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Representations of bipartite completed posets

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0. General concepts and results

0.1. A *completed poset* $\tilde{S}$ consists of a finite set S, a partial order relation $S^{\leqq} = \{(s, t) \in S^2 : s \leqq t\}$ on S and an equivalence relation $\sim$ on $S^{\leqq}$. These data are subjected to the condition that $r \leqq s \leqq t$ and $(r, t) \sim (r', t')$ imply the existence of a unique s' satisfying $r' \leqq s' \leqq t'$, $(r, s) \sim (r', s')$ and $(s, t) \sim (s', t')$.

In case $(s, s) \sim (s', s')$ we shall write $s \sim s'$, thus obtaining an equivalence relation on S. In fact, it follows from the axioms that $(s, t) \sim (s', s')$ implies $s = t$ and that $(s, t) \sim (s', t')$ implies $s \sim s'$ and $t \sim t'$.

0.2. Completed posets provide a convenient formulation of the matrix problem which is our real center of interest. We first attach two categories to the completed poset $\tilde{S}$: Let $s_1, \ldots, s_n$ be a numbering of S and k a field. The objects of our first category $\tilde{S}_k$ are the vectors $v = [v_1 \cdots v_n] \in \mathbb{N}^n$ such that $v_i = v_j$ if $s_i \sim s_j$. In order to define the morphisms, consider two objects u, v and a matrix $B \in k^{|v| \times |u|}$, where $|v| = v_1 + \cdots + v_n$ (we do accept matrices having no row or no column!). We subdivide B into rectangular blocks $\bar{B}_{ji} \in k^{v_j \times u_i}$ $(1 \leqq i, j \leqq n)$ in the usual way, and we define $\mathrm{Hom}(u, v)$ as the subspace of $k^{|v| \times |u|}$ formed by the B such that $\bar{B}_{ji} = 0$ if $s_i \nleqq s_j$ and $\bar{B}_{ji} = \bar{B}_{qp}$ if $(s_i, s_j) \sim (s_p, s_q)$. The composition of $\tilde{S}_k$ is given by matrix multiplication (the condition imposed on completed posets makes sure that $B'B \in \mathrm{Hom}(u, w)$ if $B \in \mathrm{Hom}(u, v)$ and $B' \in \mathrm{Hom}(v, w)$).

We call *dimension-vector* a pair $d = (d_0, \bar{d}) = [d_0 d_1 \ldots d_n] \in \mathbb{N} \times \mathbb{N}^n$, where $\bar{d} = [d_1 \cdots d_n] \in \tilde{S}_k$. Further, we call *representation of $\tilde{S}$ of dimension d* a pair (d, M) formed by a dimension-vector d and a matrix $M \in k^{d_0 \times |\bar{d}|}$. For $i \geqq 1$, we call d_i the *dimension of (d, M) at the point s_i*. A morphism of representations $(d, M) \to (e, N)$ is given by a pair (A, B) of matrices $A \in k^{d_0 \times e_0}$ and $B \in \mathrm{Hom}(\bar{d}, \bar{e})$ such that $AN = MB^T$. Composition is defined by $(A', B') \circ (A, B) = (AA', B'B)$. Let $\mathrm{rep}\, \tilde{S}$ denote the category thus defined.

The representations of completed posets play a central rôle in general representation theory. For information on how they fit into this broader context, we refer to [5, 9].

Our problem is to determine the isomorphism classes of $\mathrm{rep}\, \tilde{S}$. If we set

$GL_m = \{A \in k^{m\times m} : \det A \neq 0\}$ and $\operatorname{Aut} \bar{d} = \operatorname{Hom}(\bar{d}, \bar{d}) \cap GL_{|\bar{d}|}$, these classes correspond bijectively to the orbits of the groups $GL_{d_0} \times \operatorname{Aut} \bar{d}$ in the spaces $k^{d_0 \times |\bar{d}|}$ under the actions $(A, B; N) \mapsto ANB^{-T}$. We are especially interested in the case where there are only finitely many orbits for each d.

0.3. Of course, the investigation of these orbits is greatly facilitated by the observation that the category rep $\tilde{S}$ is additive. In fact, we fix and shall need a canonical construction for the direct sum of two representations. Our "canon" is illustrated with an example in Fig. 1, where $(e, P) \oplus (f, Q) = (e+f, M)$. The symbol $s \to t$ means that t is subsequent to s in S. The produced morphisms are our canonical projections. The canonical immersions are defined by the transposed matrices. The (canonical) direct sum $\oplus_{i=1}^{l} U_i$ of a sequence $U_1, \ldots, U_l$ of representations is defined recursively by $\oplus_{i=1}^{l} U_i = (\oplus_{i=1}^{l-1} U_i) \oplus U_l$.

$\tilde{S}$
$$\begin{tikzcd} \circ \arrow[r] \arrow[d] & \circ^{s_3} \arrow[d] \\ \underset{s_2}{\circ} \arrow[r] & \underset{s_4}{\circ} \end{tikzcd} \qquad (s_1, s_3) \sim (s_2, s_4)$$

$$e = [2 \quad 1 \quad 1 \quad 1 \quad 1] \qquad f = [1 \quad 1 \quad 1 \quad 0 \quad 0] \qquad e+f = [3 \quad 2 \quad 2 \quad 1 \quad 1]$$

$$P = \left[\begin{array}{c:c:c:c} a & b & c & d \\ a' & b' & c' & d' \end{array}\right] \qquad Q = [a'' \,\vdots\, b'' \,\vdots\,\vdots\,] \qquad M = \left[\begin{array}{c:c:c:c} a0 & b0 & c & d \\ a'0 & b'0 & c' & d' \\ 0a'' & 0b'' & 0 & 0 \end{array}\right]$$

$$\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \left[\begin{array}{cc:cc:c:c} 1 & 0 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 1 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 1 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \right) : (e+f, M) \to (e, P),$$

$$\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \left[\begin{array}{cc:cc:c:c} 0 & 1 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 1 & 0 & 0 \\ \hdashline \hdashline \end{array}\right] \right) : (e+f, M) \to (f, Q)$$

Figure 1

We call a representation *indecomposable* if it is not zero and not *isomorphic* to the direct sum of two non-zero representations. It is clear that each representation of $\tilde{S}$ is isomorphic to a direct sum of indecomposables. The unicity of such a decomposition up to isomorphism follows from the fact that idempotent endomorphisms of rep $\tilde{S}$ split (1.1). This reduces our classification problem to the description of the indecomposables. We are particularly interested in the case where $\tilde{S}$ is *representation-finite*, i.e. admits only finitely many isomorphism classes of indecomposables.

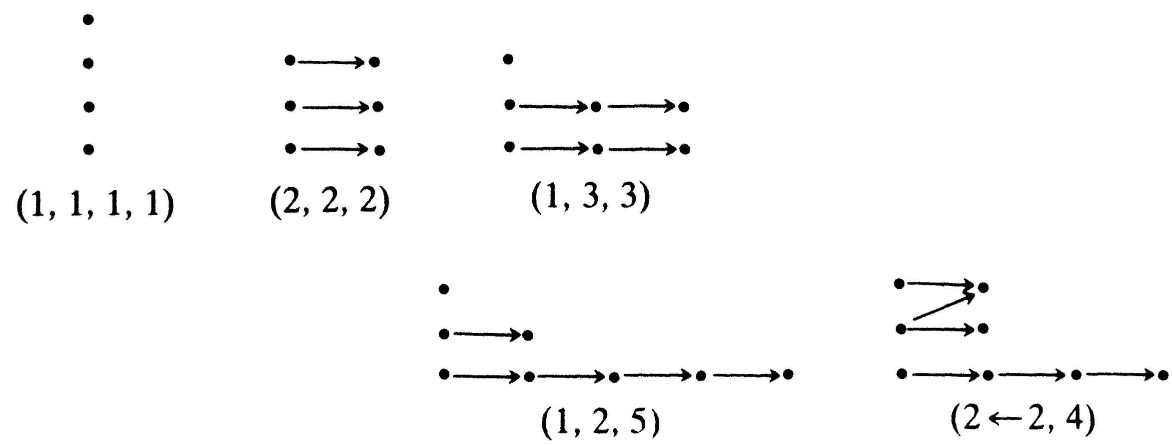


Figure 2

The representation-finite $\tilde{S}$ are determined in [1][2] when $\tilde{S}$ is a *trivially completed* poset (i.e. $\sim$ is the identity), in [6] when $(s, t) \sim (s', t')$ and $(s, t) \neq (s', t')$ imply $s = t$ and $s' = t'$. The result in the first case is that a (trivially completed) poset is representation-finite iff it does not contain a *full subposet* (= subset equipped with the induced order) of one of the 5 forms given in Fig. 2 (where the symbol $s \to t$ now means that t is subsequent to s in the subposet!).

Because of the striking simplicity of this result, our general method is to reduce the characterization of representation-finite completed posets to the trivially completed case. In the present article, we present such a reduction in a particular case which happens to be crucial for the general solution, as will be shown in a forthcoming paper.

0.4. In the case of a representation-finite $\tilde{S}$, it is easy to prove that each equivalence class of S is linearly ordered and has cardinality $\leqq 3$. From the first part of this statement and the axioms of completed posets it then follows that $(s, t) \sim (s', t)$ implies $s = s'$, and dually that $(s, t) \sim (s, t')$ implies $t = t'$. In fact, the conditions which we shall impose on $\tilde{S}$ in this article are much stronger.

Let $\{1, \ldots, m\} \subset \mathbb{N}$ be an interval and $\mu : \{1, \ldots, m\} \to \{1, \ldots, m\}$ a non-decreasing function such that $\mu(i) \geqq i + 1$ for each $i < m$. By a *μ-chain* $\mathring{P}$ in a partially ordered set P we mean a subset $\mathring{P} \subset P$ consisting of m linearly ordered elements $s_1 < \cdots < s_m$ such that for each $i \leqq m$ the interval $[a_i, a_{\mu(i)}] = \{p \in P : a_i \leqq p \leqq a_{\mu(i)}\}$. coincides with $\{s_i, s_{i+1}, \ldots, s_{\mu(i)}\}$. Whenever we refer to a *bipartite* completed poset $\tilde{S} = P \triangleleft Q$, we implicitly assume: first that we are given a function μ and two finite posets P, Q equipped with μ-chains $\mathring{P} = \{s_1 < \cdots <$

$s_m\}$, $\check{Q} = \{s'_1 < \cdots < s'_m\}$ respectively; second that $\tilde{S}$ is described in terms of the data as follows.

a) $S = P \amalg Q$ (= disjoint union)

b) $S^{\leqq} = P^{\leqq} \cup Q^{\leqq} \cup P \times Q$ (in particular, $p \in P$ and $q \in Q$ imply $p < q$)

c) $(s_i, s_j) \sim (s'_i, s'_j)$ if $i \leqq m$ and $j \leqq \mu(i)$; any other $(s, t) \in S^{\leqq}$ is equivalent only to itself.

The points of $\mathring{P}$ and $\mathring{Q}$ are called *thick*, those of $\dot{P} = P \backslash \mathring{P}$ and $\dot{Q} = Q \backslash \mathring{Q}$ *thin*. For each thick point $s \in S$, we denote by s' the point of S such that $s' \sim s \neq s'$. The quasidual of $\tilde{S} = P \triangleleft Q$ is by definition $\tilde{S}^* = Q \triangleleft P$.

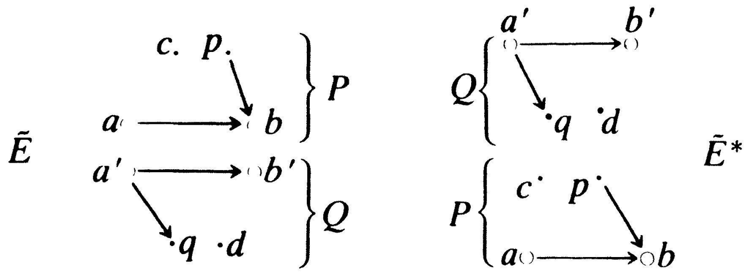


Figure 3

Figure 3 shows an example of a bipartite completed poset and its quasidual. There we have $m = 2$ and $\mu(1) = \mu(2) = 2$, the thick points are represented by ringlets and the arrows from the first to the second components of $\tilde{E}$ and $\tilde{E}^*$ are omitted.

Of course, the dual $\tilde{S}^0$ of a bipartite completed poset S can also be defined. But we have $\tilde{E}^0 \xrightarrow{\sim} \tilde{E}$ in the case of Fig. 3.

0.5. Let $\tilde{S} = P \triangleleft Q$ be a bipartite completed poset. For each $s \in S$, we set $S(s) = \{t \in S : s \nleqq t \nleqq s\}$, endow $S(s)$ with the order relation induced by S, formally add to $S(s)$ a smallest element 0 and a largest element 1 and denote the poset obtained in this way by $\bar{S}(s) = S(s) \cup \{0, 1\}$. With this notation, we attach two posets $\hat{P}$ and $\hat{Q}$ to the components P and Q: The poset $\hat{P}$ consists of the thin points $s \in \dot{P}$ and of the pairs (p, t) where $p \in \mathring{P}$ and $t \in \bar{S}(p')$. We equip the subset $\dot{P}$ of $\hat{P}$ with the order induced by $\tilde{S}$ and set $s \leqq (p, t)$ iff $s \leqq p$, $(p, t) \leqq s$ iff $p \leqq s$. We further set $(p_1, t_1) \leqq (p_2, t_2)$ in the following two cases:

a) $p_1 < p_2$ and $(p_1, p_2) \nsim (p'_1, p'_2)$.

b) $p_1 \leqq p_2$, $(p_1, p_2) \sim (p'_1, p'_2)$ and one of the conditions $p'_1 \leqq t_2 \neq 1$, $0 \neq t_1 \leqq p'_2$ or $t_1 \leqq t_2$ holds.

The description of $\hat{Q}$ is dual (and quasidual) to that of $\hat{P}$. In particular, the elements of $\hat{Q}$ have the form $t \in \dot{Q}$ or (q, s) where $q \in \mathring{Q}$ and $s \in \bar{S}(q')$.

In the case $\tilde{S} = \tilde{E}$ (Fig. 3), $\hat{P}$ and $\hat{Q}$ are given by Fig. 4.

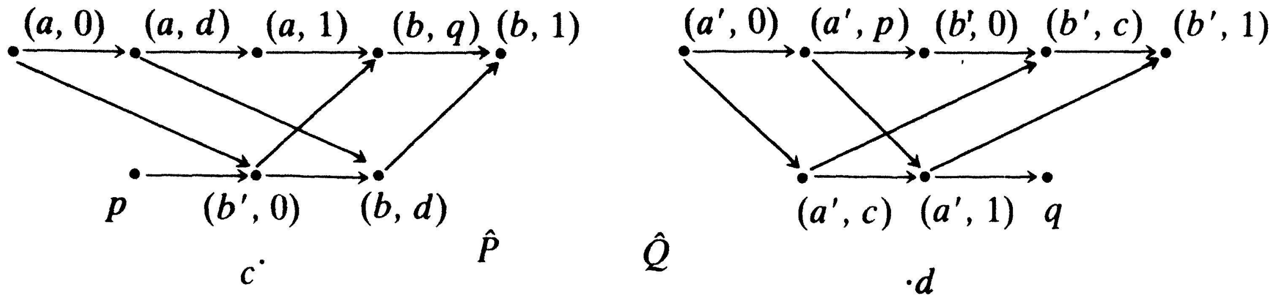


Figure 4

Now we can formulate our first main theorem:

THEOREM 1. *The bipartite completed poset $\tilde{S} = P \lhd Q$ is representation-finite iff so are the posets $\hat{P}$ and $\hat{Q}$.*

0.6. Let T be a subset of S which is *stable* under the equivalence relation of S (i.e. $s \in S$, $t \in T$ and $t \sim s$ imply $s \in T$). The structure carried by $\tilde{S}$ then naturally induces a completed poset structure $\tilde{T}$ on T. If we equip T with the numbering "induced" by that of S (0.2), we obtain a fully faithful embedding rep $\tilde{T} \to$ rep $\tilde{S}$. More precisely, we can extend each dimension vector d of $\tilde{T}$ by zero and obtain a dimension vector d^0 of $\tilde{S}$ ($d_0^0 = d_0$, $d_i^0 = d_j$ if $s_i = t_j$ and $d_i^0 = 0$ if $s_i \notin T$). The embedding functor is then simply $(d, M) \mapsto (d^0, M)$. It permits us to identify the set ind $\tilde{T}$ of isomorphism classes of rep T with a subset of ind $\tilde{S}$.

For instance, the *trivial representation* $\varnothing_0$ of S, whose dimension-vector is $[10 \cdots 0]$, is associated with a representation of $\tilde{\varnothing}$! More generally, each representation (e, M) of $\tilde{S}$ has the above form (d^0, M) if we take T to be the *support* $\{s_i \in S : e_i \neq 0\}$ of (e, M).

If the support of (e, M) equals S, we say that (e, M) is *faithful*. And we say that $\tilde{S}$ is *faithful* if $\tilde{S}$ admits a faithful indecomposable representation.

THEOREM 2. *Let $\tilde{S} = P \lhd Q$ be a faithful bipartite completed poset.*

a) *If the poset $S(s)$ is linearly ordered for each $s \in \mathring{P}$ (resp. $s \in \mathring{Q}$), then there is a natural bijection from* ind $\tilde{S} \backslash$ind $\dot{P}$ *onto* ind $\hat{Q} \backslash \{\varnothing_0\}$ *(resp. from* ind $\tilde{S} \backslash$ind $\dot{Q}$ *onto* ind $\hat{P} \backslash \{\varnothing_0\}$).

b) *If $\tilde{S}$ is representation-finite and if there exist thick points $p \in \mathring{P}$ and $q \in \mathring{Q}$ such that neither $S(p)$ nor $S(q)$ is linearly ordered, then $\tilde{S}$ is isomorphic to $\tilde{E}$ or $\tilde{E}^*$* (0.3).

1. The easy direction

Our objective in this section is to prepare the general demonstration by proving the first part of Theorem 2 and the necessity of the condition of theorem

1. *From* 1.2 *onwards, we fix* P, Q *and* $\tilde{S} = P \lhd Q$. We choose a numbering of $S = P \cup Q$ which first numbers $\mathring{P}$ (in the order of succession $s_1, \ldots, s_m$ imposed by P), then $\dot{P}$, $\mathring{Q}$ (in the order of succession $s'_1, \ldots, s'_m$) and finally $\dot{Q}$.

1.1. Let us briefly recall why representations of a completed poset $\tilde{S}$ can be "uniquely" decomposed into indecomposables.

We first notice that the category $\tilde{S}_k$ (0.2) is *k-linear* in the sense that the morphism spaces carry k-vector-space structures, that the composition is bilinear and that finite direct sums exist: In fact, we can set $u \oplus v = u + v$ if we define the canonical immersions and projections in the obvious way. Each point $t \in S$ gives rise to an indecomposable $\bar{t} \in \tilde{S}_k$ whose endomorphism-algebra is local ($\bar{t}_i = 1$ or 0 according as $s_i \sim t$ or $s_i \not\sim t$). The map $t \mapsto \bar{t}$ yields a bijection between the equivalence classes of S and the indecomposables of $\tilde{S}_k$. Each object $v \in \tilde{S}_k$ is a finite direct sum of indecomposables. Finally, for each idempotent $F \in \operatorname{Hom}(v, v)$, there exist morphisms $R \in \operatorname{Hom}(v, u)$ and $S \in \operatorname{Hom}(u, v)$ such that $F = SR$ and $\mathbb{1}_u = RS$ (since $\operatorname{Hom}(v, v)$ is a finite-dimensional algebra, F is conjugate to a sum of idempotents occurring in the natural decomposition of $\mathbb{1}_v$ into pairwise annihilating primitive idempotents).

Like $\tilde{S}_k$, the category $\operatorname{rep} \tilde{S}$ (0.2) is k-linear. Each decomposition $(d, M) \xrightarrow{\sim} (e, P) \oplus (f, Q)$ gives rise to an idempotent $(E, F) \in \operatorname{End}(d, M)$, the projection onto the first summand along the second. To prove the converse, we must supply each idempotent (E, F) with morphisms

$$(d, M) \xrightarrow{(V,R)} (e, P) \xrightarrow{(U,S)} (d, M)$$

such that $(E, F) = (VU, SR)$ and $(\mathbb{1}_{e_0}, \mathbb{1}_{|e|}) = (UV, RS)$. For this, we first construct U, V (clear!) and R, S as above; then we set $P = UMR^T$.

Since the direct sum decompositions of (d, M) corresponds to the decompositions of $\mathbb{1}_{(d,M)}$ into pairwise annihilating idempotents, (d, M) is a direct sum of indecomposable representations, which are uniquely determined up to isomorphism.

1.2. We now assume that $\tilde{S} = P \lhd Q$. Using the action of $GL_{d_0} \times \operatorname{Aut} \bar{d}$ (0.2), we can reduce each representation (d, M) of $\tilde{S}$ to the form of Fig. 5. Indeed, we can first find a matrix $A \in GL_{d_0}$ such that

$$AM = \left[\begin{array}{c:c} M_P & M' \\ \hdashline 0 & M_Q \end{array}\right],$$

where $M_P \in k^{r \times |\bar{d}_P|}$ and $r = \operatorname{rank} M_P$. Then there is a C such that $M_P C = M'$, and

AMB^{-T} is given the wanted form by setting $B = \begin{bmatrix} \mathbb{1} & 0 \\ C^T & \mathbb{1} \end{bmatrix} \in \operatorname{Aut} \bar{d}$.

$$\begin{array}{cl} \left[\begin{array}{c:c} M_P & 0 \\ \hdashline 0 & M_Q \end{array}\right] & \begin{array}{l} \}r \\ \}d_0 - r \end{array} \\ \underbrace{}_{|\bar{d}_P|}\ \underbrace{}_{|\bar{d}| - |\bar{d}_P|} & \end{array} \qquad \begin{array}{l} \bar{d}_P = [d_1 d_2 \cdots d_{|P|}] \\ r = \operatorname{rank} M_P \\ |P| = \text{cardinality of } P \end{array}$$

Figure 5

This means that each representation of $\tilde{S}$ is isomorphic to a "*reduced*" representation whose matrix has the form of Fig. 5. If $(A, B):(d, M) \to (e, N)$ is a morphism of reduced representations, we subdivide A, B into blocks adapted to those of M and N:

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}, \qquad B = \begin{bmatrix} B_1 & 0 \\ B_3 & B_4 \end{bmatrix}.$$

The condition $AN = MB^T$ then means that $A_1 N_P = M_P B_1^T$, $A_2 N_Q = M_P B_3^T$, $A_3 N_P = 0$ and $A_4 N_Q = M_Q B_4^T$. Since the rows of N_P are linearly independent by assumption, the equality $A_3 N = 0$ implies $A_3 = 0$.

In case (A, B) is an isomorphism, the condition imposed upon B_1 is to lie in the automorphism group of $\bar{d}_P$ in the category P_k associated with the (trivially completed) poset P. This means that for M_P we can choose representatives of the isomorphism classes of rep P and then restrict (A_1, B_1) to Aut M_P. The problem then stays with B_4, which must be an isomorphism of Q_k and share some "subblocks" with B_1. To examine into this condition, we introduce supplementary simplifying assumptions.

1.3. Let $\mathcal{U}$ be a sequence

$$(d_{11}, U_{11}), \ldots, (d_{1l_1}, U_{1l_1}), \ldots, (d_{ij}, U_{ij}), \ldots, (d_{m1}, U_{m1}), \ldots, (d_{ml_m}, U_{ml_m})$$

of pairwise nonisomorphic indecomposable representations of P such that (d_{ij}, U_{ij}) has dimension 1 at $s_i \in \mathring{P}$ and 0 at all other thick points of $P (1 \leqq i \leqq m, 1 \leqq j \leqq l_i)$. We denote by $\operatorname{rep}_{\mathcal{U}} \tilde{S}$ the full subcategory of rep $\tilde{S}$ formed by the reduced (1.2) representations (d, M) whose P-component has the form

$$(*) \quad (d_P, M_P) = (d_{11}, U_{11})^{\mu_{11}} \oplus \cdots \oplus (d_{ij}, U_{ij})^{\mu_{ij}} \oplus \cdots \oplus (d_{ml_m}, U_{ml_m})^{\mu_{ml_m}}$$

where $d_P = [\dot{r}\, d_1 \cdots d_{|P|}]$ (Fig. 5) and $\mu_{ij} \in \mathbb{N}$. We stress the point that this direct sum has to be constructed according to the prescribed canon (0.3).

The category $\mathrm{rep}_{\mathscr{U}}\, \tilde{S}$ is k-linear, and its indecomposables are indecomposable in $\mathrm{rep}\, \tilde{S}$: If (e, N) is a direct summand of $(d, M) \in \mathrm{rep}_{\mathscr{U}}\, \tilde{S}$, we can first reduce (e, N) to the form of Fig. 5 and then further convert the P-component to a direct sum of the form $(*)$.

Of special importance for us will be the case where, up to isomorphism, $\mathscr{U}$ exhausts the indecomposables of $\mathrm{rep}\, P$ whose support intersects $\mathring{P}$. In this case, the indecomposables of $\mathrm{rep}\, \tilde{S}$ which are not isomorphic to an indecomposable of $\mathrm{rep}_{\mathscr{U}}\, \tilde{S}$ lie in $\mathrm{rep}\, \dot{P}$ (0.6).

1.4. In order to describe $\mathrm{rep}_{\mathscr{U}}\, \tilde{S}$, we introduce a set $Q_{\mathscr{U}}$ which consists of the representations (d_{ij}, U_{ij}) and of the thin points of Q. We equip $Q_{\mathscr{U}}$ with the following *relation* $R \subset Q_{\mathscr{U}}^2$: In case $q, r \in \dot{Q}$ we set qRr (i.e. $(q, r) \in R$!) iff $q \leqq r$ in Q. Similarly, we set $qR(d_{ij}, U_{ij})$ (resp. $(d_{ij}, U_{ij})Rr$) iff $q \leqq s_i'$ (resp. $s_i' \leqq r$) in Q. In case $(s_i, s_u) \sim (s_i', s_u')$ we set $(d_{ij}, U_{ij})R(d_{uv}, U_{uv})$ iff there exists a morphism $(A, B):(d_{ij}, U_{ij}) \rightarrow (d_{uv}, U_{uv})$ such that $\bar{B}_{ui} \neq 0$ (0.2). Finally, we also set $(d_{ij}, U_{ij})R(d_{uv}, U_{uv})$ if $i \leqq u$ and $(s_i, s_u) \not\sim (s_i', s_u')$.

The following proposition uses the notations of 1.2 and 1.3. In particular, if $(d, M) \in \mathrm{rep}_{\mathscr{U}}\, \tilde{S}$, (d_P, M_P) is the direct sum of 1.3. By d_Q we denote the row

$$d_Q = [(d_0 - r)\, \mu_{11}\mu_{12} \cdots \mu_{ml_m} d_{1+|\dot{Q}|} \cdots d_n] \in \mathbb{N}^{|Q_{\mathscr{U}}|+1}$$

PROPOSITION a) *The relation R is a partial order on $Q_{\mathscr{U}}$.*

b) *If $(A, B):(d, M) \rightarrow (e, N)$ is a morphism of* $\mathrm{rep}_{\mathscr{U}}\, \tilde{S}$, *the block B_4 belongs to the morphism space* $\mathrm{Hom}\,(\tilde{d}_Q, \tilde{e}_Q)$ *of $Q_{\mathscr{U}k}$.*

c) *The reduction functor $\mathscr{R}$:*$\mathrm{rep}_{\mathscr{U}}\, \tilde{S} \rightarrow \mathrm{rep}\, Q_{\mathscr{U}}$, $(d, M) \mapsto (d_Q, M_Q)$ *which maps a morphism (A, B) onto (A_4, B_4) is an epivalence.*

The neologism *epivalence*, chosen here for a widely used notion of representation theory, means that $\mathscr{R}$ *detects* isomorphisms (μ is invertible if so is $\mathscr{R}\mu$) and induces surjections on the morphism spaces and on the isomorphism classes of the objects. It follows that $\mathscr{R}$ induces a bijection between the isomorphism classes.

Proof. a) The crucial point is to prove that $(d_{ij}, U_{ij})R(d_{uv}, U_{uv})$ and $(d_{uv}, U_{uv})R(d_{yz}, U_{yz})$ imply $(d_{ij}, U_{ij})R(d_{yz}, U_{yz})$. This is clear by definition if $(s_i, s_y) \not\sim (s_i', s_y')$. Otherwise, there are morphisms $(A, B):(d_{ij}, U_{ij}) \rightarrow (d_{uv}, U_{uv})$ and $(C, D):(d_{uv}, U_{uv}) \rightarrow (d_{yz}, U_{yz})$ such that $\bar{B}_{ui} \neq 0 \neq \bar{D}_{yu}$. It follows from 0.4 that $(\bar{D}\bar{B})_{yi} = \sum_w \bar{D}_{yw}\bar{B}_{wi} = \bar{D}_{yu}\bar{B}_{ui} \neq 0$.

With these notations, we must also prove that $i = y$, $j = z$, implies $i = u$, $j = v$. The reason is that in case $(i, j) \neq (u, v)$, (AC, DB) would be nilpotent though $(\overline{(DB)^N})_{ii} = (\bar{D}_{iu}\bar{B}_{ui})^N \neq 0$.

b) We must prove that a block of B_4 vanishes if it is associated with a pair $(x, y) \in Q^2_{\mathscr{U}}$ such that $x \nleqq y$. Since $\bar{B}_{ba} = 0$ if $s_a \nleqq s_b$, it suffices to examine the case $x = (d_{ij}, U_{ij})$, $y = (d_{uv}, U_{uv})$ where $(s_i, s_u) \sim (s'_i, s'_u)$. Then the associated block of B_4 is equal to a certain subblock of the block $\bar{B}_{ui}$ of B. We can interpret each coefficient of this subblock as the 1×1-block $\bar{D}_{ui}$ associated with a morphism $(C, D): (d_{ij}, U_{ij}) \to (d_{uv}, U_{uv})$. By definition of the order of $Q_{\mathscr{U}}$, the coefficient is zero if $x \nleqq y$.

c) By construction, $\mathscr{R}$ induces a surjection on the objects. Let now (d, M), (e, N) be two objects of $\operatorname{rep}_{\mathscr{U}} \tilde{S}$ and (C, D) a morphism $(d_Q, M_Q) \to (e_Q, N_Q)$. We must find an (A, B) such that $C = A_4$ and $D = B_4$. Of course, we will set $A_2 = A_3 = B_2 = B_3 = 0$ (1.2). The problem is to find an $(A_1, B_1): (d_P, M_P) \to (e_P, N_P)$ such that B_1 shares appropriate blocks with B_4. More precisely, each pair $(x, y) \in Q^2_{\mathscr{U}}$ such that $x = (d_{ij}, U_{ij}) \leqq y = (d_{uv}, U_{uv})$ and $(s_i, s_u) \sim (s'_i, s'_u)$ determines a subblock of $(\bar{B}_1)_{ui} = \bar{B}_{ui}$ which is prescribed by the datum of B_4. So it is enough to prove the existence of an (A_1, B_1) for which all these subblocks are arbitrarily prescribed. As in b) above, this follows from the interpretation of the coefficients of these subblocks as 1×1-blocks associated with morphisms between direct summands of (d_P, M_P) and (e_P, N_P) of type (d_{ij}, U_{ij}) and (d_{uv}, U_{uv}).

It remains to prove that $\mathscr{R}$ detects isomorphisms: Consider a morphism $\mu: X \to Y$ such that $\mathscr{R}\mu$ is invertible, and choose a $\nu: Y \to X$ such that $\mathscr{R}\nu = (\mathscr{R}\mu)^{-1}$. The kernel K of $\operatorname{End} X \to \operatorname{End} \mathscr{R}X$ then contains $\mathbb{1}_X - \nu\mu$. Since $0 \neq Z \in \operatorname{rep}_{\mathscr{U}} \tilde{S}$ implies $\mathscr{R}Z \neq 0$, K contains no primitive idempotent. We infer that K and $\mathbb{1}_X - \nu\mu$ are nilpotent. Hence $\nu\mu$ is invertible and so is $\mu\nu$.

1.5. *Proof of the necessity in Theorem* 1. Each thick point $t \in \mathring{P}$ gives rise to two indecomposable representations of P supported by t: Their dimension-vectors are $[0\, \bar{t}_P]$ and $[1\, \bar{t}_P]$ where $\bar{t}_P \in \mathbb{N}^{|P|}$ satisfies $\bar{t}_{Pi} = 1$ if $s_i = t$ and $\bar{t}_{Pi} = 0$ if $s_i \neq t$; we denote them by $\{t\}_0$ and $\{t\}_1$.

Similarly, if $t \in \mathring{P}$ and $s \in \dot{P}$ are incomparable, we denote by $\{t, s\}_0$ "the" indecomposable representation of P with support $\{s, t\}$ and dimension-vector $[1\, \bar{s}_P + \bar{t}_P]$.

Now, if the sequence $\mathscr{U}$ of 1.2 runs through all indecomposables of $\operatorname{rep} P$ of the form $\{t\}_0$, $\{t\}_1$ and $\{t, s\}_0$, the poset $Q_{\mathscr{U}}$ of 1.4 is obviously identified with $\hat{Q}$. By Proposition 1.4c), $\hat{Q} \simeq Q_{\mathscr{U}}$ is representation-finite if so is $\tilde{S}$.

1.6. *Proof of Theorem* 2, *part a*). If $S(t)$ is linearly ordered for each $t \in \mathring{P}$, the sequence $\mathscr{U}$ chosen in 1.5 exhausts (up to isomorphism) the indecomposables of $\operatorname{rep} P$ whose support intersects $\mathring{P}$. The statement to be proved therefore follows from the last sentence of 1.3 and the Proposition 1.4c).

2. The poset $\mathring{Q}$ associated with $\tilde{S} = P \lhd Q$

The progress made in Section 2 reduces the proof of our Theorems 1 and 2 to the following combinatorial statement. Its demonstration will spread over the rest of the article, where *$\hat{P}$ and $\hat{Q}$ are always supposed to be representation-finite.*

THEOREM 3. *Suppose that $\tilde{S} = P \lhd Q$ is faithful, that $\hat{P}$ and $\hat{Q}$ are representation-finite and that there exist points $p \in \mathring{P}$ and $q \in \mathring{Q}$ such that neither $S(p)$ nor $S(q)$ are linearly ordered. Then $\tilde{S}$ is isomorphic to $\tilde{E}$ or to $\tilde{E}^*$*(0.4).

2.1. We first recall the classification of the indecomposable representations of a representation-finite poset T. According to [3] the support of a non-trivial (0.6) indecomposable is a full subset of T which is isomorphic to one of the 13 posets of Fig. 6. The number below the symbol of a listed poset is the number of its isoclasses of faithful indecomposables.

So each *supporting subposet* Σ of T (i.e. each full subposet of the form of Fig. 6) yields the indicated number of non-trivial indecomposables of T. We denote these indecomposables by $\Sigma_0, \Sigma_1 \cdots$. For instance, each "monad" $\{t_i\}$ yields 2 indecomposables, the representation $\{t_i\}_0$ whose dimension vector d satisfies $d_0 = 0$, $d_i = |\bar{d}| = 1$, and a representation $\{t_i\}_1$ with matrix [1]. Each "dyad" $\{t_i, t_j\}$ yields 1 indecomposable $\{t_i, t_j\}_0$ with matrix $[1 \,\vdots\, 1]$. Each "triad" $\{t_i, t_j, t_k\}$ yields 2 indecomposables, the first $\{t_i, t_j, t_k\}_0$ with matrix $[1 \,\vdots\, 1 \,\vdots\, 1]$, the second $\{t_i, t_j, t_k\}_1$ with matrix $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \cdots$.

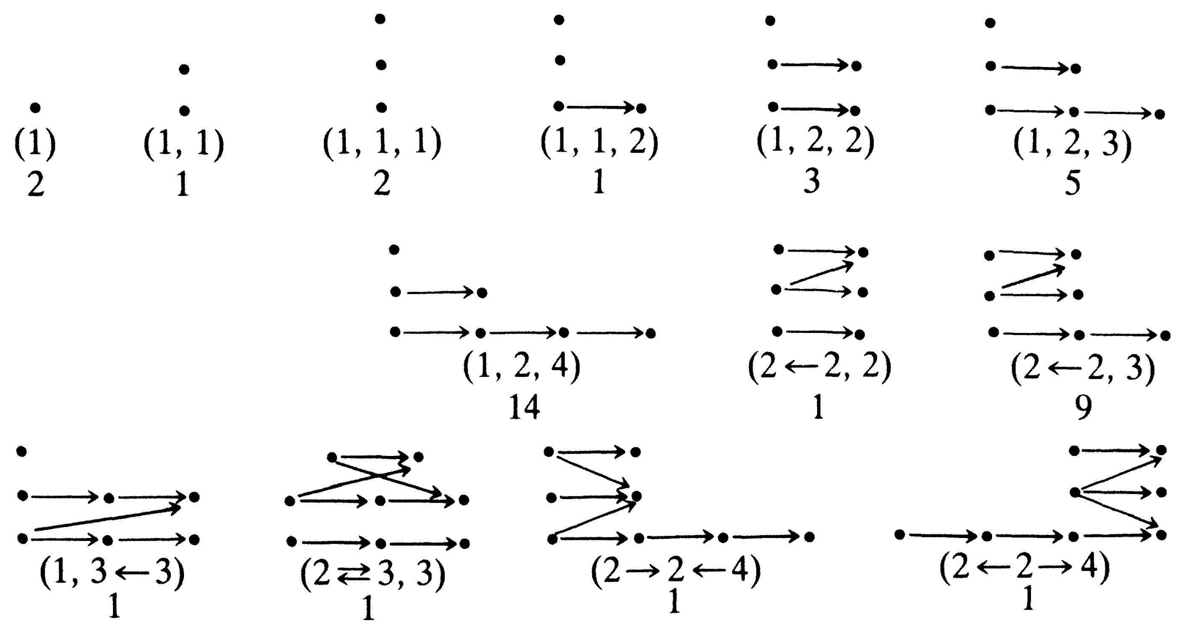


Figure 6

2.2. EXAMPLE. In the case $\tilde{S} = \tilde{E}$ (0.4), all the indecomposable representations of P whose support intersects $\mathring{P}$ are listed in Fig. 7. For each of them, the intersection consists of 1 point, and the dimension at this point is 1. Therefore, we can let the sequence $\mathcal{U}$ of 1.3 run through all the indecomposables of Fig. 7, which describes the poset $Q_{\mathcal{U}}$ of 1.4 in this particular case.

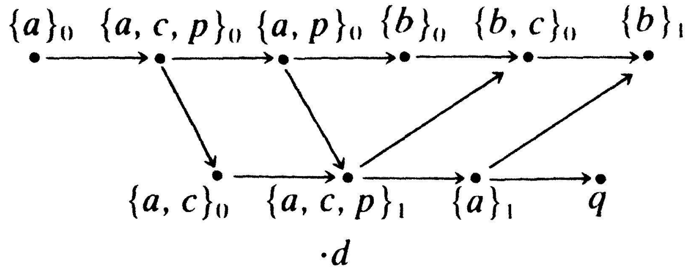


Figure 7

The poset of Figure 7 “fully” contains 11 monads, 18 dyads, 8 triads, 12 copies of $\vdots \to \cdot$ and 1 of $\vdots \rightrightarrows \vdots$, which yield 22, 18, 16, 12 and 3 indecomposables respectively. Together with $\varnothing_0$ and the five nontrivial indecomposables located in $\dot{P}$, $\tilde{E}$ therefore has 77 indecomposables and is representation-finite. Among the 50 “supporting” subposets enumerated above, there is just one which involves all the points of $\tilde{E}$ up to equivalence, namely $\{d, \{b\}_0, \{a, c, p\}_1 < q\}$. This means that E has exactly 1 faithful indecomposable, whose matrix is

$$\begin{array}{c}\left[\begin{array}{cccc:cccc} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right] \\ \begin{array}{cccccccc} a & b & c & p & a' & b' & d & q \end{array}\end{array}$$

Figure 8

2.3. Returning to the general case, we denote by $T_{t\times e}$ the poset obtained from a representation-finite poset T by substituting a chain $t_1 \to t_2 \to \cdots \to t_e$ for a point $t \in T$ as shown in Fig. 9 ($e \geqq 1$). We say that t has *multiplicity* $\geqq e$ in T if $T_{t\times e}$ is

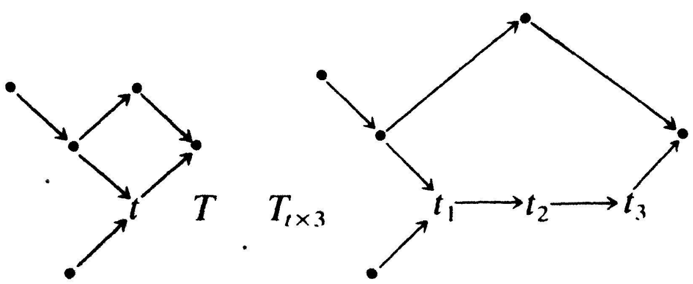


Figure 9

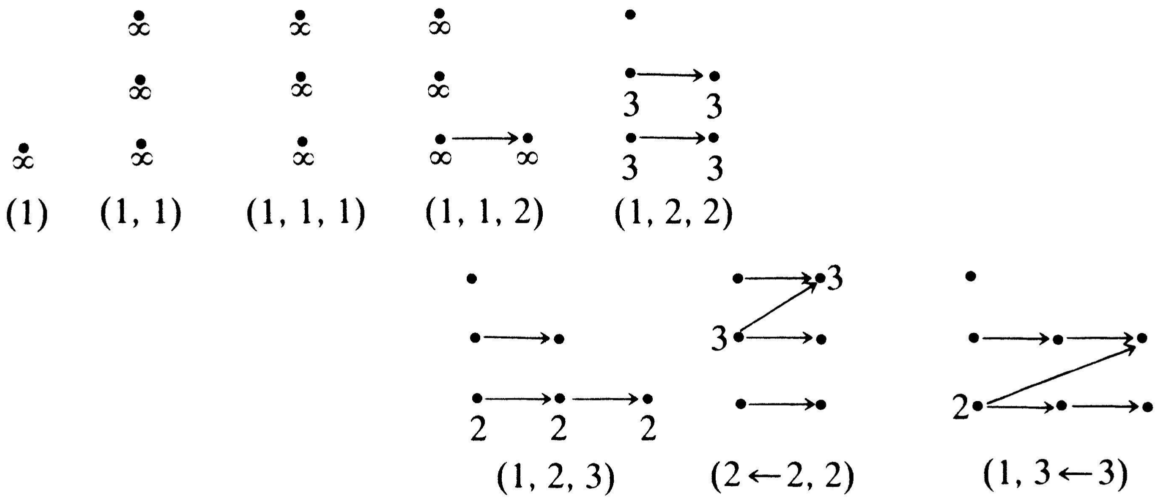
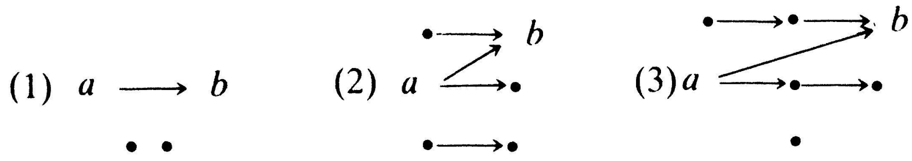


Figure 10

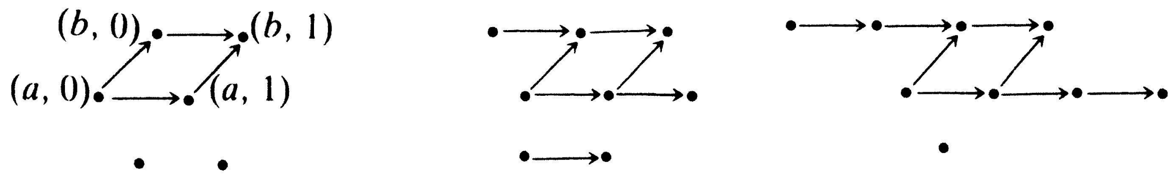
representation-finite. The multiplicities $\geqq 2$ occurring in the posets of Fig. 6 are listed in Fig. 10.

We apply the construction above in case $T = P$ and $t \in \mathring{P}$. If Q contains a chain $q_1 \to \cdots \to q_c$ of elements incomparable with $t' \in \mathring{Q}$, then $\hat{P}$ contains the full subposet formed by the elements $p \in \dot{P}$, $(r, 0)$ for $r \in \check{P}$ and $r \leqq t$, (t, q_i) for $1 \leqq i \leqq c$, and $(s, 1)$ for $s \in \mathring{P}$ and $t \leqq s$. This subposet is naturally isomorphic to $P_{t \times (c+2)}$ and is representation-finite. It follows that t has multiplicity $\geqq c+2$ in each full subposet of P containing t. In particular, a supporting subposet Σ of P (2.1) which intersects $\mathring{P}$ must be isomorphic to one of the 8 posets of Fig. 10; and $\Sigma \cap \mathring{P}$ contains only points of multiplicity $\geqq 2$ in Σ.

2.4. LEMMA. *P contains no full subposet of one of the following three forms, where a and b are supposed to be thick.*



Proof. We first assume that $(a, b) \sim (a', b')$. Then, if P contained 1), 2) or 3), $\hat{P}$ would contain a full subposet of one of the following forms, hence would not be representation-finite



In case $(a, b) \not\sim (a', b')$, we introduce the point $c \in \mathring{P}$ subsequent to a, which satisfies $a < c < b$ and $(a, c) \sim (a', c')$. In subcase 1) P then contains the full subposet $\overset{\circ}{a} \rightarrow \overset{\circ}{c}$ in contradiction to the first part of the proof. In subcase 2) or 3), P contains a full subposet of the form $\begin{tikzcd}[row sep=small] d \cdot \arrow[r] & \circ b & \cdot f \\ a \circ \arrow[r] \arrow[ur] & \cdot e & \end{tikzcd}$. Since P cannot contain a full subposet of the form $\begin{tikzcd}[row sep=small] c \circ \arrow[r] & \circ b \\ e \cdot & \cdot f \end{tikzcd}$, c must be comparable with e. This implies $c < e$ because $e \not< b$. By duality, we also obtain that $d < c$, in contradiction to $d \not< e$.

2.5. THEOREM 4. *Let Σ be the support of an indecomposable representation (d, U) of P. If Σ intersects, $\mathring{P}$, $\Sigma \cap \mathring{P}$ has exactly one point, and the dimension of (d, U) at this point is* 1.

Proof. By 2.3 Σ is isomorphic to one of the 8 posets of Fig. 10, and $\Sigma \cap \mathring{P}$ consists of points of multiplicity $\geqq 2$ in Σ. In case $|\Sigma \cap \mathring{P}| \geqq 2$, it follows from Fig. 10 that Σ contains a full subposet of one of the three forms excluded by Lemma 2.4. So we must have $|\Sigma \cap \mathring{P}| \leqq 1$.

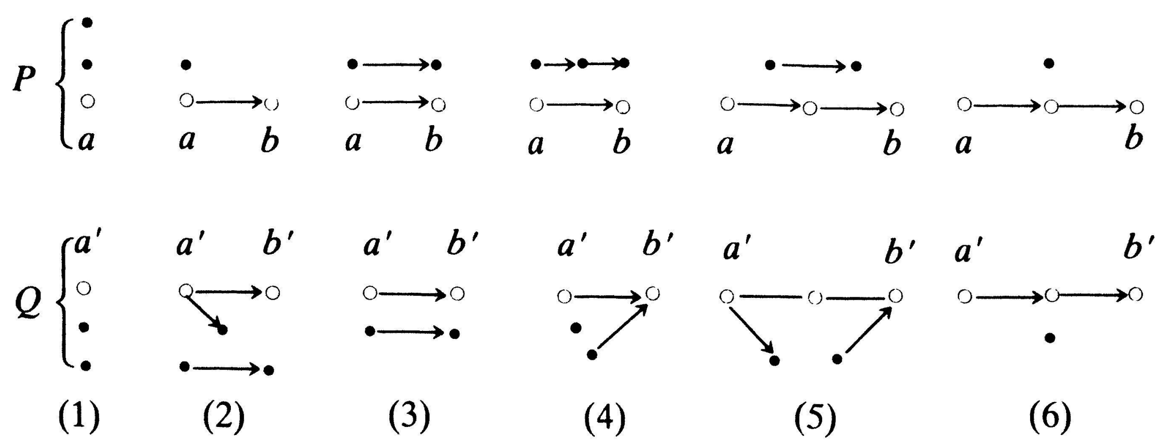
It now remains for us to go through the list of the faithful indecomposable representations of the posets of Fig. 10 and to check that the dimension at a point of multiplicity $\geqq 2$ is always 1.

2.6. The proof of Theorem 4 only uses the representation-finiteness of $\hat{P}$, not that of $\hat{Q}$. Therefore, if $\hat{P}$ is representation-finite, we can let the sequence $\mathcal{U}$ of 1.3 run through representatives of all the indecomposables of rep P whose support intersects $\mathring{P}$. Proposition 1.4c) then reduces the representation-theory of $\tilde{S}$ to the representation-theory of a poset $Q_{\mathcal{U}}$ which in the case considered here will be further denoted by $\mathring{\hat{Q}}$.

In other words, if $\hat{P}$ is representation-finite, all the required information is contained in the poset $\mathring{\hat{Q}}$ and not in $\hat{Q}$, which is identified with a full subposet of $\mathring{\hat{Q}}$. The problem is that the structure of $\mathring{\hat{Q}}$ is much more intricate than that of $\hat{Q}$.

In Section 3 below, we collect the information about $\mathring{\hat{Q}}$ used in the further demonstration, at various places of which we also need statements of the following lemma.

LEMMA. *S contains no stable subset T such that the induced completed poset $\tilde{T}$* (0.6) *has one of the following forms* (*where* $(a, b) \sim (a', b')$).



Proof. Construct the associated posets $\hat{Q}$ and check that they contain full subposets of the forms described by Fig. 2.

3. On the structure of the poset $\hat{\hat{Q}}$

Denote by $\hat{\hat{Q}}_a$ the subset of $\hat{\hat{Q}}$ formed by the indecomposable representations of P chosen in 2.6 whose support contains a thick point $a \in \mathring{P}$. Our purpose is to compare $\hat{\hat{Q}}_a$ with $\hat{\hat{Q}}_b$ under the assumption, valid throughout this section, that $a < b$ and $(a, b) \sim (a', b')$.

3.1. Our first lemma uses the following notation: If $V = (d, M)$ is a representation of a completed poset $\tilde{T} = \{t_1, t_2, \ldots\}^{\sim}$, we denote by $V(s_1) \in k^{d_0 \times d_1}$ the matrix consisting of the first d_1 columns of M, by $V(s_2) \in k^{d_0 \times d_2}$ the matrix formed by the following d_2 columns $\cdots$. In particular, if $\tilde{T} = P$ and $V \in \hat{\hat{Q}}_a$, we know by 2.5 that $V(a)$ is reduced to 1 column.

LEMMA. *A representation $V \in \hat{\hat{Q}}_a$ is smaller than the minimal element $\{b\}_0$ of $\hat{\hat{Q}}_b$ iff $V(a)$ is a linear combination of the columns of the "strips" $V(s)$ where $s \in P$ and $s < b$.*

Proof. Set $V = (d, M)$. If $(A, B): V \to \{b\}_0$ is a morphism of rep P, then B is a row and A the "empty" matrix. The condition $AN = MB^T$ of 0.2 therefore means that $0 = MB^T = \sum_{s \in P} M(s)B(s)^T$ if we define j by $s_j = b$ and set $B(s_i) = \bar{B}_{ji}$ (0.2). In the occurring sum, we have $M(b) = 0$ by 2.5 and $B(s) = 0$ if $s \nleq b$ (0.2).

Now, if $V < \{b\}_0$, we can choose B so that $B(a) \in k$ is non-zero. It follows that $M(a) = -\sum_{s \neq a, s < b} M(s)B(s)^T B(a)^{-1}$.

The converse should be clear.

3.2. LEMMA. *If P contains no element which is incomparable with a and b, then $\{a\}_1$ is the only element of $\hat{Q}_a$ which is incomparable with $\{b\}_0$.*

Proof. The lemma follows from 3.1 and Lemma 3.3 below.

3.3. LEMMA. *Let T be a finite poset, $t \in T$ a point and V an indecomposable representation of T which is not isomorphic to $\{t\}_1$. Then each column of $V(t)$ is a linear combination of the columns of the strips $V(s)$ where $s \not\geq t$.*

Proof. Assume that the conclusion of our lemma is wrong for $V = (d, M)$. Then there is a row $x \in k^{d_0}$ such that $xV(t) \neq 0$ and $xV(s) = 0$ whenever $s \not\geq t$. Setting $y = xM$, we infer that y^T is a non-zero morphism from $\bar{t}$ (1.1) to $\bar{d}$ (0.2) in $\tilde{S}_k$ and $(x, y^T): \{t\}_1 \to V$ a non-zero morphism in rep T.

The row $xV(t) \neq 0$ has d_i entries, where i is defined by $s_i = t$. We choose a row $w \in k^{d_i}$ such that $xV(t)w^T \neq 0$ and set

$$z = \left[\underbrace{0 \cdots 0}_{d_1 + \cdots + d_{i-1}} \; w_1 \cdots w_{d_i} 0 \cdots 0\right] \in k^{|\bar{d}|}$$

In this way, we obtain morphisms

$$\{t\}_1 \xrightarrow{(x,\, y^T)} V \xrightarrow{(Mz^T,\, z)} \{t\}_1$$

with composition $(xMz^T, zy^T) = (yz^T, zy^T) = (xV(t)w^T, wV(t)^T x^T) \neq 0$. We infer that $\{t\}_1$ is a direct summand of V in contradiction with the assumptions of the lemma.

3.4. From now on, we write $s \asymp t$ if $s, t \in S$ are incomparable, and we say that a thick point c is *normal* if $S(c) = \{s \in S : s \asymp c\}$ is a linearly ordered subset of S.

LEMMA. *Assume that $b \in \mathring{P}$ is normal and that there is a $d \in \dot{Q}$ such that $a' \asymp d \asymp b'$. Then the elements of $\hat{Q}_b$ which are incomparable with $\{a\}_1$ are $\{b\}_0$ and $\{b, c\}_0$, where $a \asymp c \asymp b$. The elements V of $\hat{Q}_a$ which are incomparable wtih $\{b\}_0$ are $\{a\}_1$, $\{a, c\}_0$ where $a \asymp c \asymp b$ and $\{a, c, s\}_1$ where $c \asymp a \asymp s \asymp c \asymp b > s$.*

Proof. It is clear that the listed indecomposables have the required properties. And the elements of $\hat{Q}_b$ which are incomparable with $\{a\}_1$ are the listed ones, because $\hat{Q}_b$ consists of $\{b\}_0$, $\{b\}_1$ and indecomposables of the form $\{b, c\}_0$ where $b \asymp c$.

It remains for us to examine the indecomposables $V \in \hat{Q}_a$ whose support Σ does not have the form (1) or (1, 1) of Fig. 10. The existence of d implies that a has multiplicity $\geqq 3$ in Σ (2.3) and excludes the posets (1, 2, 3) and $(1, 3 \leftarrow 3)$ of Fig. 10. We shall consider the 4 remaining cases separately.

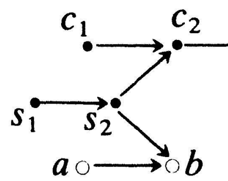
If $\Sigma = \{a, c, s\}$ has the form (1, 1, 1), V equals $\{a, c, s\}_1$ or $\{a, c, s\}_0 = (d, M)$ where $d = [1111]$ and $M = [111]$. The first evantuality is "accepted" by our lemma. In the second one, c or s is comparable with b (2.4(1)), say $s < b$. But then $V(a) = [1] = V(s)$, and we have $V < \{b\}_0$ by 3.1.

If $\Sigma = \{x_1 \rightarrow x_2, y, z\}$ has the form (1, 1, 2), V has the dimension-vector $d = [21111]$ and the matrix $M = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$. Then three cases are possible. (1) In case $a \in \{y, z\}$, say $a = y$, we must have $x_1 < x_2 < b$ or $x_1 < b > z$ because of 2.4(1) and of the dual of 2.6(2). Accordingly, $T(a)$ is a linear combination of $T(x_1)$, $T(x_2)$ in the first subcase, of $T(x_1)$, $T(z)$ in the second. (2) In case $a = x_1$, we have $x_2 \in \dot{P}$ by 2.5 and $x_2 \not< b$ by 0.4. By 2.4(1) this implies $y < b$ and $z < b$. The associated columns $T(y)$ and $T(z)$ generate $T(a)$. (3) In case $a = x_2$, b is comparable with y or z, say $y < b$ (2.4(1)). Then $T(a)$ is a linear combination of $T(x_1)$ and $T(y)$.

If $\Sigma = \{x_1 \rightarrow x_2, y, z_1 \rightarrow z_2\}$ has the form (1, 2, 2), two cases are to be considered (2.3): (1) In case $a = x_2$, we have $z_1 < z_2 < b$ or $z_1 < b > y$ (2.4(1) and 2.6(2)). If we let T run through the 3 faithful representations with support Σ [3], it remains to check that $T(a)$ is a linear combination of $T(x_1)$, $T(z_1)$, $T(z_2)$ in the first subcase, of $T(x_1)$, $T(z_1)$, $T(y)$ in the second. (2) In case $a = x_1$, we have $x_2 \in \dot{P}$ by 2.5 and $x_2 \not< b$ by 0.4. Since b is normal, we have $z_1 < z_2 < b > y$, and $T(a)$ is a linear combination of $T(y)$, $T(z_1)$, $T(z_2)$ by 3.3.

Finally, if $\Sigma = \{x_1 \rightarrow x_2 \leftarrow y_1 \rightarrow y_2, z_1 \rightarrow z_2\}$, a equals x_2 or y_1 (Fig. 10). The two cases are treated like case 1) and 2) of (1, 2, 2).

3.5. By $\hat{Q}_{ab}$ we denote the full subposet of $\hat{Q}$ formed by the representations $V \in \hat{Q}_a \cup \hat{Q}_b$ which are incomparable with $\{b\}_0$ or with $\{a\}_1$. By P_{ab} we denote the union of their supports equipped with the order induced by P.



$\bar{P}_{ab}$

Figure 11

LEMMA. *Under the assumptions of Lemma* 3.4, P_{ab} *is equal to* $\{a, b\}$ *or isomorphic to a full subposet of* $\bar{P}_{ab}$ (*Fig.* 11) *containing* $\{a, b, c_1\}$. *The poset* $\hat{Q}_{ab}$

is identified with the full subposet of $\bar{Q}_{ab}$ (*Fig.* 12) *formed by the vertices which involve only points of* P_{ab}.

$$\begin{tikzcd}[column sep=small]
\{a, c_1\}_0 \arrow[r] & \{a, c_1, s_1\}_1 \arrow[r] & \{a, c_1, s_2\}_1 \arrow[r] \arrow[dr] & \{a, c_2\}_0 \arrow[r] \arrow[dr] & \{a, c_3\}_0 \arrow[r] \arrow[dr] & \{a\}_1 \\
 & \bar{Q}_{ab} & \{b\}_0 \arrow[r] & \{b, c_1\}_0 \arrow[r] & \{b, c_2\}_0 \arrow[r] & \{b, c_3\}_0
\end{tikzcd}$$

Figure 12

Proof. By 2.4(1), the points $c \in P$ such that $a \bowtie c \bowtie b$ form a linearly ordered set $c_1 \to c_2 \to \cdots \to c_k$. If k was $\geqq 4$, $\hat{P}$ would contain the full subposet

$$\begin{tikzcd}[row sep=small]
(a, d) \arrow[r] \arrow[dr] & (a, 1) \\
(b, 0) \arrow[r] & (b, d)
\end{tikzcd}
\qquad c_1 \to c_2 \to c_2 \to c_4$$

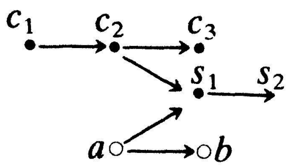
If there was an $s \in P$ such that $a \bowtie s \bowtie c_i$ for some $i \geqq 2$, we would have $c_1 \to s \to b$ by (2.4)(1) and the dual of 2.6(2).

Finally, the points $s \in P$ such that $a \bowtie s \bowtie c_1$ form a linearly ordered set $s_1 \to s_2 \to \cdots \to s_l$ (0.3). If l was $\geqq 3$, $\hat{P}$ would contain the full subposet

$$s_1 \to s_2 \to s_3 \qquad (a, 0) \to (a, d) \to (a, 1) \qquad c_1$$

The rest should be clear.

3.6. LEMMA. *Assume that* $a \in \mathring{P}$ *is normal and that there is a* $d \in \dot{Q}$ *such that* $a' \bowtie d \bowtie b'$. *Then* P_{ab} *is equal to* $\{a, b\}$ *or isomorphic to a full subposet of* $\underline{P}_{ab}$ (*Fig.* 13) *containing* $\{a, b, c_3\}$. *The poset* $\hat{Q}_{ab}$ *is identified with the full subposet of* $\underline{Q}_{ab}$ (*Fig.* 14) *formed by the vertices which involve only points of* P_{ab}.



$\underline{P}_{ab}$

Figure 13

$$\begin{tikzcd}[column sep=small]
\{a, c_1\}_0 \arrow[r] \arrow[dr] & \{a, c_2\}_0 \arrow[r] \arrow[dr] & \{a, c_3\}_0 \arrow[r] \arrow[dr] & \{a\}_1 & & \underline{Q}_{ab} \\
\{b\}_0 \arrow[r] & \{b, c_1\}_0 \arrow[r] & \{b, c_2\}_0 \arrow[r] & \{b, c_3, s_1\}_0 \arrow[r] & \{b, c_3, s_2\}_0 \arrow[r] & \{b, c_3\}_0
\end{tikzcd}$$

Figure 14

Proof. This is "the dual of the quasi-dual" of lemma 3.5. Since duality theory is screened by the use of matrices, we sketch the essentials: For each representation $V = (d, M)$ of P, we choose a matrix $K \in k^{|\bar{d}| \times e_0}$ such that $MK = 0$ and rank $K = e_0 = |\bar{d}|$-rank M. Setting $e = [e_0 \bar{d}] \in \mathbb{N}^{n+1}$, we then interpret the pair $\mathscr{D}V = (e, K^T)$ as a representation of the opposite poset P^0, and we assemble a contravariant functor $\mathscr{D}: \operatorname{rep} P \to \operatorname{rep}(P^0)$ by piecing out the map $V \mapsto \mathscr{D}V$ as follows: First we notice that each morphism $B \in \operatorname{Hom}(\bar{d}, \bar{d}')$ of P_k (0.2) produces a morphism $B^T \in \operatorname{Hom}(\bar{d}', \bar{d})$ of $(P^0)_k$. Our second observation is that, for each morphism $(A, B): (d, M) \to (d', M')$ of rep P, there is a unique matrix $C \in k^{e_0 \times e_0'}$ such that $KC = B^T K'$, where $\mathscr{D}(d', M') = (e', K')$. This means that $(C^T, B^T) = \mathscr{D}(A, B)$ is a morphism of $\operatorname{rep}(P^0)$ from (e', K') to (e, K). The contravariant functor thus defined induces an antiequivalence from $\operatorname{rep}_0 P$ (the full subcategory of rep P formed by the (d, M) such that $d_0 = \operatorname{rank} M$) to $\operatorname{rep}_0(P^0)$. For instance, we have $\mathscr{D}\{a\}_0 = \{a\}_1$, $\mathscr{D}\{a, c\}_0 = \{a, c\}_0$, $\mathscr{D}\{a, c, s\}_1 = \mathscr{D}\{a, c, s\}_0 \cdots$

3.7. LEMMA. *Assume that there is a $d \in \dot{Q}$ satisfying $a' \asymp d \asymp b'$ and that a or b is normal. Let further $z \in \mathring{P}$ be such that $b < z$ and $(b, z) \sim (b', z')$. Then $\hat{Q}_{ab} \cap \hat{Q}_{bz}$ consists of the representations $\{b, c\}_0$ where c is incomparable with a, b and z. If there is only one such c, then $\{a\}_1$ is the only element of $\hat{Q}_a$ which is incomparable with $\{b, c\}_0$.*

Proof. The first statement directly follows from 3.5 if b is normal. If a is normal, we must prove that $\{b, c_3, s_i\}_0 \notin \hat{Q}_{ab} \cap \hat{Q}_{bz}$ (3.6). But this follows from the validity of $c_3 < z$ or of $s_i < z$ (2.4(1)).

Now, the points incomparable with a and b form a chain $c_1 \to c_2 \to \cdots \to c_l$. If there is only one c as above, we must have $c = c_1$. Our second statement therefore follows from Fig. 12 or Fig. 14.

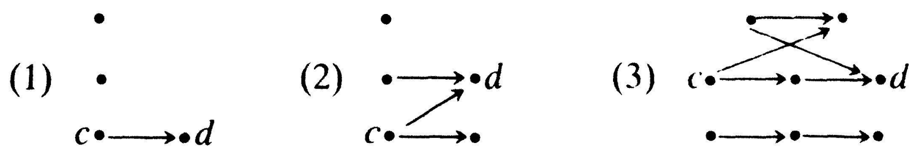
Remark. By duality and quasi-duality, the first statement of the lemma is also true under the assumption that there is a q satisfying $b' \asymp q \asymp z'$ and that b or z is normal. If, moreover, there is only one c, then $\{z\}_0$ is the unique element of $\hat{Q}_z$ such that $\{z\}_0 \asymp \{b, a\}_0$.

4. Mixed edges

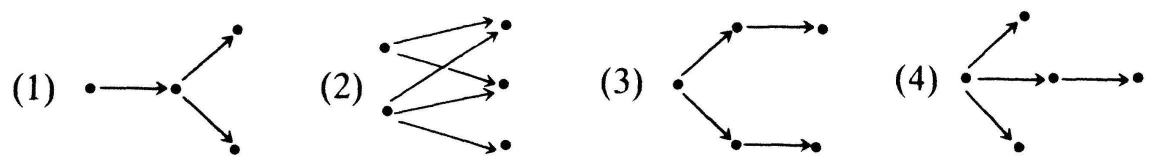
From now onwards, we suppose that $\tilde{S}$ admits a faithful indecomposable representation $U = (d, M)$ (0.6). We denote by Σ_U the support of the associated representation $U_Q = (d_Q, M_Q)$ of $\hat{Q}$ (1.4, 2.6). We investigate Σ_U under the following assumption, valid throughout section 4: $a \in \mathring{P}$ is thick, $b \in \mathring{P}$ is subsequent to a, $a' \in \mathring{Q}$ is normal and $b' \in \mathring{Q}$ is not. By $\bar{a}$ and $\bar{b}$ we denote a maximal and a minimal element of $\hat{Q}_a \cap \Sigma_U$ and $\hat{Q}_b \cap \Sigma_U$ respectively.

The lemmas 4.1–4.5 are preliminary and follow directly from 2.1, Fig. 6. As in 2.1, we denote by Σ the support of an indecomposable representation of a representation-finite poset T.

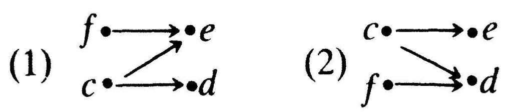
4.1. LEMMA. *Suppose that Σ has at least three points. Then, for any two points c and d (comparable or not), there is an x such that $c \mathrel{>\!\!\!<} x \mathrel{>\!\!\!<} d$. In case $c < d$, $\{c, d\}$ is contained in a full subposet of Σ having one of the following three forms.*



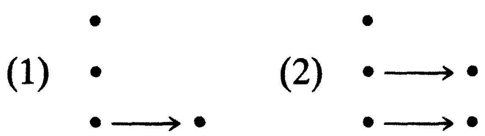
4.2. LEMMA. *Σ contains no full subposet which is isomorphic or dual to one of the following posets.*



4.3. LEMMA. *If $d \in \Sigma$ and $e \in \Sigma$ are subsequent to $c \in \Sigma$ and satisfy $d \mathrel{>\!\!\!<} e$, then Σ contains a full subposet of one of the following two forms. Moreover, we have $g \mathrel{>\!\!\!<} f$ whenever $g \in \Sigma$ is incomparable with c, d and e.*

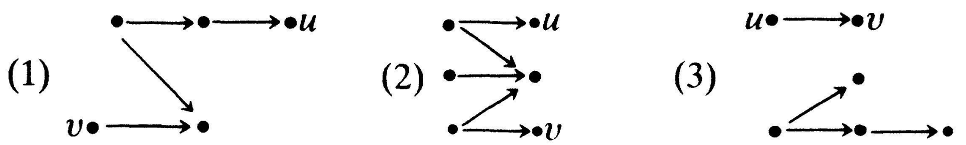


4.4. LEMMA. *A proper full subposet Σ of the form* (1) *below is contained in a full subposet of Σ isomorphic to* (2).



4.5. In case $v \in T$, we call *duplicate of v in T* an element $w \in T$ which is comparable with v and such that, for any $t \in T \backslash \{v, w\}$, the inequality $v < t$ is equivalent to $w < t$ and $t < v$ to $t < w$.

LEMMA. *If X is a full subposet of Σ (resp. of Σ^0) of one of the three forms below, then $\Sigma\backslash X$ (resp. $\Sigma^0\backslash X$) contains a duplicate of u in Σ or of v.*



4.6. LEMMA. *$\dot{Q}$ contains a point q subsequent to a' and a point d such that $b' \asymp q \asymp d$ and $a' \asymp d \asymp b'$.*

Proof. Consider the full subposet $Q(b', \cdot)$ of Q formed by the points $s \in Q$ which can be incorporated into a triad $\{b', c, s\}$ of three pairwise incomparable elements of Q. This subposet contains at least two minimal elements, say q_1 and q_2. If q_1 is incomparable with a', we can set $d = q_1$ and $q = q_2$: Indeed, 2.4(1) implies $a' < q_2$; if q_2 was not subsequent to a', each element q_3 such that $a' < q_3 < q_2$ should be incomparable with b' (which is subsequent to a'); accordingly, $q_3 < q_2$ would imply $q_3 < q_1$ and Σ_U would contain $\begin{tikzcd}[row sep=small] & q_1 \\ \bar{a} \arrow[r] & q_3 \arrow[u] \arrow[d] \\ & q_2 \end{tikzcd}$ in contradiction with 4.2(1).

In case $a' < q_1$ and $a' < q_2$, the same argument shows that q_1 and q_2 are both subsequent to a'. By lemma 4.3, Σ_U contains a full subposet of the form, say $\begin{tikzcd}[row sep=small] \bar{a} \arrow[r] \arrow[dr] & q_1 \\ x \arrow[r] & q_2 \end{tikzcd}$, which satisfies $x \asymp \bar{b}$ if $\bar{a} \asymp \bar{b}$. We claim that $x \in \dot{Q}$, because $y \in \mathring{Q}$, $x \in \hat{\mathring{Q}}_y$ and $x < q_2$ would imply $y \leqq a$, hence $x < q_1$. Furthermore, we have $x < \bar{b}$, because q_2 is minimal in $Q(b', \cdot)$. We infer that $\bar{a} < \bar{b}$ and that Σ_U contains the full subposet of Fig. 15 in contradiction with 4.2(2).

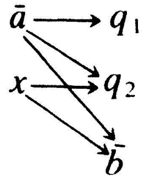


Figure 15

4.7. LEMMA. *Let P contain a point which is incomparable with all points of $\mathring{P}$. Then $\bar{a} \in \hat{\mathring{Q}}_a \cap \Sigma_U$ and $\bar{b} \in \hat{\mathring{Q}}_b \cap \Sigma_U$ can be chosen so that $\bar{a} \asymp \bar{b}$.*

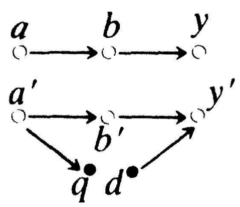
Proof. Otherwise, we can apply 4.3 to the subset $\begin{tikzcd}[row sep=small] d & q \\ \bar{a} \arrow[r] \arrow[ur] & \bar{b} \end{tikzcd}$ of Σ_U and find an $x \in \Sigma_U$ such that $\bar{a} \asymp x \asymp d$ and that either $q \asymp x < \bar{b}$ or $\bar{b} \asymp x < q$. In both cases we have $x \notin \dot{Q}$ since a' is normal. So let x be in $\hat{\mathring{Q}}_y$, $y \in \mathring{P}$, and first suppose that $y \leqq a$: Then $x < q$, $x >\!\!< \bar{b}$ and $y < a$ since $x \in \hat{\mathring{Q}}_a \cap \Sigma_U$ is supposed to imply $x < \bar{b}$; it follows that $(y, b) \sim (y', b')$, and we obtain a contradiction with 2.6(6), which reduces our proof to the case $y \geqq b$. But then we have $q >\!\!< x < \bar{b}$, hence $y = b$ and the contradiction $\bar{a} < x$ (since $x \in \hat{\mathring{Q}}_b \cap \Sigma_U$ is supposed to imply $\bar{a} < x$).

4.8. LEMMA. *Let $\mathring{P}$ contain a point which is not normal and P a point which is incomparable with all points of $\mathring{P}$. If Σ_U has at least 5 points, it contains a full subposet of the following form, where $\bar{a} \in \hat{\mathring{Q}}_a$, $\{\bar{b}, \bar{\bar{b}}\} \subset \hat{\mathring{Q}}_b$ and $\{q, d\} \subset \dot{Q}$.*

$$\begin{array}{lll} d \bullet & & \\ \bar{b} \bullet & \longrightarrow & \bullet\, \bar{\bar{b}} \\ \bar{a} \bullet & \longrightarrow & \bullet\, q \end{array}$$

Proof. We apply 4.4 to the full subposet $\begin{array}{lll} d \bullet & & \bullet\, \bar{b} \\ \bar{a} \bullet & \longrightarrow & \bullet\, q \end{array}$ of Σ_U which is provided by 4.6 and 4.7. By 4.4 there is an $x \in \Sigma_U$ such that $\bar{a} >\!\!< x >\!\!< q$ and that $d >\!\!< x$ or $\bar{b} >\!\!< x$. If x was in $\dot{Q}$, it would be comparable with d (because a' is normal) and provide a contradiction to 2.6(2).

Therefore, we have $x = \bar{y} \in \hat{\mathring{Q}}_y$ for some $y \geqq b$ (because $x >\!\!< q$). In case $y = b$ the proof is perfect. So it remains for us to exclude the case $y > b$. In this case, 2.6(6) implies $d < y'$, and S, Σ_U contain the full subposets of Fig. 16, where $(a, y) \sim (a', y')$. By 2.6(1), b is normal; by 2.6(6), y is subsequent to b; by 2.6(5), there is at most one $c \in \dot{P}$ such that $a >\!\!< c >\!\!< y$; by 3.7 and the assumptions of the lemma, we have $\bar{b} = \{b, c\}_0$, $\bar{a} = \{a\}_1$ and $\bar{y} = \{y\}_0$, where $c >\!\!< z$ for all $z \in P$.



$$\begin{array}{lll} d\bullet & \longrightarrow & \bullet\, \bar{y} \\ \bar{b}\bullet & & \\ \bar{a}\bullet & \longrightarrow & \bullet\, q \end{array}$$

Figure 16

By assumption, $\dot{P}$ contains a point $e >\!\!< x$. Since U is faithful and b normal, e belongs to the support of some $\bar{t} \in \hat{\mathring{Q}}_t \cap \Sigma_U$, where $t \neq b$. Up to duality and quasi-duality, we may assume that $t < b$. Let us then compare $\bar{t}$ with $\bar{a}$, q, $\bar{b}$, d, $\bar{y}$: Obviously, $\bar{t} < \bar{a} = \{a\}_1$. It follows that $\bar{t} >\!\!< d$, because $\bar{t} < d$ would contradict 4.2(3). We claim that $\bar{t} < \bar{b}$: Indeed, this follows from 3.7 if $t = a$; and the case $t < a$, $\bar{t} >\!\!< \bar{b}$ is excluded by 2.6(6) (since c is incomparable with t, a, b and d with t', a', b'). Finally, we have $\bar{t} >\!\!< \bar{y}$ because $\bar{t} < \bar{y}$ would contradict 4.2(4).

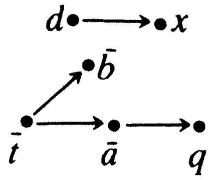


Figure 17

Now, by 4.5(3) d or x has a duplicate z in Σ_U. If $z \in \dot{Q}$, $\tilde{S}$ contains the stable subset of Fig. 18 in contradiction to 2.6(2). If $z = \bar{r} \in \hat{\mathring{Q}}_r$, $q \bowtie z$ implies $b \leqq r$, and $b' \bowtie d$ implies $y \leqq r$. Then the dual of the quasi-dual of the argument applied to $\bar{t}$ above yields the contradiction $\bar{b} < \bar{r}$.

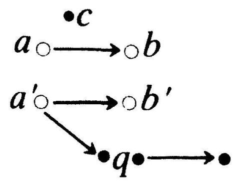


Figure 18

Remark. Our proof involves quasi-duality. This may need some explanation since we formally use the fact that the "dual of the quasi-dual" $\tilde{S}^{*0} = P^0 \lhd Q^0$ also admits a faithful indecomposable representation. In fact, the antiequivalence $\mathscr{D}: \text{rep}\, P \to \text{rep}\,(P^0)$ of 3.6 induces an isomorphism of $(\hat{\mathring{Q}})^0$ onto the poset $(Q^0)^{\hat{}}$ attached to $\tilde{S}^{*0} = P^0 \lhd Q^0$. The needed faithful indecomposable representation V of $\tilde{S}^{*0}$ is defined by $V_{Q^0} \xrightarrow{\sim} \mathscr{D}(U_Q)$.

Using similar arguments, one shows that the dual $\tilde{S}^0 = Q^0 \lhd P^0$ and the quasi-dual $\tilde{S}^* = Q \lhd P$ admit faithful indecomposable representations.

5. Proof of theorem 3

As in section 4, we denote by U a faithful indecomposable representation of the bipartite completed poset $\tilde{S} = P \lhd Q$. We suppose that $\mathring{P}$ and $\mathring{Q}$ contain non-normal points.

5.1. *We first prove theorem* 3 *under the assumption that* $\mathring{P}$ *has cardinality* 2. Using quasi-duality and 2.6(1), we may assume that $\mathring{P} = \{a < b\}$ and $\mathring{Q} = \{a' < b'\}$, where a', b are normal and a, b' not. By lemma 4.6 and its dual, S then contains the full subposet of Fig. 19. If there is any other point which is incomparable with a and b or with a' and b', we may by duality assume that it lies in Q, hence that c is the unique point which is incomparable with a and b (2.6(3)).

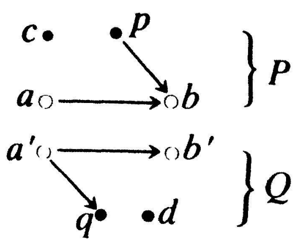


Figure 19

Let us now assume that Q contains more points and find the contradiction. The dual argument will then show that P also has 4 points, and our proof will be complete.

By lemma 4.8, Σ_U contains a full subposet of the form $\begin{array}{l} \bar{a} \bullet\!\to\!\bullet\, q \\ \bar{b} \bullet\!\to\!\bullet\, \bar{\bar{b}} \end{array} \bullet\, d$, where $\bar{a} \in \hat{Q}_a$ and $\{\bar{\bar{b}}, \bar{b}\} \subset \hat{Q}_b$. By lemma 3.5, $\bar{a} = \{a\}_1$, $\bar{b} = \{b\}_0$ and $\bar{\bar{b}} = \{b, c\}_0$. Accordingly, since U has dimension $\geqq 1$ at p, Σ_U must contain some other $\bar{\bar{a}} \in \hat{Q}_a$. Lemma 3.5 implies $\bar{\bar{a}} < \bar{\bar{b}}$ and lemma 4.2(3) $\bar{\bar{a}} \bowtie \bar{b}$ (Fig. 20).

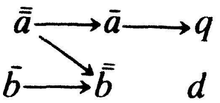


Figure 20

We now apply 4.5: A duplicate of $\bar{b}$ in Σ_U cannot belong to $\dot{Q}$ because a' is normal, to $\hat{Q}_b$ because of 3.5, to $\hat{Q}_a$ because $b \bowtie q$. Accordingly, only q can admit a duplicate in Σ_U. Therefore, Σ_U contains the full subposet of Fig. 21, where q is one of the non-specified points. By 2.1, Fig. 6 it follows that Σ_U has the form of Fig. 22. But y cannot belong to $\dot{Q}$ because $\bar{b} \bowtie y < \bar{\bar{b}}$. And y cannot belong to $\hat{Q}_a \cup \hat{Q}_b$, i.e. in fact to $\hat{Q}_{ab}$, because of 3.5.

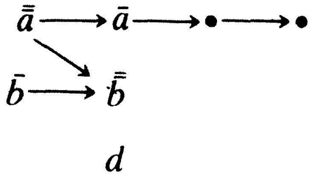


Figure 21

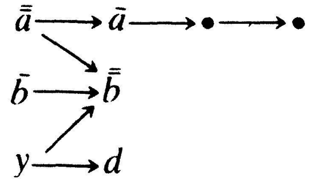


Figure 22

5.2. LEMMA. *Suppose that $\mathring{P}$ contains points $a < b < c$ and $\dot{P}$ a point d such that $(a, c) \sim (a', c')$ and $a \bowtie d \bowtie c$. Then b is normal.*

Proof. Otherwise, $\dot{P}$ contains two points d_1, d_2 such that $d_1 \bowtie b \bowtie d_2 \bowtie d_1$. Since $a < d_i < c$ is excluded by 0.4, each point d_i satisfies $a \bowtie d_i$ or $d_i \bowtie c$. By

2.4(1) it follows that we have, say $a \mathrel{>\!\!<} d_1 < c$ and $a < d_2 \mathrel{>\!\!<} c$. By 2.6(1), d_1 and d_2 are both comparable with d. So we obtain $d_1 < d$ ($d < d_1$ would imply $d < c$!), $d < d_2$ and the contradiction $d_1 < d_2$.

5.3. LEMMA. *$\dot{P}$ contains a point s incomparable with all $u \in \mathring{P}$.*

Proof. By the dual of 4.6 we can assume that $\mathring{P}$ has cardinality $\geqq 3$.

Let a be the minimal and c the maximal point of $\mathring{P}$. Choose $\bar{a}$ in $\hat{\mathring{Q}}_a \cap \Sigma_U$, $\bar{c}$ in $\hat{\mathring{Q}}_c \cap \Sigma_U$. By 4.1, Σ_U contains a point x such that $\bar{a} \mathrel{>\!\!<} x \mathrel{>\!\!<} \bar{c}$. If $x \in \mathring{Q}$, it follows that $a \mathrel{>\!\!<} x \mathrel{>\!\!<} c$, and we can set $s = x$. Hence we may suppose that $x \in \hat{\mathring{Q}}_b$ for some $b \in \mathring{P}$; if $b \neq c$, $\dot{P}$ contains an element incomparable with b and c, since the contrary would imply $x = \{b\}_1 \geqq \{a\}_1 \geqq \bar{a}$ (3.2). Similarly, if $a \neq b$, $\dot{P}$ contains an element incomparable with a and b.

This solves our problem in case $a = b$ or $b = c$. In general, it implies that, whenever $v \in \mathring{P}$ is subsequent to $w \in \mathring{P}$, there is a point incomparable with v and w. By duality, the same statement holds for $\mathring{Q}$.

Now suppose that $a < b < c$. Let $v \in \mathring{P}$ be subsequent to b, and b to $u \in \mathring{P}$. We claim that one of the points u, b, v is normal: Indeed, by 5.2 u is normal if $a \neq u$, and v is if $c \neq v$; if neither u nor v is normal, we have $a = u$, $c = v$, and b must be normal (otherwise, all points a', b', c' of $\mathring{Q}$ would be normal by 2.6(1)).

So we can apply 3.7 to u, b, w. Since $x \in \hat{\mathring{Q}}_b$ satisfies $\{a\}_1 \mathrel{>\!\!<} x \mathrel{>\!\!<} \{c\}_0$, it satisfies $\{u\}_1 \mathrel{>\!\!<} x \mathrel{>\!\!<} \{v\}_0$ and has the form $x = \{b, s\}_0$. But $\{a\}_1 \mathrel{>\!\!<} \{b, s\}_0 \mathrel{>\!\!<} \{c\}_0$ implies $a \mathrel{>\!\!<} s \mathrel{>\!\!<} c$.

5.4. LEMMA. *Let $b \in \mathring{P}$ and $p \in \dot{P}$ be such that b is normal and $p < b$. Then there is a point $x \in \mathring{P}$ such that $x < b$ and a representation $t \in \hat{\mathring{Q}}_x \cap \Sigma_U$ with dimension $\geqq 1$ at p.*

Proof. Since U is faithful, there is an $x \in \mathring{P}$ and a $t \in \hat{\mathring{Q}}_x \cap \Sigma_U$ with dimension $\geqq 1$ at p. Since b is normal, $\hat{\mathring{Q}}_b$ consists of $\{b\}_0$, $\{b\}_1$ and of representations $\{b, c\}_0$ with $b \mathrel{>\!\!<} c$. We infer that $x \neq b$. Suppose that $x > b$. By 4.1, the support Σ of t contains a full subposet X or Y as shown in Fig. 23 (the case (3) of 4.1 is excluded because x has multiplicity $\geqq 2$). In case $\Sigma \supset X$, the inequalities $p < b < x$

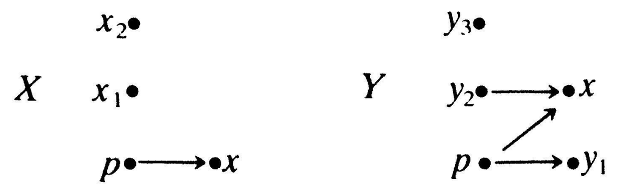


Figure 23

imply $x_1 \asymp b \asymp x_2$ in contradiction to the normality of b. In case $\Sigma \supset Y$, this normality and the condition $p \asymp y_3 \asymp x$ imply that b is comparable with y_1 and y_2, hence that $b < y_1$ ($x \asymp y_1$ implies $b \not> y_1$) and $y_2 < b$. This leads us to the contradiction $y_2 < y_1$.

5.5. *Proof of theorem* 3. Applying 2.6(6), 5.3 and the dual of 5.3, we first observe that y must be subsequent to x if $(x, y) \sim (x', y')$ and $x \neq y$.

By lemma 2.6(1), out of two equivalent points at least one is normal. We choose two equivalence classes $\{u \sim u'\}$, $\{b \sim b'\}$ such that $u, b \in \mathring{P}$ and $u', b' \in \mathring{Q}$, that u', b are normal and that u, b' are not. Furthermore, we suppose that all points of $\mathring{P}$ between u and b (if there are any) are normal, as well as all points of $\mathring{Q}$ between u' and b'. Up to quasi-duality, we may also suppose that $u < b$. We then denote by $v \in \mathring{P}$ the point subsequent to u, by $a \in \mathring{P}$ the point to which b is subsequent ($u \leqq a$, $v \leqq b$). To the pairs (a, b) and (u, v) thus constructed we apply the lemmas 4.6, 4.8 and their duals, which provide us with the full subposets of P, Q and $\hat{Q}$ described in Fig. 24.

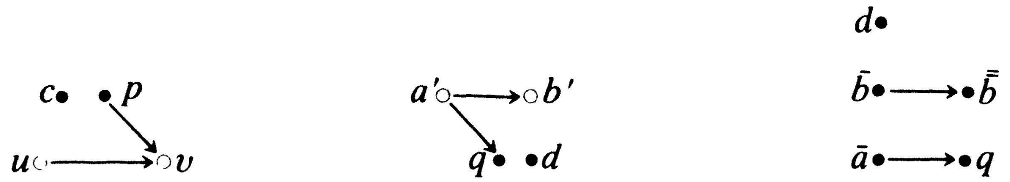


Figure 24

By 2.6(4), there are at most two points incomparable with a and b. Using 3.5, we infer that $\bar{a} = \{a\}_1$ or $\bar{a} = \{a, c'\}_0$ where $a \asymp c' \asymp b$. Accordingly, $\bar{a}$ has dimension 0 at p, and is distinct from the point $t \in \hat{Q}_x$, $x \leqq a$, constructed in 5.4 and obviously subjected to $t < q$.

Let us suppose that $t < \bar{b}$. Then we have $\bar{a} \asymp t \asymp d$ by 4.2((3) and (4)) and can apply 4.5(1) to $\begin{smallmatrix} t \to \bar{b} \to \bar{\bar{b}} \\ \searrow \quad \\ \bar{a} \to q \quad \end{smallmatrix}$. But there is no way of obtaining a duplicate z of $\bar{\bar{b}}$ in Σ_U from $\dot{Q}$ because a' is normal; from $\hat{Q}_b$ because $\bar{b}$, $\bar{\bar{b}}$ exhaust the elements of $\hat{Q}_b$ incomparable with $\bar{a}$ ($\bar{a} \asymp t$ implies $\bar{a} = \{a, c_2\}_0$, $\bar{\bar{b}} = \{b, c_1\}_0$, $\bar{b} = \{b\}_0$ in 3.5); from $\hat{Q}_y$, $y > b$, because $a < b < y$ implies $(a, y) \not\sim (a', y')$ and $\bar{a} < z$. Nor can we obtain a duplicate z of $\bar{a}$ from $\dot{Q}$ (q is subsequent to a' and $z < \bar{a}$ would imply $z < \bar{b}$); from $\hat{Q}_a$ by 3.5; from $\hat{Q}_y$, $y < a$, because $y < a < b$ implies $z < \bar{b}$.

So we are reduced to the case $t \asymp \bar{b}$, hence $x = a$. By 3.5, t is comparable with $\bar{a}$. It is $< \bar{a}$ because $\bar{a}$ is supposed to be maximal in $\hat{Q}_a \cap \Sigma_U$. The case $\bar{a} = \{a\}_1$, $t = \{a, c_2\}_0$, $\bar{\bar{b}} = \{b, c_1\}_0$, $\bar{b} = \{b\}_0$ is excluded by the assumption that t has dimension $\geqq 1$ at p. By 3.5 this implies that $t < \bar{b}$. In this case, we obtain Fig. 20

and can repeat the argument produced in the last paragraph of 5.1. Theorem 3 is proved.

6. Appendix

Our objective in this section is to expose a more synthetical point of view for the reduction used in section 1. The following is due to P. Gabriel.

6.1. Let $k^m = k^{1\times m}$ be the space of m-rows and $\operatorname{mod} k$ the category of finite dimensional vector spaces. The category $\tilde{S}_k$ (0.2) is naturally equipped with a functor $F:\tilde{S}_k \to \operatorname{mod} k$, $v \mapsto k^{|v|}$ which maps the morphism $B \in \operatorname{Hom}(u, v)$ onto $x \mapsto xB^T$. Using F, we can interprete a representation (d, M) as a pair $(\bar{d}, f)$ consisting of an object $\bar{d} \in \tilde{S}_k$ and a linear map $f: k^{d_0} \to F\bar{d}$, $y \mapsto yM$. In this way, we obtain an *equivalence between* $\operatorname{rep} \tilde{S}$ *and the* following *F-subspace category* $\operatorname{sub} F$ [5]: An object of $\operatorname{sub} F$ is an "F-subspace", i.e. a pair (v, f) formed by an object $v \in \tilde{S}_k$ and a morphism $f: V \to Fv$ of $\operatorname{mod} k$. A morphism $(u, e) \to (v, f)$ is given by a pair of morphisms $B \in \operatorname{Hom}(u, v)$ and $A \in \operatorname{Hom}(U, V)$ such that $fA = (FB)e$.

The natural decompositions of the rows $v \in \tilde{S}_k$ and $x \in Fv$ into "blocks" $v_P = [v_1 \cdots v_{|P|}]$, $v_Q = [v_{|P|+1} \cdots v_n]$ and $x_P = [x_1 \cdots x_{|v_P|}]$, $x_Q = [x_{|v_P|+1} \cdots x_{|v|}]$ yield an exact sequence of functors

$$0 \longrightarrow F_Q \xrightarrow{\iota} F \xrightarrow{\pi} F_P \longrightarrow 0,$$

where $F_P v = k^{|v_P|}$, $\pi v: x \mapsto x_P$, $F_Q v = k^{|v_Q|}$ and $(\iota v)[y_1 \cdots y_{|v_Q|}] = [0 \cdots 0 y_1 \cdots y_{|v_Q|}]$. The residue-functor F_P gives rise to an F_P-subspace category $\operatorname{sub} F_P$, which is defined like $\operatorname{sub} F$ and contains the full subcategory $\operatorname{sub}_0 F_P$ formed by the *proper F_P-subspaces,* i.e. by the pairs (v, g) such that g is injective. The subcategory $\operatorname{sub}_0 F_P$ finally provides us with the wanted *reduction-functor*

$$\mathcal{R}: \operatorname{sub} F \longrightarrow \operatorname{sub} F'_Q$$

$$(v, V \xrightarrow[f]{} Fv) \mapsto ((v, \operatorname{Im}(\pi v)f \xrightarrow[f_P]{} F_P v), \operatorname{Ker}(\pi v)f \xrightarrow[f_Q]{} F_Q v)$$

where f_P, f_Q are induced by f and

$$F'_Q: \operatorname{sub}_0 F_P \to \operatorname{mod} k$$

maps $(v, U \xrightarrow[g]{} F_P v)$ onto $F_Q v$.

PROPOSITION. *The reduction-functor* $\mathcal{R}: \mathrm{sub}\, F \to \mathrm{sub}\, F'_Q$ *induces a bijection between the isomorphism classes of* sub F *and of* sub F'_Q.

It follows that $\mathcal{R}$ also induces a bijection between the isomorphism classes of indecomposables.

Proof. It is easy to show that $\mathcal{R}$ hits each isomorphism class of sub F'_Q. Indeed, each object $((v, U \xrightarrow[g]{} F_P v), W \xrightarrow[h]{} F_Q v)$ of sub F'_Q is isomorphic to the image of the object $(v, U \oplus W \xrightarrow[{[sg\ h]}]{} Fv)$ of sub F, where $s: F_P v \to Fv$ denotes an arbitrary linear section of πv.

To prove the injectivity of the map induced by $\mathcal{R}$, we first remark that, for each linear map $e: F_P u \to F_Q v$, there is a morphism $E: u \to v$ of $\tilde{S}_k$ such that $FE = (\iota v)e(\pi u)$ (if u and v are indecomposable in $\tilde{S}_k$, this immediately follows from (0.4b)). We then consider two objects $(v, V \xrightarrow{f} Fv)$ and $(v, V' \xrightarrow{f'} Fv)$ of sub F having isomorphic images in sub F'_Q. This means that there are isomorphisms B, C and D which make commutative the first two squares of Fig. 7. We extend D to an isomorphism $A: V \xrightarrow{\sim} V'$ which induces C. Then $f'A - (FB)f$ vanishes on $\mathrm{Ker}\,(\pi v)f$ and factors through $\iota v: F_Q v \to Fv$. In other words, $f'A - (fB)f$ can be written as a composition

$$V \xrightarrow{(\pi v)f} F_P v \xrightarrow{e} F_Q v \xrightarrow{\iota v} Fv$$

We infer that $f'A - (FB)f = (\iota v)e(\pi v)f = (fE)f$ for some $E: v \to v$ such that $(FE)^2 = 0$, hence $E^2 = 0$. So we finally obtain the isomorphism $(A, B + E)$: $(v, f) \xrightarrow{\sim} (v, f')$.

$$\begin{tikzcd}
\mathrm{Ker}(\pi v)f \arrow[r, "f_Q"] \arrow[d, "D"', "\wr"] & F_Q v \arrow[d, "F_Q B", "\wr"'] \\
\mathrm{Ker}(\pi v)f' \arrow[r, "f'_q"] & F_Q v
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Im}(\pi v)f \arrow[r, "f_p"] \arrow[d, "C"', "\wr"] & F_p v \arrow[d, "F_p B"] \\
\mathrm{Im}(\pi v)f' \arrow[r, "f'_p"] & F_p v
\end{tikzcd}$$

$$\begin{tikzcd}
V \arrow[r, "f"] \arrow[d, "A"', "\wr"] & Fv \arrow[d, "FB", "\wr"'] \\
V' \arrow[r, "f'"] & Fv
\end{tikzcd}$$

Fig. 7

6.2. The construction of the subspace category sub F'_Q considered in 6.1 is based on the category $\mathrm{sub}_0\, F_Q$ which, in general, does not have the form $\tilde{T}_k$. We therefore insert some remarks about general subspace categories [5] [9].

Let K be a k-linear category such that the dimensions of the morphism spaces are finite and that each object is a finite direct sum of indecomposables with local endomorphism algebras. If $\Phi: K \to \operatorname{mod} k$ is a k-linear functor, sub Φ is related in a simple way to the category of representations of a poset: Let $U_1, \ldots, U_s$ be pairwise non-isomorphis indecomposables such that $\dim \Phi U_i = 1$. Define a partial order on the set $\mathscr{V} = \{U_1, \ldots, U_s\}$ by setting $U_i \leqq U_j$ is $\Phi\mu \neq 0$ for some $\mu: U_i \to U_j$. Denote by $\bar{\Phi}: K/\mathrm{Ket}\,\Phi = \bar{K} \to \operatorname{mod} k$ the functor induced by Φ, where Ker Φ denotes the ideal of K formed by the morphisms v such that $\Phi v = 0$. We then have the following *comparism diagram*

$$\operatorname{sub} \Phi \xrightarrow{\gamma} \operatorname{sub} \bar{\Phi} \xleftarrow{\varepsilon} \operatorname{rep} \mathscr{V},$$

where γ is the functor $(N, f) \mapsto (N, f)$ and ε is determined by the choice of a basis vector in each ΦU_i. The functor γ induces a bijection between the "isoclasses" of indecomposables of sub $\bar{\Phi}$ and the isoclasses of indecomposables of sub Φ which are not of the form $(N, 0)$ with $\Phi N = 0$. The functor ε is fully faithful; it is an equivalence if $U_1, \ldots, U_s$ exhaust the indecomposables of K. This takes place for instance in case $K = \operatorname{sub}_0 F_P$ and $\Phi = F'_P$, when $\tilde{S} = P \lhd Q$ is representation-finite (2.5).

6.3. With proposition 6.1 we can also prove that, *if $\tilde{S} = P \lhd Q$ is faithful, the subsets P and Q are uniquely determined by $\tilde{S}$.* Indeed, suppose that $\tilde{S} = P \lhd Q = P' \lhd Q'$ and that, say, $P \cap Q' \neq \varnothing$. The (trivially completed) poset P then has the form $P = (P \backslash Q') \lhd (P \cap Q')$. From 6.1 we infer that an indecomposable representation of P has its support in $P \backslash Q'$ or in $P \cap Q'$. In particular, there is no indecomposable of rep P whose support intersects $\mathring{P} \subset P \backslash Q'$ and $P \cap Q'$. From 6.1 it then follows that there is no indecomposable of $\tilde{S}$ whose support intersects $\mathring{P}$ and $P \cap Q'$.

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