Zeitschrift: Commentarii Mathematici Helvetici

Herausgeber: Schweizerische Mathematische Gesellschaft

Band: 63 (1988)

Artikel: Two-knot groups with torsion free abelian normal subgroups of rank two.

Autor: Hillman, Jonathan A.

DOI: https://doi.org/10.5169/seals-48224

Comment. Math. Helvetici 63 (1988) 664–671
0010–2571/88/040664–08$01.50 + 0.20/0

Two-knot groups with torsion free abelian normal subgroups of rank two

JONATHAN A. HILLMAN

Abstract. We show that a torsion free abelian normal subgroup of rank two of a two-knot group which is contained in the commutator subgroup must be free abelian, the centralizer of the abelian subgroup is not contained in the commutator subgroup, and neither of the latter two groups is finitely generated. Furthermore, we characterize algebraically the groups of 2-knots which are cyclic branched covers of twist spun torus knots.

In [7] we showed that if a 2-knot group G has a torsion free abelian normal subgroup A of rank $r \geqslant 2$ then it is an orientable Poincaré duality group of formal dimension 4 (or PD_4^+-group), so $r \leqslant 4$, and we determined the groups admitting such a subgroup with $r = 3$ or 4 (in which case A must be finitely generated). All the known examples with $r = 2$ derive from cyclic branched covers of twist spun torus knots (or branched twist spun torus knots, for short) and have A free abelian and not contained in the commutator subgroup G'. In Section 1 of this paper we shall show that if $r = 2$ and A is contained in G' then A must be finitely generated and that three of the four cases left open by Theorem 9 of [7] cannot occur. (The remaining case seems highly unlikely, but we have been unable to settle this). Our arguments to this end are similar in spirit to those of [7], particularly in their reliance on the results in [1] on subgroups of PD-groups and on groups with large abelian normal subgroups and small cohomological dimension. In Section 2 we shall use work of Zimmermann [20] on realizing geometrically outer automorphism classes of the fundamental groups of Seifert fibred 3-manifolds to give an algebraic characterization of the groups of branched twist spun torus knots, and we give a version of a theorem of Plotnick [13] characterizing the closed 4-manifolds obtained by surgery on such knots among fibred 4-manifolds.

1. Further restrictions on the groups

THEOREM 1. *Let G be a 2-knot group with a torsion free abelian normal subgroup A of rank 2 which is contained in G'. Then A is finitely generated.*

Proof. We may assume that A is maximal and is not finitely generated. For if A_1 is a maximal abelian subgroup containing A then it is torsion free (since G is a PD_4^+-group), has rank 2 and is contained in G', for otherwise G would have a torsion free abelian normal subgroup of rank at least 3, and such 2-knot groups do not have rank 2 abelian normal subgroups. (Cf. Section 3 of [7]). Now $c.d.A = 3 = c.d.C_{G'}(A)$, so $A = C_{G'}(A)$ by [1; Theorem 8.8]. If $c.d.C_G(A) = 4$ then $[G' : A] = [G' : C_{G'}(A)] \leq [G : C_G(A)] < \infty$ so G would be virtually metabelian, hence poly-($\mathbb{Z}$ or finite), and A would be finitely generated. Thus we may assume that $c.d.C_G(A) = 3$ and so $A = C_G(A)$. Therefore G/A is a finitely generated subgroup of $\mathrm{Aut}\,(A)$ and hence of $GL(2, \mathbb{Q})$. Now if G'/A were a torsion group then it would be locally finite [9; page 105], so $h.d.G' = h.d.A = 2$ and hence $h.d.G = 3$, which would be absurd, as G is a PD_4^+-group. Therefore there is an element g of G' which is of infinite order modulo A. The subgroup B generated by g and A is solvable and $c.d.B = 3$ since $A \subset B \subseteq G'$. The Hirsch–Plotkin radical N of B contains A, so is not finitely generated, and $c.d.N \leq c.d.B = 3$. Therefore N is abelian, so $N = A$ and we may apply Theorem 7.15 of [1]. Thus $L = H_2(A; \mathbb{Z})$ is the underlying abelian group of a subring $\mathbb{Z}[m^{-1}]$ of $\mathbb{Q}$, and the action of g on L is multiplication by a rational number a/b, where ab and m have the same prime divisors. (Note that the action of B on A by conjugation is determined by that of g). But g acts on A as an element of $GL(2, \mathbb{Q})' < SL(2, \mathbb{Q})$ and so acts on $L \simeq A \wedge A$ (cf. [15: page 334]) as $\det(g) = +1$. Therefore $L \simeq \mathbb{Z}$. It follows that our assumption was false and A must be finitely generated. □

With Theorem 6 of [7] it follows that if the centre of every PD_3-group is finitely generated then all torsion free abelian normal subgroups of 2-knot groups are finitely generated in the rank 2 case also. (Recall however that the group of Fox's Example 10 has commutator subgroup $\mathbb{Z}[\frac{1}{2}]$ and so finite generation does not always hold in the rank 1 case).

If there is such a 2-knot group G as in Theorem 1, then the quotient G/A is finitely presentable and maps onto $\mathbb{Z}$, and so by a theorem of Bieri and Strebel it is an HNN extension with finitely generated base and associated subgroups [2]. The preimage of the base in G is a finitely generated group of cohomological dimension 3 which contains A as a normal subgroup and therefore the base is finitely generated free-by-finite, by [1; Theorem 8.4]. Our strategy below is to attempt to show that $v.c.d.G/A$ is finite, and then to deduce a contradiction. (Cf. the proof of Theorem 3. Note that it would suffice to show that G/A contains a subgroup of finite cohomological dimension which is not free, for then its preimage in G would have cohomological dimension at least 4 by [1; Theorem 5.6], and so be of finite index in G, by [18]. Peter M. Neumann has constructed an HNN extension with base $\langle a, b \mid a^5 = 1\rangle$ and associated subgroups free of rank 5, and such that every subgroup of finite index contains the base, so some further assumption is needed).

Theorem 2 is a substantial extension of the observation in [7] that there are no such examples which are the groups of fibred 2-knots, or even with G' a PD_3-group.

THEOREM 2. *Let G be a 2-knot group with an abelian normal subgroup $A \simeq \mathbb{Z}^2$ which is contained in G'. Then G' is not finitely generated. If $C_G(A)$ has infinite index in G then it is not finitely generated either.*

Proof. We may assume that A is maximal. Now $c.d.G' = 3$, so G'/A is infinite. If G' is finitely generated then G'/A is finitely generated free-by-finite by [1: Theorem 8.4], that is, there is a subgroup B of finite index in G' such that B/A is free. We may assume that B is characteristic in G' and so normal in G. There is then a subgroup H of finite index in G which contains B and is such that H/B is infinite cyclic. The quotient H/A is then finitely generated free-by-infinite cyclic and so $c.d.H/A \leq 2$. But then H/A is a PD_2-group by [1: Theorem 9.11] (and a spectral sequence corner argument to identify the dualizing module). Since the only PD_2-groups with nontrivial finitely generated free normal subgroups are solvable, G is virtually solvable. But then $C_{G'}(A)$ would be abelian of rank 3, by the argument of the first two paragraphs of Theorem 9 of [7], which contradicts the maximality of A.

If $C_G(A)$ is finitely generated and of infinite index in G, so it has cohomological dimension 3, then $C_G(A)/A$ is finitely generated free-by-finite, as above, while $G/C_G(A)$ is a subgroup of $GL(2, \mathbb{Z})$ and so is also finitely generated free-by-finite. Thus there is a subgroup B of finite index in $C_G(A)$ which is normal in G and such that B/A is finitely generated free, and G/B is finite-by-free-by-finite. But such a group G/B is in fact free-by-finite, and so G/A is f.g.free-by-f.g.free-by-finite. Therefore $v.c.d.G/A$ is finite and G/A has a nontrivial finitely generated free normal subgroup, so we reach the same contradiction. Since $C_G(A)/C_{G'}(A)$ is cyclic it follows that $C_{G'}(A)$ cannot be finitely generated either. □

In the next theorem we show that the first three cases allowed by Theorem 9 of [7] are impossible.

THEOREM 3. *Let G be a 2-knot group with a maximal abelian normal subgroup $A \simeq \mathbb{Z}^2$ which is contained in G'. Then $C_G(A)$ has infinite index in G, but is not contained in G'.*

Proof. Suppose otherwise. Then either $C_G(A)=G'$ or $c.d.C_G(A)=4$, by Theorem 9 of [7]. In each case $C=C_{G'}(A)$ has finite index in G', and we may assume that C' is a nonabelian free group, so $A\cap C'=1$. Suppose first that $C=G'$. Then A maps injectively to $M=G'/G''$. Since M is finitely generated as a module over $\mathbb{Z}[G/G']\simeq\mathbb{Z}[t,t^{-1}]$, and since A maps to a submodule of M, it has a submodule M_1 of finite index such that $M_1\simeq A\oplus(M_1/A)$ as an abelian group. Let G_1 be the preimage of M_1 in G. Then G_1 is normal in G and $G_1\simeq A\times(G_1/A)$. Hence $c.d.G_1/A=1$ by [1: Theorem 5.6] so is free and $v.c.d.G/A$ is finite. If $C\neq G'$ then $[G':C]=3$ and we may apply the same argument to the $\mathbb{Z}[G/C]$-module $N=C/C'$, obtaining a submodule $N_1\simeq A\times(N_1/A)$, a subgroup $C_1\simeq A\times(C_1/A)$ and hence again $v.c.d.G/A$ is finite. In each case we conclude that G/A is virtually a PD_2-group. Now Eckmann and Müller have shown that each virtual PD_2-group with infinite abelianization maps onto some planar discontinuous group, with finite kernel [4]. In this case the planar groups are virtually surface groups and so have compact fundamental region. On considering the presentations of such planar discontinuous groups, as given for instance in Theorem 4.5.6 of Zieschang, Vogt and Coldewey [19], we see that no such group has infinite cyclic abelianization. This gives a contradiction and so proves the theorem. □

All our work thus far has required that the 2-knot group admit a *torsion free* abelian normal subgroup. In [8] we shall show that the strategy of [7] may be adapted to the case when the quotient of the knot group by its maximal locally finite normal subgroup has such an abelian subgroup.

2. Branched twist spun torus knots

The commutator subgroup of the group of a cyclic branched cover of the r-twist spun (p,q)-torus knot (with p,q relatively prime) is the fundamental group of a Brieskorn manifold $M(p,q,r)$, and is therefore finite only when $p^{-1}+q^{-1}+r^{-1}$ is greater than 1 [7]. The triples $(2,q,2)$ for q odd give rise to groups with commutator subgroup cyclic of odd order, and these groups each have an unique weight class. (Recall that a weight class in a group is the conjugacy class of an element whose normal closure is the whole group). The remaining 6 such triples lead to 3 nontrivial groups, with commutator subgroups Q, T^*, I^* and having 1, 2 or 4 weight classes (up to inversion) respectively. Of the 7 possibilities, 6 are realized by twist spun torus knots; the 7th by the 2-fold cyclic branched cover of the 5-twist spun trefoil knot [14]. With these observations, the next theorem determines the groups of branched twist spun torus knots.

THEOREM 4. *A group G which is not virtually solvable is the group of a 2-knot which is a cyclic branched cover of a twist spun torus knot if and only if it is a high dimensional knot group, a PD_4^+-group with centre of rank 2, some nonzero power of a weight element being central, and such that G' is virtually representable onto $\mathbb{Z}$.*

Proof. If K is a cyclic branched cover of the r-twist spun (p, q)-torus knot then the closed 4-manifold $M(K)$ obtained by surgery on the knot fibres over the circle with fibre $M(p, q, r)$ and monodromy of order r, and so the rth power of a meridian is central [12]. Moreover the monodromy commutes with the natural circle action on $M(p, q, r)$ (cf. Lemma 1.1 of [10]) and hence preserves a Seifert fibration. It follows that the meridian commutes with the centre of $\pi_1(M(p, q, r))$, which is therefore also central in G. Since (with the above exceptions) $\pi_1(M(p, q, r))$ is a PD_3-group with infinite cyclic centre and which is virtually representable onto $\mathbb{Z}$, the necessity of the conditions is evident.

Conversely, if G is such a group then G' is the fundamental group of a Seifert fibred 3-manifold, M say, by the argument of [7; Theorem 6] with [6]. Moreover M is "sufficiently complicated" in the sense of [20], since G' is not virtually solvable. Let t be an element of G whose normal closure is the whole group, and such that t^n is central for some positive n (which we may assume minimal). Then $\hat{G} = G/\langle t^n \rangle$ is a semidirect product, $\hat{G} \simeq G' \rtimes \mathbb{Z}/n\mathbb{Z}$. By [20; Korollar 3.3] there is a fibre-preserving selfhomeomorphism τ of M inducing the outer automorphism of G' determined by t, which moreover can be so chosen that its lifts to the universal covering space $\tilde{M}$ together with the covering group generate a group of homeomorphisms isomorphic to $\hat{G}$. The automorphism of $\tilde{M}$ corresponding to the image of t in $\hat{G}$ has connected 1-dimensional fixed point set by Smith theory. (Note that $\tilde{M} \simeq \mathbb{R}^3$). Therefore the fixed point set of the map τ in M is nonempty. Let P be a fixed point. Then P determines a cross-section $\gamma = P \times S^1$ of the mapping torus of τ. We may perform surgery on γ to obtain a 2-knot with group G which is fibred with monodromy of finite order. We may then apply Proposition 6.1 of Plotnick [12] to conclude that the 2-knot is a branched twist spin of a knot in a homotopy 3-sphere. Since the monodromy repects the Seifert fibration and leaves the centre of G' invariant, the branch set must be a fibre, and the orbit manifold a Seifert fibred homotopy 3-sphere. Therefore the orbit knot is a fibre of a Seifert fibration of S^3 and so is a torus knot. Thus the 2-knot is a branched twist spin of a torus knot. □

If G is a virtually solvable 2-knot group in which some power of a weight element is central then by Theorems 13 and 14 of [7] either G' is finite or G is torsion free and virtually poly-$\mathbb{Z}$; in the latter cases G must be either the group of the 6-twist spin of the trefoil knot or of the 3-twist spin of the figure eight knot, or of the 2-twist spins of certain 3-branch Montesinos knots.

We may also apply the work of Zimmermann and Plotnick when G is a high dimensional knot group and some power of a weight element is central to show that if the 3-dimensional Poincaré conjecture holds and if also G' is the group of a virtually Haken 3-manifold then G is the group of a branched twist spun prime knot, while if instead G' is the group of a hyperbolic 3-manifold then G is the group of a branched twist spun simple (non-torus) knot. However we do not yet have purely algebraic characterizations of such 2-knot groups. (Note moreover that in Theorem 4 we were able to avoid the Poincaré Conjecture). We raise the following question. Is a high dimensional knot group G which is a PD_4^+-group in which some nonzero power of a weight element is central the group of a branched twist spun prime knot?

Plotnick's argument uses properties of S^1 actions on homotopy 4-spheres. We might hope to avoid this by a homological argument to show that τ has connected fixed point set. The projection onto the orbit space would then be a branched cyclic covering, branched over a knot, and a standard argument using the fact that the normal closure of t in G is the whole group would then show that the orbit space is simply connected. Since M is Seifert fibred and τ is fibre preserving, the branch locus would then be a torus knot in the standard 3-sphere. However I have been unable thus far to establish this connectedness directly.

We may ask to what extent does the group G determine the triple p, q, r and the degree of the branched covering. Moreover, does any 2-knot group with a rank 2 abelian normal subgroup satisfy the hypotheses of Theorem 4? In particular, must such a subgroup be central? In [11], Plotnick shows that there are infinitely many distinct conjugacy classes of weight elements in the group of a Brieskorn homology sphere $M(p, q, r)$, where p, q, r are relatively prime (and excluding the case $M(2, 3, 5)$). Such groups are central extensions of triangle groups, and the latter are virtually torsion free [10]. Plotnick's lemma 1 then implies that there are weight elements such that no power is central. Now if H is a perfect group of weight 1 and h is a weight element, then $H \times \mathbb{Z}$ also has weight 1, and ht is a weight element for it (where t generates the $\mathbb{Z}$ factor). Moreover this gives a bijection between weight elements of the two groups, and also between conjugacy classes of such elements. (The groups of r-twist spun (p, q)-torus knots, for such $p, q, r,$ are such direct products). Therefore we may obtain many 2-knots whose groups are as in Theorem 4 but which are not themselves branched twist spun, by surgery on such weight classes in the closed 4-manifolds obtained by surgery on branched twist spun torus knots. Is there a 2-knot group which contains a rank 2 free abelian normal subgroup and for which no nonzero power of any weight element is central?

The closed 4-manifold $M(K)$ obtained by surgery on a 2-knot K with group G as in Theorem 4 is aspherical [7], and so the theorem implies that it is homotopy equivalent to $M(K_1)$ for some branched twist spun torus knot K_1. It is a well known open question as to whether homotopy equivalent aspherical closed manifolds must be homeomorphic. We shall show next that if K if fibred then $M(K)$ and $M(K_1)$ are homeomorphic. This is a version of Plotnick's Proposition 6.1, starting from more algebraic hypotheses.

THEOREM 5. *Let K be a fibred 2-knot whose group G has centre of rank 2, some power of a weight element being central, and has G' virtually representable onto $\mathbb{Z}$. Suppose that the fibre is irreducible. Then the closed 4-manifold obtained by surgery on K is homeomorphic to one obtained by surgery on a branched twist spun torus knot.*

Proof. Let F be the closed fibre and $\phi: F \to F$ the characteristic map. Then F is a Seifert manifold by [6] and [16] (cf. Theorem 5). Now the automorphism of F constructed as in Theorem 5 induces the same outer automorphism of $\pi_1(F)$ as ϕ, and so these maps must be homotopic. Therefore they are in fact isotopic, by [3, 17]. The theorem now follows. □

The closed fibre of any fibred 2-knot with such a group is aspherical, and so is a connected sum $F \# P$, where F is irreducible and P is a homotopy 3-sphere. If we could show that the characteristic map may be isotoped so that the fake 3-cell is carried onto itself then we would have $K = K_1 \# K_2$ where K_1 is fibred with irreducible fibre and where K_2 has group $\mathbb{Z}$. We could then use Freedman's Unknotting Theorem [5] to sidestep the 3-dimensional Poincare conjecture.

Using the Rigidity Theorem of Mostow, we can extend Theorem 5 to show that if K is a fibred 2-knot with hyperbolic fibre and group G such that some power of a weight element is central then $M(K)$ is homeomorphic to $M(K_1)$ for some branched twist spun simple non-torus knot K_1. We raise the following question. If K is a 2-knot such that G' has one end and some nonzero power of a weight element is central then is $M(K)$ homeomorphic to $M(K_1)$ for some branched twist spun prime knot K_1? Ideally we might expect that any 2-knot whose group is a PD_4^+-group and has finitely generated commutator subgroup might be fibred, but at present we can only determine the potential fibre from the group in the Seifert case. Work in progress suggests that there may be new obstructions to fibering when the group has torsion.

REFERENCES

[1] BIERI, R. *Homological Dimension of Discrete Groups,* Queen Mary College Mathematics Notes, London (1976).

[2] —— and STREBEL, R. *Almost finitely presentable soluble groups,* Comment. Math. Helv. *53* (1978), 258–278.

[3] BOILEAU, M. and OTAL, J. P. *Groupe des difféotopie de certaines variétés de Seifert,* C.R. Acad. Sci. Paris *303* (1986), 19–22.

[4] ECKMANN, B. and MÜLLER, H. *Plane motion groups and virtual Poincaré duality of dimension two,* Inventiones Math. *69,* (1982), 293–310.

[5] FREEDMAN, M. H. The disc theorem for four-dimensional manifolds, Proc. I.C.M. Warsaw, Poland (1983), 647–663.

[6] HILLMAN, J. A. *Seifert fibre spaces and Poincaré duality groups,* Math. Z. *190* (1985), 365–369.

[7] —— *Abelian normal subgroups of two-knot groups,* Comment. Math. Helv. *61* (1986), 122–148.

[8] —— *Poly-(solvable or locally finite) two-knot groups,* Pre-print, Macquarie University (1987).

[9] KAPLANSKY, I. *Fields and Rings,* Chicago Lectures in Mathematics, Chicago University Press, Chicago–London (1969).

[10] MILNOR, J. W. On the 3-dimensional Brieskorn manifolds $M(p, q, r)$, in *Knots, Groups and 3-Manifolds* (ed. L. P. Neuwirth), Ann. Math. Study *84,* Princeton University Press, Princeton N.J. (1975), 175–225.

[11] PLOTNICK, S. P. *Infinitely many disk knots with the same exterior,* Math. Proc. Cambridge Phil. Soc. *93* (1983), 67–72.

[12] —— Fibred knots in S^4-twisting, spinning, rolling, surgery and branching, in *Four-Manifold Theory,* CONM 35, Amer. Math. Soc. (1984), 437–459.

[13] —— *Equivalent intersection forms, knots in S^4, and rotations in 2-spheres,* Trans. Amer. Math. Soc. *296* (1986), 543–575.

[14] —— and SUCIU, A. I. *Fibred knots and spherical space forms,* J. London Math. Soc. *35* (1987), 514–526.

[15] ROBINSON, D. J. S. *A Course in the Theory of Groups,* Graduate Texts in Mathematics 80, Springer-Verlag, Berlin–Heidelberg–New York (1982).

[16] SCOTT, P. *There are no fake Seifert fibre spaces with infinite π_1,* Ann. Math. *117* (1983), 35–70.

[17] —— *Homotopy implies isotopy for some Seifert fibre spaces,* Topology *24* (1985), 341–351.

[18] STREBEL, R. *A remark on subgroups of infinite index in Poincaré duality groups,* Comment, Math. Helv. *52* (1977), 317–324.

[19] ZIESCHANG, H., VOGT, E. and COLDEWEY, H. D. *Surfaces and Planar Discontinuous Groups,* Lecture Notes in Mathematics 835, Springer-Verlag, Berlin–Heidelberg–New York (1980).

[20] ZIMMERMANN, B. *Periodische Homöomorphismen Seifertscher Faserräume,* Math. Z. *166,* (1979), 289–297.

School of Mathematics, Physics, Computing and Electronics
Macquarie University
NSW 2109 Australia

Received September 4, 1986/December 15, 1987