Comment. Math. Helvetici 63 (1988) 579–586 0010-2571/88/040579-08$01.50 + 0.20/0

On values of the Gauss map of complete minimal surfaces in R^3

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1. Introduction

Let M be a c.m.s. (complete minimal surface) in R^3 and $g: M \to \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, be the Gauss map; g is the composition of the normal mapping $M \to S^2$, with stereographic projection from the north pole, and g is conformal.

In this note we shall prove:

THEOREM 1.1. *Assume M is of finite topological type and not a plane. Then either g assumes every value infinitely often, with the possible exception of six values, or M is of finite total curvature and g omits at most three values.*

Henceforth, assume M is a c.m.s. in R^3 which is not a plane. R. Osserman proved the image of g is dense and that if M has infinite total curvature (denoted by $c(M)$) then g takes on every value infinitely often with the possible exception of a set of capacity zero. If $c(M)$ is finite then g omits at most three values [2].

F. Xavier has proved that g can miss at most six points (the Gauss map of Scherk's surface misses four points and it is unknown if four is sharp).

We shall use Xavier's techniques to prove 1.1 [3]. It seems reasonable that finite topological type is not necessary. That is, if $c(M) = \infty$, then g should take on every value infinitely often, except perhaps for six values.

2. A theorem of Yau

Let M be a complete Riemannian manifold, and consider the equation

$$\Delta \log u = F$$

* The first author was partially supported by the CNPQ of Brasil.

where u is a nonnegative function on M. Assume the zeros L of u are discrete, u is smooth on $M - L$ and Lipschitz in a neighborhood of L.

Yau has proved that if F is bounded below by a constant (where $u \neq 0$), and is Lebesgue integrable with $0 < \int_M F$, then $\int_M u^p = \infty$ for all $p > 0$, unless u is constant. If F is zero almost everywhere, the same conclusion holds [4]. We shall only use this last statement.

3. Localizing the problem

Let M be a c.m.s. of finite topological type with Gauss map g; M is topologically a compact Riemann surface $\bar{M}$ minus a finite number of points $q_1, \ldots, q_r$. If this is the conformal structure of M as well, then 1.1 follows from Osserman's work and Picard's theorem. More precisely, consider g in a punctured disc neighborhood of one of the points q_j. If q_j is an essential singularity of g then 1.1 follows from Picard's theorem. Otherwise g extends meromorphically to q_j. If this happens at each q_j then g extends meromorphically to $\bar{M}$ and M has finite total curvature. Then $g(M)$ misses at most three points by Osserman's theorem.

Hence we may assume the conformal structure at some end of M is that of an annulus

$$A = \{z/0 < r^{-1} \leq |z| < r < \infty\}.$$

The Weierstrass representation of M in A is (g, w) where $w = f(z)\,dz$, f is holomorphic in A, and the zeros of f coincide with the poles of g.

The metric on M in A is given by

$$ds = \lambda(z)\,|dz|,$$

where $2\lambda = |f(z)|\,(1 + |g(z)|^2)$. Since M is complete we have

$$\int_\gamma ds = \infty$$

for every path γ in A tending to the boundary $|z| = r$.

Suppose that g does not take on every value infinitely often, with the possible exception of six values. Then there are seven points $p_1, \ldots, p_7$ on the sphere that have a finite number of preimages in M. Since these preimages are in a compact subset of M, we can choose the end A so that g does not take any of the values $p_1, \ldots, p_7$ in A.

Now 1.1 will follow from:

THEOREM 3.1. *g can omit at most six points in an annulus end A of M.*

The proof of 3.1 will occupy the rest of the paper and will be by contradiction. So, we assume g misses seven points $p_1, \ldots, p_7$ in A. After a rotation of M we can assume $p_7 = \infty$, so that $g : A \to \mathbb{C} - \{p_1, \ldots, p_6\}$.

Consider the function on A:

$$h(z) = \frac{g'(z)}{f^q(z) \prod_{j=1}^{6} (g(z) - p_j)^\alpha}$$

where q and α will be chosen later. Since g has no poles in A, f has no zeros in A and h is holomorphic in A.

Using Yau's theorem, we will prove

$$\int_A |h|^p = \infty \qquad \mathrm{I}$$

for all $p > 0$. Then (by generalizing Xavier's technique) we will prove this latter integral is finite, thus proving 3.1.

4. How we apply Yau's theorem

Since h is holomorphic in A, we have $\Delta \log |h| = 0$ almost everywhere; however, we cannot directly conclude I is true since the metric on A is not complete at the boundary component $|z| = r^{-1}$.

Consider the metric $d\tau$ on A defined by:

$$d\tau = \lambda(z)\lambda\left(\frac{1}{z}\right) |dz|.$$

It is easy to check $d\tau$ is a complete metric on $\mathring{A}$. Define $\hat{h}(z) = h(z)h(1/z)$. Then $\Delta \log |\hat{h}| = 0$ on $\mathring{A}$ so Yau's theorem applies and we have

$$\infty = \int_A |\hat{h}| = \int_A \left|h(z)h\left(\frac{1}{z}\right)\right| \lambda^2(z)\lambda^2\left(\frac{1}{z}\right) dx\, dy.$$

Write $\mathring{A} = A_1 + A_2 + B$ where A_1, A_2 are small annular neighborhoods of

$|z|=r^{-1}$, $|z|=r$, respectively, and B is their compact complement. Moreover, choose A_1, A_2 so that A_2 is the image of A_1 by the map $z \mapsto z^{-1}$.

Since $\int_B |\hat{h}| < \infty$, we have $\int_{A_1} |\hat{h}| = \infty$ or $\int_{A_2} |\hat{h}| = \infty$. By change of variables we have

$$\int_{A_1} |\hat{h}| = \int_{A_2} |\hat{h}| \left|\frac{1}{z}\right|^4;$$

hence

$$\int_{A_1} |\hat{h}| = \int_{A_2} |\hat{h}| = \infty.$$

On A_2, $\lambda(1/z)$ and $h(1/z)$ are both bounded; hence

$$\begin{aligned}\int_{A_2} \left|h(z)h\left(\frac{1}{z}\right)\right| \lambda^2(z)\lambda^2\left(\frac{1}{z}\right) dx\, dy &\leq C\int_{A_2} |h(z)|\, \lambda^2(z)\, dx\, dy \\ &= C\int_{A_2} |h|\end{aligned}$$

and we conclude $\int_A |h|^p = \infty$, for all $p > 0$ (since $\Delta \log |h|^p = p\Delta \log |h|$).

The only point to check is that if $\hat{h}$ is a nonzero constant, then $\int_{\mathring{A}} |\hat{h}| = \infty$. This follows from the fact that the volume of $\mathring{A}$ is infinite with respect to any complete metric: just apply Yau's theorem to $|z|$ on $\mathring{A}$.

5. Another Estimate of $\int_A |h|^p$

F. Xavier proved the integral in question is finite on the disc H, when h is defined on all of H, i.e., when g missed seven points on M. His proof uses some estimates coming from normal families in the disc. We will do the same analysis in the annulus.

LEMMA 5.1. *Let $g: A \to \mathbb{C}$ be holomorphic and omit k points $p_1, \ldots, p_k$, $k \geq 6$. Then there exists α and p, $0 < p < 1$, such that*

$$\int_A \frac{|g'(z)|^p (1+|g(z)|^2)^2}{\prod_{j=1}^{k} |g(z)-p_j|^{p\alpha}}\, dx\, dy < \infty.$$

We claim 5.1 proves $\int_A |h|^p$ is finite for some $p > 0$, thus establishing 3.1.

To see this, notice that the volume form on H, in terms of (g, f) is given by $\lambda^2\,dx\,dy$, where $2\lambda = |f|\,(1+|g|^2)$. Hence the integral of 5.1 is in fact $\int_A |h|^p$, where $pq = 2$.

The remainder of this section will be devoted to obtaining a majorant for the integrand of 5.1:

$$J = \frac{|g'(z)|^p(1+|g(z)|^2)^2}{\prod_{j=1}^{k} |g(z)-p_j|^{p\alpha}}.$$

Then in section 6 we will prove this majorant has a finite integral on A.

LEMMA 5.2. *Assume $kp\alpha = 5$, $p\alpha \le 1$, $\alpha \le 1$ and $p(2-\alpha) \le 1$. Then there are constants C_j, such that*

$$J \le K = \frac{C_j\,|g'|^p}{(|g(z)-p_j|^{\alpha} + |g(z)-p_j|^{2-\alpha})^p}.$$

More precisely, we shall decompose A into measurable sets, on each of which this estimate is valid.

Proof of 5.2. Let $T_j(z) = |g(z)-p_j|$ and $D_j = \{z \in A/|g(z)-p_j| \le \epsilon < 1\}$, where ϵ is chosen so that the D_j are pairwise disjoint for $1 \le j \le k$.

On each D_j, we have g bounded so there is a constant C_j such that for $z \in D_j$,

$$J \le \frac{C_j\,|g'|^p}{T_j^{p\alpha}}.$$

Now if $\alpha \le 1$, then on D_j, we have

$$\frac{1}{T_j^{\alpha}} \le \frac{2}{T_j^{\alpha} + T_j^{2-\alpha}}.$$

Hence on $V = \bigcup_{j=1}^{k} D_j$, 5.2 is established.

Now consider J on $A - V$. On the set where $\epsilon \le T_j \le 1$, we have $|g|$ and T_i^{-1} bounded, so there is a constant C such that

$$J \le \frac{C\,|g'|^p}{T_j^{p\alpha}}$$

for $\alpha \le 1$, so the same reasoning as above establishes 5.2.

So assume each $T_j \geq 1$. Then if $kp\alpha = 5$, $p\alpha \leq 1$, $p(2-\alpha) \leq 1$, and $a_0, \ldots, a_4$ are positive numbers, then it is clear that

$$\frac{\sum_{i=0}^{4} a_i T_j^i}{T_j^{kp\alpha}} \leq \frac{C_1}{T_j} \leq \frac{C_2}{(T_j^\alpha + T_j^{2-\alpha})^p}$$

for some constants C_1, C_2.

On the set in $A - V$ where $1 \leq T_i \leq K$, for some K, we have g bounded and

$$J \leq \frac{C\,|g'|^p}{T_i}$$

for some constant C. So 5.2 follows on this set.

We know $T_1, \ldots, T_k$ differ by bounded numbers that depend only on $p_1, \ldots, p_k$. Hence there is a constant $K > 1$ and $C > 0$ such that if each $T_i \geq K$, then

$$\frac{1}{\prod_{i=1}^{k} T_i^{p\alpha}} \leq \frac{C}{T_j^{p\alpha k}}$$

for each j.

Hence

$$\begin{aligned} J &\leq \frac{C\,|g'|^p}{T_j^{kp\alpha}} (1 + |g|^2)^2 \\ &\leq C\,|g'|^p \frac{(1 + (|g(z) - p_j| + |p_j|)^2)^2}{T_j^{kp\alpha}} \\ &= C\,|g'|^p \frac{(a_0 + a_1 T_j + a_2 T_j^2 + a_3 T_j^3 + a_4 T_j^4)}{T_j^{kp\alpha}}. \end{aligned}$$

Now by our previous remark, 5.2 is proved.

6. An estimate for $\int_A K\,dx\,dy$

We will use Xavier's idea of employing normal families to estimate this integral.

LEMMA 6.1. *Let $g:\Omega\to\mathbb{C}$ be a holomorphic map on a domain $\Omega\subset\mathbb{C}$, such that g omits two values 0 and $a\neq 0$. Let $\phi_\lambda: H\to\Omega$ be a family of conformal maps indexed by $\lambda\in\Gamma$. There is a $C>0$ such that for $z=\phi_\lambda(0)$,*

$$\frac{|g'(z)|}{1+|g(z)|^2}\leq\frac{C}{|\phi'_\lambda(0)|},$$

$$\frac{|g'(z)|}{|g(z)|^\alpha+|g(z)|^{2-\alpha}}\leq\frac{Ck}{|\phi'_\lambda(0)|}$$

for $k\in Z$ and $\alpha=1-(1/k)$.

Proof of 6.1. The family $g\phi_\lambda: H\to\mathbb{C}$ is a normal family. Hence, by [1],

$$\frac{|(g\phi_\lambda)'(0)|}{1+|g\phi_\lambda(0)|^2}\leq C.$$

Now to obtain the second inequality, apply the above to the family $(g\phi_\lambda)^{1/k}$.

We will apply 6.1 to $\Omega=$ the annulus $A\subset H$, $A=\{e^{-\pi}<|z|<1\}$, and the family of conformal covering maps $\phi_z: H\to A$, given by

$$\phi_z(w)=\exp\left(i\log\left(\frac{w\bar{d}-d}{w-1}\right)\right),$$

where $w\in H$, $z\in A$, $z=\exp(i\log d)$, $\phi_z(0)=z$.

Calculating the derivative we obtain:

$$|\phi'_z(0)|=|z|\left|1-\frac{\bar{d}}{d}\right|.$$

We have $\bar{d}/d=e^{i\log|z|^2}$.

Let $t=\log|z|^2$. Then

$$\left|1-\frac{\bar{d}}{d}\right|=|1-\cos t-i\sin t|=\sqrt{2-2\cos t}.$$

In a neighborhood of ∂A (i.e., $t=0$), we have

$$C_1\leq\frac{\sqrt{2-2\cos t}}{\sqrt{t^2}}\leq C_2,$$

hence

$$\int_A \frac{1}{|\phi_z'(0)|^p}\,dx\,dy \le C\int_A \frac{1}{|z|^p\,(-\log|z|)^p}\,dx\,dy$$
$$\le C_1\int_A \frac{1}{|z|^2\,(-\log|z|)^p}\,dx\,dy$$
$$= C_2\int_c^1 \frac{rdr}{r^2(-\log r)^p} = C_2[\log^{1-p}(r)]_c^1,$$

and this is finite for $p<1$.

Hence $\int_A J\,dx\,dy$ is finite, provided $kp\alpha = 5$, $p\alpha<1$, $\alpha<1$ and $(2-\alpha)p\le 1$. So take $k=6$, $\alpha=\frac{11}{12}$, $p=\frac{60}{66}$ to conclude.

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Received January 14, 1987