r-2})(P \otimes F_2 + F_2 \otimes P).$$

Similarly by Lemma 3.8 and Lemma 3.1(a) if

$$\sigma^*x \otimes \sigma^*x \in Sq^{2^{r+1}k} Sq^{2^r-2+2^rk}(P \otimes F_2 + F_2 \otimes P)$$

then

$$(B \otimes B) \cap Sq^{2^r-2+2^rk}(P \otimes F_2 + F_2 \otimes P) \neq 0.$$

But this is also impossible for excess reasons. Finally, consider the indeterminacy $Sq^{2^{r+1}k} Sq^{2^r-2}$ and $Sq^{2^{r+1}-2}$.

LEMMA 3.14.

(a) $A \cap Sq^6 F_2 = 0$.

(b) $B \cap Sq^6 F_2 = 0$.

*Proof.* Since $Sq^1 H^*(\Omega X) = 0$, $Sq^6 F_2 = Sq^2 Sq^4 F_2$. For (a), by equation (3.4), if $Sq^6 F_2 \cap A \neq 0$ then $Sq^2 Sq^4 \gamma_2(y)$ is nontrivial for some $y$ with degree $y = 2^{r-1} + 2^r k - 4$. By (3.4) $y = z^2$. Therefore, since $Sq^2(z^2) = 0$, it follows that $Sq^4 \gamma_2(y)$ is primitive plus a decomposable by (3.3). We conclude $Sq^2 Sq^4 \gamma_2(y)$ is decomposable. An analogous argument proves (b). Q.E.D.

PROPOSITION 3.15. *If* $r = 3$ *and*

$$\sigma^*x \otimes \sigma^*x \in (Sq^{16k} Sq^6 + Sq^{14})(P \otimes F_2 + F_2 \otimes P)$$

*then* $\sigma^*x \in \operatorname{im} Sq^8$.

*Proof.* If $\sigma^*x \otimes \sigma^*x \in Sq^{16k} Sq^6(P \otimes F_2 + F_2 \otimes P)$ then by Lemma 3.8 and Proposition 3.6

$$\sigma^*x \otimes \sigma^*x \in (Sq^{8k} \otimes Sq^{8k}) Sq^6(P \otimes F_2 + F_2 \otimes P).$$

Since $Sq^2 P$ is decomposable we have

$$\begin{aligned} \sigma^*x \otimes \sigma^*x \in\ & (Sq^{8k} \otimes Sq^{8k})(P \otimes Sq^6 F_2 + Sq^4 P \otimes Sq^2 F_2) \\ & + (Sq^{8k} \otimes Sq^{8k})(Sq^6 F_2 \otimes P + Sq^2 F_2 \otimes Sq^4 P). \end{aligned}$$

It follows that

$$\sigma^*x \in Sq^{8k}Sq^4P + Sq^{8k}Sq^6F_2.$$

By equation (3.11) and Proposition 3.4, if

$$P^{16k+6} \cap Sq^{8k}Sq^4P \subseteq \operatorname{im} Sq^8.$$

By Lemma 3.14

$$Sq^{8k}Sq^6F_2 \cap P^{16k+6} = 0.$$

We conclude $\sigma^*x \in \operatorname{im} Sq^8$. Q.E.D.

Now suppose $r > 3$ and

$$\sigma^*x \otimes \sigma^*x \in (Sq^{2^{r+1}k}Sq^{2^r-2})(P \otimes F_2 + F_2 \otimes P).$$

By Lemma 3.8

$$(\sigma^*x \otimes \sigma^*x) \in (Sq^{2^rk} \otimes Sq^{2^rk})Sq^{2^r-2}(P \otimes F_2 + F_2 \otimes P).$$

Hence

$$(B \otimes B) \cap Sq^{2^r-2}(P \otimes F_2 + F_2 \otimes P) \neq 0.$$

By Proposition 3.6,

$$\begin{aligned}(B \otimes B) &\cap Sq^{2^r-2}(P \otimes F_2 + F_2 \otimes P)\\ &\subseteq (P \otimes Sq^{2^r-2}F_2) + (Sq^{2^r-2}F_2 \otimes P)\\ &\quad + \sum_{i=2}^{r-1} Sq^{2^i}P \otimes Sq^{2^r-2^i-2}F_2 + Sq^{2^r-2^i-2}F_2 \otimes Sq^{2^i}P.\end{aligned}$$

But $Sq^{2^r-2}F_2 \subseteq Sq^6F_2$ and if $i > 2$

$$Sq^{2^r-2^i-2}F_2 \subseteq Sq^6F_2$$

therefore by Lemma 3.14, we have

$$\begin{aligned}(B \otimes B) \cap Sq^{2^r-2}&(P \otimes F_2 + F_2 \otimes P)\\ &\subseteq (Sq^4P \otimes Sq^{2^r-6}F_2) + (Sq^{2^r-6}F_2 \otimes Sq^4P).\end{aligned}$$

But now $B \cap Sq^{2^r-6}F_2$ is spanned by elements of the form $Sq^{2^r-6}\gamma_2(\sigma^* x_{2^{r-1}k+3})$. Now since $r>3$, $x_{2^{r-1}k+3}$ is annihilated by all Steenrod operations of degree congruent to four mod 8 since $Sq^4 QH^{8l+3}(X)=0$. Therefore

$$Sq^{2^{r-1}-4}\sigma^*(x_{2^{r-1}k+3})=0.$$

It follows that

$$\begin{aligned} Sq^{2^r-6}\gamma_2(\sigma^* x_{2^{r-1}k+3}) &= Sq^2(Sq^{2^r-8})\gamma_2(\sigma^* x_{2^{r-1}k+3}) \\ &\in Sq^2(P+\text{decomposables}) \text{ by equation (3.3)} \\ &\in \text{decomposables by equation (3.4).} \end{aligned}$$

Therefore

$$B \cap S^{2^r-6}F_2 = 0.$$

This proves

PROPOSITION 3.16.

$$\sigma^* x \otimes \sigma^* x \notin Sq^{2^{r+1}k} Sq^{2^r-2}(P \otimes F_2 + F_2 \otimes P).$$

Now suppose

$$\sigma^* x \otimes \sigma^* x \in Sq^{2^{r+1}-2}(P \otimes F_2 + F_2 \otimes P).$$

Again by Proposition 3.6,

$$\begin{aligned} &(A \otimes A) \cap Sq^{2^{r+1}-2}(P \otimes F_2 + F_2 \otimes P) \\ &\quad \subseteq (P \otimes Sq^{2^{r+1}-2}F_2) + (Sq^{2^{r+1}-2}F_2 \otimes P) \\ &\qquad + \sum_{i=2}^{r} (Sq^{2^i}P \otimes Sq^{2^{r+1}-2^i-2}F_2) + (Sq^{2^{r+1}-2^i-2}F_2 \otimes Sq^{2^i}P) \\ &\quad \subseteq (Sq^4 P \otimes Sq^{2^{r+1}-6}F_2) + (Sq^{2^{r+1}-6}F_2 \otimes Sq^4 P) \end{aligned}$$

by Lemma 3.14.

$$A \cap Sq^{2^{r+1}-6}F_2 = 0$$

by an argument analogous to the proof in Proposition 3.16.

It follows that

PROPOSITION 3.17. $\sigma^*x \otimes \sigma^*x \notin Sq^{2^{r+1}-2}(P \otimes F_2 + F_2 \otimes P)$.

We can now prove

THEOREM 1.1(r). $\sigma^*QH^{2^r+2^{r+1}k-1}(X) \subseteq Sq^{2^r}PH^*(\Omega X)$ *and*

$Sq^{2^r}QH^{2^r+2^{r+1}k-1}(X) = 0$.

*Proof.* By the previous theorems, we conclude

$$\sigma^*x \otimes \sigma^*x \in Sq^{2^r}P \otimes P + P \otimes Sq^{2^r}P.$$

Therefore $\sigma^*x \in Sq^{2^r}P$. Now since $Sq^{2^r}Sq^{2^r}$ is a doubleton of degree less than $2^r$, $Sq^{2^r}\sigma^*x = \sigma^*(Sq^{2^r}x) = 0$. But $\sigma^*$ is a monomorphism on odd indecomposables. Therefore $Sq^{2^r}x$ is decomposable. Q.E.D.

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