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A non-immersion theorem for spaceforms

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Introduction

This note gives a uniform treatment of some non-existence results concerning isometric immersions of spaceforms into spaceforms. Let $M, \tilde{M}$ be spaceforms of $\dim M = m$, $\dim \tilde{M} = 2m - 1$ and curvatures $c, \tilde{c}$. Then we prove the following

THEOREM. *Let M be complete. Then there exists no isometric immersion $f: M \to \tilde{M}$ if*

(1) $c < \tilde{c}$, $c < 0$ *and M is fuchsian,*

(2) $0 < c < \tilde{c}$.

A hyperbolic spaceform is called fuchsian if the limit set of the fundamental group $\Pi_1(M)$ in the sphere at infinity of the universal cover contains more than two (and hence infinitely many) points [2]. Part (1) of our result extends recent work of F. Xavier [6] where the same statement is proven by a different method in case $\tilde{M} = \mathbb{R}^{2m-1}$ with the standard flat metric. Since every compact hyperbolic spaceform is fuchsian our result also relates to early work of S. S. Chern and N. H. Kuiper [1] where, in order to use compactness of M, $\tilde{M}$ has to be diffeomorphic to $\mathbb{R}^{2m-1}$. Behind all of this one clearly has in mind the problem of whether m-dimensional hyperbolic space H^m can be isometrically immersed into $\mathbb{R}^{2m-1}$. The case $m = 2$ was treated by D. Hilbert who showed that there are no isometric immersions of H^2 into $\mathbb{R}^3$ [4]. Part (2) proves again a result of J. D. Moore [3] and we only add it since it fits canonically into our approach.

It is well known [4] that in the case $c \geq \tilde{c}$ one always has isometric immersions $f: M \to \tilde{M}$ for complete M.

Our method is based on the observation that an isometric immersion $f: M \to \tilde{M}$ always comes with a flat metric if $\tilde{c} > c$. If M is complete this metric will be complete and hence put restrictions on the fundamental group of M.

Basic facts

Let $I, \tilde{I}$ be the riemannian metrics of constant curvatures $c, \tilde{c}$ on $M, \tilde{M}$ and assume $\kappa = \tilde{c} - c > 0$. If $f: M \to \tilde{M}$ is an isometric immersion then $f^*\tilde{I} = I$ and we denote by $II \in \Gamma(\odot^2 T^*M, \perp_f M)$ the 2nd fundamental form of f. Applying E. Cartan's Theorem on exteriorly orthogonal symmetric bilinear forms to $II \oplus \kappa I \in \Gamma(\odot^2 T^*M, \perp_f M \oplus \mathbb{R})$ one gets [3]:

For every $p \in M$ there exists an open neighbourhood $p \in U \subset M$, an orthonormal coframe $(\omega^1, \ldots, \omega^m)$ on U and an orthonormal frame $(e_{m+1}, \ldots, e_{2m-1})$ of $\perp_f M$ over U such that

$$I = \sum_i (\omega^i)^2$$

$$II = \sum_{i,\alpha} b_i^\alpha (\omega^i)^2 e_\alpha$$

with smooth functions $b_i^\alpha : U \to \mathbb{R}$. Moreover, setting

$$a_i = \frac{1}{\sqrt{\kappa}} \sqrt{\kappa + b_i^2}, \; b_i^2 = \sum_\alpha (b_i^\alpha)^2,$$

the 1-forms $a_i \omega^i \in \Omega^1(U)$ are closed. Hence there exist (principal curvature) coordinates $x = (x^1, \ldots, x^m) : U \to \mathbb{R}^m$ with $dx^i = a_i \omega^i$.

Let $III \in \Gamma(\odot^2 T^*M, \mathbb{R})$ be the 3rd fundamental form of f, i.e.,

$$III(X, Y) = \mathrm{tr}_I \, \tilde{I}(II(X, -), II(-, Y)), \qquad X, Y \in TM,$$

then we have

LEMMA. *Let M be complete. Then $I_0 = \frac{1}{\kappa} III + I$ is a complete flat riemannian metric on M.*

Proof. Since $\kappa > 0$ and III is positive semi-definite $I_0 \geq I$ and so I_0 is complete. To show flatness of I_0 we express I_0 in the above coordinates on $U \subset M$:

$$III = \sum_{i,\alpha} (b_i^\alpha)^2 (\omega^i)^2 = \sum_i b_i^2 (\omega^i)^2$$

and so

$$I_0 = \frac{1}{\kappa} III + I = \frac{1}{\kappa} \sum_i (b_i^2 + \kappa)(\omega^i)^2 = \sum_i (a_i \omega^i)^2 = \sum_i (dx^i)^2.$$

Hence I_0 is flat. □

Proof of the theorem

Assume there is an isometric immersion $f : M \to \tilde{M}$ with M complete and $\tilde{c} - c > 0$. Then by the Lemma (M, I_0) is a flat complete riemannian manifold.

Hence its fundamental group satisfies the exact sequence

$$0 \to \mathbb{Z}^r \to \Pi_1(M) \to G \to 1$$

where $0 \leq r \leq m$ and G is finite [5].

Now let $c < 0$ and M be fuchsian. Then $\Pi_1(M)$ contains a free group on at least 2 generators $a, b \in \Pi_1(M)$ [2], [6]. Since G is finite there are $p, q \in \mathbb{N}$ with $a^p, b^q \in \mathbb{Z}^r$. But $\mathbb{Z}^r$ is abelian, so $a^p b^q = b^q a^p$ which contradicts that a, b generate a free group.

In case $c > 0$ we know that the universal cover of M is $S^m(1/c)$. But we also have that the universal cover is $\mathbb{R}^m$ which gives a contradiction. □

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