

**Zeitschrift:** Commentarii Mathematici Helvetici  
**Herausgeber:** Schweizerische Mathematische Gesellschaft  
**Band:** 63 (1988)

**Artikel:** A non-immersion theorem for spaceforms.  
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**DOI:** <https://doi.org/10.5169/seals-48225>

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## A non-immersion theorem for spaceforms

F. J. PEDIT

### Introduction

This note gives a uniform treatment of some non-existence results concerning isometric immersions of spaceforms into spaceforms. Let  $M, \tilde{M}$  be spaceforms of  $\dim M = m, \dim \tilde{M} = 2m - 1$  and curvatures  $c, \tilde{c}$ . Then we prove the following

**THEOREM.** *Let  $M$  be complete. Then there exists no isometric immersion  $f : M \rightarrow \tilde{M}$  if*

- (1)  $c < \tilde{c}, c < 0$  and  $M$  is fuchsian,
- (2)  $0 < c < \tilde{c}$ .

A hyperbolic spaceform is called fuchsian if the limit set of the fundamental group  $\Pi_1(M)$  in the sphere at infinity of the universal cover contains more than two (and hence infinitely many) points [2]. Part (1) of our result extends recent work of F. Xavier [6] where the same statement is proven by a different method in case  $\tilde{M} = \mathbb{R}^{2m-1}$  with the standard flat metric. Since every compact hyperbolic spaceform is fuchsian our result also relates to early work of S. S. Chern and N. H. Kuiper [1] where, in order to use compactness of  $M, \tilde{M}$  has to be diffeomorphic to  $\mathbb{R}^{2m-1}$ . Behind all of this one clearly has in mind the problem of whether  $m$ -dimensional hyperbolic space  $H^m$  can be isometrically immersed into  $\mathbb{R}^{2m-1}$ . The case  $m = 2$  was treated by D. Hilbert who showed that there are no isometric immersions of  $H^2$  into  $\mathbb{R}^3$  [4]. Part (2) proves again a result of J. D. Moore [3] and we only add it since it fits canonically into our approach.

It is well known [4] that in the case  $c \geq \tilde{c}$  one always has isometric immersions  $f : M \rightarrow \tilde{M}$  for complete  $M$ .

Our method is based on the observation that an isometric immersion  $f : M \rightarrow \tilde{M}$  always comes with a flat metric if  $\tilde{c} > c$ . If  $M$  is complete this metric will be complete and hence put restrictions on the fundamental group of  $M$ .

### Basic facts

Let  $I, \tilde{I}$  be the riemannian metrics of constant curvatures  $c, \tilde{c}$  on  $M, \tilde{M}$  and assume  $\kappa = \tilde{c} - c > 0$ . If  $f : M \rightarrow \tilde{M}$  is an isometric immersion then  $f^*\tilde{I} = I$  and we

denote by  $II \in \Gamma(\odot^2 T^*M, \perp_f M)$  the 2nd fundamental form of  $f$ . Applying E. Cartan's Theorem on exteriorly orthogonal symmetric bilinear forms to  $II \oplus \kappa I \in \Gamma(\odot^2 T^*M, \perp_f M \oplus \mathbb{R})$  one gets [3]:

For every  $p \in M$  there exists an open neighbourhood  $p \in U \subset M$ , an orthonormal coframe  $(\omega^1, \dots, \omega^m)$  on  $U$  and an orthonormal frame  $(e_{m+1}, \dots, e_{2m-1})$  of  $\perp_f M$  over  $U$  such that

$$I = \sum_i (\omega^i)^2$$

$$II = \sum_{i,\alpha} b_i^\alpha (\omega^i)^2 e_\alpha$$

with smooth functions  $b_i^\alpha: U \rightarrow \mathbb{R}$ . Moreover, setting

$$a_i = \frac{1}{\sqrt{\kappa}} \sqrt{\kappa + b_i^2}, \quad b_i^2 = \sum_\alpha (b_i^\alpha)^2,$$

the 1-forms  $a_i \omega^i \in \Omega^1(U)$  are closed. Hence there exist (principal curvature) coordinates  $x = (x^1, \dots, x^m): U \rightarrow \mathbb{R}^m$  with  $dx^i = a_i \omega^i$ .

Let  $III \in \Gamma(\odot^2 T^*M, \mathbb{R})$  be the 3rd fundamental form of  $f$ , i.e.,

$$III(X, Y) = \text{tr}_I \tilde{I}(II(X, -), II(-, Y)), \quad X, Y \in TM,$$

then we have

**LEMMA.** *Let  $M$  be complete. Then  $I_0 = \frac{1}{\kappa} III + I$  is a complete flat riemannian metric on  $M$ .*

*Proof.* Since  $\kappa > 0$  and  $III$  is positive semi-definite  $I_0 \geq I$  and so  $I_0$  is complete. To show flatness of  $I_0$  we express  $I_0$  in the above coordinates on  $U \subset M$ :

$$III = \sum_{i,\alpha} (b_i^\alpha)^2 (\omega^i)^2 = \sum_i b_i^2 (\omega^i)^2$$

and so

$$I_0 = \frac{1}{\kappa} III + I = \frac{1}{\kappa} \sum_i (b_i^2 + \kappa) (\omega^i)^2 = \sum_i (a_i \omega^i)^2 = \sum_i (dx^i)^2.$$

Hence  $I_0$  is flat.  $\square$

### Proof of the theorem

Assume there is an isometric immersion  $f: M \rightarrow \tilde{M}$  with  $M$  complete and  $\tilde{c} - c > 0$ . Then by the Lemma  $(M, I_0)$  is a flat complete riemannian manifold.

Hence its fundamental group satisfies the exact sequence

$$0 \rightarrow \mathbb{Z}^r \rightarrow \Pi_1(M) \rightarrow G \rightarrow 1$$

where  $0 \leq r \leq m$  and  $G$  is finite [5].

Now let  $c < 0$  and  $M$  be fuchsian. Then  $\Pi_1(M)$  contains a free group on at least 2 generators  $a, b \in \Pi_1(M)$  [2], [6]. Since  $G$  is finite there are  $p, q \in \mathbb{N}$  with  $a^p, b^q \in \mathbb{Z}^r$ . But  $\mathbb{Z}^r$  is abelian, so  $a^p b^q = b^q a^p$  which contradicts that  $a, b$  generate a free group.

In case  $c > 0$  we know that the universal cover of  $M$  is  $S^m(1/c)$ . But we also have that the universal cover is  $\mathbb{R}^m$  which gives a contradiction.  $\square$

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Received November 4, 1987