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Invariant theory of G_2 and Spin_7

GERALD W. SCHWARZ*

§0. Introduction

(0.0) Let G be a semisimple complex algebraic group. An *invariant theory* for G is a faithful representation $\phi: G \to GL(V)$ together with generators and relations for the algebras of invariants $\mathbb{C}[nV]^G$, $n \in \mathbb{N}$, where nV denotes the direct sum of n copies of V. Given an invariant theory for G, one can use the symbolic method [W] to garner information about the invariants of any representation of G.

If G is one of the classical groups SL_m, SO_m, etc. with its standard representation on $V = \mathbb{C}^m$, then classical invariant theory (CIT) tells us generators and relations for $\mathbb{C}[nV]^G$, $n \in \mathbb{N}$, i.e. CIT is an invariant theory for the classical groups. There remains the problem of finding an invariant theory for the non-classical simple connected complex algebraic groups, i.e. for the exceptional groups G_2, F_4, E_6, E_7 and E_8, and the spin groups Spin_m, $m \geq 7$. The first cases to consider are G_2 and Spin_7 (also denoted B_3) which have faithful irreducible 7-dimensional and 8-dimensional representations, respectively. In this paper we establish an invariant theory for G_2 and B_3. Our results for G_2 were announced in [S4].

(0.1) The invariant theory for G_2 fits into the following general framework: Let A be a finite dimensional central $\mathbb{C}$-algebra, not necessarily associative. Let G denote the group of algebra automorphisms of A, and let $\operatorname{tr}(a) = (\dim A)^{-1} \operatorname{trace}(R_a)$ where $R_a: A \to A$ is right multiplication by $a \in A$. Then the trace (i.e. tr) of any product of elements in nA, $n \in \mathbb{N}$, is an element of $\mathbb{C}[nA]^G$.

Suppose that $A = M_k(\mathbb{C}) = k \times k$ complex matrices. Then $G = PSL_k$ acting on A by conjugation. In this case, Procesi ([Pr1], [Pr2]) and Rasmyslov [R] showed that traces of products give all the generators of $\mathbb{C}[nA]^G$. Moreover, the relations among these generators all result from the Cayley–Hamilton identity – the "standard" identity for $M_k(\mathbb{C})$.

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(0.2) Suppose that A is the usual (complex) Cayley algebra. Then the automorphism group G is G_2. We will show that traces of products of at most 4 elements give generators of $\mathbb{C}[nA]^G$. (J. Ferrar has informed us that he has also proved this result.) Moreover, in analogy with the case of $M_k(\mathbb{C})$, we show that the relations among these generators are a result of the standard quadratic identity and alternative laws for the Cayley algebra. The alternative laws are the Cayley algebra's analogue to the associativity of $M_k(\mathbb{C})$. The faithful 7-dimensional representation of G_2 is its action on the trace zero Cayley numbers.

(0.3) $\mathrm{Spin}_7 = \mathsf{B}_3$ is also connected to the Cayley algebra A: There is a non-degenerate quadratic form δ on A and a 4-form ϵ on A, both G_2-invariant, such that B_3 is isomorphic to the subgroup of $GL(A)$ preserving δ and ϵ. The algebras $\mathbb{C}[nA]^{\mathsf{B}_3}$ have generators of degrees 2 and 4 corresponding to δ and ϵ (see §2), and the relations are a consequence of the identities of A.

(0.4) Let $\phi: G \to GL(V)$ be faithful. A *first main theorem* (FMT) for ϕ (or G) gives generators for the algebras $\mathbb{C}[nV]^G$, $n \in \mathbb{N}$. A *second main theorem* (SMT) is a determination of the relations among these generators. We use the tools of "modern" invariant theory and commutative algebra to determine a FMT and SMT for G_2 and for B_3. Then we show that the generators and relations arise, as sketched above, from the structure of the Cayley algebra. We would be surprised and pleased if there were a purely "Cayley theoretic" way to establish an invariant theory for G_2 and B_3.

(0.5) The contents of this paper are as follows: In §§1–2 we recall the construction and basic properties of the Cayley algebra A and the actions of G_2 and B_3 on A. We list the generators which figure in the FMT's for G_2 and B_3. In §3 we recall general results on FMT's, and we apply them to establish the FMT's for G_2 and B_3. In §§4–5 we recall results on SMT's, and we list proposed SMT's for G_2 and B_3. We show that our proposed SMT for G_2 (resp. B_3) is correct if it is correct for 6 (resp. 7) copies of the fundamental representation $\phi: G \to GL(V)$. In §6 we show that our proposed SMT's result from the identities of A.

To establish the SMT for small numbers of copies of ϕ we used Poincaré series techniques. For example, consider the case $S = \mathbb{C}[6V]^G$ where $G = \mathsf{G}_2$. In §7 we show that S is a finite free graded module over a subalgebra generated by 18 elements of degree 2 and 10 elements of degree 3. Thus the Poincaré series $P_t(S)$ is $(1-t^2)^{-18}(1-t^3)^{-10}\sum_{i=0}^{l} a_i t^i$ where the a_i are in $\mathbb{N}$ and we assume that $a_l \neq 0$. Moreover, $l = 24$, and $a_i = a_{l-i}$, $0 \leq i \leq 24$.

Let S' denote the algebra given by the generators and proposed relations for S. In §9 we compute the Poincaré series of an algebra S'' which maps onto a certain associated graded algebra to S'. (This involves finding finite free resolutions of certain modules over polynomial rings.) We find that $P_t(S'') = (1-t^2)^{-18}(1-t^3)^{-10}\sum_{i=0}^{24} b_i t^i$ where $b_i = b_{24-i}$, $0 \le i \le 24$. In §10 we compute (rather easily) that $a_i = b_i$ for $i \le 12$. It follows that $P_t(S'') = P_t(S') = P_t(S)$, establishing the SMT for G_2. The techniques used in the case of B_3 are similar.

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§1. The Cayley algebra, G_2 and trace invariants

(1.0) We recall the construction and properties of the Cayley algebra and G_2. ([Sf] is a general reference for what follows.) We exhibit the trace invariants which play a fundamental role in the invariant theory of G_2.

(1.1) Let A be a finite dimensional simple central algebra over $\mathbb{C}$ (not necessarily associative). Assume that A is an *alternative algebra*, i.e.

$$(1.2)\quad x(xy) = (x^2)y; \qquad (yx)x = y(x^2) \qquad x, y \in A.$$

Let $\operatorname{tr}(x) = (\dim A)^{-1} \operatorname{trace}(R_x)$ as in (0.1), and let $b(x, y) = (\dim A)^{-1} \operatorname{trace}(R_x \circ R_y)$ for $x, y \in A$. Then b is a non-degenerate symmetric bilinear form satisfying $b(xy, z) = b(x, yz)$ for all $x, y, z \in A$ ([Sf] p. 44). Since $\operatorname{tr}(x) = b(x, 1)$, we have $b(x, y) = \operatorname{tr}(xy)$, hence

$$(1.3)\quad \operatorname{tr}(xy) = \operatorname{tr}(yx) \qquad x, y \in A$$

$$(1.4)\quad \operatorname{tr}((xy)z) = \operatorname{tr}(x(yz)) \qquad x, y, z \in A.$$

Define an endomorphism of A, $x \mapsto \bar{x}$, by

$$(1.5)\quad \bar{x} = 2\operatorname{tr}(x) - x, \qquad x \in A,$$

where we have identified $\operatorname{tr}(x)$ with $\operatorname{tr}(x) \cdot 1 \in A$. Assume further that A is a *quadratic algebra*, i.e. assume that $x\bar{x}$ lies in the center of A for every $x \in A$.

Define norm (x) to be tr $(x\bar{x})$. Then

(1.6) $x^2 - 2\,\mathrm{tr}\,(x)x + \mathrm{norm}\,(x) = 0 \qquad x \in A.$

(1.7) $\overline{xy} = \bar{y}\bar{x} \qquad x, y \in A.$

The identity (1.6) is called the *standard quadratic identity,* and it is immediate from our assumptions. One can derive (1.7) from (1.6).

(1.8) Up to isomorphism, there is only one non-commutative, non-associative algebra A as above ([Sf] pp. 70, 73), the 8-dimensional Cayley algebra. It can be constructed directly as follows: Let $A_{\mathbb{R}}$ denote the set of ordered pairs of quaternions with co-ordinatewise addition and the following multiplication:

$$(a, b)(c, d) = (ac - \bar{d}b, da + b\bar{c}),$$

where $a \mapsto \bar{a}$ is the usual conjugation of quaternions. Then $A_{\mathbb{R}}$ is a central simple non-associative, non-commutative algebra of dimension 8 over $\mathbb{R}$ satisfying the same identities as A. If $x = (a, b) \in A_{\mathbb{R}}$, then $\bar{x} = (\bar{a}, -b)$ and $\mathrm{tr}\,(x) = \mathrm{Re}\, a$, the real part of a. Our algebra A can be taken to be the complexification of $A_{\mathbb{R}}$.

(1.9) Let K denote the group of algebra automorphism of A. Then K acts trivially on $\mathbb{C} \cdot 1$, hence faithfully on $A' = \mathrm{Ker}\,\mathrm{tr}$. The Lie algebra of K is isomorphic to that of G_2 ([Sf] p. 82), hence K^0, the identity component of K, is isomorphic to G_2, and the representation $G_2 \simeq K^0 \to GL(A')$ must be the irreducible 7-dimensional representation. Since G_2 has no outer automorphisms, Schur's lemma implies that K/K^0 is generated by scalar multiplications. But the only scalar which gives an automorphism of A is 1. Hence we have:

(1.10) PROPOSITION. $G_2 \simeq \mathrm{Aut}\,(A)$ *acts irreducibly and faithfully on* A'.

From now on we will identify G_2 with Aut (A).

(1.11) The form $B(x, y) = \mathrm{tr}\,(x\bar{y})$ is symmetric, non-degenerate and preserved by G_2. Hence the representation of G_2 on A' is orthogonal. The form B is positive definite on $A_{\mathbb{R}}$, hence a compact real form of G_2 is Aut $(A_{\mathbb{R}})$.

(1.12) We end this section by exhibiting some important trace invariants of A': Let $n \in \mathbb{N}$ and let $(x_1, \ldots, x_n) \in nA'$ be arbitrary. Define functions

(1.12.1) $\qquad \alpha_{ij} = -\mathrm{tr}\,(x_i x_j) \qquad 1 \leq i, j \leq n,$

(1.12.2) $\beta_{ijk} = -\mathrm{tr}\,(x_i(x_jx_k)) \qquad 1 \le i, j, k \le n,$

(1.12.3) $\gamma_{ijkl} = \text{skew tr}\,(x_i(x_j(x_kx_l))) \qquad 1 \le i, j, k, l \le n,$

where the last function is skew symmetrized with respect to its arguments. Clearly the α_{ij}, etc. are in $\mathbb{C}[nA']^{G_2}$. Note that $\alpha_{ij} = \alpha_{ji}$. Since the x_i are in A', we have $\bar{x}_i = -x_i$. Thus

$$\begin{aligned}\beta_{123} &= -\mathrm{tr}\,(x_1(x_2x_3)) = -\mathrm{tr}\,\overline{(x_1(x_2x_3))} = \mathrm{tr}\,((x_3x_2)x_1)\\ &= \mathrm{tr}\,(x_1(x_3x_2)) = -\beta_{132}.\end{aligned}$$

Similarly, $\beta_{123} = -\beta_{213}$. Hence the β_{ijk} are skew symmetric in their indices as are the γ_{ijkl}, by definition.

In §6 we give a "Cayley theoretic" proof that the α_{ij}, etc. generate the "trace" invariants of nA'. In §3 we show that the α_{ij}, etc. generate $\mathbb{C}[nA']^{G_2}$.

(1.13) Let W denote the dual $(A')^*$ of A'. We let $\alpha \in (S^2W)^{G_2}$, $\beta \in \Lambda^3(W)^{G_2}$ and $\gamma \in (\Lambda^4W)^{G_2}$ denote non-zero elements corresponding to the invariants α_{ij}, etc.

(1.14) *Remark.* Let H be the subgroup of $GL(A')$ preserving α and β (or α and γ), and extend H to $GL(A)$ so that H preserves 1. Then one easily shows that H consists of automorphisms of A, i.e. $H = G_2$.

§2. Spin$_7$ and the Cayley algebra

(2.0) We show that there is a natural action of B_3 on the Cayley algebra A. We exhibit generators of the B_3-invariants of several copies of A.

(2.1) We consider various Lie subalgebras of $\mathfrak{so}(A)$ (resp. Lie subgroups of $SO(A)$) where A is given the symmetric bilinear form $x, y \mapsto \mathrm{tr}\,(x\bar{y})$. Let $\mathfrak{l} = \{L_a : a \in A'\}$ and $\mathfrak{r} = \{R_a : a \in A'\}$ where L_a (resp. R_a) denotes left (resp. right) multiplication by a. Let $\mathfrak{g}_2 = \mathrm{Der}\,(A) =$ derivations of A. Clearly, $\mathfrak{g}_2 \subseteq \mathfrak{so}(A)$ and $\mathfrak{g}_2$ is the Lie algebra of G_2. We have ([Sf] p. 81)

(2.2) $\mathfrak{so}(A) = \mathfrak{g}_2 \oplus \mathfrak{l} \oplus \mathfrak{r}.$

Let

(2.3) $\mathfrak{a} = \mathfrak{g}_2 + \{L_a - R_a : a \in A'\}.$

(2.4) $\mathfrak{b} = \mathfrak{g}_2 + \{2L_a + R_a : a \in A'\}.$

By (2.2), the sums in (2.3) and (2.4) are direct. Let $\eta : \mathfrak{b} \to \mathfrak{a}$ be the linear map which is the identity on $\mathfrak{g}_2$ and sends $2L_a + R_a$ to $L_a - R_a$, $a \in A'$.

(2.5) LEMMA. *$\mathfrak{a}$ and $\mathfrak{b}$ are Lie subalgebras of $\mathfrak{so}(A)$, and $\eta : \mathfrak{b} \to \mathfrak{a}$ is a Lie algebra isomorphism.*

Proof. Let $x, y \in A$, and let z denote $yx - xy$. From equations (3.2), (3.67), (3.68) and (3.70) of [Sf] one obtains

$$\begin{aligned} 3[L_x, R_y] = 3[R_x, L_y] &= R_z - L_z - D, \\ 3[L_x, L_y] &= -2R_z - L_z + 2D, \\ 3[R_x, R_y] &= R_z + 2L_z + 2D, \end{aligned}$$

where $D \in \mathfrak{g}_2$, and

$$D = [L_x, L_y] + [L_x, R_y] + [R_x, R_y].$$

(The formulas above actually differ in sign from Schafer's since he writes operators on the right rather than on the left.) We then obtain

$$\begin{aligned} [L_x - R_x, L_y - R_y] &= L_z - R_z + 2D, \\ [2L_x + R_x, 2L_y + R_y] &= L_z + 2R_z + 2D, \end{aligned}$$

and the lemma follows easily. □

(2.6) *Remarks.* (1) There is another Lie subalgebra $\mathfrak{c} = \mathfrak{g}_2 + \{L_a + 2R_a : a \in A'\}$, and $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{c}$ are permuted by the "principle of triality" of $\mathfrak{so}(A)$ ([Sf] p. 88).

(2) $\mathfrak{a}$ annihilates $1 \in A$, hence $\mathfrak{a}$ can be considered as a subalgebra of $\mathfrak{so}(A')$. Since $\dim \mathfrak{a} = \dim \mathfrak{so}(A') = 21$, we have $\mathfrak{a} = \mathfrak{so}(A')$.

The Lie algebra $\mathfrak{b} \simeq \mathfrak{so}(7)$ acts irreducibly on A (since $\mathfrak{g}_2 \subseteq \mathfrak{b}$ already acts irreducibly on A' and A' is not $\mathfrak{b}$-stable), hence there is an irreducible representation $\rho : \mathsf{B}_3 \to SO(A)$ such that the induced mapping $\rho_* : \mathfrak{so}(7) \to \mathfrak{so}(A)$ has image $\mathfrak{b}$. By the classification of representations of B_3, ρ is the (8-dimensional) spin representation.

(2.7) We now construct a B_3-invariant function on $A \times A \times A \times A$: Let

$$M: A \times A \to A'$$
$$(x, y) \mapsto \tfrac{1}{2}(xy - yx) + \operatorname{tr}(x)y - \operatorname{tr}(y)x.$$

A rather tedious computation shows that

$$M(bx, y) + M(x, by) = \eta(b)M(x, y); \qquad b \in \mathfrak{b}; \qquad x, y \in A.$$

Since $\eta(\mathfrak{b}) = \mathfrak{so}(A')$, we obtain a $\mathfrak{b}$-invariant (hence B_3-invariant) function

$$F: A \times A \times A \times A \to \mathbb{C}$$
$$y_1, y_2, y_3, y_4 \mapsto \operatorname{tr}(M(y_1, y_2)M(y_3, y_4)).$$

(2.8) We now exhibit the generators of $\mathbb{C}[nA]^{\mathsf{B}_3}$: Let $(y_1, \ldots, y_n) \in nA$ be arbitrary. Then there are B_3-invariant functions

(2.8.1) $\qquad \delta_{ij} = \operatorname{tr}(y_i\bar{y}_j) \qquad 1 \leq i, j \leq n,$

(2.8.2) $\qquad \epsilon_{ijkl} = \operatorname{skew} F(y_i, y_j, y_k, y_l) \qquad 1 \leq i, j, k, l \leq n,$

where the last invariant is skew symmetrized with respect to its arguments.

Write $y_i = x_i + \operatorname{tr}(y_i) \cdot 1$, so that $x_i \in A'$; $i = 1, \ldots, n$. Then the B_3-invariants can be expressed in terms of the G_2-invariants of the x_i. We obtain the following two formulas, where the first is obvious and the second follows from proposition (6.8) of §6:

(2.9) $\qquad \delta_{ij} = \operatorname{tr}(y_i)\operatorname{tr}(y_j) + \alpha_{ij}.$

(2.10) $\qquad \epsilon_{ijkl} = \gamma_{ijkl} - \operatorname{tr}(y_i)\beta_{jkl} + \operatorname{tr}(y_j)\beta_{ikl} - \operatorname{tr}(y_k)\beta_{ijl} + \operatorname{tr}(y_l)\beta_{ijk}.$

Note that (2.10) implies that the ϵ_{ijkl} are not zero! In §3 we will show that the δ_{ij} and ϵ_{ijkl} generate $\mathbb{C}[nA]^{\mathsf{B}_3}$.

(2.11) Let $\delta \in (S^2A^*)^{\mathsf{B}_3}$ and $\epsilon \in (\Lambda^4A^*)^{\mathsf{B}_3}$ denote non-zero elements corresponding to the δ_{ij} and ϵ_{ijkl}.

(2.12) *Remark.* B_3 is the subgroup of $GL(A)$ preserving δ and ϵ: Since $\mathfrak{b}$ maps $1 \in A$ onto A', one easily sees that the orbit $\mathsf{B}_3 \cdot 1$ is open and closed in $X = \{x \in A : \operatorname{norm}(x) = 1\}$. Since X is irreducible, $\mathsf{B}_3 \cdot 1 = X$. The isotropy group

H of B_3 at 1 acts orthogonally on A' and preserves $\beta \in \Lambda^3((A')^*)^{\mathsf{G}_2}$ by (2.10). Then $H = \mathsf{G}_2$ by (1.14). It follows that no subgroup of $GL(A)$ strictly larger than B_3 can preserve δ and ϵ.

§3. First main theorems

(3.0) We begin by recalling properties of integral representations of GL_n (those lying in tensor powers of $\mathbb{C}^n$) and some results of classical invariant theory. We then establish the FMT's for G_2 and B_3.

(3.1) Let $\psi_1(n)$ denote the standard representation of GL_n on $\mathbb{C}^n$, and let $\psi_i(n) = \Lambda^i(\psi_1(n))$, $i \geq 0$. Note that $\psi_i(n) = 0$ for $i > n$ and that $\psi_0(n)$ is the trivial 1-dimensional representation. Let $\mathbb{N}^\infty$ denote the sequences in $\mathbb{N}$ which are eventually 0. If $(a) = (a_1, a_2, \ldots) \in \mathbb{N}^\infty$, let $\psi_{(a)}(n)$ denote the highest weight (Cartan) component in $S^{a_1}(\psi_1(n)) \otimes \cdots \otimes S^{a_k}(\psi_k(n))$ where k is minimal such that $a_j = 0$ for $j > k$. If $k \leq n$ (hence $\psi_{(a)}(n) \neq 0$), we will also use the notation $\psi_1^{a_1} \cdots \psi_n^{a_n}(n)$ or $\psi_1^{a_1} \cdots \psi_k^{a_k}(n)$ to denote $\psi_{(a)}(n)$. *If* (a) is the zero sequence, then $\psi_{(a)}(n) = \psi_0(n)$. We will confuse the $\psi_{(a)}(n)$ with their corresponding representation spaces, and similarly for representations $\psi_{(a)}$ defined below.

(3.2) We embed $\mathbb{C}^n \subseteq \mathbb{C}^{n+1}$ as the subspace of vectors with last component zero. Then for $(a) \in \mathbb{N}^\infty$ we have inclusions $\psi_{(a)}(n) \subseteq \psi_{(a)}(n+1)$ compatible with the actions of $GL_n \subseteq GL_{n+1}$. Thus $GL = \varinjlim GL_n$ acts on $\psi_{(a)} = \varinjlim \psi_{(a)}(n)$. Let U_n denote the subgroup of GL_n consisting of upper triangular matrices with 1's on the diagonal, and set $U = \varinjlim U_n$. We identify GL_n, U_n and $\psi_{(a)}(n)$ with their images in GL, U and $\psi_{(a)}$, respectively. If $\psi_{(a)}(n) \neq 0$, then $\psi_{(a)}^U = \psi_{(a)}(n)^{U_n}$ is the space of highest weight vectors of $\psi_{(a)}(n)$.

(3.3) Let $(a) \in \mathbb{N}^\infty$. We define $\deg(a) = \sum ia_i$, width $(a) = \sum a_i$, and ht (a) (the height of (a)) is the least $j \geq 0$ such that $a_i = 0$ for $i > j$. The height, etc. of $\psi_{(a)}$ and $\psi_{(a)}(n)$ are defined to be the height, etc. of (a). (In the language of Young diagrams ([W]) Ch IV), our notions of width and height correspond to the width and height of diagrams, and degree just counts the number of boxes in diagrams.) If $(b) \in \mathbb{N}^\infty$, then $(a) + (b)$ denotes $(a_1 + b_1, \ldots)$ and $\psi_{(a)}\psi_{(b)}$ denotes $\psi_{(a)+(b)}$. We write $(a) < (b)$ (and $\psi_{(a)} < \psi_{(b)}$, etc.) if $a_l < b_l$ for the greatest l such that $a_l \neq b_l$.

(3.4) Let $(a), (b), (c) \in \mathbb{N}^\infty$. We say that $\psi_{(c)}(n)$ *occurs in* $\psi_{(a)}(n) \otimes \psi_{(b)}(n)$ (resp. $\psi_{(c)}$ *occurs in* $\psi_{(a)} \otimes \psi_{(b)}$) if the latter representation contains a subrepresentation isomorphic to $\psi_{(c)}(n)$ (resp. $\psi_{(c)}$). We identify isomorphic representations of GL (and GL_n). Hence, for example, we have equalities $\psi_1(n) \otimes \psi_1(n) = \psi_1^2(n) + \psi_2(n)$ for all n, and the equality $\psi_1 \otimes \psi_1 = \psi_1^2 + \psi_2$.

(3.5) PROPOSITION (see [S5], [V3]). *Let $(a), (b), (c) \in \mathbb{N}^\infty$. Suppose that $\psi_{(c)}(n)$ occurs in $\psi_{(a)}(n) \otimes \psi_{(b)}(n)$ for some n. Then*

(1) $\deg \psi_{(c)} = \deg \psi_{(a)} + \deg \psi_{(b)}$.

(2) $\mathrm{ht}\, \psi_{(a)}, \mathrm{ht}\, \psi_{(b)} \leq \mathrm{ht}\, \psi_{(c)} \leq \mathrm{ht}\, \psi_{(a)} + \mathrm{ht}\, \psi_{(b)}$.

(3) $\mathrm{width}\, \psi_{(a)}, \mathrm{width}_{\psi(b)} \leq \mathrm{width}\, \psi_{(c)} \leq \mathrm{width}\, \psi_{(a)} + \mathrm{width}\, \psi_{(b)}$.

(4) *The multiplicity of $\psi_{(c)}(n)$ in $\psi_{(a)}(n) \otimes \psi_{(b)}(n)$ is independent of n as long as $n \geq \mathrm{ht}\, \psi_{(c)}$.*

(3.6) COROLLARY. *There are $(c^1), \ldots, (c^r) \in \mathbb{N}^\infty$ (not necessarily distinct) such that*

$$\psi_{(a)} \otimes \psi_{(b)} = \bigoplus_{i=1}^{r} \psi_{(c^i)}.$$

(3.7) Let $\phi : G \to GL(V)$ be a representation of the complex reductive algebraic group G. At times we denote ϕ by (V, G), (ϕ, G) or $\phi(G)$. We sometimes confuse ϕ with V, so that $\mathbb{C}[\phi]^G = \mathbb{C}[V]^G$ denotes the G-invariant polynomial functions on V.

Let $(a) \in \mathbb{N}^\infty$. As in (3.1), we obtain an irreducible representation (space) $\psi_{(a)}(V)$ of $GL(V)$, and by composition a representation $\phi_{(a)}$ of G on $\psi_{(a)}(V)$.

(3.8) Let $\phi = (V, G)$ where $\dim V = m$. Let $P = S^{\cdot}(\psi_1 \otimes V^*)$ and $P(n) = S^{\cdot}(\psi_1(n) \otimes V^*) \subseteq P$. Then P (resp. $P(n)$) is a graded direct sum of $GL \times G$ (resp. $GL_n \times G$) representations. Let $R = P^G$ and $R(n) = P(n)^G$. Note that $P(n) \simeq \mathbb{C}[nV]$, $R(n) \simeq \mathbb{C}[nV]^G$ and that $P = \varinjlim P(n)$, $R = \varinjlim R(n)$. We will use notation $R(\phi)$ or $R(n, G)$, etc. if it is necessary to emphasize the relevant representation or group involved.

Cauchy's formula ([Pr2], [S5]) gives us

$$(3.9) \quad S^d(\psi_1 \otimes V^*) = \bigoplus_{\deg(a)=d} \psi_{(a)} \otimes \psi_{(a)}(V^*).$$

Since $\psi_{(a)}(V^*) = 0$ if $\mathrm{ht}\,(a) > m$, we may restrict the sum in (3.9) to those (a) with $\mathrm{ht}\,(a) \leq m$. We then obtain

$$(3.10) \quad P = \bigoplus_{\mathrm{ht}(a) \leq m} \psi_{(a)} \otimes \psi_{(a)}(V^*),$$

(3.11) $R = \bigoplus_{\mathrm{ht}(a)\le m} \psi_{(a)} \otimes \psi_{(a)}(V^*)^G,$

and similarly for $P(n)$ and $R(n)$.

(3.12) Let $R(n)^+$ (resp. R^+) denote the elements of $R(n)$ (resp. R) with zero constant term. Since $R(n)$ is finitely generated ([Kft] p. 95), $R(n)^+/(R(n)^+)^2$ is a finite dimensional GL_n-representation $\bigoplus_{i=1}^p \psi_{(a^i)}(n)$. We can find invariants $f_i \in \psi_{(a^i)}(V^*)^G$ such that the representation space $\psi_{(a^i)}(n) \otimes f_i \subseteq R(n)$ maps onto the copy of $\psi_{(a^i)}(n)$ in $R(n)^+/(R(n)^+)^2$; $1 \le i \le p$. Then the subspaces $\psi_{(a^i)}(n) \otimes f_i$ minimally generate $R(n)$, i.e. bases of these subspaces are a minimal generating set for $R(n)$.

(3.13) THEOREM ([W], [S5], [V3]). *Suppose that* $\{\psi_{(a^i)}(n) \otimes f_i\}_{i=1}^p$ *minimally generates* $R(n)$. *If* $k \le n$ *or* $n \ge m = \dim V$, *then* $\{\psi_{(a^i)}(k) \otimes f_i\}$ *minimally generates* $R(k)$. *In particular, if* $n \ge m$, *then* $\{\psi_{(a^i)} \otimes f_i\}_{i=1}^p$ *minimally generates* R. □

Note that if $\mathrm{ht}\,(a^i) > k$, then $\psi_{(a^i)}(k) \otimes f_i$ is zero.

(3.14) Let $\{\psi_{(a^i)}(n) \otimes f_i\}_{i=1}^p$ be as in (3.13). We say that the elements lying in $\psi_{(a^i)}(n) \otimes f_i$ *transform by* $\psi_{(a^i)}(n)$, and their height, width and degree are defined to be the height, etc. of (a^i). We say that the *minimal generators of* $R(n)$ *transform by* $\psi_{(a^1)}(n), \dots, \psi_{(a^p)}(n)$. If $n \ge m$, then R is generated by $\{\psi_{(a^i)} \otimes f_i\}$, and we say that the *minimal generators* of R *transform by* $\psi_{(a^1)}, \dots, \psi_{(a^p)}$. Note that the representations $\psi_{(a^i)}(n)$ are well-defined but that the subspaces $\psi_{(a^i)}(n) \otimes f_i$, etc. are usually not.

Let $0 \ne \omega_i \in \psi_{(a^i)}^U$ be a highest weight vector, and define $h_i = \omega_i \otimes f_i$, $i = 1, \dots, p$. We call the h_i (*minimal*) *highest weight generators* of R (and of $R(n)$, $n \ge \mathrm{ht}\,(a^i)$). All elements of $\psi_{(a^i)}(n) \otimes f_i$ can be obtained from h_i via the action of the Lie algebra of strictly lower triangular $n \times n$ matrices. In Weyl's language [W], a minimal generating set of $R(n)$ can be obtained from the elements h_i by polarization.

(3.15) Let $0 \ne h = \omega \otimes f \in \psi_{(a)}(n)^{U_n} \otimes \psi_{(a)}(V^*)^G$. Identifying $R(n)$ with $\mathbb{C}[nV]^G$ in the standard way, one sees that h corresponds to an invariant homogeneous of degree $\sum_{i \ge j} a_i$ in the jth copy of V.

(3.16) EXAMPLE. Let $G = O_m$ act as usual on $V = \mathbb{C}^m$, and let $(x_1, \dots, x_n) \in nV$ be arbitrary. Then CIT tells us that the G-invariant functions are generated by the inner product invariants $p_{ij} = x_i \cdot x_j$. These invariants are a basis for a copy of $\psi_1^2(n) \simeq \psi_1^2(n) \otimes (S^2V^*)^G = S^2(\psi_1(n) \otimes V^*)^G \subseteq R(n)$. A highest weight generator h is p_{11}. Note that h has the degree of homogeneity given in (3.15), and that the other generators obviously are obtained from h by polarization.

(3.17) From (3.11) we see that the minimal highest weight generators of R have height at most $m = \dim V$. Sometimes one can improve on this estimate: We say that a representation $\psi_{(a)}(n)$ is *irrelevant* (for ϕ) if $\psi_{(a)}(n) = 0$ or $\psi_{(a)}(n)$ does not occur as a subrepresentation of $P(n)/R(n)^+P(n)^+$. One similarly defines when $\psi_{(a)}$ is irrelevant, and if $\mathrm{ht}\,(a) \leq n$, then $\psi_{(a)}$ is irrelevant if and only if $\psi_{(a)}(n)$ is also. Clearly, if $\mathrm{ht}\,(a) > m$, then $\psi_{(a)}$ is irrelevant.

(3.18) *Remarks* ([S5]). (1) If $\psi_{(a)}$ is irrelevant, then no elements of a minimal generating set of R transform by $\psi_{(a)}$.

(2) If $\psi_{(a)}$ is irrelevant and $(b) \in \mathbb{N}^\infty$, then any irreducible representation occurring in $\psi_{(a)} \otimes \psi_{(b)}$ is irrelevant. In particular, $\psi_{(a)+(b)}$ is irrelevant.

(3) If ψ_k is irrelevant, then ψ_n is irrelevant for any $n > k$, and minimal generators of R transform by representations of height $<k$.

(3.19) THEOREM ([W], c.f. [S5]). *Let $\phi = (V, G)$ where* $\dim V = m$.

(1) *Suppose that V admits a non-degenerate G-invariant skew form (i.e. ϕ is symplectic) and $m > 2$. Then ψ_{k+1} is irrelevant, where $m = 2k$.*

(2) *Suppose that V admits a non-degenerate G-invariant symmetric bilinear form (i.e. ϕ is orthogonal) and $m > 1$. Then the representation $\psi_k\psi_l$ is irrelevant if $k + l > m$.*

(3.20) LEMMA ([S5]). *The representation ψ_k is irrelevant if and only if*

$$\Lambda^k V^* \subseteq \sum_{1 \leq i < k} \Lambda^i(V^*)^G \wedge \Lambda^{k-i}(V^*).$$

In particular, ψ_m is irrelevant if and only if $\Lambda^i(V^*)^G \neq 0$ for some i with $1 \leq i < m$. □

(3.21) We now apply the results above to the determination of the FMT for G_2 (see (3.24) for B_3). Let ϕ_1 (resp. ϕ_2) denote the irreducible 7-dimensional (resp. adjoint) representation of G_2. If $a, b \in \mathbb{N}$, let $\phi_1^a\phi_2^b$ or $\phi_1^a\phi_2^b(\mathsf{G}_2)$ or $(\phi_1^a\phi_2^b, \mathsf{G}_2)$ denote the Cartan component in $S^a\phi_1 \otimes S^b\phi_2$. The $\phi_1^a\phi_2^b$ exhaust the irreducible representations of G_2. We use θ_j to denote a trivial representation of dimension j.

(3.22) PROPOSITION. *The representation ψ_5 is irrelevant for (ϕ_1, G_2).*

Proof. From ([S1] Table 5b) we have

$$(1)\quad S^2\phi_1 = \phi_1^2 + \theta_1, \qquad \Lambda^2\phi_1 = \phi_2 + \phi_1$$
$$\Lambda^3\phi_1 = \phi_1^2 + \phi_1 + \theta_1, \qquad \Lambda^4\phi_1 = \Lambda^3\phi_1.$$

Let $\beta \in (\Lambda^3 V^*)^G$ and $\gamma \in (\Lambda^4 V^*)^G$ be as in (1.13), where $(V, G) = (\phi_1, \mathrm{G}_2)$. Let L_β denote left multiplication by β in $\Lambda^\cdot V^*$. Clearly $L_\beta(\gamma) = \beta \wedge \gamma$ is a generator of $\Lambda^7 V^* \simeq \theta_1$. We show that $L_\beta(\Lambda^2 V^*) = \Lambda^5 V^*$, and then (3.20) establishes the proposition.

Recall that $(V, G) \simeq (V^*, G)$ is orthogonal. By (1), $(\Lambda^2 V^*, G) = (W_1 \oplus W_2, G)$ where $(W_i, G) = (\phi_i, G)$. Let b be a non-degenerate $SO(V)$-invariant symmetric bilinear form on $\Lambda^2 V$. Since the (W_i, G) are non-isomorphic, b decomposes as a direct sum $b_1 \oplus b_2$ where $b_i \in (S^2 W_i)^G$ is non-degenerate. Let σ denote the orthogonal projection from $S^2(\Lambda^2 V^*)$ to $\Lambda^4 V^*$. Then $\sigma(b) = 0$ since $(\Lambda^4 V^*)^{SO(V)} = 0$. If $\sigma(b_1) = 0$ or $\sigma(b_2) = 0$, then both are zero, which would imply that γ is zero, since γ must be a linear combination of $\sigma(b_1)$ and $\sigma(b_2)$. Hence $\sigma(b_i) \neq 0$; $i = 1, 2$.

Suppose that $0 = L_\beta(W_1) \subseteq \Lambda^5 V^*$. Since γ lies in the image of $W_1 \otimes W_1$ in $\Lambda^4 V^*$, it would follow that $L_\beta(\gamma) = 0$, a contradiction. By Schur's lemma, $L_\beta : W_1 \to \Lambda^5 V^*$ is injective. Similarly, $L_\beta : W_2 \to \Lambda^5 V^*$ is injective, and hence $L_\beta : \Lambda^2 V^* \to \Lambda^5 V^*$ is an isomorphism. □

(3.23) FIRST MAIN THEOREM FOR G_2. *Let $(V, G) = (\phi_1, \mathrm{G}_2)$. Then a minimal generating set of R transforms by ψ_1^2, ψ_3 and ψ_4 with corresponding highest weight generators α_{11}, β_{123} and γ_{1234}, respectively. In other words, for any $n \in \mathbb{N}$, $R(n)$ is generated by the α_{ij}, β_{ijk} and γ_{ijkl} of* (1.12).

Proof. Suppose that part of a minimal generating set of R transforms by $\psi_{(a)}$. If $\mathrm{ht}(a) \leq 3$, then $\psi_{(a)} = \psi_1^2$ or ψ_3, since in [S1] we showed that $R(3)$ is a polynomial algebra on $\alpha_{11}, \ldots, \alpha_{33}$ and β_{123}. By (3.19.2) and (3.22) the remaining possibility is $\psi_{(a)} = \psi_{(b)}\psi_4$ where $\mathrm{ht}(b) \leq 3$. In this case, a corresponding highest weight generator h lies in $\mathbb{C}[4V]^G$ and has degree exactly one in the last copy of V (see (3.15)). Thus h corresponds to a covariant in $\mathbb{C}[3V]$ transforming by $V = \phi_1$. By ([S2] Table 4, Theorem 1.1), as a module over $\mathbb{C}[3V]^G$, the ϕ_1-covariants of $\mathbb{C}[3V] \simeq S^\cdot(3\phi_1)$ are a free module on 7 generators. It is easy to see that there are three generators each in $S^1(3\phi_1)$ and in $3\Lambda^2\phi_1 \subseteq S^2(3\phi_1)$, and that there is a generator in $\Lambda^3\phi_1 \subseteq S^3(3\phi_1)$ (see (3.22.1)). Hence $\deg(a) \leq 4$, i.e. $\psi_{(a)} = \psi_4$. □

(3.24) We now consider the case $G = \mathsf{B}_3$. Let ϕ_1 denote the usual representation of $\mathsf{B}_3 = \mathrm{Spin}_7$ on $\mathbb{C}^7$, set $\phi_2 = \Lambda^2\phi_1$ and let ϕ_3 denote the (8-dimensional) spin representation. As in (3.21), we denote the irreducible representations of B_3 by $\phi_1^a\phi_2^b\phi_3^c$ or $(\phi_1^a\phi_2^b\phi_3^c, \mathsf{B}_3)$, etc. where $\phi_1^a\phi_2^b\phi_3^c$ is the Cartan component of $S^a\phi_1 \otimes S^b\phi_2 \otimes S^c\phi_3$; $a, b, c \in \mathbb{N}$.

(3.25) PROPOSITION. *The representation* ψ_6 *is irrelevant for* (ϕ_3, B_3).

Proof. We have ([S1] Table 3b)

$$
\begin{aligned}
(1)\quad & S^2\phi_3 = \phi_3^2 + \theta_1, && \Lambda^2\phi_3 = \phi_2 + \phi_1 \\
& \Lambda^3\phi_3 = \phi_1\phi_3 + \phi_3, && \Lambda^4\phi_3 = \phi_3^2 + \phi_1^2 + \phi_1 + \theta_1 \\
& \phi_3^2 = \Lambda^3\phi_1. &&
\end{aligned}
$$

Let $\epsilon \in (\Lambda^4 V^*)^G$ be as in (2.13), where $(V, G) = (\phi_3, \mathsf{B}_3)$. Exactly as in (3.22) one can show that $L_\epsilon \Lambda^2 V^* = \Lambda^6 V^*$. $\square$

(3.26) FIRST MAIN THEOREM FOR B_3. *Let* $(V, G) = (\phi_3, \mathsf{B}_3)$. *Then a minimal generating set of* R *transforms by* ψ_1^2 *and* ψ_4 *with corresponding highest weight generators* δ_{11} *and* ϵ_{1234}, *respectively. In other words, for any* $n \in \mathbb{N}$, $R(n)$ *is generated by the* δ_{ij} *and* ϵ_{ijkl} *of* (2.8).

Proof. Let $\psi_{(a)}$ correspond to a subset of a minimal generating set of R. In [S1] we showed that the δ_{ij} and ϵ_{1234} generate $R(4)$, hence by (3.19.2) and (3.25) we may assume that $\psi_{(a)} = \psi_{(b)}\psi_5$ where $\mathrm{ht}(b) \leq 3$. A corresponding highest weight generator h then lies in $\mathbb{C}[5V]^G$ and is skew and of degree 1 in the last two copies of V. Thus h corresponds to a covariant in $\mathbb{C}[3V] \simeq S^{\cdot}(3\phi_3)$ transforming by ϕ_1 or ϕ_2 (since $\Lambda^2\phi_3 = \phi_1 + \phi_2$). From ([S2] Table 2, Theorem 1.1) we obtain that the ϕ_1-covariants are a free module on three generators, which are easily seen to be the three copies of ϕ_1 in $3\Lambda^2\phi_3 \subseteq S^2(3\phi_3)$. In this case, then, $\deg \psi_{(a)} \leq 4$, which is impossible.

The ϕ_2-covariants are a free module on six generators [S2]. There are three generators in $3\Lambda^2\phi_3 \subseteq S^2(3\phi_3)$ (which again lead to the contradiction $\deg \psi_{(a)} \leq 4$). Now $\phi_3^2 = \Lambda^3\phi_1$, hence $\phi_3^2 \otimes \phi_1 = \Lambda^3\phi_1 \otimes \phi_1$ contains a copy of $\Lambda^2\phi_1 = \phi_2$ (by contraction). Thus there are three copies of ϕ_2 in $3(S^2\phi_3 \otimes \Lambda^2\phi_3) \subseteq S^4(3\phi_3)$, and these are the other three generators of the ϕ_2-covariants. Hence $\deg \psi_{(a)} = 6$ and $\psi_{(a)} = \psi_1\psi_5$.

Now $\psi_1\psi_5 \subseteq \psi_1 \otimes \psi_5$ and $\Lambda^5\phi_3 = \Lambda^3\phi_3 = \phi_1\phi_3 + \phi_3$. Hence our highest weight generator h is the contraction of the first copy of $\phi_3 = V$ with the copy lying in the exterior product of the five copies of V. But (3.25) implies that

$L_\epsilon(V^*)$ is the copy of V^* in $\Lambda^5 V^*$, hence h is not part of a minimal generating set. (Brutally, h is a multiple of $\delta_{11}\epsilon_{2345} - \delta_{12}\epsilon_{1345} + \cdots + \delta_{15}\epsilon_{1234}$.) □

§4. Second main theorems

(4.0) We discuss some general facts concerning second main theorems. In §5 we apply them to the cases of G_2 and B_3.

Let $\phi = (V, G)$ and $m = \dim V$ as in (3.8)–(3.15). Let $R = S^\cdot(\psi_1 \otimes V^*)^G$ be minimally generated by subspaces $\psi_{(a^i)} \otimes f_i$, $i = 1, \ldots, p$. Let $T = S^\cdot(\bigoplus \psi_{(a^i)})$, and let $\pi: T \to R$ be the canonical (given our choice of the f_i) GL-equivariant surjection. Define $T(n) = S^\cdot(\bigoplus \psi_{(a^i)}(n)) \subseteq T$. Then π induces $\pi(n): T(n) \to R(n)$, and $I(n) = \operatorname{Ker} \pi(n)$ lies in $I = \operatorname{Ker} \pi$. We give elements of $\psi_{(a^i)} \supseteq \psi_{(a^i)}(n)$ their natural degree ($= \deg(a^i)$) so that π and $\pi(n)$ are degree preserving homomorphisms of graded algebras. We use notation $T(G)$, $I(n, \phi)$, etc. if it is necessary to emphasize the group or representation involved.

(4.1) Given $\pi: T \to R$ a relation is, of course, an element of I. It will be convenient for us to use the same term to apply to irreducible subspaces of I: A *relation* (of $\pi: T \to R$) is an equivariant injection $\eta: \psi_{(b)} \to I$ for some (b). Note that $\eta: \psi_{(b)} \to T$ has image in I if and only if $\eta(h) \in I$ where h is a highest weight vector of $\psi_{(b)}$. We call such elements $\eta(h) \in I$ *highest weight relations*. We also refer to equivariant injections $\sigma: \psi_{(c)}(n) \to I(n)$ as relations (of $\pi(n): T(n) \to R(n)$). A relation $\eta: \psi_{(b)} \to I$ induces relations $\eta(n): \psi_{(b)}(n) \to I(n)$ by restriction, and if $\sigma: \psi_{(c)}(n) \to I(n)$ is a relation with $n \geq \operatorname{ht}(c)$, then there is a unique relation $\eta: \psi_{(c)} \to I$ with $\eta(n) = \sigma$. We use the notation $(\psi_{(b)}, \eta)$ to denote relations $\eta: \psi_{(b)} \to I$, and similarly for $I(n)$.

(4.2) Let $\eta: \psi_{(b)} \to T$ be an equivariant inclusion. If $\operatorname{ht}(b) > m$, then $\operatorname{Im} \eta \subseteq I$ by (3.11), hence η is a relation. We call such relations *general*. General relations are ones which arise for dimensional reasons. We call a relation $\eta: \psi_{(b)} \to I$ *special* if $\operatorname{ht}(b) \leq m$.

(4.3) It is now natural to consider a second main theorem for ϕ to be a collection of relations $(\psi_{(b^j)}, \eta_j)$ whose images $\eta_j(\psi_{(b^j)})$ generate I. Equivalently, the images $\eta_j(n)(\psi_{(b^j)}(n))$ should generate $I(n)$ for all n.

(4.4) THEOREM ([S5]). *Let V, G, $T = S^\cdot(\bigoplus_{i=1}^p \psi_{(a^i)})$ etc. be as above. Then* *there are a finite number of relations* $(\psi_{(b^j)}, \eta_j)$, $j = 1, \ldots, q$ *such that*

(1) $\bigoplus_j \eta_j(\psi_{(b^j)})$ *minimally generates I.*

(2) $\operatorname{ht}(b^j) \leq m + \max \operatorname{ht}(a^i)$, $j = 1, \ldots, q$.

(4.5) COROLLARY. *I is generated by any collection of relations* $(\psi_{(c^j)}, \sigma_j)$ *such that* $\sum \sigma_j(k)(\psi_{(c^j)}(k))$ *generates* $I(k)$ *for some* $k \geq m + \max \deg(a^i)$. *In particular,* $k = 2m$ *suffices.*

(4.6) Let Ht_r denote the direct sum of the subspaces of T transforming by representations of height $\geq r$. By (3.5), Ht_r is an ideal of T. Let Spc denote the subideal of I generated by the special relations. Then $I = Spc + Ht_{m+1}$.

Assume now that generators of Spc (i.e. of $I(m)$) are known, and consider the problem of finding generators of Ht_{m+1}, or more generally, of some Ht_r, $r \in \mathbb{N}$.

(4.7) THEOREM ([S5]). *Let* $T = S^{\cdot}(\bigoplus_{i=1}^{p} \psi_{(a^i)})$ *and* $r \in \mathbb{N}$. *Then the generators of* Ht_r *lie in the sum of subspaces* $S^{d_1}\psi_{(a^1)} \otimes \cdots \otimes S^{d_p}\psi_{(a^p)}$ *where* $\sum d_i \leq 1 + \max\{0, r - t\}$, *and* $t = \max\{\operatorname{ht}(a^i) : d_i > 0\}$.

(4.8) COROLLARY ([V2]). *The generators of* Ht_r *lie in the sum of subspaces* $S^{d_1}\psi_{(a^1)} \otimes \cdots \otimes S^{d_p}\psi_{(a^p)}$ *where* $\sum d_i \leq r$.

(4.9) EXAMPLE. Let $(V, G) = (\mathbb{C}^m, O_m)$ as in (3.16). Then $T = S^{\cdot}\psi_1^2$. By [W] (or lemma (7.3) below) there are no special relations, and by (3.5), representations occurring in $S^j\psi_1^2$ have height $\leq j$. Thus (4.8) implies that $I = Ht_{m+1}$ is generated by the height $m+1$ representations in $S^{m+1}(\psi_1^2)$, namely $\psi_{m+1}^2 \subseteq S^{m+1}(\psi_1^2)$ ([S1] Prop. 2.4). A highest weight relation is $\det(p_{ij})_{i,j=1}^{m+1}$ where the p_{ij} are the basis of $\psi_1^2(m+1)$ given in (3.16). Note that theorem (4.4) also shows that I is generated by relations of height $\leq m+1$.

§5. Some relations

(5.0) We first consider the case $(V, G) = (\phi_1, \mathrm{G}_2)$. Then there is a surjection $\pi: T \to R$ where $T = S^{\cdot}(\psi_1^2 + \psi_3 + \psi_4)$. We exhibit six irreducible subspaces $\mathrm{Rel}_1, \ldots, \mathrm{Rel}_6$ of $I = \operatorname{Ker} \pi$. We use them to show that I is generated by Rel_6 and relations of height ≤ 6. In §§9 and 10 we show that $\mathrm{Rel}_1(6), \ldots, \mathrm{Rel}_5(6)$ generate $I(6)$, giving our SMT for G_2. In §6 we show that $\mathrm{Rel}_1, \ldots, \mathrm{Rel}_6$ follow from the identities of the Cayley algebra. We derive similar results for (ϕ_3, B_3) beginning in (5.11).

(5.1) Let $\eta : \psi_{(a)} \to S^b\psi_1^2 \otimes S^c\psi_3 \otimes S^d\psi_4$ be an equivariant inclusion. In the cases we consider, η will almost always be determined up to scalars by (a), b, c and d (we will note the exceptions), so we will use the notation $\psi_{(a)}(\alpha^b\beta^c\gamma^d)$ to denote $\eta(\psi_{(a)})$, and we will denote a highest weight vector of $\eta(\psi_{(a)})$ by $\lambda(\psi_{(a)}(\alpha^b\beta^c\gamma^d))$. For example, there are copies $\psi_1\psi_5(\alpha\gamma)$ and $\psi_1\psi_5(\beta^2)$ of $\psi_1\psi_5$ in T, and one can compute that they have highest weight vectors:

$$\lambda(\psi_1\psi_5(\alpha\gamma)) = \alpha_{11}\gamma_{2345} - \alpha_{12}\gamma_{1345} + \alpha_{13}\gamma_{1245} - \alpha_{14}\gamma_{1235} + \alpha_{15}\gamma_{1234}.$$
$$\lambda(\psi_1\psi_5(\beta^2)) = \beta_{123}\beta_{145} - \beta_{124}\beta_{135} + \beta_{125}\beta_{134}.$$

(5.2) Let $r(\psi_{(a)})$ (resp. $t(\psi_{(a)})$) denote the multiplicity of $\psi_{(a)}$ in R (resp. T). Then $\psi_{(a)}$ occurs in I with multiplicity $t(\psi_{(a)}) - r(\psi_{(a)})$. Consider, for example, the cases of $\psi_1\psi_5$ and ψ_4^2. Using the Littlewood–Richardson rule [McD] and ([S1] Table 2b) one can verify that $\psi_1\psi_5(\alpha\gamma)$ and $\psi_1\psi_5(\beta^2)$ account for all the occurrences of $\psi_1\psi_5$ in T, i.e. $t(\psi_1\psi_5) = 2$. Similarly, there are occurrences $\psi_4^2(\alpha^4)$, $\psi_4^2(\alpha\beta^2)$ and $\psi_4^2(\gamma^2)$ of ψ_4^2, and $t(\psi_4^2) = 3$.

To compute the multiplicities $r(\psi_{(a)})$ we use the fact that $r(\psi_{(a)}) = \dim \psi_{(a)}(V^*)^G = \dim \psi_{(a)}(V)^G$ (use 3.11). Now $\psi_1 \otimes \psi_5 = \psi_1\psi_5 + \psi_6$ where $(\psi_6(V), G) = (\phi_1, \mathrm{G}_2)$ and $(\psi_5(V), G) = (\Lambda^2\phi_1, \mathrm{G}_2) = (\phi_2 + \phi_1, \mathrm{G}_2)$. Thus $\dim \psi_1\psi_5(V)^G = \dim(\phi_1 \otimes (\phi_1 + \phi_2))^{\mathrm{G}_2} - \dim \phi_1^{\mathrm{G}_2} = 1 - 0 = 1$ and hence $r(\psi_1\psi_5) = 1$. Also, $r(\psi_4^2) = \dim \psi_4^2(V)^G = \dim \psi_3^2(V)^G = r(\psi_3^2) = t(\psi_3^2)$ since $\mathbb{C}[3V]^G$ is regular. Clearly $t(\psi_3^2) = 2$, hence $r(\psi_4^2) = 2$.

(5.3) Our computations show that I contains single copies of $\psi_1\psi_5$ and ψ_4^2. We can specify these subspaces by computing the corresponding highest weight relations, which we now do for the case of $\psi_1\psi_5$. Note that the highest weight relation must be a linear combination of the highest weight vectors $\lambda(\psi_1\psi_5(\alpha\gamma))$ and $\lambda(\psi_1\psi_5(\beta^2))$ given in (5.1).

Let 1, i, j, k denote the usual basis of the quaternions, and let $1_0, i_0, \ldots$ denote the Cayley numbers $(0, 1), (0, i), \ldots$ (see (1.8)). As in (1.12), the α_{ij}, etc. are functions of Cayley numbers $x_1, x_2, \ldots$. Let $x_1 = i$, $x_2 = j$, $x_3 = k$, $x_4 = 1_0$, and $x_5 = i_0$. Then $\lambda(\psi_1\psi_5(\alpha\gamma))$ has value -1 and $\lambda(\psi_1\psi_5(\beta^2))$ has value 1. Hence $\lambda(\psi_1\psi_5(\alpha\gamma)) + \lambda(\psi_1\psi_5(\beta^2))$ is our highest weight relation.

(5.4) Using the techniques above we computed the highest weight relations given in (5.4.1) through (5.4.10) below. They are presented as linear combinations of highest weight vectors $\lambda(\psi_{(a)}(\alpha^b\beta^c\gamma^d))$ which we list in Table I. We use the notation Rel_j or $\mathrm{Rel}_j(\mathrm{G}_2)$ (resp. $\mathrm{Rel}_j(n)$ or $\mathrm{Rel}_j(n, \mathrm{G}_2)$) to refer to the subrepresentation of I (resp. $I(n)$) with highest weight vector given in (5.4.j).

Table I

1	$\lambda(\psi_1\psi_5(\beta^2)) = \sum_{3\le i<j\le 5} (-1)^{i+j+1}\beta_{1ij}\hat{\beta}_{ij}$
2	$\lambda(\psi_1\psi_5(\alpha\gamma)) = \sum_{i=1}^{5} (-1)^{i+1}\alpha_{1i}\hat{\gamma}_i$
3	$\lambda(\psi_2\psi_5(\beta\gamma)) = \sum_{i=3}^{5} (-1)^{i}\beta_{12i}\hat{\gamma}_i$
4	$\lambda(\psi_2\psi_5(\alpha^2\beta)) = \sum_{1\le i<j\le 5} (-1)^{i+j}(\alpha_{1i}\alpha_{2j} - \alpha_{1j}\alpha_{2i})\hat{\beta}_{ij}$
5	$\lambda(\psi_1\psi_6(\beta\gamma)) = \sum_{2\le i<j\le 6} (-1)^{i+j}\beta_{1ij}\hat{\gamma}_{ij}$
6	$\lambda(\psi_4^2(\gamma^2)) = \gamma_{1234}^2$
7	$\lambda(\psi_4^2(\alpha\beta^2)) = \sum_{i,j=1}^{4} (-1)^{i+j}\alpha_{ij}\hat{\beta}_i\hat{\beta}_j$
8	$\lambda(\psi_4^2(\alpha^4)) = \det(\alpha_{ij})_{i,j=1}^4$
9	$\lambda(\psi_2\psi_6(\gamma^2)) = \sum_{3\le i<j\le 5} (-1)^{i+j}\gamma_{12ij}\hat{\gamma}_{ij}$
10	$\lambda(\psi_2\psi_6(\alpha^2\gamma)) = \sum_{1\le i<j\le 6} (-1)^{i+j}(\alpha_{1i}\alpha_{2j} - \alpha_{1j}\alpha_{2i})\hat{\gamma}_{ij}.$
11	$\lambda(\psi_8(\gamma^2)) = \sum_{1\le i<j<k<l\le 8} (-1)^{i+j+k+l}\gamma_{ijkl}\hat{\gamma}_{ijkl}.$
12	$\lambda(\psi_5^2(\alpha^5)) = \det(\alpha_{ij})_{i,j=1}^5$
13	$\lambda(\psi_5^2(\alpha\gamma^2)) = \sum_{i,j=1}^{5} (-1)^{i+j}\alpha_{ij}\hat{\gamma}_i\hat{\gamma}_j$
14	$\lambda(\psi_5^2(\alpha^2\beta^2)) = \sum_{1\le i<j\le 5}\ \sum_{1\le k<l\le 5} (-1)^{i+j+k+l}(\alpha_{ki}\alpha_{lj} - \alpha_{li}\alpha_{kj})\hat{\beta}_{ij}\hat{\beta}_{kl}$
15	$\lambda(\psi_4\psi_5(\alpha\beta\gamma)) = \sum_{i=1}^{4}\sum_{j=1}^{5} (-1)^{i+j}\alpha_{ij}\hat{\beta}_i\hat{\gamma}_j$
16	$\lambda(\psi_3\psi_7(\alpha^3\gamma)) = \sum_{1\le i<j<k\le 7} (-1)^{i+j+k}\hat{\gamma}_{ijk}\det(\alpha_{pq})_{p=1,2,3}^{q=i,j,k}$
17	$\lambda(\psi_3\psi_7(\beta^2\gamma)) = \beta_{123}\sum_{1\le i<j<k\le 7} (-1)^{i+j+k}\beta_{ijk}\hat{\gamma}_{ijk}$

where $\psi_3\psi_7 \subseteq \psi_3^2 \otimes \psi_4 \subseteq S^2\psi_3 \otimes \psi_4$.

In Table I, we use $\hat{\gamma}_i$ (resp. $\hat{\gamma}_{ij}$, $i<j$) to denote γ_{abcd} where $a<b<c<d$ and $\{a, b, c, d, i\} = \{1, 2, 3, 4, 5\}$ (resp. $\{a, b, c, d, i, j\} = \{1, 2, 3, 4, 5, 6\}$). Symbols $\hat{\gamma}_{ijk}$, $\hat{\beta}_i$ and $\hat{\beta}_{ij}$ have analogous meanings.

(5.4.1) $\lambda(\psi_1\psi_5(\beta^2)) + \lambda(\psi_1\psi_5(\alpha\gamma))$.

(5.4.2) $\lambda(\psi_2\psi_5(\beta\gamma)) + \lambda(\psi_2\psi_5(\alpha^2\beta))$.

(5.4.3) $\lambda(\psi_1\psi_6(\beta\gamma))$.

(5.4.4) $\lambda(\psi_4^2(\gamma^2))+\lambda(\psi_4^2(\alpha\beta^2))-\lambda(\psi_4^2(\alpha^4))$.

(5.4.5) $\lambda(\psi_2\psi_6(\gamma^2))+\lambda(\psi_2\psi_6(\alpha^2\gamma))$.

(5.4.6) $\lambda(\psi_8(\gamma^2))$.

(5.4.7) $\lambda(\psi_5^2(\alpha^5))-\lambda(\psi_5^2(\alpha\gamma^2))$.

(5.4.8) $2\lambda(\psi_5^2(\alpha^5))-\lambda(\psi_5^2(\alpha^2\beta^2))$.

(5.4.9) $\lambda(\psi_4\psi_5(\alpha\beta\gamma))$.

(5.4.10) $7\lambda(\psi_3\psi_7(\alpha^3\gamma))-\lambda(\psi_3\psi_7(\beta^2\gamma))$,

where $\psi_3\psi_7(\beta^2\gamma)\subseteq\psi_3^2\otimes\psi_4\subseteq S^2\psi_3\otimes\psi_4$.

(5.5) Let $J(\mathrm{G}_2)$ or just J (resp. $J(n, \mathrm{G}_2)$ or just $J(n)$) denote the ideal in T (resp. $T(n)$) generated by $\mathrm{Rel}_1,\ldots,\mathrm{Rel}_6$ (resp. $\mathrm{Rel}_1(n),\ldots,\mathrm{Rel}_6(n)$). Now we can state, but not yet prove:

(5.6) SECOND MAIN THEOREM FOR G_2. $I(\mathrm{G}_2)=J(\mathrm{G}_2)$.

Note that $\mathrm{Rel}_1,\ldots,\mathrm{Rel}_5$ are special relations, while Rel_6 is general. Thus "most" of the relations for G_2 are special. For the classical groups most, if not all, relations are general.

(5.7) At first glance, the relations Rel_j are somewhat bewildering. However, one can make the following rough statements immediately. Since $S^2\psi_4=\psi_4^2+\psi_2\psi_6+\psi_8$, relations Rel_4, Rel_5 and Rel_6 imply that the γ invariants all satisfy quadratic equations over the subalgebra generated by the α and β invariants. In other words, any monomial in the α's, β's and γ's can be reduced to ones of degree 0 or 1 in the γ's. Note that theorem (3.19) says that there must be relations like Rel_4 and Rel_5 showing that $\psi_4^2(\gamma^2)$ and $\psi_2\psi_6(\gamma^2)$ are in the ideal of $\psi_1^2(\alpha)$.

Taking $\mathrm{Rel}_1,\ldots,\mathrm{Rel}_5$ into account one can see that, if $n\le 6$, there is a surjection

(5.8) $$\bigoplus_{\substack{2i+3j+4k=d\\ k\le 1}} S^i\psi_1^2(n)\otimes\psi_3^j\psi_4^k(n)\to R(n)_d,$$

where $R(n)_d$ denotes the elements of $R(n)$ of degree d. The mapping in (5.8) is not injective, and in §§9–10 we will see that Rel_8 and Rel_9 account for the kernel.

The relations $\mathrm{Rel}_7,\ldots,\mathrm{Rel}_{10}$ will be useful in our proof of theorem (5.6). They are consequences of $\mathrm{Rel}_1,\ldots,\mathrm{Rel}_6$, i.e.

(5.9) PROPOSITION. Rel_7, Rel_8, Rel_9 *and* Rel_{10} *are in* J.

Proof. We first consider Rel_9. From Littlewood–Richardson we know that there is a unique copy of $\psi_4\psi_5(\alpha\beta\gamma) \subseteq \psi_3(\beta) \otimes \psi_1\psi_5(\alpha\gamma)$. Thus tensoring Rel_1 with $\psi_3(\beta)$ and considering the subspace transforming by $\psi_4\psi_5$ we obtain a relation involving $\psi_4\psi_5(\alpha\beta\gamma)$ (in a non-trivial way) and copies of $\psi_4\psi_5$ in $S^3\psi_3(\beta)$. But $S^3\psi_3$ contains no copy of $\psi_4\psi_5$ (see [S1] Table 2b), hence $\psi_4\psi_5(\alpha\beta\gamma) \subseteq J$.

Similarly, the subspaces transforming by ψ_5^2 in $\mathrm{Rel}_1 \otimes \psi_4(\gamma)$, $\mathrm{Rel}_2 \otimes \psi_3(\beta)$ and $\mathrm{Rel}_4 \otimes \psi_1^2(\alpha)$ give relations indicating that $\psi_5^2(\alpha^5)$, $\psi_5^2(\alpha\gamma^2)$, $\psi_5^2(\alpha^2\beta^2)$ and $\psi_5^2(\beta^2\gamma)$ all have the same image in R. Thus Rel_7 and Rel_8 are in J. The subspace of $\mathrm{Rel}_3 \otimes \psi_3(\beta)$ transforming by $\psi_3\psi_7$ is a nontrivial relation between the copies of $\psi_3\psi_7$ in $\psi_3^2(\beta^2) \otimes \psi_4(\gamma)$ and $\psi_1\psi_5(\beta^2) \otimes \psi_4(\gamma)$ (neither copy is a relation by itself.) Then from $\mathrm{Rel}_1 \otimes \psi_4(\gamma)$ and $\mathrm{Rel}_5 \otimes \psi_1^2(\alpha)$ we see that $\psi_3\psi_7(\alpha^3\gamma)$ and $\psi_3\psi_7(\beta^2\gamma) \subseteq \psi_1\psi_5(\beta^2) \otimes \psi_4(\gamma)$ have the same image in R. Hence $\mathrm{Rel}_{10} \subseteq J$. □

(5.10) THEOREM. *$I(\mathrm{G}_2)$ is generated by* Rel_6 *and relations of height* ≤ 6. *Hence $I(\mathrm{G}_2) = J(\mathrm{G}_2)$ if $I(6, \mathrm{G}_2) = J(6, \mathrm{G}_2)$.*

Proof. From theorem (4.7) and Littlewood–Richardson, one sees that, modulo J, the ideal Ht_7 is generated by:

$$\psi_7^2(\alpha^7),\ \psi_4\psi_7(\alpha^4\beta),\ \psi_3\psi_7(\alpha^3\gamma) \quad \text{and} \quad \psi_7(\beta\gamma).$$

Note that a highest weight vector of $\psi_7(\beta\gamma)$ is the determinant $\det \in R(7)$. As in the proof of (5.9), the ψ_7^2 subrepresentations of $\mathrm{Rel}_{10} \otimes \psi_4(\gamma)$ and $\mathrm{Rel}_7 \otimes \psi_2^2(\alpha^2)$ show that, $\bmod J$, $\psi_7^2(\alpha^7)$ has $\det^2$ as highest weight vector. Similarly, $\mathrm{Rel}_4 \otimes \psi_3(\beta)$, $\mathrm{Rel}_1 \otimes \psi_1\psi_4(\alpha\beta)$ and $\mathrm{Rel}_2 \otimes \psi_2^2(\alpha^2)$ show that $\psi_4\psi_7(\alpha^4\beta)$ has highest weight vector $\gamma_{1234}\det$, $\bmod J$, and Rel_{10} shows that $\psi_3\psi_7(\alpha^3\gamma)$ has highest weight vector $\beta_{123}\det$, $\bmod J$. Hence any representation in T/J of height ≥ 7 is in the ideal of $\psi_7(\beta\gamma)$. In particular, any relation of height 7 is a consequence of relations of height ≤ 6. Also, one easily sees that elements of height >7 in T/J lie in the ideal of

$$\psi_1\psi_8(\alpha\beta\gamma) \subseteq \psi_3(\beta) \otimes \psi_1\psi_5(\alpha\gamma).$$

Now $\mathrm{Rel}_1 \otimes \psi_3(\beta)$ shows that $\psi_1\psi_8(\alpha\beta\gamma)$ is a sum of copies of $\psi_1\psi_8$ lying in $S^3\psi_3(\beta)$, $\bmod J$. But one can check that $S^3\psi_3$ contains no copies of $\psi_1\psi_8$, hence $\psi_1\psi_8(\alpha\beta\gamma) \subseteq J$. □

(5.11) We now describe analogous results for the case of B_3. We omit all proofs since they are similar and even easier than in the case of G_2.

Let $(V, G) = (\phi_3, \mathsf{B}_3)$. Then there is a surjection $\pi: T \to R$ with kernel I, where $T = S^{\cdot}(\psi_1^2 + \psi_4)$. As in (5.1), irreducible representations $\psi_{(a)}$ in $S^b\psi_1^2 \otimes S^c\psi_4$ are denoted $\psi_{(a)}(\delta^b\epsilon^c)$, and their highest weight vectors are denoted $\lambda(\psi_{(a)}(\delta^b\epsilon^c))$. Using the techniques of (5.2) and (5.3) we found relations with the following highest weight vectors:

(5.11.1) $\lambda(\psi_2\psi_6(\epsilon^2)) + \lambda(\psi_2\psi_6(\delta^2\epsilon))$.

(5.11.2) $\lambda(\psi_5^2(\delta^5)) - \lambda(\psi_5^2(\delta\epsilon^2))$.

(5.11.3) $\lambda(\psi_{12}(\epsilon^3))$.

(5.11.4) $\lambda(\psi_1\psi_9(\delta\epsilon^2))$.

The expressions for $\lambda(\psi_2\psi_6(\epsilon^2))$, $\lambda(\psi_2\psi_6(\delta^2\epsilon))$, $\lambda(\psi_5^2(\delta^5))$ and $\lambda(\psi_5^2(\delta\epsilon^2))$ are as in Table I, just replace α's by δ's and γ's by ϵ's. We leave it to the reader to write out expressions for $\lambda(\psi_{12}(\epsilon^3))$ and $\lambda(\psi_1\psi_9(\delta\epsilon^2))$.

(5.12) We use Rel_j or $\mathrm{Rel}_j\,(\mathsf{B}_3)$ to refer to the relations with highest weight vector (5.11.j), $1 \le j \le 4$; and similarly for $\mathrm{Rel}_j\,(n)$, etc. Let $J = J(\mathsf{B}_3)$ denote the ideal in T generated by $\mathrm{Rel}_1, \ldots, \mathrm{Rel}_4$; and similarly define $J(n) = J(n, \mathsf{B}_3)$.

(5.13) SECOND MAIN THEOREM FOR B_3. $I(\mathsf{B}_3) = J(\mathsf{B}_3)$.

In §§8–10 we show that $I(7) = J(7)$. This is sufficient to establish the SMT for B_3 because of the following result.

(5.14) THEOREM. *$I(\mathsf{B}_3)$ is generated by $J(\mathsf{B}_3)$ and relations of height ≤ 7. Hence $I(\mathsf{B}_3) = J(\mathsf{B}_3)$ if $I(7, \mathsf{B}_3) = J(7, \mathsf{B}_3)$.* □

§6. Generators and relations via the Cayley algebra

(6.0) We show that the relations $\mathrm{Rel}_i\,(\mathsf{G}_2)$ and $\mathrm{Rel}_j\,(\mathsf{B}_3)$ are consequences of the identities satisfied by the Cayley algebra A. We also use these identities to show that all trace invariants of several copies of A' are generated by the ones of type α, β, γ. The only part of this section used in the rest of the paper is proposition (6.8) which we used in establishing (2.10).

(6.1) Let a, b, c be elements of A. Then polarizing the identities (1.2) and

(1.6) we obtain:

(6.2) $a(bc)+b(ac)=(ab+ba)c.$

(6.3) $(ab)c+(ac)b=a(bc+cb).$

(6.4) $ab+ba=2\operatorname{tr}(a)b+2\operatorname{tr}(b)a-2\operatorname{tr}(a\bar{b}).$

We use these identities to study monomial mappings from n copies of A' to A: Let $(x_1,\ldots,x_n)\in nA'$ be arbitrary. As in (1.12), we set $\alpha_{ij}=-\operatorname{tr}(x_ix_j)$, etc. Let e and f be two expressions which are sums of terms p and $\operatorname{tr}(p)q$ where p and q are products of the x_i's. We write $e\sim f$ if the identities of A show that $e-f$ equals a sum of products each of which has a factor α_{ij}. For example, (6.4) gives

(6.5) $x_i(x_jx_k)+(x_jx_k)x_i\sim -2\beta_{ijk},$

and from (6.2), (6.3) and (6.4) one derives that

(6.6) $x_i(x_jx_k)$ and $(x_ix_j)x_k$ are skew in i, j, and k, modulo $\sim$.

(6.7) Let $a, b, c\in A$. We use $[a, b]$ to denote $ab-ba$ and (a, b, c) to denote $(ab)c-a(bc)$. It follows from the alternative laws (1.2) that (a, b, c) is skew in its arguments ([Sf]).

In the following proposition, the terms $\hat{\beta}_i$, etc. have the same meaning as in Table I of §5.

(6.8) PROPOSITION.

(1) $\gamma_{1234}\sim\operatorname{tr}(x_1(x_2(x_3x_4))).$

(2) $x_1(x_2(x_3x_4))\sim\gamma_{1234}+\sum_{i=1}^{4}(-1)^i\hat{\beta}_ix_i.$

(3) $(x_1x_2)(x_3x_4)\sim\gamma_{1234}-\beta_{234}x_1+\beta_{134}x_2+\beta_{124}x_3-\beta_{123}x_4.$

(4) $x_1((x_2x_3)x_4)\sim-\gamma_{1234}-\beta_{234}x_1-\beta_{134}x_2+\beta_{124}x_3-\beta_{123}x_4.$

(5) $\sum_{i=1}^{5}(-1)^i\hat{\gamma}_ix_i\sim-\frac{1}{4}\sum_{1\le i<j\le 5}(-1)^{i+j}\hat{\beta}_{ij}[x_i, x_j],$

in fact there is equality.

Proof. It follows from (6.2) and (6.4) that, modulo $\sim$, $x_1(x_2(x_3x_4))$ is skew in its arguments, hence (1) holds. We also have

(6) $x_1(x_2(x_3x_4)) \sim -x_1((x_2x_3)x_4) - 2\beta_{234}x_1,$

(7) $x_1((x_2x_3)x_4) \sim -(x_2x_3)(x_1x_4) - 2\beta_{123}x_4,$

(8) $x_1(x_2(x_3x_4)) \sim (x_2x_3)(x_1x_4) + 2\beta_{123}x_4 - 2\beta_{234}x_1,$

(9) $x_2(x_1(x_4x_3)) \sim (x_1x_4)(x_2x_3) - 2\beta_{124}x_3 + 2\beta_{134}x_2,$

where (6) follows from (6.5) and (6.6), we obtain (7) from (6.2) and (6.5), equation (8) combines (6) and (7), and (9) results from (8) by interchanging x_1 with x_2 and x_3 with x_4. Now

(10) $(x_1x_4)(x_2x_3) + (x_2x_3)(x_1x_4) \sim 2\gamma_{1234}$

by (6.4), (8) and (1). The left hand sides of (8) and (9) are equal, mod $\sim$, hence (8), (9) and (10) combine to give (2). One then easily obtains (3) and (4) from (8) and (6) after switching indices.

To establish (5) we need the following identity for alternative algebras ([Sf] p. 79):

(11) $[a, (b, c, d)] = (bc, a, d) + (cd, a, b) + (db, a, c) \quad a, b, c, d \in A.$

Substitute $a = x_1x_2$, $b = x_3$, $c = x_4$, and $d = x_5$, and let *RHS* (resp. *LHS*) denote the resulting right hand side (resp. left hand side) of (11). Then

(12) $RHS = (x_3x_4, x_1x_2, x_5) + (x_4x_5, x_1x_2, x_3) + (x_5x_3, x_1x_2, x_4).$

(13) $LHS = [x_1x_2, (x_3, x_4, x_5)] \sim (x_1x_2)((x_3x_4)x_5) + (x_1x_2)((x_4x_5)x_3 + 2\beta_{345}) + (x_5(x_3x_4) + 2\beta_{345})(x_1x_2) + (x_3(x_4x_5))(x_1x_2).$

Skewing the term $(x_1x_2)((x_3x_4)x_5)$ of (13) with respect to x_1, x_2, x_3 and x_4 we obtain $\gamma_{1234}x_5$ by (3), and similarly for the terms $(x_1x_2)((x_4x_5)x_3)$, etc. Thus

(14) $\operatorname{skew} LHS \sim -\frac{4}{5}\sum_{i=1}^{5}(-1)^i\hat{\gamma}_i x_i - \frac{2}{5}\sum_{1\le i<j\le 5}(-1)^{i+j}\hat{\beta}_{ij}x_ix_j$

where we skew with respect to $x_1, \ldots, x_5$. Clearly skew $RHS = 0$, hence skew $LHS = 0$ and the right hand side of (14) is 0, yielding (5). □

Let $\mathbb{C}[nA']^{\mathrm{tr}}$ denote the subalgebra of $\mathbb{C}[nA']$ generated by all functions tr (p), where p is a product of the x_i. We give a "Cayley theoretic" proof of

(6.9) THEOREM. *$\mathbb{C}[nA']^{\mathrm{tr}}$ is generated by the invariants of type α, β and γ.*

Proof. Let p be a product of k of the variables $x_1, \ldots, x_n$. We may assume that the x_i occurring in p are distinct. If $k \leq 3$, then tr (p) is in the subalgebra of the α and β invariants. If $k > 4$, then using (6.8.2) through (6.8.4) one can easily show that p, modulo α, β, and γ invariants, is of the form qr where q or r is a product of k' of the x_i, with $4 \leq k' < k$. For example, $(x_1x_2)(x_3(x_4x_5)) = -x_3((x_1x_2)(x_4x_5))$ modulo α and β invariants. By induction, we reduce to the case $p = qr$ where q or r is a product of 4 of the x_i's. But (6.8.2), (6.8.3) and (6.8.4) and the corresponding conjugated equations show that any product of 4 of the x_i's is zero modulo the α, β and γ invariants. □

(6.10) THEOREM. *The relations* $\mathrm{Rel}_1(\mathrm{G}_2), \ldots, \mathrm{Rel}_6(\mathrm{G}_2)$ *are consequences of the identities of* A.

Proof. It is enough to derive the highest weight relations (5.4.1), ..., (5.4.6). Let LHS (resp. RHS) denote the left (resp. right) hand side of (6.8.5). Then (5.4.1) is the relation $-\mathrm{tr}\,(x_1(LHS)) + \mathrm{tr}\,(x_1(RHS)) = 0$, and $\mathrm{tr}\,((x_1x_2)LHS) - \mathrm{tr}\,((x_1x_2)RHS) = 0$ combined with (6.8.3) yields the relation $-\frac{1}{2}\lambda(\psi_2\psi_5(\beta\gamma)) \sim 0$. The representation $\psi_2\psi_5$ has multiplicity two in T, generated by $\psi_2\psi_5(\beta\gamma)$ and $\psi_2\psi_5(\alpha^2\beta)$. Thus the identities of A imply (5.4.2).

From (6.8.5) again, we obtain $\mathrm{tr}\,(x_1(LHS)x_6) - \mathrm{tr}\,(x_1(RHS)x_6) = 0$. Skewing over the indices 2 through 6 and using (6.8), one obtains that $\frac{1}{10}\lambda(\psi_1\psi_6(\beta\gamma)) \sim 0$. Since no subspace of $S^\cdot(\psi_1^2 + \psi_3 + \psi_4)$ of positive degree in ψ_1^2 transforms by $\psi_1\psi_6$, we see that (5.4.3) is obtainable from the identities of A.

Relations (5.4.4), (5.4.5) and (5.4.6) are obtained as follows: Let $a = x_1x_2$, $b = x_3x_4$, $c = x_5x_6$. Then by (6.8.3),

$$\gamma_{1234}\gamma_{1256} \sim \mathrm{tr}\,((ba)(ac))$$

and by (1.2) through (1.4),

$$\mathrm{tr}\,((ba)(ac)) = \mathrm{tr}\,(b(a(ac)) = \mathrm{tr}\,(b(a^2c)),$$

where $a^2 = (x_1x_2)(x_1x_2) \sim 0$. Thus $\gamma_{1234}\gamma_{1256} \sim 0$. The same argument works to show that $\gamma_{1234}^2 \sim 0$, and as above, we see that (5.4.4) and (5.4.5) follow from the identities of A.

Now

$$\gamma_{1234}\gamma_{5678} \sim \mathrm{tr}\,[\gamma_{1234}x_5(x_6(x_7x_8))].$$

Skewing in $x_1, \ldots, x_5$ and applying (6.8.5) we see that $\lambda(\psi_8(\gamma^2))$ must be an expression in the α and β invariants. But no such expression can transform by ψ_8, and we obtain (5.4.6). □

(6.11) THEOREM. *The relations* $\mathrm{Rel}_1(\mathsf{B}_3), \ldots, \mathrm{Rel}_4(\mathsf{B}_3)$ *are consequences of the identities of* A.

Proof. Let $y_i = x_i + \mathrm{tr}\,(y_i)\cdot 1$, $i = 1, \ldots, n$ be as in (2.8)–(2.10). By (2.9) and (2.10) the relations (5.11.j) can be expressed as polynomials in the $\mathrm{tr}\,(y_i)$ multipled by relations of the α, β and γ invariants of $x_1, \ldots, x_n$. It is easy to see that the relations of the G_2 invariants thus obtained are in $J(\mathsf{G}_2)$, (or use the SMT for G_2), and then we need only apply theorem (6.10). □

§7. Poincaré series of algebras of invariants

(7.0) We recall some general properties of algebras of invariants and their Poincaré series. We examine closely the cases of $R(n, \mathsf{G}_2)$ and $R(n, \mathsf{B}_3)$.

(7.1) Let E be a graded $\mathbb{C}$-algebra. We use E_n to denote the elements of E homogeneous of degree n. Assuming that $\dim_{\mathbb{C}} E_n < \infty$ for all n, we define the Poincaré series $P_t(E)$ to be $\sum_{n=0}^{\infty} (\dim_{\mathbb{C}} E_n)t^n$.

(7.2) Let H be a reductive complex algebraic group and W a representation space of H. Let $D = \mathbb{C}[W]^H$ and set $d = \dim D$.

(7.3) LEMMA (see [Kft] pp. 100–101, [S3] p. 68). *If H^0 is semisimple or the representation (W, H^0) is orthogonal, then $d = \dim W - \max_{w\in W}(\dim Hw)$.* □

(7.4) If $f_1, \ldots, f_r \in D$, let $(f_1, \ldots, f_r)$ denote the ideal $f_1D + \cdots + f_rD$, and let $Z(f_1, \ldots, f_r)$ denote the zero set of the f_i in W. A *homogeneous sequence of parameters* (HSOP) for D is a sequence $f_1, \ldots, f_d$ of non-constant homogeneous elements of D such that $\dim D/(f_1, \ldots, f_d) = 0$. Noether normalization implies that D always has an HSOP.

(7.5) THEOREM. *Let $f_1, \ldots, f_r$ be non-constant homogeneous elements of D.*

(1) *D is a free graded $\mathbb{C}[f_1, \ldots, f_r]$-module if and only if $\dim D/(f_1, \ldots, f_r) = d - r$.*

(2) *If $f_1, \ldots, f_d$ is an* HSOP *for D, then*

$$D \simeq \mathbb{C}[f_1, \ldots, f_d] \otimes_{\mathbb{C}} D^0$$

as graded $\mathbb{C}[f_1, \ldots, f_d]$-module, where $D^0 = D/(f_1, \ldots, f_d)$.

(3) *If $Z(f_1, \ldots, f_r)$ has codimension r in W, then D is a free $\mathbb{C}[f_1, \ldots, f_r]$-module.*

Proof. Part (1) is the fact that D is Cohen–Macaulay ([HR], [St1]), and (2) follows from (1). The hypothesis of (3) implies that $\mathbb{C}[W]$ is a free $\mathbb{C}[f_1, \ldots, f_r]$-module (use (1) with H = trivial group), and projecting equivariantly from $\mathbb{C}[W]$ to $\mathbb{C}[W]^H$ we obtain (3). □

(7.6) Fix an HSOP $f_1, \ldots, f_d$ for D. Then $D^0 = D/(f_1, \ldots, f_d)$ is a finite dimensional algebra, hence

$$(7.7) \quad P_t(D^0) = \sum_{i=0}^{l} a_i t^i$$

for some $a_i \in \mathbb{N}$, where we assume that $a_l \neq 0$. Let $e_i = \deg f_i$, $i = 1, \ldots, d$. Then (7.5.2) shows that

$$(7.8) \quad P_t(D) = \prod_{i=1}^{d} (1 - t^{e_i})^{-1} P_t(D^0).$$

(7.9) PROPOSITION. *Assume that H is connected and semisimple. Then*

(1) $a_i = a_{l-i}$, $0 \leq i \leq l$.

(2) $d \leq -l + \sum e_i \leq \dim W$.

(3) $l = -\dim W + \sum e_i$ *if* $\operatorname{codim}_W (W - W') \geq 2$, *where W' denotes the union of the orbits in W with finite isotropy.*

Proof. By Murthy [Mur], D is Gorenstein, which implies (1) (c.f. [St1]). Parts (2) and (3) are recent work of Knop [Kn] (c.f. [St2]). □

(7.10) After some preliminaries we find HSOP's for the algebras $R(n, \mathsf{G}_2)$ and $R(n, \mathsf{B}_3)$: Let $B(n) = \mathbb{C}[n\mathbb{C}^7]^{\mathsf{O}_7}$ and $C(n) = \mathbb{C}[n\mathbb{C}^4]^{\mathsf{O}_4}$, $n \geq 1$. By CIT (see (4.9)), $B(n) \simeq S^{\cdot}\psi_1^2(n)/(\psi_8^2(n))$ where $(\psi_8^2(n))$ denotes the ideal of $\psi_8^2(n) \subseteq S^{\cdot}\psi_1^2(n)$, and similarly $C(n) \cong S^{\cdot}\psi_1^2(n)/(\psi_5^2(n))$. The canonical surjection

$$\sigma : B(n) \cong S^{\cdot}\psi_1^2(n)/(\psi_8^2(n)) \to S^{\cdot}\psi_1^2(n)/(\psi_5^2(n)) \simeq C(n)$$

is induced by the standard inclusion of $\mathbb{C}^4$ into $\mathbb{C}^7$. Let p_{ij}, $1 \le i, j \le n$ denote the usual inner product generators of $B(n)$.

(7.11) LEMMA. *Let $n \ge 3$ and set $k = 4n - 6$. Then* $\dim C(n) = k$ *and there are k linear combinations $h_1, \ldots, h_k$ of the p_{ij} such that*

(1) *$Z(h_1, \ldots, h_k)$ has codimension k in $n\mathbb{C}^7$.*

(2) *The $\sigma(h_i)$ are an* HSOP *for $C(n)$.*

Proof. If $n \geqslant 3$, then there are orbits in $n\mathbb{C}^4$ of dimension $6 = \dim O_4$, and lemma (7.3) shows that $\dim C(n) = k$. The techniques of ([S2] pp. 8–10) show that the zero set of all the p_{ij} has codimension k. Let

$$s = \binom{n+1}{2},$$

and let $Z = (\mathbb{C}^s)^k = \{z_{ijr} : 1 \le i \le j \le n \text{ and } 1 \le r \le k\}$. If $\{z_{ijr}\} \in Z$, let $h_r = \sum z_{ijr} p_{ij}$, $r = 1, \ldots, k$. Then there is a non-empty Zariski open subset Z' of Z such that all the corresponding $\{h_r\}$ have a zero set of codimension k (see [ZS] Vol. I pp. 266–267). Similarly, there is a Z'' such that the corresponding $\{\sigma(h_r)\}$ are HSOP's for $C(n)$. Choosing coefficients in $Z' \cap Z''$ gives the required $h_1, \ldots, h_k$. $\square$

(7.12) *Remark.* One may replace $B(n)$ by $B'(n) = \mathbb{C}[n\mathbb{C}^8]^{O_8}$ in (7.11) and obtain the same conclusions.

(7.13) THEOREM. *Let $n \ge 4$. Then*

(1) $\dim R(n, \mathrm{G}_2) = 7n - 14$.

(2) *$R(n, \mathrm{G}_2)$ has an* HSOP *consisting of* $4n - 6$ *elements of degree* 2 *and* $3n - 8$ *elements of degree* 3.

(3) *degree* $P_t(R(n, \mathrm{G}_2)^0) = 10n - 36$.

Proof. It follows from ([S3] Cor. 7.4, Table V) that $(n\phi_1, \mathrm{G}_2)$ satisfies the hypothesis of (7.9.3) when $n \ge 4$. Hence (2) implies (3), and (7.3) shows that $\dim R(n, \mathrm{G}_2) = 7n - \dim \mathrm{G}_2 = 7n - 14$, establishing (1).

Let $k = \dim C(n) = 4n - 6$, and choose $h_1, \ldots, h_k$ as in (7.11) (identifying $\phi_1(\mathrm{G}_2)$ with $\mathbb{C}^7$ orthogonally so that the α_{ij} and p_{ij} are identified). Let $R(n)'$ (resp. $R(n)''$) denote the subalgebra of $R(n)$ generated by the α and β invariants (resp.

β invariants and the h_i). As observed in (5.7), $R(n)$ is finite over $R(n)'$, and by $\mathrm{Rel}_8(n)$ and our choice of the h_i, $R(n)'$ is finite over $R(n)''$. Thus $R(n)/(h_1, \ldots, h_k)$ is finite over $R(n)''/(h_1, \ldots, h_k)$, where both have dimension $7n - 14 - k = 3n - 8$ (use (7.11.1) and (7.5)). Now $R(n)''/(h_1, \ldots, h_k)$ is generated by the β invariants, hence by Noether normalization there are $q = 3n - 8$ linear combinations $h_{k+1}, \ldots, h_{k+q}$ of the β invariants which are an HSOP for $R(n)''/(h_1, \ldots, h_k)$. Thus $h_1, \ldots, h_k, h_{k+1}, \ldots, h_{k+q}$ is our required HSOP for $R(n)$. $\square$

(7.14) *Remarks.* We consider the Poincaré series of $R(n, \mathsf{G}_2)$ for $3 \leq n \leq 6$.

(1) $P_t(R(3, \mathsf{G}_2)) = (1 - t^2)^{-6}(1 - t^3)^{-1}$.

(2) $P_t(R(4, \mathsf{G}_2)) = (1 - t^2)^{-10}(1 - t^3)^{-4}(1 + t^4)$.

(3) $$P_t(R(5, \mathsf{G}_2)) = (1 - t^2)^{-14}(1 - t^3)^{-7}(1 + t^3 + 3t^3 + 6t^4 + 3t^5 + 7t^6 + 8t^7 + 7t^8 + \cdots + t^{14}).$$

(4) $$P_t(R(6, \mathsf{G}_2)) = (1 - t^2)^{-18}(1 - t^3)^{-10}(1 + 3t^2 + 10t^3 + 21t^4 + 30t^5 + 75t^6 + 120t^7 + 165t^8 + 220t^9 + 315t^{10} + 330t^{11} + 330t^{12} + 330t^{13} + \cdots + t^{24}).$$

We will establish (3) and (4) in §10. Since $R(3, \mathsf{G}_2)$ is regular, (1) is immediate. Note that the conclusion of (7.9.3) fails in this case. When $n = 4$, the α_{ij} and β_{ijk} form an HSOP, and $P_t(R(4, \mathsf{G}_2)^0) = 1 + t^4$ by $\mathrm{Rel}_4(4)$. Hence (2) is as claimed.

Using techniques as above one establishes:

(7.15) THEOREM. *Let* $n \geq 5$. *Then*

(1) $\dim R(n, \mathsf{B}_3) = 8n - 21$.

(2) *There is an* HSOP *for* $R(n, \mathsf{B}_3)$ *consisting of* $4n - 6$ *elements of degree* 2 *and* $4n - 15$ *elements of degree* 4.

(3) *degree* $P_t(R(n, \mathsf{B}_3)^0) = 16n - 72$. $\square$

(7.16) *Remarks.* We will show that

(1) $$P_t(R(6, \mathsf{B}_3)) = (1 - t^2)^{-18}(1 - t^4)^{-9}(1 + 3t^2 + 12t^4 + 28t^6 + 57t^8 + 78t^{10} + 92t^{12} + 78t^{14} + \cdots + t^{24}).$$

(2) $$P_t(R(7, \mathsf{B}_3)) = (1 - t^2)^{-22}(1 + t^4)^{-13}(1 + 6t^2 + 43t^4 + 188t^6 + 701t^8 + 1966t^{10} + 4621t^{12} + 8708t^{14} + 13818t^{16} + 17976t^{18} + 19782t^{20} + 17976t^{22} + \cdots + t^{40}).$$

§8. Partial resolutions I

(8.0) Let $R = R(\mathsf{B}_3)$. Then $R = T/I$ where $T = S^{\cdot}(\psi_1^2 + \psi_4)$. We want to show that $I = J$, where $J = J(\mathsf{B}_3)$ is generated by $\mathrm{Rel}_1(\mathsf{B}_3), \ldots, \mathrm{Rel}_4(\mathsf{B}_3)$ (see (5.11)).

Let M_j be the ideal in T generated by $S^j\psi_1^2$. Then the M_j induce decreasing filtrations of R and T/J, and the associated graded algebras satisfy the relations

(8.1) $0 = \psi_2\psi_6 \subseteq S^2\psi_4$,

(8.2) $0 = \psi_5^2 \subset \psi_1^2 \otimes S^2\psi_4$

which result from $\mathrm{Rel}_1(\mathsf{B}_3)$ and $\mathrm{Rel}_2(\mathsf{B}_3)$.

Let K denote the ideal in T generated by the representations in (8.1) and (8.2). We show that the Poincaré series of $T(7)/K(7)$ is the one given in (7.16.2). By (7.15.3), $P_t(R(7)^0)$ has degree 40, and in §10 we show that $P_t(R(7))$ equals $P_t(T(7)/K(7))$ up to degree 20. Thus, by (7.9.1) and (7.15.2), $P_t(R(7)) = P_t(T(7)/K(7))$, and it follows that $P_t(R(7)) = P_t(T(7)/J(7))$, establishing the SMT for B_3. In §§9–10 we use similar techniques to establish the SMT for G_2.

(8.3) From now on we will often use the notation $\phi_{(a)}$, ϕ_1^2, etc. for $\psi_{(a)}(n)$, $\psi_1^2(n)$, etc. Usually, n will be specified or clear from the context.

(8.4) Let D denote $T(7)/K(7)$. Then $D = \bigoplus_{j\geq 0} D_j$ where D_j is the $S^{\cdot}\phi_1^2$-submodule of D generated by $S^j\phi_4$ ($n = 7$!). It follows from (8.1) and theorem (4.7) (or from the CIT of SL_4) that D_j is generated by $\phi_4^j \subseteq S^j\phi_4$.

In order to compute $P_t(D)$, we compute resolutions of the $S^{\cdot}\phi_1^2$-modules D_j. These resolutions and those of §9 are among ones established in [PW]. The particular cases we need follow easily and directly from Bott's theorem on the cohomology of homogeneous vector bundles, as formulated by Lascoux [L], and for completeness we sketch the details involved.

(8.5) Let S be a polynomial algebra over $\mathbb{C}$, and $f_1, \ldots, f_r$ elements of S. If $(f_1, \ldots, f_r) \neq S$ and $\dim S/(f_1, \ldots, f_r) = \dim S - r$, then we say that $f_1, \ldots, f_r$ is a *regular sequence* in S. (For non-polynomial rings the definition above must be changed.)

Let ϕ be a representation of GL_n and S as above. We denote the free S-module $S \otimes_{\mathbb{C}} \phi$ by $\{\phi\}$. If GL_n acts on S (e.g. $S = S^{\cdot}\phi_1^2$), then we may single out those S-module morphisms $\{\phi\} \to \{\phi'\}$ which are equivariant.

(8.6) THEOREM. *Let $n = 7$ and $S = S^{\cdot}\phi_1^2$. There are equivariant free resolu-*

tions of the modules D_j as follows:

(1) $$0 \to \{\phi_0\} \to D_0 \to 0$$

(2) $$0 \to \{\phi_4\} \to D_1 \to 0.$$

(3) $$0 \to \{\phi_5^2\} \to \{\phi_4^2\} \to D_2 \to 0.$$

(4) $$0 \to \{\phi_6^3\} \to \{\phi_5^2\phi_6\} \to \{\phi_4\phi_5^2\} \to \{\phi_4^3\} \to D_3 \to 0.$$

(5) $$0 \to \{\phi_4^{j-4}\phi_7^4\} \to \{\phi_4^{j-4}\phi_6^2\phi_7^2\} \to \{\phi_4^{j-4}\phi_5\phi_6^2\phi_7\} \to \{\phi_4^{j-4}\phi_5^3\phi_7 + \phi_4^{j-3}\phi_6^3\}$$
$$\to \{\phi_4^{j-3}\phi_5^2\phi_6\} \to \{\phi_4^{j-2}\phi_5^2\} \to \{\phi_4^j\} \to D_j \to 0, \quad j \geq 4.$$

(8.7) LEMMA. *Let $n=3$, let S be a polynomial algebra over $\mathbb{C}$, and let $b:\{\phi_1^2\} \to \{\phi_0\} = S$ be a morphism. Then, canonically associated to b, there are complexes*:

(1) $0 \to \{\phi_3^4\} \to \{\phi_2^2\phi_3^2\} \to \{\phi_1\phi_2^2\phi_3\} \to \{\phi_1^3\phi_3 + \phi_2^3\} \to \{\phi_1^2\phi_2\} \to \{\phi_1^2\} \xrightarrow{b} \{\phi_0\}.$

(2) $0 \to \{\phi_2\phi_3^2\} \to \{\phi_1\phi_2\phi_3\} \to \{\phi_2^2\} \to \{\phi_0\}.$

(3) $0 \to \{\phi_3^3\} \to \{\phi_1^2\phi_3\} \to \{\phi_1\phi_2\} \to \{\phi_1\}.$

Let $[b_{ij}]$ be the (symmetric) matrix of b relative to a basis of ϕ_1, and assume that $\sum b_{ij}S \neq S$. Then (1) *is exact if and only if the b_{ij}, $i \leq j$, are a regular sequence in S, and* (2) *and* (3) *are exact if and only if the ideal of 2×2 minors $[b_{ij}]_2$ of $[b_{ij}]$ contains a regular sequence of length* 3.

Proof. The Kozul complex of the b_{ij} has $\{\Lambda^m\phi_1^2\}$ in the mth position. One easily computes that $\Lambda^2\phi_1^2 = \phi_1^2\phi_2$, $\Lambda^3\phi_1^2 = \phi_1^3\phi_3 + \phi_2^3$, etc., yielding (1). The exactness criterion is well-known. The complex (2) and its exactness criterion can be found in [J]. Moreover, if (2) is exact, then $S/[b_{ij}]_2$ is Cohen–Macaulay of dimension $\dim S - 3$, and (2) is a resolution of $S/[b_{ij}]_2$. It follows that the dual (2)* of (2) is exact. But, modulo a character of GL_3, (2)* is (3). □

(8.8) *Remark.* Let $S = S^{\cdot}\phi_1^2$ in (8.7). Then there is a canonical morphism $b:\{\phi_1^2\} \to \{\phi_0\}$, and the complexes of (8.7) are exact and equivariant (the "generic" case).

(8.9) We use Bott's theorem and the sequences in (8.7) to establish (8.6): Let $W = \mathbb{C}^7$ and $Y = S^2W^*$. Then $\mathbb{C}[Y] = S^{\cdot}\phi_1^2$. Let M denote the trivial vector bundle $Y \times W$, let $X = \mathrm{Grass}_3(M)$ and $\rho: X \to Y$ the canonical projection. There is an exact sequence of vector bundles

$$0 \to L \to \rho^*M \to Q \to 0$$

where L is the tautological bundle of X. Let $\mathscr{L}$ and $\mathscr{Q}$ denote the sheaves of $\mathcal{O}_X$-modules corresponding to L and Q, and let $\mathscr{M}$ similarly correspond to M. Given $(a) \in \mathbb{N}^\infty$ we may construct vector bundles $\psi_{(a)}(L)$ and $\psi_{(a)}(Q)$ on X and $\psi_{(a)}(M)$ on Y (cf. (3.7)), and there are corresponding locally free sheaves $\psi_{(a)}(\mathscr{L})$, etc.

Let $x \in X$. Then $\rho(x)$ induces a symmetric bilinear form on $L_x \subseteq W$. Hence there is a canonical section of $(S^2L)^*$, and using (8.7.1) we form a complex of sheaves $\mathscr{C}_1$:

$$0 \to \psi_3^4(\mathscr{L}) \to \psi_2^2\psi_3^2(\mathscr{L}) \to \cdots$$

Similarly, there are complexes $\mathscr{C}_2$ and $\mathscr{C}_3$ corresponding to (8.7.2) and (8.7.3).

(8.10) LEMMA. *The complexes $\mathscr{C}_1$, $\mathscr{C}_2$ and $\mathscr{C}_3$ are exact.*

Proof. It is enough to show that the complexes of global sections are exact on affine open sets covering X. Let $e_1, \ldots, e_7$ be the standard basis of $\mathbb{C}^7$. Let $w = (w_1, w_2, w_3)$ where $w_i = w_{i4}e_4 + \cdots + w_{i7}e_7$, $i = 1, 2, 3$, and let $b \in Y$. Let $L_{b,w} = \mathrm{span}\,\{e_1 + w_1, e_2 + w_2, e_3 + w_3\} \subseteq W$ and $x(b, w)$ the corresponding point of X above b. Then the $x(b, w)$ form an affine open subset X' of X. Let $b_{ij} = b(e_i, e_j)$. Then the bilinear form on $L_{b,w}$ has matrix $[b'_{ij}]$, $1 \le i, j \le 3$, where $b'_{ij} = b(e_i + w_i, e_j + w_j) = b_{ij} + f_{ij}$ and f_{ij} is a polynomial homogeneous of degree 2 in the w_{pq} and b_{rs} with r or $s \ge 4$. Clearly, the b'_{ij} form a regular sequence in $\mathcal{O}(X')$ and the 2×2 minors of $[b'_{ij}]$ contain a regular scquence of length 3 (since this is true for $[b_{ij}]$, $1 \le i, j \le 3$). Lemma (8.7) then gives the required exactness on X'. Finally, we need only observe that the GL_7 translates of X' cover X. □

(8.11) Let ρ_* denote the mapping of coherent sheaves of $\mathcal{O}_X$-modules to coherent sheaves of $\mathcal{O}_Y$-modules induced by ρ, and as usual let $\mathscr{R}^i\rho_*$ denote the right derived functors of ρ_*. Let $\mathscr{F}_{(a),j}$ denote $\psi_{(a)}(\mathscr{L}) \otimes_{\mathcal{O}_X} \psi_4^j(\mathscr{Q})$ where $(a) = (a_1, a_2, a_3, 0, \ldots)$. If $w = \mathrm{width}\,(a) \le j$, then $\psi_{(a)} \otimes \psi_4^j$ contains exactly one factor of width j, namely $\psi_4^{j-w}\psi_5^{a_1}\psi_6^{a_2}\psi_7^{a_3}$, which we denote by $\psi_{(a')}$.

(8.12) PROPOSITION. *Let $j \ge 0$ and let $\psi_{(a)}(\mathscr{L})$ be one of the sheaves occurring in $\mathscr{C}_1$. Then*

$$\mathscr{R}^i\rho_*\mathscr{F}_{(a),j} = \psi_{(a')}(\mathscr{M}) \text{ if } i = 0 \text{ and width } (a) \le j$$
$$= 0 \text{ otherwise.}$$

Proof. Lascoux [L] gives a formula for the sheaves $\mathscr{R}^i\rho_*\psi_{(a)}(\mathscr{L}) \otimes_{\mathcal{O}_X} \psi_{(b)}(\mathscr{Q})$ which yields our proposition as a special case. □

Proof of Theorem (8.6). Take the exact sheaf sequence $\mathscr{C}_1$, tensor it with $\psi_4^j(\mathscr{Q})$ and apply proposition (8.12). Then we obtain that the $\rho_*\mathscr{F}_{(a),j}$ form an exact sequence of sheaves of $\mathcal{O}_Y$-modules. Taking global sections we obtain the free parts of (8.6.3), (8.6.4) and (8.6.5) when $j=2$, $j=3$ and $j\geq 4$, respectively. Note that exactness forces the morphisms $\tau_j:\{\phi_4^{j-2}\phi_5^2\}\to\{\phi_4^j\}$ to be non-trivial, and the τ_j are unique up to scalars since

$$(*)\quad \phi_1^2\otimes\phi_4^j=\phi_1^2\phi_4^j+\phi_1\phi_4^{j-1}\phi_5+\phi_4^{j-2}\phi_5^2,\qquad j\geq 2,$$

i.e. $\{\phi_4^j\}$ contains only one copy of $\phi_4^{j-2}\phi_5^2$.

For $j\geq 2$, D_j is $\{\phi_4^j\}$ modulo the submodule generated by the product of $\phi_5^2\subseteq\{\phi_4^2\}$ with ϕ_4^{j-2} in $S^\cdot\phi_1^2\otimes(\bigoplus_{j\geq 0}\phi_4^j)$ (see (8.4)). The product ends up in $\phi_1^2\otimes\phi_4^j$, and by $(*)$ the product is $\phi_4^{j-2}\phi_5^2\subseteq\{\phi_4^j\}$. Hence D_j is the cokernel of τ_j, $j\geq 2$. □

(8.13) *Remark.* We used Bott's theorem to establish the complexes of (8.6) and their exactness. To merely show that the complexes exist is easy. The exact sequence $0\to\{\phi_3^4\}\to\cdots$ of (8.7.1) (with $S=S^\cdot\phi_1^2$ and $n=3$) canonically gives rise to a complex $0\to\{\phi_3^4\}\to\cdots$ for $n=7$. Let $\{\phi\}$ be a term in the complex, and let $\phi'\subseteq\phi\otimes\phi_4^j$ be the sum of the subrepresentations of width j. Then the complex $\cdots\to\{\phi\}\otimes\phi_4^j\to\cdots$ has $\cdots\to\{\phi'\}\to\cdots$ as a subcomplex, and in this way one obtains (8.6.3), etc.

(8.14) It is now not difficult to compute the Poincaré series of $D=\bigoplus D_j$. Let $E_0=\phi_0+\phi_4+\phi_4^2+\cdots$, $E_1=\phi_5^2+\phi_4\phi_5^2+\cdots$, etc. Then $S^\cdot\phi_1^2\otimes E_k$ is the direct sum of the kth terms of our resolution of D, $0\leq k\leq 6$. Note that each E_k is a module over $E_0=S^\cdot\phi_4/(\phi_2\phi_6)=\bigoplus_{j\geq 0}\phi_4^j$. By CIT, $E_0\simeq\mathbb{C}[7\mathbb{C}^4]^{SL_4}$, hence $\dim E_0=13$ (use (7.3)), and E_0 has an HSOP consisting of 13 elements of degree 4.

(8.15) PROPOSITION. *Let* $Q(t)=(1-t^4)^{13}$. *Then*

(1) $Q(t)P_t(E_0)=1+22t^4+113t^8+190t^{12}+113t^{16}+22t^{20}+t^{24}$.

(2) $Q(t)P_t(E_1)=196t^{10}+980t^{14}+1176t^{18}+392t^{22}+28t^{26}$.

(3) $Q(t)P_t(E_2)=882t^{16}+3234t^{20}+2436t^{24}+378t^{28}$.

(4) $Q(t)P_t(E_3)=84t^{18}+1008t^{22}+2352t^{26}+1176t^{30}$
$$+1176t^{22}+2352t^{26}+1008t^{30}+84t^{34}.$$

(5) $Q(t)P_t(E_4)=378t^{24}+2436t^{28}+3234t^{32}+882t^{36}$.

(6) $Q(t)P_t(E_5) = 28t^{26} + 392t^{30} + 1176t^{34} + 980t^{38} + 196t^{42}$.

(7) $P_t(E_6) = t^{28}P_t(E_0)$.

(8.16) *Remark.* The proposition suggests that the E_k are Cohen–Macaulay E_0-modules.

Proof of (8.15). It follows from theorem (7.9) that $Q(t)P_t(E_0)$ is a polynomial of degree 24. Using the Weyl dimension formula ([Hu] pp. 139–140) and (7.9.1) one easily computes that $Q(t)P_t(E_0)$ and $P_t(E_6)$ are as claimed.

We now show how to compute $Q(t)P_t(E_1)$; the other cases are similar. Let $h(j) = (1/12!)(12+j)\cdots(1+j)$ for $j \in \mathbb{Z}$. Then

$$Q(t)^{-1}(196t^{10} + \cdots + 28t^{26}) = \sum_{i\geq 0} t^{10+4i}(196h(i) + \cdots + 28h(i-4)).$$

Set $f(i) = \dim \phi_4^i\phi_5^2$. Our formula for $P_t(E_1)$ is equivalent to the claim that

$$(*) \quad f(i) = 196h(i) + \cdots + 28h(i-4), \qquad i \geq 0.$$

By the Weyl dimension formula,

$$f(i) = (2^3 3^3 4^3 5^2)^{-1}(i+1)(i+2)(i+3)(i+4)^2(i+5)^2(i+6)^2(i+7)^2(i+8).$$

Now $f(i)$ and the right hand side of $(*)$ are polynomials of degree 12 in i, and both are evenly divisible by $(i+1)\cdots(i+8)$. Thus equality holds in $(*)$ if it holds when $i = 0, \ldots, 4$, and this is easily checked. □

(8.17) THEOREM. $P_t(D) = P_t(T(7)/K(7))$ *is the series given in* (7.16.2).

Proof. We know that

$$P_t(D) = (1-t^2)^{-28} \sum_{k=0}^{6} (-1)^k P_t(E_k).$$

Using (8.15) one computes that $P_t(D)$ is as claimed. □

(8.18) *Remark.* From (8.6) one immediately obtains a resolution for $T(6)/K(6)$, and one computes as above that $P_t(T(6)/K(6))$ equals the series given in (7.16.1).

§9. Partial resolutions II

(9.0) Let $R = R(\mathrm{G}_2)$. Then $R = T/I$ where $T = S^{\cdot}(\psi_1^2 + \psi_3 + \psi_4)$. We want to show that $I = J$, where $J = J(\mathrm{G}_2)$ is generated by $\mathrm{Rel}_1(\mathrm{G}_2), \ldots, \mathrm{Rel}_6(\mathrm{G}_2)$ (see (5.4)). Filtering R and T/J as in (8.0), we obtain associated graded algebras which are quotients of T/K, where K is generated by:

(9.1) $\psi_1\psi_5 \subseteq S^2\psi_3$.

(9.2) $\psi_2\psi_5 + \psi_1\psi_6 \subseteq \psi_3 \otimes \psi_4$.

(9.3) $S^2\psi_4$.

(9.4) $\psi_5^2 \subseteq S^2\psi_1^2 \otimes S^2\psi_3$.

(9.5) $\psi_4\psi_5 \subseteq \psi_1^2 \otimes \psi_3 \otimes \psi_4$.

We compute the Poincaré series of $E = T(6)/K(6)$ using the techniques of §8. We omit most of the proofs since they involve no new ideas.

(9.6) Let $n = 6$, and let C_j be the $S^{\cdot}\phi_1^2$-submodule of E generated by $\phi_3^j \subseteq S^j\phi_3$, and let D_j be the submodule generated by $\phi_3^j\phi_4 \subseteq S^j\phi_3 \otimes \phi_4$. Set $C = \bigoplus_{j\geq 0} C_j$ and $D = \bigoplus_{j\geq 0} D_j$. It follows from (9.1) through (9.5) that $E = C \oplus D$. From (9.4), C_j is isomorphic to $\{\phi_3^j\}$ modulo the submodule generated by $\phi_3^{j-2}\phi_5^2 \subseteq \phi_2^2 \otimes \phi_3^j \subseteq S^2\phi_1^2 \otimes \phi_3^j$, $j \geq 2$. From (9.5) and (9.4), we see that D_j is isomorphic to $\{\phi_3^j\phi_4\}$ modulo the submodules generated by $\phi_3^{j-1}\phi_4\phi_5 \subseteq \phi_1^2 \otimes \phi_3^j\phi_4$, $j \geq 1$, and $\phi_3^{j-2}\phi_4\phi_5^2 \subseteq \phi_2^2 \otimes \phi_3^j\phi_4$, $j \geq 2$. However, when $j \geq 2$, there is a copy of $\phi_3^{j-2}\phi_4\phi_5^2$ in $\phi_1^2 \otimes \phi_3^{j-1}\phi_4\phi_5 \subseteq \phi_1^2 \otimes \phi_1^2 \otimes \phi_3^j\phi_4$, and its image in $\{\phi_3^j\phi_4\}$ is non-zero. Hence we need only divide $\{\phi_3^j\phi_4\}$ by the submodule of $\phi_3^{j-1}\phi_4\phi_5$ to obtain D_j.

(9.7) THEOREM. *There are equivariant free resolutions of the $S^{\cdot}\phi_1^2$-modules C_j and D_j as follows*:

(1) $$0 \to \{\phi_0\} \to C_0 \to 0.$$

(2) $$0 \to \{\phi_3\} \to C_1 \to 0.$$

(3) $$0 \to \{\phi_5^2\} \to \{\phi_3^2\} \to C_2 \to 0.$$

(4) $$0 \to \{\phi_3^{j-3}\phi_5\phi_6^2\} \to \{\phi_3^{j-3}\phi_4\phi_5\phi_6\} \to \{\phi_3^{j-2}\phi_5^2\} \to \{\phi_3^j\} \to C_j \to 0, \quad j \geq 3.$$

(5) $$0 \to \{\phi_4\} \to D_0 \to 0.$$

(6) $$0 \to \{\phi_4\phi_5\} \to \{\phi_3\phi_4\} \to D_1 \to 0.$$

(7) $$0 \to \{\phi_3^{j-2}\phi_6^3\} \to \{\phi_3^{j-2}\phi_4^2\phi_6\} \to \{\phi_3^{j-1}\phi_4\phi_5\} \to \{\phi_3^j\phi_4\} \to D_j \to 0, \quad j \geq 2.$$

Proof. Let $W = \mathbb{C}^6$, and construct $\rho: X \to Y$ and vector bundles L, M and Q as in (8.9), where now both L and Q have fiber dimension 3. As in (8.10), we have exact sheaf sequences

$$0 \to \psi_2\psi_3^2(\mathscr{L}) \to \psi_1\psi_2\psi_3(\mathscr{L}) \to \psi_2^2(\mathscr{L}) \to \psi_0(\mathscr{L}),$$
$$0 \to \psi_3^3(\mathscr{L}) \to \psi_1^2\psi_3(\mathscr{L}) \to \psi_1\psi_2(\mathscr{L}) \to \psi_1(\mathscr{L}).$$

Let $\psi_{(a)}(\mathscr{L})$ be a sheaf occurring above, and let $j \geq 0$. If $w = \text{width}\,(a) \leq j$, let $(a') = (0, 0, j - w, a_1, a_2, a_3)$. Then, by [L], $\mathscr{R}^i\rho_*\psi_{(a)}(\mathscr{L}) \otimes_{\mathcal{O}_X} \psi_3^j(\mathscr{Q})$ is $\psi_{(a')}(\mathscr{M})$ if $i = 0$ and width $(a) \leq j$, else 0. The proof concludes as that for (8.6). $\square$

(9.8) Let M_k (resp. N_k) be the direct sum of the kth summands of our resolutions of $\oplus C_j$ (resp. $\oplus D_j$), $0 \leq k \leq 3$. As in §8 one establishes

(9.9) PROPOSITION. *Let* $Q(t) = (1 - t^3)^{10}$. *Then*

(1) $Q(t)P_t(M_0) = 1 + 10t^3 + 20t^6 + 10t^9 + t^{12}$.
(2) $Q(t)P_t(M_1) = 21t^{10} + 126t^{13} + 105t^{16}$.
(3) $Q(t)P_t(M_2) = 70t^{15} + 196t^{18} + 70t^{21}$.
(4) $Q(t)P_t(M_3) = 6t^{17} + 45t^{20} + 60t^{23} + 15t^{26}$.
(5) $Q(t)P_t(N_0) = 15t^4 + 60t^7 + 45t^{10} + 6t^{13}$.
(6) $Q(t)P_t(N_1) = 70t^9 + 196t^{12} + 70t^{15}$.
(7) $Q(t)P_t(N_2) = 105t^{14} + 126t^{17} + 21t^{20}$.
(8) $P_t(N_3) = t^{18}P_t(M_0)$.

(9.10) THEOREM. $P_t(E) = P_t(T(6)/K(6))$ *is the series given in* (7.14.4).

(9.11) *Remark.* One similarly obtains that $P_t(T(5))/K(5))$ is the series of (7.14.3).

§10. Comparison of Poincaré series

(10.0) We show that $R(6, \mathrm{G}_2)$ has the same Poincaré series (7.14.4) as $T(6)/K(6)$, establishing the SMT for G_2 (see (10.4) for B_3). The most straightforward approach would be to use Weyl's formulas to compute $P_t(R(6))$ (see [St2]), but the integrals involved are not easy to evaluate. We adopt a less direct approach.

(10.1) Let $Q(t)=(1-t^2)^{18}(1-t^3)^{10}$. Then $P_t(R(6))=Q(t)^{-1}\sum_{i=0}^{24} a_i t^i$, $P_t(T(6)/J(6))=Q(t)^{-1}\sum_{i\geq 0} b_i t^i$ and $P_t(T(6)/K(6))=Q(t)^{-1}\sum_{i=0}^{24} c_i t^i$ where the c_i are as in (7.14.4) and $\dim(T(6)/J(6))_j \leq \dim(T(6)/K(6))_j$ for all j. If $a_j=c_j$ for $j\leq 12$, then one easily sees that $a_j=b_j=c_j$ for all j, establishing the SMT for G_2.

Let $\eta: T(6)/J(6)\to R(6)$ be the canonical surjection, and let $E=\mathbb{C}[f_1,\ldots,f_{28}]$ where the $f_i\in T(6)$ are homogeneous and map onto the HSOP of $R(6)$ given by Theorem (7.13). Then $R(6)\simeq E\otimes_{\mathbb{C}} R(6)^0$ is a free E-module, hence there is an isomorphism of E-modules:

(10.2) $T(6)/J(6)\simeq \operatorname{Ker}\eta \oplus E\otimes_{\mathbb{C}} R(6)^0$.

Let $(\operatorname{Ker}\eta)_i$ denote the part of $\operatorname{Ker}\eta$ homogeneous of degree i, and suppose that $a_i=c_i$ for $i<j$. Then $a_i=b_i$ and $(\operatorname{Ker}\eta)_i=0$ for $i<j$, and it follows from (10.2) that $a_j+\dim(\operatorname{Ker}\eta)_j=b_j\leq c_j$. Now $(\operatorname{Ker}\eta)_j$ is a direct sum of irreducible representations of GL_6 of degree j, and we obtain

(10.3) PROPOSITION. *Suppose that every irreducible representation ϕ of GL_6 with $\deg\phi=j\leq 12$ and $\dim\phi\leq c_j$ occurs with the same multiplicity in $T(6)/K(6)$ and $R(6)$. Then $I(6)=J(6)$.* □

Given ϕ, let $r(\phi)$ denote its multiplicity in $R(6)$ and $s(\phi)$ its multiplicity in $T(6)/K(6)$. In Table II we list the relevant ϕ and their multiplicities $r(\phi)$. In each case we computed that $r(\phi)=s(\phi)$, establishing the SMT for G_2. We used the techniques of (5.2) to compute the $r(\phi)$ and the resolutions (9.7) to compute the $s(\phi)$.

(10.4) We now establish the SMT for B_3: It is easy to use the method of (10.3) to show that $I(6,\mathsf{B}_3)=J(6,\mathsf{B}_3)$. Table III lists the relevant multiplicities, where the c_j now are the coefficients given in (7.16.1). Unfortunately, the method is impractical for showing that $I(7)=J(7)$ since the coefficients c_j are then in the thousands! Instead, we begin by showing that the canonical map $\sigma: T(7,\mathsf{B}_3)/J(7,\mathsf{B}_3)\to R(7,\mathsf{G}_2)$ is injective in degrees $\leq 20=\frac{1}{2}\deg P_t(R(7,\mathsf{B}_3)^0)$. It follows that $R(7,\mathsf{B}_3)\simeq T(7,\mathsf{B}_3)/J(7,\mathsf{B}_3)$ in degrees ≤ 20. Finally, we show that $P_t(T(7)/K(7))=P_t(T(7)/J(7))$ in degrees ≤ 20, and the SMT for B_3 follows.

(10.5) LEMMA. *The natural mapping $T(7,\mathsf{B}_3)\to T(7,\mathsf{G}_2)$ induces an injection $\tau: R(7,\mathsf{B}_3)\to R(7,\mathsf{G}_2)$.*

Proof. Both $R(7,\mathsf{B}_3)$ and $R(7,\mathsf{G}_2)$ are integral domains of dimension 35. Thus τ is injective if and only if the quotient field $QR(7,\mathsf{G}_2)$ of $R(7,\mathsf{G}_2)$ is finite

Table II

Degree j	c_j	Representation	Multiplicity
0	1	ϕ_0	1
1	0		
2	3		
3	10		
4	21	ϕ_4	1
5	30	ϕ_5	0
6	75	ϕ_6	0
		$\phi_1\phi_5$	1
7	120	$\phi_1^2\phi_5$, $\phi_1\phi_6$	0
		$\phi_2\phi_5$	1
8	165	$\phi_3\phi_5$, $\phi_1^2\phi_6$	0
		$\phi_2\phi_6$	1
		ϕ_4^2	2
9	220	$\phi_4\phi_5$, $\phi_1\phi_2\phi_6$, $\phi_1^3\phi_6$	0
		$\phi_3\phi_6$	1
10	315	$\phi_1^4\phi_6$, $\phi_2^2\phi_6$, $\phi_4\phi_6$	0
		ϕ_5^2, $\phi_1^2\phi_2\phi_6$, $\phi_1\phi_3\phi_6$	1
11	330	$\phi_1\phi_5^2$, $\phi_1^5\phi_6$, $\phi_5\phi_6$	0
		$\phi_2\phi_3\phi_6$, $\phi_1\phi_4\phi_6$	1
12	330	$\phi_2\phi_5^2$, $\phi_1^2\phi_4\phi_6$, $\phi_1\phi_5\phi_6$	0
		$\phi_3^2\phi_6$, ϕ_6^2	1
		$\phi_2\phi_4\phi_6$	2

over $QR(7, \mathsf{B}_3)$, i.e. if and only if $QR(7, \mathsf{G}_2)$ is finite over the subfield generated by the α_{ij} and γ_{ijkl}.

Let det denote the determinant invariant in $R(7, \mathsf{G}_2)$. Then $\mathrm{Rel}_{10}(\mathsf{G}_2)$ (see (5.4)) shows that all the elements β_{ijk} det are in $\tau(R(7, \mathsf{B}_3))$, and we know that $\det^2$ is a polynomial in the α_{ij}. Thus $QR(7, \mathsf{G}_2) \simeq QR(7, \mathsf{B}_3)[\det]$ where det satisfies a quadratic equation over $R(7, \mathsf{B}_3)$. □

Table III

Degree j	c_j	Representation	Multiplicity
0	1	ϕ_0	1
2	3		
4	12		
6	28	ϕ_6	0
8	57	$\phi_1^2\phi_6$	0
		$\phi_2\phi_6$	1
10	78	$\phi_4\phi_6$	0
		ϕ_5^2	1
12	92	$\phi_1\phi_5\phi_6$	0
		ϕ_6^2	1

Combining the lemma with the fact that $I(6, \mathsf{B}_3) = J(6, \mathsf{B}_3)$, we obtain

(10.6) COROLLARY. $R(6, \mathsf{B}_3)$ *naturally embeds in* $R(6, \mathsf{G}_2)$. *The subalgebra of* $R(6, \mathsf{G}_2)$ *generated by the* α_{ij} and γ_{ijkl} *has relations* $\mathrm{Rel}_5(\mathsf{G}_2)$ *and* $\mathrm{Rel}_7(\mathsf{G}_2)$.

(10.7) We say that a morphism of GL_7-modules is injective mod $\sim$ (resp. surjective mod $\sim$) if it is injective (resp. surjective) modulo representations ϕ with height $\phi < 7$ or $\deg \phi > 20$. By (10.6), $\sigma: T(7, \mathsf{B}_3)/J(7, \mathsf{B}_3) \to R(7, \mathsf{G}_2)$ is injective in degrees ≤ 20 if and only if σ is injective mod $\sim$.

If ϕ is a representation of GL_7, let $[\phi]$ denote the sum of the representations in $\{\phi\} = S^{\cdot}\phi_1^2 \otimes \phi$ of height ≤ 6. Let $[\phi]\phi_7^k$ denote $\oplus \phi^{(i)}\phi_7^k$ where $[\phi] = \oplus \phi^{(i)}$. Let E_j denote the $S^{\cdot}\phi_1^2$-module generated by the image of $\phi_4^j \subseteq S^j\phi_4 \subseteq T(7, \mathsf{B}_3)$ in $T(7, \mathsf{B}_3)/J(7, \mathsf{B}_3)$.

(10.8) LEMMA. *There are complexes of* GL_7*-modules which are exact mod* $\sim$, *except perhaps at their middle positions, as follows*:

(1) $0 \to [\phi_4\phi_5]\phi_7 \to [\phi_3\phi_4]\phi_7 \oplus [\phi_0]\phi_7^2 \oplus \phi_3^2\phi_7^2 \to E_0 + E_2 \to 0.$

(2) $0 \to \phi_4^2\phi_5\phi_7 \oplus \phi_3\phi_5^2\phi_7 \to [\phi_3\phi_4^2]\phi_7 \oplus [\phi_3]\phi_7 \oplus [\phi_4]\phi_7^2 \to E_1 + E_3 \to 0.$

Proof. We will give the details for establishing (1) and leave (2) to the highly motivated reader. It is clear that the sequence $0 \to [\phi_0]\phi_7^2 \to E_0 \to 0$ is exact mod $\sim$ since ([S1] p. 171)

(3) $S^{\cdot}\phi_1^2 = \oplus \phi_1^{2a_1}\phi_2^{2a_2} \cdots \phi_7^{2a_7}; \qquad a_1, \ldots, a_7 \in \mathbb{N}.$

From theorem (4.7) we see that the height 7 elements of E_2 are generated by the images of subspaces

$$\phi_2\phi_5\phi_7 + \phi_1\phi_6\phi_7 + \phi_7^2 + \phi_3\phi_4\phi_7 \subseteq \phi_3^2 \otimes \phi_4^2 \subseteq \{\phi_4^2\},$$

while $\mathrm{Rel}_2(7, \mathsf{B}_3)$ (see 5.11) shows that the leftmost 3 factors already land in E_0. Thus $\phi_3\phi_4\phi_7 \subseteq E_2$ generates the height 7 elements, modulo E_0.

From (3) there is clearly a complex

$$\{\phi_1\phi_3\phi_4\} \to \{\phi_3^2\} \to \{\phi_0\},$$

and tensoring with ϕ_4^2 and decomposing via Littlewood–Richardson, one sees that there is a subcomplex

(4) $\{\phi_4\phi_5\phi_7\}\to\{\phi_3\phi_4\phi_7\}\to\{\phi_4^2\}$.

It follows from the injectivity of $\{\phi_4\phi_5\}\to\{\phi_3\phi_4\}$ when $n=6$ (see (9.7.6)) that (4) yields a complex $0\to[\phi_4\phi_5]\phi_7\to[\phi_3\phi_4]\phi_7\to E_2$ which is exact at $[\phi_4\phi_5]\phi_7$.

Finally, consider the elements of $\{\phi_3\phi_4\phi_7\}$ of the form $\phi\phi_7^2$ with ht $\phi\le 6$. Using (4.7) again we see that the possibilities in degree ≤ 20 are

$$\phi_2\phi_4\phi_7^2+\phi_1\phi_5\phi_7^2+\phi_6\phi_7^2+\phi_3^2\phi_7^2\subseteq\phi_7^2\otimes\phi_3\phi_4\phi_7.$$

The leftmost three terms are in $\phi_2^2\otimes\phi_4\phi_5\phi_7$, hence by (4) we gain only the term $\phi_3^2\phi_7^2$. □

(10.9) PROPOSITION. *σ is injective* mod $\sim$, *and the complexes* (10.8.1) *and* (10.8.2) *are exact* mod $\sim$.

Proof. Let $\phi_3\phi_4^2(\alpha\beta^3)$ denote the copy of $\phi_3\phi_4^2$ in $R(7,\mathrm{G}_2)$ which is of degree 1 in the α_{ij} and degree 3 in the β_{ijk} (c.f. (5.1)), and let $(\phi_3\phi_4^2(\alpha\beta^3))$ det denote the determinant invariant multiplied by the sum of the representations of height ≤ 6 in the $S^{\cdot}\phi_1^2$-submodule of $R(7,\mathrm{G}_2)$ generated by $\phi_3\phi_4^2(\alpha\beta^3)$. Terms $(\phi_4(\gamma))\,\det^2$, etc. are defined similarly.

Let $0\to F_1\to F_0$ be the leftmost part of (10.8.1). The relations of $R(6,\mathrm{G}_2)$ (see also (9.7)) and the construction of (10.8.1) show that the canonical map $T(7,\mathrm{B}_3)\to T(7,\mathrm{G}_2)$ induces an injection mod $\sim$:

(1) $F_0/F_1\to(\phi_3\phi_4(\beta\gamma))\det+(\phi_0)\det^2+(\phi_3^2(\beta^2))\det^2$.

Thus (10.8.1) is exact mod $\sim$ and σ restricted to E_0+E_2 is injective mod $\sim$. Similarly, (10.8.2) is exact mod $\sim$ and

(2) $\sigma\colon E_1+E_3\to(\phi_3\phi_4^2(\alpha\beta^3))\det+(\phi_3(\beta))\det+(\phi_4(\gamma))\det^2$

is injective mod $\sim$. Now representations in E_4, E_5, etc. are zero mod $\sim$, and the right hand sides of (1) and (2) have zero intersection (use the resolutions (9.7)). Thus σ is injective mod $\sim$. □

(10.10) PROPOSITION. *As GL_7-modules, $T(7)/K(7)$ and $T(7)/J(7)$ are isomorphic* mod $\sim$.

Proof. Using the relations (8.1) and (8.2) and proceeding as in lemma (10.8), one constructs a complex

$$(1)\quad 0\to[\phi_4\phi_5]\phi_7\to[\phi_3\phi_4]\phi_7\oplus[\phi_0]\phi_7^2\oplus\phi_3^2\phi_7^2\to D_0\oplus D_2\to 0$$

which is exact mod $\sim$, except perhaps at its middle position, where $D_j\subseteq T(7)/K(7)$ is defined analogously to E_j, $j\geq 0$. Since there is an equivariant surjection from $D_0\oplus D_2$ onto E_0+E_2 and since (10.8.1) is exact mod $\sim$, we see that $D_0\oplus D_2\simeq E_0+E_2$ mod $\sim$. Similarly, $D_1\oplus D_3\approx E_1+E_3$ mod $\sim$, hence $T(7)/K(7)\simeq T(7)/J(7)$ mod $\sim$. □

Our proof that $I(6,\mathsf{B}_3)=J(6,\mathsf{B}_3)$ showed that $K(6,B_3)\approx J(6,B_3)$ as GL_6-representations. Hence (10.10) implies that $P_t(T(7)/K(7))=P_t(T(7)/J(7))$ in degrees ≤ 20, and as noted in (10.4), the SMT for B_3 follows from the injectivity mod $\sim$ of σ.

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Dept. of Mathematics
Brandeis University
Waltham, Massachusetts 02254
USA

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