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Representations of bipartite completed posets

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0. General concepts and results

0.1. A *completed poset* \tilde{S} consists of a finite set S , a partial order relation $S^{\leq} = \{(s, t) \in S^2 : s \leq t\}$ on S and an equivalence relation \sim on S^{\leq} . These data are subjected to the condition that $r \leq s \leq t$ and $(r, t) \sim (r', t')$ imply the existence of a unique s' satisfying $r' \leq s' \leq t'$, $(r, s) \sim (r', s')$ and $(s, t) \sim (s', t')$.

In case $(s, s) \sim (s', s')$ we shall write $s \sim s'$, thus obtaining an equivalence relation on S . In fact, it follows from the axioms that $(s, t) \sim (s', s')$ implies $s = t$ and that $(s, t) \sim (s', t')$ implies $s \sim s'$ and $t \sim t'$.

0.2. Completed posets provide a convenient formulation of the matrix problem which is our real center of interest. We first attach two categories to the completed poset \tilde{S} : Let s_1, \dots, s_n be a numbering of S and k a field. The objects of our first category \tilde{S}_k are the vectors $v = [v_1 \cdots v_n] \in \mathbb{N}^n$ such that $v_i = v_j$ if $s_i \sim s_j$. In order to define the morphisms, consider two objects u, v and a matrix $B \in k^{|\nu| \times |\mu|}$, where $|\nu| = v_1 + \cdots + v_n$ (we do accept matrices having no row or no column!). We subdivide B into rectangular blocks $\tilde{B}_{ji} \in k^{v_j \times u_i}$ ($1 \leq i, j \leq n$) in the usual way, and we define $\text{Hom}(u, v)$ as the subspace of $k^{|\nu| \times |\mu|}$ formed by the B such that $\tilde{B}_{ji} = 0$ if $s_i \not\sim s_j$ and $\tilde{B}_{ji} = \tilde{B}_{qp}$ if $(s_i, s_j) \sim (s_p, s_q)$. The composition of \tilde{S}_k is given by matrix multiplication (the condition imposed on completed posets makes sure that $B'B \in \text{Hom}(u, w)$ if $B \in \text{Hom}(u, v)$ and $B' \in \text{Hom}(v, w)$).

We call *dimension-vector* a pair $d = (d_0, \vec{d}) = [d_0 d_1 \dots d_n] \in \mathbb{N} \times \mathbb{N}^n$, where $\vec{d} = [d_1 \cdots d_n] \in \tilde{S}_k$. Further, we call *representation of \tilde{S} of dimension d* a pair (d, M) formed by a dimension-vector d and a matrix $M \in k^{d_0 \times |d|}$. For $i \geq 1$, we call d_i the *dimension of (d, M) at the point s_i* . A morphism of representations $(d, M) \rightarrow (e, N)$ is given by a pair (A, B) of matrices $A \in k^{d_0 \times e_0}$ and $B \in \text{Hom}(\vec{d}, \vec{e})$ such that $AN = MB^T$. Composition is defined by $(A', B') \circ (A, B) = (AA', B'B)$. Let $\text{rep } \tilde{S}$ denote the category thus defined.

The representations of completed posets play a central rôle in general representation theory. For information on how they fit into this broader context, we refer to [5, 9].

Our problem is to determine the isomorphism classes of $\text{rep } \tilde{S}$. If we set

$GL_m = \{A \in k^{m \times m} : \det A \neq 0\}$ and $\text{Aut } \bar{d} = \text{Hom}(\bar{d}, \bar{d}) \cap GL_{|\bar{d}|}$, these classes correspond bijectively to the orbits of the groups $GL_{d_0} \times \text{Aut } \bar{d}$ in the spaces $k^{d_0 \times |\bar{d}|}$ under the actions $(A, B; N) \mapsto ANB^{-T}$. We are especially interested in the case where there are only finitely many orbits for each d .

0.3. Of course, the investigation of these orbits is greatly facilitated by the observation that the category $\text{rep } \bar{S}$ is additive. In fact, we fix and shall need a canonical construction for the direct sum of two representations. Our ‘‘canon’’ is illustrated with an example in Fig. 1, where $(e, P) \oplus (f, Q) = (e + f, M)$. The symbol $s \rightarrow t$ means that t is subsequent to s in S . The produced morphisms are our canonical projections. The canonical immersions are defined by the transposed matrices. The (canonical) direct sum $\bigoplus_{i=1}^l U_i$ of a sequence U_1, \dots, U_l of representations is defined recursively by $\bigoplus_{i=1}^l U_i = (\bigoplus_{i=1}^{l-1} U_i) \oplus U_l$.

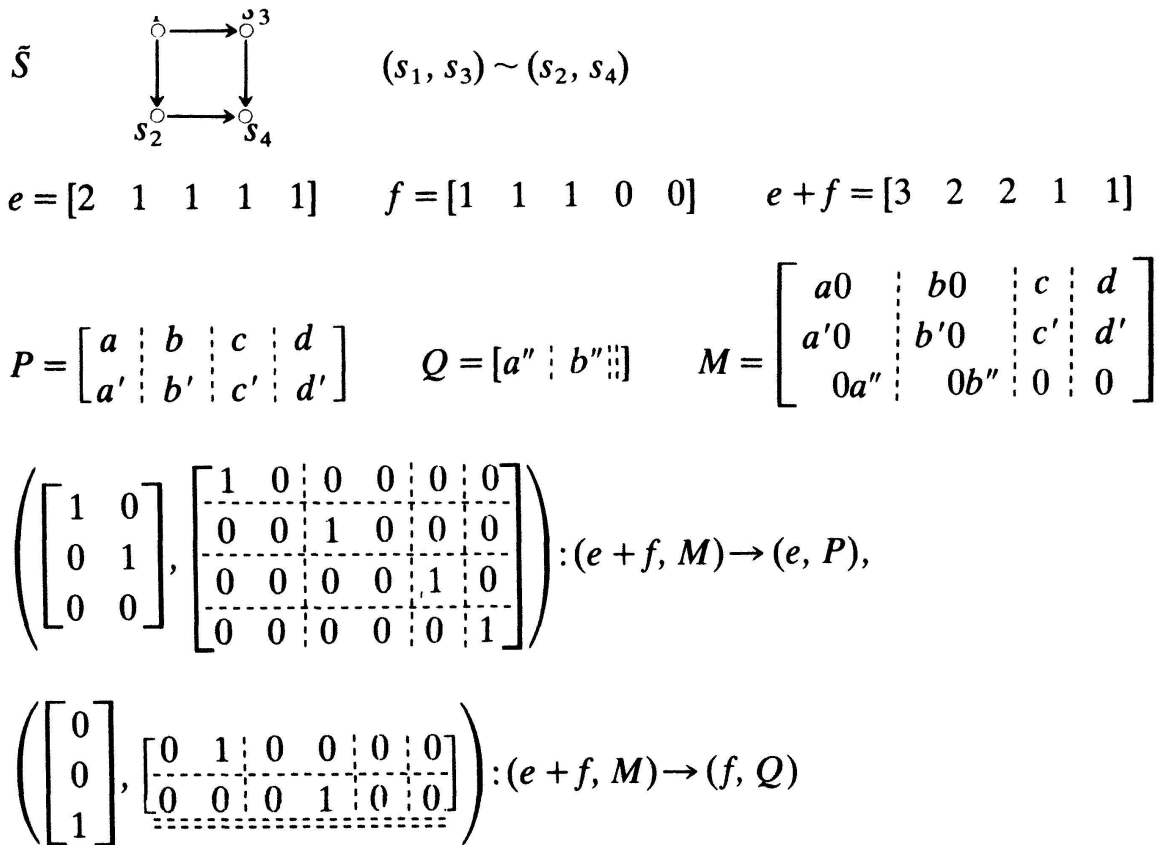


Figure 1

We call a representation *indecomposable* if it is not zero and not *isomorphic* to the direct sum of two non-zero representations. It is clear that each representation of \bar{S} is isomorphic to a direct sum of indecomposables. The unicity of such a decomposition up to isomorphism follows from the fact that idempotent endomorphisms of $\text{rep } \bar{S}$ split (1.1). This reduces our classification problem to the description of the indecomposables. We are particularly interested in the case

where \tilde{S} is *representation-finite*, i.e. admits only finitely many isomorphism classes of indecomposables.

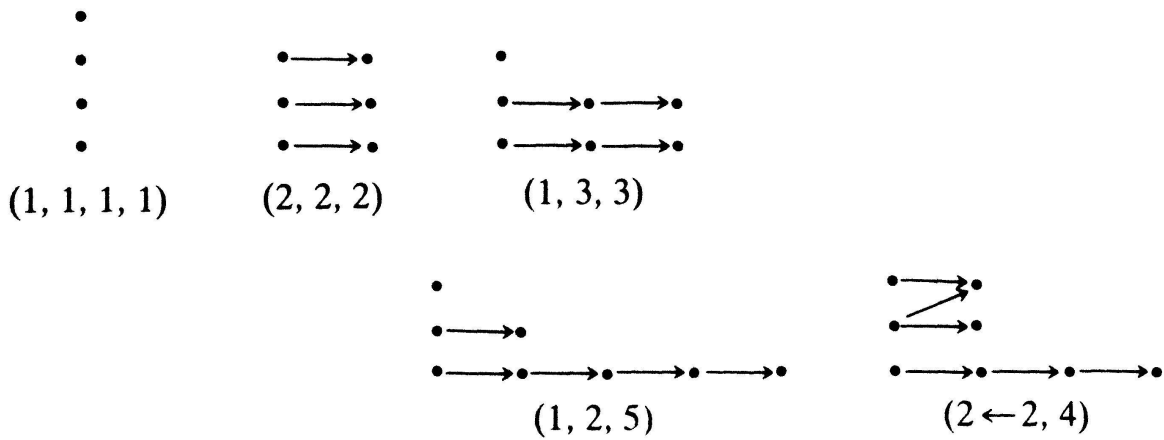


Figure 2

The representation-finite \tilde{S} are determined in [1][2] when \tilde{S} is a *trivially completed* poset (i.e. \sim is the identity), in [6] when $(s, t) \sim (s', t')$ and $(s, t) \neq (s', t')$ imply $s = t$ and $s' = t'$. The result in the first case is that a (trivially completed) poset is representation-finite iff it does not contain a *full subposet* (= subset equipped with the induced order) of one of the 5 forms given in Fig. 2 (where the symbol $s \rightarrow t$ now means that t is subsequent to s in the subposet!).

Because of the striking simplicity of this result, our general method is to reduce the characterization of representation-finite completed posets to the trivially completed case. In the present article, we present such a reduction in a particular case which happens to be crucial for the general solution, as will be shown in a forthcoming paper.

0.4. In the case of a representation-finite \tilde{S} , it is easy to prove that each equivalence class of \tilde{S} is linearly ordered and has cardinality ≤ 3 . From the first part of this statement and the axioms of completed posets it then follows that $(s, t) \sim (s', t)$ implies $s = s'$, and dually that $(s, t) \sim (s, t')$ implies $t = t'$. In fact, the conditions which we shall impose on \tilde{S} in this article are much stronger.

Let $\{1, \dots, m\} \subset \mathbb{N}$ be an interval and $\mu: \{1, \dots, m\} \rightarrow \{1, \dots, m\}$ a non-decreasing function such that $\mu(i) \geq i + 1$ for each $i < m$. By a μ -chain \dot{P} in a partially ordered set P we mean a subset $\dot{P} \subset P$ consisting of m linearly ordered elements $s_1 < \dots < s_m$ such that for each $i \leq m$ the interval $[a_i, a_{\mu(i)}] = \{p \in P: a_i \leq p \leq a_{\mu(i)}\}$ coincides with $\{s_i, s_{i+1}, \dots, s_{\mu(i)}\}$. Whenever we refer to a *bipartite* completed poset $\tilde{S} = P \triangleleft Q$, we implicitly assume: first that we are given a function μ and two finite posets P, Q equipped with μ -chains $\dot{P} = \{s_1 < \dots <$

$s_m\}$, $\tilde{Q} = \{s'_1 < \dots < s'_m\}$ respectively; second that \tilde{S} is described in terms of the data as follows.

- a) $S = P \sqcup Q$ (= disjoint union)
- b) $S^{\cong} = P^{\cong} \cup Q^{\cong} \cup P \times Q$ (in particular, $p \in P$ and $q \in Q$ imply $p < q$)
- c) $(s_i, s_j) \sim (s'_i, s'_j)$ if $i \leq m$ and $j \leq \mu(i)$; any other $(s, t) \in S^{\cong}$ is equivalent only to itself.

The points of \hat{P} and \hat{Q} are called *thick*, those of $\dot{P} = P \setminus \hat{P}$ and $\dot{Q} = Q \setminus \hat{Q}$ *thin*. For each thick point $s \in S$, we denote by s' the point of S such that $s' \sim s \neq s'$. The quasidual of $\tilde{S} = P \triangleleft Q$ is by definition $\tilde{S}^* = Q \triangleleft P$.

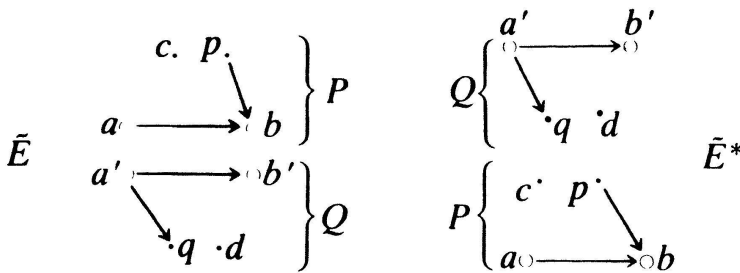


Figure 3

Figure 3 shows an example of a bipartite completed poset and its quasidual. There we have $m = 2$ and $\mu(1) = \mu(2) = 2$, the thick points are represented by ringlets and the arrows from the first to the second components of \tilde{E} and \tilde{E}^* are omitted.

Of course, the dual \tilde{S}^0 of a bipartite completed poset S can also be defined. But we have $\tilde{E}^0 \xrightarrow{\sim} \tilde{E}$ in the case of Fig. 3.

0.5. Let $\tilde{S} = P \triangleleft Q$ be a bipartite completed poset. For each $s \in S$, we set $S(s) = \{t \in S : s \not\leq t \leq s\}$, endow $S(s)$ with the order relation induced by S , formally add to $S(s)$ a smallest element 0 and a largest element 1 and denote the poset obtained in this way by $\tilde{S}(s) = S(s) \cup \{0, 1\}$. With this notation, we attach two posets \hat{P} and \hat{Q} to the components P and Q : The poset \hat{P} consists of the thin points $s \in \dot{P}$ and of the pairs (p, t) where $p \in \hat{P}$ and $t \in \tilde{S}(p')$. We equip the subset \dot{P} of \hat{P} with the order induced by \tilde{S} and set $s \leq (p, t)$ iff $s \leq p$, $(p, t) \leq s$ iff $p \leq s$. We further set $(p_1, t_1) \leq (p_2, t_2)$ in the following two cases:

- a) $p_1 < p_2$ and $(p_1, p_2) \neq (p'_1, p'_2)$.
- b) $p_1 \leq p_2$, $(p_1, p_2) \sim (p'_1, p'_2)$ and one of the conditions $p'_1 \leq t_2 \neq 1$, $0 \neq t_1 \leq p'_2$ or $t_1 \leq t_2$ holds.

The description of \hat{Q} is dual (and quasidual) to that of \hat{P} . In particular, the elements of \hat{Q} have the form $t \in \dot{Q}$ or (q, s) where $q \in \hat{Q}$ and $s \in \tilde{S}(q')$.

In the case $\tilde{S} = \tilde{E}$ (Fig. 3), \hat{P} and \hat{Q} are given by Fig. 4.

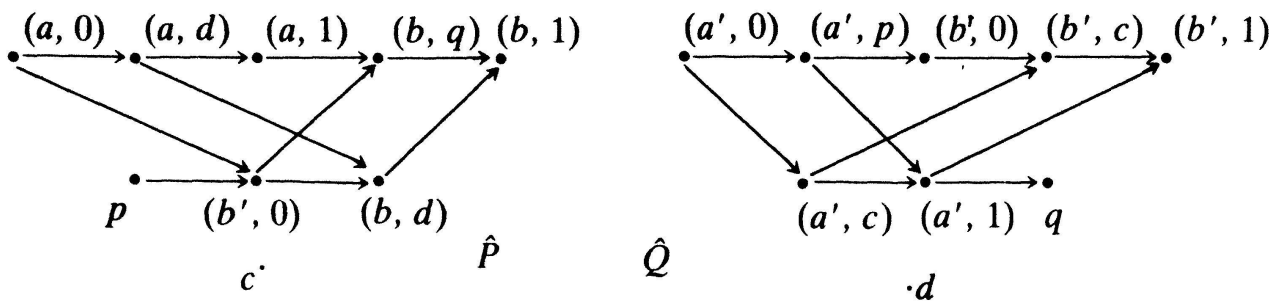


Figure 4

Now we can formulate our first main theorem:

THEOREM 1. *The bipartite completed poset $\tilde{S} = P \triangleleft Q$ is representation-finite iff so are the posets \hat{P} and \hat{Q} .*

0.6. Let T be a subset of S which is *stable* under the equivalence relation of S (i.e. $s \in S, t \in T$ and $t \sim s$ imply $s \in T$). The structure carried by \tilde{S} then naturally induces a completed poset structure \tilde{T} on T . If we equip T with the numbering “induced” by that of S (0.2), we obtain a fully faithful embedding $\text{rep } \tilde{T} \rightarrow \text{rep } \tilde{S}$. More precisely, we can extend each dimension vector d of \tilde{T} by zero and obtain a dimension vector d^0 of \tilde{S} ($d_0^0 = d_0, d_i^0 = d_j$ if $s_i = t_j$ and $d_i^0 = 0$ if $s_i \notin T$). The embedding functor is then simply $(d, M) \mapsto (d^0, M)$. It permits us to identify the set $\text{ind } \tilde{T}$ of isomorphism classes of $\text{rep } T$ with a subset of $\text{ind } \tilde{S}$.

For instance, the *trivial representation* \mathcal{O}_0 of S , whose dimension-vector is $[10 \cdots 0]$, is associated with a representation of $\tilde{\mathcal{O}}$! More generally, each representation (e, M) of \tilde{S} has the above form (d^0, M) if we take T to be the *support* $\{s_i \in S : e_i \neq 0\}$ of (e, M) .

If the support of (e, M) equals S , we say that (e, M) is *faithful*. And we say that \tilde{S} is *faithful* if \tilde{S} admits a faithful indecomposable representation.

THEOREM 2. *Let $\tilde{S} = P \triangleleft Q$ be a faithful bipartite completed poset.*

a) *If the poset $S(s)$ is linearly ordered for each $s \in \hat{P}$ (resp. $s \in \hat{Q}$), then there is a natural bijection from $\text{ind } \tilde{S} \setminus \text{ind } \hat{P}$ onto $\text{ind } \hat{Q} \setminus \{\mathcal{O}_0\}$ (resp. from $\text{ind } \tilde{S} \setminus \text{ind } \hat{Q}$ onto $\text{ind } \hat{P} \setminus \{\mathcal{O}_0\}$).*

b) *If \tilde{S} is representation-finite and if there exist thick points $p \in \hat{P}$ and $q \in \hat{Q}$ such that neither $S(p)$ nor $S(q)$ is linearly ordered, then \tilde{S} is isomorphic to \tilde{E} or \tilde{E}^* (0.3).*

1. The easy direction

Our objective in this section is to prepare the general demonstration by proving the first part of Theorem 2 and the necessity of the condition of theorem

1. From 1.2 onwards, we fix P, Q and $\tilde{S} = P \triangleleft Q$. We choose a numbering of $S = P \cup Q$ which first numbers \dot{P} (in the order of succession s_1, \dots, s_m imposed by P), then \dot{Q} (in the order of succession s'_1, \dots, s'_m) and finally \dot{Q} .

1.1. Let us briefly recall why representations of a completed poset \tilde{S} can be “uniquely” decomposed into indecomposables.

We first notice that the category \tilde{S}_k (0.2) is k -linear in the sense that the morphism spaces carry k -vector-space structures, that the composition is bilinear and that finite direct sums exist: In fact, we can set $u \oplus v = u + v$ if we define the canonical immersions and projections in the obvious way. Each point $t \in S$ gives rise to an indecomposable $\bar{t} \in \tilde{S}_k$ whose endomorphism-algebra is local ($\bar{t}_i = 1$ or 0 according as $s_i \sim t$ or $s_i \not\sim t$). The map $t \mapsto \bar{t}$ yields a bijection between the equivalence classes of S and the indecomposables of \tilde{S}_k . Each object $v \in \tilde{S}_k$ is a finite direct sum of indecomposables. Finally, for each idempotent $F \in \text{Hom}(v, v)$, there exist morphisms $R \in \text{Hom}(v, u)$ and $S \in \text{Hom}(u, v)$ such that $F = SR$ and $\mathbb{1}_u = RS$ (since $\text{Hom}(v, v)$ is a finite-dimensional algebra, F is conjugate to a sum of idempotents occurring in the natural decomposition of $\mathbb{1}_v$ into pairwise annihilating primitive idempotents).

Like \tilde{S}_k , the category $\text{rep } \tilde{S}$ (0.2) is k -linear. Each decomposition $(d, M) \xrightarrow{\sim} (e, P) \oplus (f, Q)$ gives rise to an idempotent $(E, F) \in \text{End}(d, M)$, the projection onto the first summand along the second. To prove the converse, we must supply each idempotent (E, F) with morphisms

$$(d, M) \xrightarrow{(V,R)} (e, P) \xrightarrow{(U,S)} (d, M)$$

such that $(E, F) = (VU, SR)$ and $(\mathbb{1}_{e_0}, \mathbb{1}_{|e|}) = (UV, RS)$. For this, we first construct U, V (clear!) and R, S as above; then we set $P = UMR^T$.

Since the direct sum decompositions of (d, M) corresponds to the decompositions of $\mathbb{1}_{(d,M)}$ into pairwise annihilating idempotents, (d, M) is a direct sum of indecomposable representations, which are uniquely determined up to isomorphism.

1.2. We now assume that $\tilde{S} = P \triangleleft Q$. Using the action of $GL_{d_0} \times \text{Aut } \bar{d}$ (0.2), we can reduce each representation (d, M) of \tilde{S} to the form of Fig. 5. Indeed, we can first find a matrix $A \in GL_{d_0}$ such that

$$AM = \left[\begin{array}{c|c} M_P & M' \\ \hline 0 & M_Q \end{array} \right],$$

where $M_P \in k^{r \times |\bar{d}_P|}$ and $r = \text{rank } M_P$. Then there is a C such that $M_P C = M'$, and

AMB^{-1} is given the wanted form by setting $B = \begin{bmatrix} \uparrow & 0 \\ C^T & \uparrow \end{bmatrix} \in \text{Aut } \bar{d}$.

$$\left[\begin{array}{c|c} M_P & 0 \\ \hline 0 & M_Q \end{array} \right] \begin{array}{l} \} r \\ \} d_0 - r \end{array} \quad \begin{array}{l} \bar{d}_P = [d_1 d_2 \cdots d_{|P|}] \\ r = \text{rank } M_P \end{array}$$

$$|\bar{d}_P| |\bar{d}| - |\bar{d}_P| \quad |P| = \text{cardinality of } P$$

Figure 5

This means that each representation of \bar{S} is isomorphic to a “reduced” representation whose matrix has the form of Fig. 5. If $(A, B): (d, M) \rightarrow (e, N)$ is a morphism of reduced representations, we subdivide A, B into blocks adapted to those of M and N :

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 & 0 \\ B_3 & B_4 \end{bmatrix}.$$

The condition $AN = MB^T$ then means that $A_1 N_P = M_P B_1^T$, $A_2 N_Q = M_P B_3^T$, $A_3 N_P = 0$ and $A_4 N_Q = M_Q B_4^T$. Since the rows of N_P are linearly independent by assumption, the equality $A_3 N = 0$ implies $A_3 = 0$.

In case (A, B) is an isomorphism, the condition imposed upon B_1 is to lie in the automorphism group of \bar{d}_P in the category P_k associated with the (trivially completed) poset P . This means that for M_P we can choose representatives of the isomorphism classes of $\text{rep } P$ and then restrict (A_1, B_1) to $\text{Aut } M_P$. The problem then stays with B_4 , which must be an isomorphism of Q_k and share some “subblocks” with B_1 . To examine into this condition, we introduce supplementary simplifying assumptions.

1.3. Let \mathcal{U} be a sequence

$$(d_{11}, U_{11}), \dots, (d_{l_1}, U_{l_1}), \dots, (d_{ij}, U_{ij}), \dots, (d_{m_1}, U_{m_1}), \dots, (d_{ml_m}, U_{ml_m})$$

of pairwise nonisomorphic indecomposable representations of P such that (d_{ij}, U_{ij}) has dimension 1 at $s_i \in \mathring{P}$ and 0 at all other thick points of $P (1 \leq i \leq m, 1 \leq j \leq l_i)$. We denote by $\text{rep}_{\mathcal{U}} \bar{S}$ the full subcategory of $\text{rep } \bar{S}$ formed by the reduced (1.2) representations (d, M) whose P -component has the form

$$(*) \quad (d_P, M_P) = (d_{11}, U_{11})^{\mu_{11}} \oplus \cdots \oplus (d_{ij}, U_{ij})^{\mu_{ij}} \oplus \cdots \oplus (d_{ml_m}, U_{ml_m})^{\mu_{ml_m}}$$

where $d_P = [r d_1 \cdots d_{|P|}]$ (Fig. 5) and $\mu_{ij} \in \mathbb{N}$. We stress the point that this direct sum has to be constructed according to the prescribed canon (0.3).

The category $\text{rep}_{\mathcal{Q}} \tilde{S}$ is k -linear, and its indecomposables are indecomposable in $\text{rep} \tilde{S}$: If (e, N) is a direct summand of $(d, M) \in \text{rep}_{\mathcal{Q}} \tilde{S}$, we can first reduce (e, N) to the form of Fig. 5 and then further convert the P -component to a direct sum of the form (*).

Of special importance for us will be the case where, up to isomorphism, \mathcal{Q} exhausts the indecomposables of $\text{rep} P$ whose support intersects \dot{P} . In this case, the indecomposables of $\text{rep} \tilde{S}$ which are not isomorphic to an indecomposable of $\text{rep}_{\mathcal{Q}} \tilde{S}$ lie in $\text{rep} \dot{P}$ (0.6).

1.4. In order to describe $\text{rep}_{\mathcal{Q}} \tilde{S}$, we introduce a set $Q_{\mathcal{Q}}$ which consists of the representations (d_{ij}, U_{ij}) and of the thin points of Q . We equip $Q_{\mathcal{Q}}$ with the following relation $R \subset Q_{\mathcal{Q}}^2$: In case $q, r \in \dot{Q}$ we set qRr (i.e. $(q, r) \in R$!) iff $q \leq r$ in Q . Similarly, we set $qR(d_{ij}, U_{ij})$ (resp. $(d_{ij}, U_{ij})Rr$) iff $q \leq s'_i$ (resp. $s'_i \leq r$) in Q . In case $(s_i, s_u) \sim (s'_i, s'_u)$ we set $(d_{ij}, U_{ij})R(d_{uv}, U_{uv})$ iff there exists a morphism $(A, B): (d_{ij}, U_{ij}) \rightarrow (d_{uv}, U_{uv})$ such that $\bar{B}_{ui} \neq 0$ (0.2). Finally, we also set $(d_{ij}, U_{ij})R(d_{uv}, U_{uv})$ if $i \leq u$ and $(s_i, s_u) \not\sim (s'_i, s'_u)$.

The following proposition uses the notations of 1.2 and 1.3. In particular, if $(d, M) \in \text{rep}_{\mathcal{Q}} \tilde{S}$, (d_P, M_P) is the direct sum of 1.3. By d_Q we denote the row

$$d_Q = [(d_0 - r) \mu_{11} \mu_{12} \cdots \mu_{m_l m} d_{1+|\dot{Q}|} \cdots d_n] \in \mathbb{N}^{|\mathcal{Q}_{\mathcal{Q}}|+1}$$

PROPOSITION a) *The relation R is a partial order on $Q_{\mathcal{Q}}$.*

b) *If $(A, B): (d, M) \rightarrow (e, N)$ is a morphism of $\text{rep}_{\mathcal{Q}} \tilde{S}$, the block B_4 belongs to the morphism space $\text{Hom}(\bar{d}_Q, \bar{e}_Q)$ of $Q_{\mathcal{Q}k}$.*

c) *The reduction functor $\mathcal{R}: \text{rep}_{\mathcal{Q}} \tilde{S} \rightarrow \text{rep} Q_{\mathcal{Q}}$, $(d, M) \mapsto (d_Q, M_Q)$ which maps a morphism (A, B) onto (A_4, B_4) is an epivalence.*

The neologism *epivalence*, chosen here for a widely used notion of representation theory, means that \mathcal{R} detects isomorphisms (μ is invertible if so is $\mathcal{R}\mu$) and induces surjections on the morphism spaces and on the isomorphism classes of the objects. It follows that \mathcal{R} induces a bijection between the isomorphism classes.

Proof. a) The crucial point is to prove that $(d_{ij}, U_{ij})R(d_{uv}, U_{uv})$ and $(d_{uv}, U_{uv})R(d_{yz}, U_{yz})$ imply $(d_{ij}, U_{ij})R(d_{yz}, U_{yz})$. This is clear by definition if $(s_i, s_y) \not\sim (s'_i, s'_y)$. Otherwise, there are morphisms $(A, B): (d_{ij}, U_{ij}) \rightarrow (d_{uv}, U_{uv})$ and $(C, D): (d_{uv}, U_{uv}) \rightarrow (d_{yz}, U_{yz})$ such that $\bar{B}_{ui} \neq 0 \neq \bar{D}_{yu}$. It follows from 0.4 that $(\bar{D}\bar{B})_{yi} = \sum_w \bar{D}_{yw} \bar{B}_{wi} = \bar{D}_{yu} \bar{B}_{ui} \neq 0$.

With these notations, we must also prove that $i = y, j = z$, implies $i = u, j = v$. The reason is that in case $(i, j) \neq (u, v)$, (AC, DB) would be nilpotent though $((DB)^N)_{ii} = (\bar{D}_{iu} \bar{B}_{ui})^N \neq 0$.

b) We must prove that a block of B_4 vanishes if it is associated with a pair $(x, y) \in Q_{\mathcal{Q}}^2$ such that $x \not\leq y$. Since $\bar{B}_{ba} = 0$ if $s_a \not\leq s_b$, it suffices to examine the case $x = (d_{ij}, U_{ij})$, $y = (d_{uv}, U_{uv})$ where $(s_i, s_u) \sim (s'_i, s'_u)$. Then the associated block of B_4 is equal to a certain subblock of the block \bar{B}_{ui} of B . We can interpret each coefficient of this subblock as the 1×1 -block \bar{D}_{ui} associated with a morphism $(C, D): (d_{ij}, U_{ij}) \rightarrow (d_{uv}, U_{uv})$. By definition of the order of $Q_{\mathcal{Q}}$, the coefficient is zero if $x \not\leq y$.

c) By construction, \mathcal{R} induces a surjection on the objects. Let now $(d, M), (e, N)$ be two objects of $\text{rep}_{\mathcal{Q}} \bar{S}$ and (C, D) a morphism $(d_Q, M_Q) \rightarrow (e_Q, N_Q)$. We must find an (A, B) such that $C = A_4$ and $D = B_4$. Of course, we will set $A_2 = A_3 = B_2 = B_3 = 0$ (1.2). The problem is to find an $(A_1, B_1): (d_P, M_P) \rightarrow (e_P, N_P)$ such that B_1 shares appropriate blocks with B_4 . More precisely, each pair $(x, y) \in Q_{\mathcal{Q}}^2$ such that $x = (d_{ij}, U_{ij}) \leq y = (d_{uv}, U_{uv})$ and $(s_i, s_u) \sim (s'_i, s'_u)$ determines a subblock of $(\bar{B}_1)_{ui} = \bar{B}_{ui}$ which is prescribed by the datum of B_4 . So it is enough to prove the existence of an (A_1, B_1) for which all these subblocks are arbitrarily prescribed. As in b) above, this follows from the interpretation of the coefficients of these subblocks as 1×1 -blocks associated with morphisms between direct summands of (d_P, M_P) and (e_P, N_P) of type (d_{ij}, U_{ij}) and (d_{uv}, U_{uv}) .

It remains to prove that \mathcal{R} detects isomorphisms: Consider a morphism $\mu: X \rightarrow Y$ such that $\mathcal{R}\mu$ is invertible, and choose a $\nu: Y \rightarrow X$ such that $\mathcal{R}\nu = (\mathcal{R}\mu)^{-1}$. The kernel K of $\text{End } X \rightarrow \text{End } \mathcal{R}X$ then contains $\mathbb{1}_X - \nu\mu$. Since $0 \neq Z \in \text{rep}_{\mathcal{Q}} \bar{S}$ implies $\mathcal{R}Z \neq 0$, K contains no primitive idempotent. We infer that K and $\mathbb{1}_X - \nu\mu$ are nilpotent. Hence $\nu\mu$ is invertible and so is $\mu\nu$.

1.5. *Proof of the necessity in Theorem 1.* Each thick point $t \in \mathring{P}$ gives rise to two indecomposable representations of P supported by t : Their dimension-vectors are $[0 \bar{t}_P]$ and $[1 \bar{t}_P]$ where $\bar{t}_P \in \mathbb{N}^{|\mathring{P}|}$ satisfies $\bar{t}_{P_i} = 1$ if $s_i = t$ and $\bar{t}_{P_i} = 0$ if $s_i \neq t$; we denote them by $\{t\}_0$ and $\{t\}_1$.

Similarly, if $t \in \mathring{P}$ and $s \in \mathring{P}$ are incomparable, we denote by $\{t, s\}_0$ "the" indecomposable representation of P with support $\{s, t\}$ and dimension-vector $[1 \bar{s}_P + \bar{t}_P]$.

Now, if the sequence \mathcal{U} of 1.2 runs through all indecomposables of $\text{rep } P$ of the form $\{t\}_0, \{t\}_1$ and $\{t, s\}_0$, the poset $Q_{\mathcal{Q}}$ of 1.4 is obviously identified with \hat{Q} . By Proposition 1.4c), $\hat{Q} \simeq Q_{\mathcal{Q}}$ is representation-finite if so is \bar{S} .

1.6. *Proof of Theorem 2, part a).* If $S(t)$ is linearly ordered for each $t \in \mathring{P}$, the sequence \mathcal{U} chosen in 1.5 exhausts (up to isomorphism) the indecomposables of $\text{rep } P$ whose support intersects \mathring{P} . The statement to be proved therefore follows from the last sentence of 1.3 and the Proposition 1.4c).

2. The poset \hat{Q} associated with $\tilde{S} = P \triangleleft Q$

The progress made in Section 2 reduces the proof of our Theorems 1 and 2 to the following combinatorial statement. Its demonstration will spread over the rest of the article, where \hat{P} and \hat{Q} are always supposed to be representation-finite.

THEOREM 3. *Suppose that $\tilde{S} = P \triangleleft Q$ is faithful, that \hat{P} and \hat{Q} are representation-finite and that there exist points $p \in \hat{P}$ and $q \in \hat{Q}$ such that neither $S(p)$ nor $S(q)$ are linearly ordered. Then \tilde{S} is isomorphic to \tilde{E} or to $\tilde{E}^*(0.4)$.*

2.1. We first recall the classification of the indecomposable representations of a representation-finite poset T . According to [3] the support of a non-trivial (0.6) indecomposable is a full subset of T which is isomorphic to one of the 13 posets of Fig. 6. The number below the symbol of a listed poset is the number of its isoclasses of faithful indecomposables.

So each *supporting subposet* Σ of T (i.e. each full subposet of the form of Fig. 6) yields the indicated number of non-trivial indecomposables of T . We denote these indecomposables by $\Sigma_0, \Sigma_1 \dots$. For instance, each ‘‘monad’’ $\{t_i\}$ yields 2 indecomposables, the representation $\{t_i\}_0$ whose dimension vector d satisfies $d_0 = 0, d_i = |\bar{d}| = 1$, and a representation $\{t_i\}_1$ with matrix $[1]$. Each ‘‘dyad’’ $\{t_i, t_j\}$ yields 1 indecomposable $\{t_i, t_j\}_0$ with matrix $[1 \mid 1]$. Each ‘‘triad’’ $\{t_i, t_j, t_k\}$ yields 2 indecomposables, the first $\{t_i, t_j, t_k\}_0$ with matrix $[1 \mid 1 \mid 1]$, the second $\{t_i, t_j, t_k\}_1$ with matrix $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \dots$

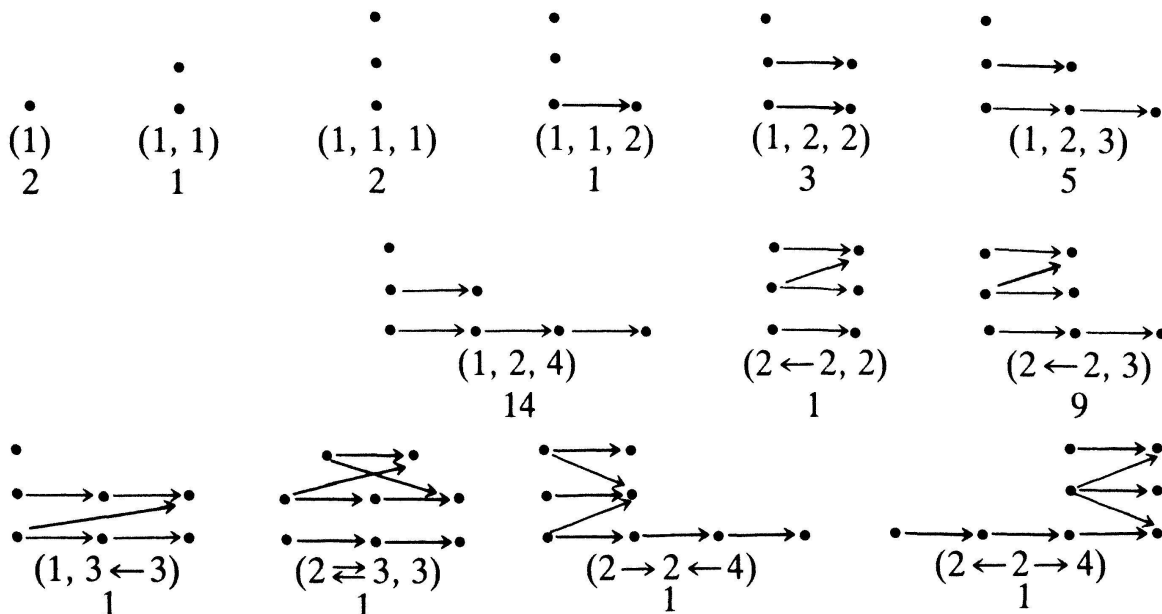


Figure 6

2.2. EXAMPLE. In the case $\tilde{S} = \tilde{E}$ (0.4), all the indecomposable representations of P whose support intersects \dot{P} are listed in Fig. 7. For each of them, the intersection consists of 1 point, and the dimension at this point is 1. Therefore, we can let the sequence \mathcal{U} of 1.3 run through all the indecomposables of Fig. 7, which describes the poset $Q_{\mathcal{U}}$ of 1.4 in this particular case.

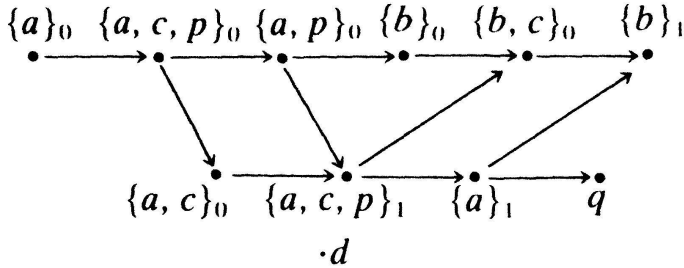


Figure 7

The poset of Figure 7 “fully” contains 11 monads, 18 dyads, 8 triads, 12 copies of \rightarrow and 1 of \rightarrow , which yield 22, 18, 16, 12 and 3 indecomposables respectively. Together with \emptyset_0 and the five nontrivial indecomposables located in \dot{P} , \tilde{E} therefore has 77 indecomposables and is representation-finite. Among the 50 “supporting” subposets enumerated above, there is just one which involves all the points of \tilde{E} up to equivalence, namely $\{d, \{b\}_0, \{a, c, p\}_1 < q\}$. This means that E has exactly 1 faithful indecomposable, whose matrix is

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

$a \quad b \quad c \quad p \quad a' \quad b' \quad d \quad q$

Figure 8

2.3. Returning to the general case, we denote by $T_{t \times e}$ the poset obtained from a representation-finite poset T by substituting a chain $t_1 \rightarrow t_2 \rightarrow \dots \rightarrow t_e$ for a point $t \in T$ as shown in Fig. 9 ($e \geq 1$). We say that t has multiplicity $\geq e$ in T if $T_{t \times e}$ is

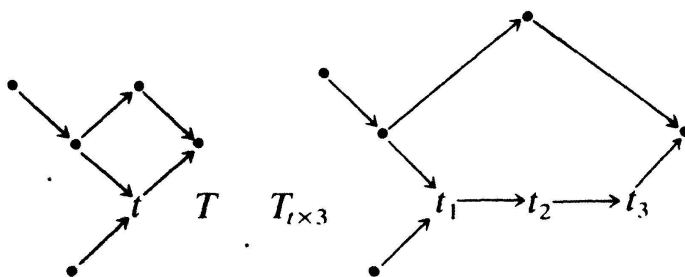


Figure 9

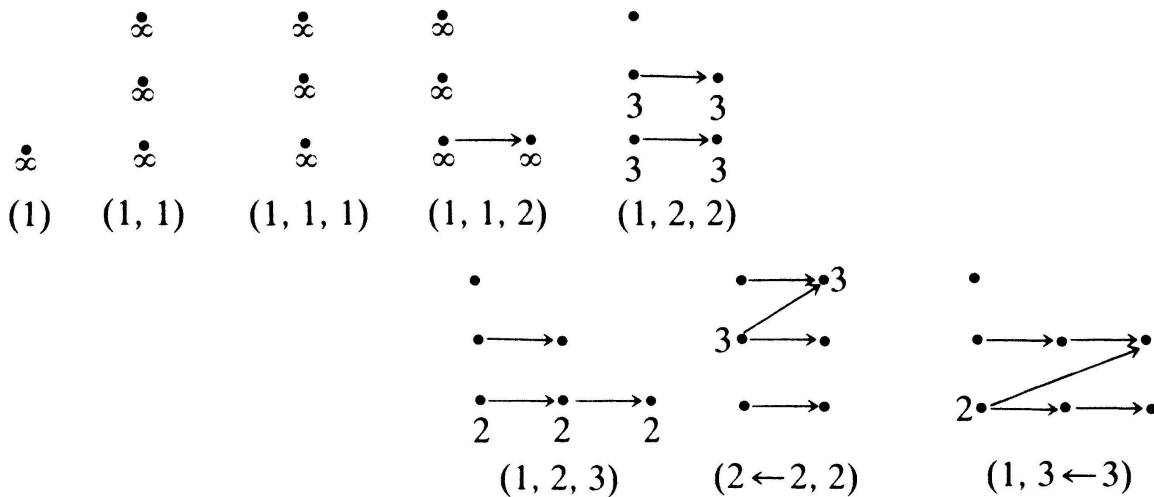
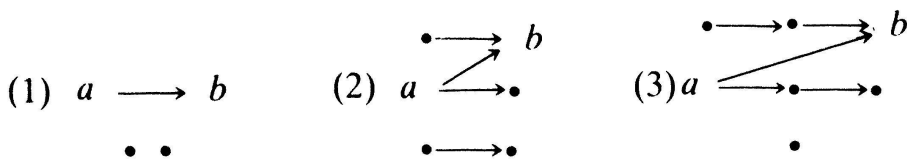


Figure 10

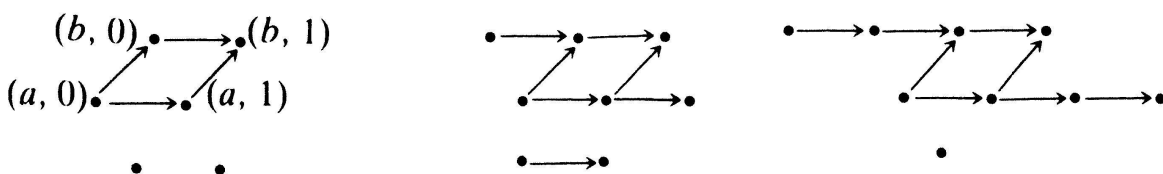
representation-finite. The multiplicities ≥ 2 occurring in the posets of Fig. 6 are listed in Fig. 10.

We apply the construction above in case $T = P$ and $t \in \mathring{P}$. If Q contains a chain $q_1 \rightarrow \dots \rightarrow q_c$ of elements incomparable with $t' \in \mathring{Q}$, then \hat{P} contains the full subposet formed by the elements $p \in \mathring{P}$, $(r, 0)$ for $r \in \mathring{P}$ and $r \leq t$, (t, q_i) for $1 \leq i \leq c$, and $(s, 1)$ for $s \in \mathring{P}$ and $t \leq s$. This subposet is naturally isomorphic to $P_{t \times (c+2)}$ and is representation-finite. It follows that t has multiplicity $\geq c + 2$ in each full subposet of P containing t . In particular, a supporting subposet Σ of P (2.1) which intersects \mathring{P} must be isomorphic to one of the 8 posets of Fig. 10; and $\Sigma \cap \mathring{P}$ contains only points of multiplicity ≥ 2 in Σ .

2.4. LEMMA. P contains no full subposet of one of the following three forms, where a and b are supposed to be thick.



Proof. We first assume that $(a, b) \sim (a', b')$. Then, if P contained 1), 2) or 3), \hat{P} would contain a full subposet of one of the following forms, hence would not be representation-finite



In case $(a, b) \not\sim (a', b')$, we introduce the point $c \in \hat{P}$ subsequent to a , which satisfies $a < c < b$ and $(a, c) \sim (a', c')$. In subcase 1) P then contains the full subposet $a \circ \rightarrow \circ c$ in contradiction to the first part of the proof. In subcase 2) or 3),

P contains a full subposet of the form $\begin{matrix} d \cdot & \longrightarrow & \circ b \\ & \nearrow & \\ a \circ & \longrightarrow & \cdot e \end{matrix} \cdot f$. Since P cannot contain a full subposet of the form $\begin{matrix} c \circ & \longrightarrow & \circ b \\ & & \\ e \cdot & & \cdot f \end{matrix}$, c must be comparable with e . This implies $c < e$

because $e \not\leftarrow b$. By duality, we also obtain that $d < c$, in contradiction to $d \not\leftarrow e$.

2.5. THEOREM 4. *Let Σ be the support of an indecomposable representation (d, U) of P . If Σ intersects, \hat{P} , $\Sigma \cap \hat{P}$ has exactly one point, and the dimension of (d, U) at this point is 1.*

Proof. By 2.3 Σ is isomorphic to one of the 8 posets of Fig. 10, and $\Sigma \cap \hat{P}$ consists of points of multiplicity ≥ 2 in Σ . In case $|\Sigma \cap \hat{P}| \geq 2$, it follows from Fig. 10 that Σ contains a full subposet of one of the three forms excluded by Lemma 2.4. So we must have $|\Sigma \cap \hat{P}| \leq 1$.

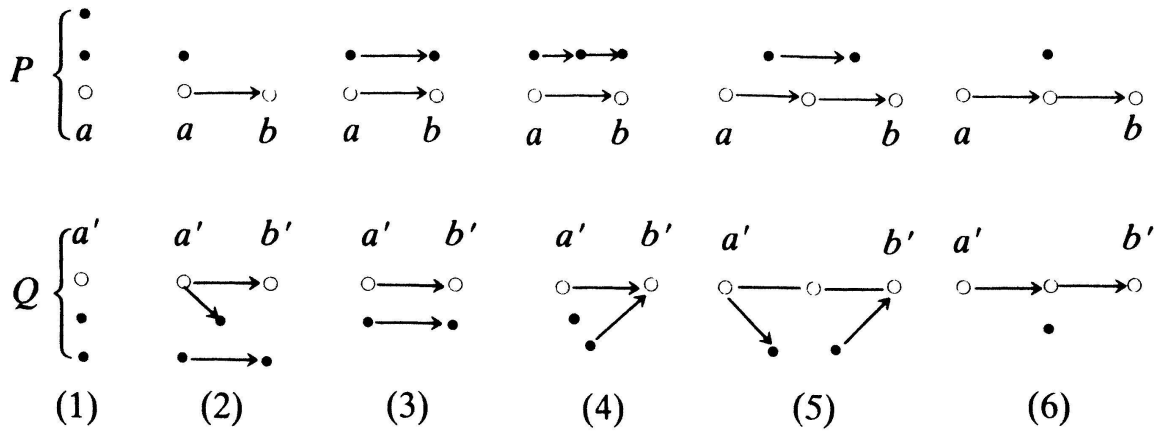
It now remains for us to go through the list of the faithful indecomposable representations of the posets of Fig. 10 and to check that the dimension at a point of multiplicity ≥ 2 is always 1.

2.6. The proof of Theorem 4 only uses the representation-finiteness of \hat{P} , not that of \hat{Q} . Therefore, if \hat{P} is representation-finite, we can let the sequence \mathcal{U} of 1.3 run through representatives of all the indecomposables of $\text{rep } P$ whose support intersects \hat{P} . Proposition 1.4c) then reduces the representation-theory of \tilde{S} to the representation-theory of a poset $Q_{\mathcal{U}}$ which in the case considered here will be further denoted by \hat{Q} .

In other words, if \hat{P} is representation-finite, all the required information is contained in the poset \hat{Q} and not in \hat{Q} , which is identified with a full subposet of \hat{Q} . The problem is that the structure of \hat{Q} is much more intricate than that of \hat{Q} .

In Section 3 below, we collect the information about \hat{Q} used in the further demonstration, at various places of which we also need statements of the following lemma.

LEMMA. *S contains no stable subset T such that the induced completed poset \tilde{T} (0.6) has one of the following forms (where $(a, b) \sim (a', b')$).*



Proof. Construct the associated posets \hat{Q} and check that they contain full subposets of the forms described by Fig. 2.

3. On the structure of the poset \hat{Q}

Denote by \hat{Q}_a the subset of \hat{Q} formed by the indecomposable representations of P chosen in 2.6 whose support contains a thick point $a \in \hat{P}$. Our purpose is to compare \hat{Q}_a with \hat{Q}_b under the assumption, valid throughout this section, that $a < b$ and $(a, b) \sim (a', b')$.

3.1. Our first lemma uses the following notation: If $V = (d, M)$ is a representation of a completed poset $\tilde{T} = \{t_1, t_2, \dots\}^\sim$, we denote by $V(s_1) \in k^{d_0 \times d_1}$ the matrix consisting of the first d_1 columns of M , by $V(s_2) \in k^{d_0 \times d_2}$ the matrix formed by the following d_2 columns \dots . In particular, if $\tilde{T} = P$ and $V \in \hat{Q}_a$, we know by 2.5 that $V(a)$ is reduced to 1 column.

LEMMA. *A representation $V \in \hat{Q}_a$ is smaller than the minimal element $\{b\}_0$ of \hat{Q}_b iff $V(a)$ is a linear combination of the columns of the “strips” $V(s)$ where $s \in P$ and $s < b$.*

Proof. Set $V = (d, M)$. If $(A, B): V \rightarrow \{b\}_0$ is a morphism of rep P , then B is a row and A the “empty” matrix. The condition $AN = MB^T$ of 0.2 therefore means that $0 = MB^T = \sum_{s \in P} M(s)B(s)^T$ if we define j by $s_j = b$ and set $B(s_i) = \bar{B}_{ji}$ (0.2). In the occurring sum, we have $M(b) = 0$ by 2.5 and $B(s) = 0$ if $s \not\leq b$ (0.2).

Now, if $V < \{b\}_0$, we can choose B so that $B(a) \in k$ is non-zero. It follows that $M(a) = -\sum_{s \neq a, s < b} M(s)B(s)^T B(a)^{-1}$.

The converse should be clear.

3.2. LEMMA. *If P contains no element which is incomparable with a and b , then $\{a\}_1$ is the only element of \hat{Q}_a which is incomparable with $\{b\}_0$.*

Proof. The lemma follows from 3.1 and Lemma 3.3 below.

3.3. LEMMA. *Let T be a finite poset, $t \in T$ a point and V an indecomposable representation of T which is not isomorphic to $\{t\}_1$. Then each column of $V(t)$ is a linear combination of the columns of the strips $V(s)$ where $s \not\preceq t$.*

Proof. Assume that the conclusion of our lemma is wrong for $V = (d, M)$. Then there is a row $x \in k^{d_0}$ such that $xV(t) \neq 0$ and $xV(s) = 0$ whenever $s \not\preceq t$. Setting $y = xM$, we infer that y^T is a non-zero morphism from \bar{t} (1.1) to \bar{d} (0.2) in \tilde{S}_k and $(x, y^T): \{t\}_1 \rightarrow V$ a non-zero morphism in $\text{rep } T$.

The row $xV(t) \neq 0$ has d_i entries, where i is defined by $s_i = t$. We choose a row $w \in k^{d_i}$ such that $xV(t)w^T \neq 0$ and set

$$z = \left[\underbrace{0 \cdots 0}_{d_1 + \cdots + d_{i-1}} \quad w_1 \cdots w_{d_i} \quad 0 \cdots 0 \right] \in k^{|\bar{d}|}$$

In this way, we obtain morphisms

$$\{t\}_1 \xrightarrow{(x, y^T)} V \xrightarrow{(Mz^T, z)} \{t\}_1$$

with composition $(xMz^T, zy^T) = (yz^T, zy^T) = (xV(t)w^T, wV(t)^T x^T) \neq 0$. We infer that $\{t\}_1$ is a direct summand of V in contradiction with the assumptions of the lemma.

3.4. From now on, we write $s \asymp t$ if $s, t \in S$ are incomparable, and we say that a thick point c is *normal* if $S(c) = \{s \in S : s \asymp c\}$ is a linearly ordered subset of S .

LEMMA. *Assume that $b \in \hat{P}$ is normal and that there is a $d \in \hat{Q}$ such that $a' \asymp d \asymp b'$. Then the elements of \hat{Q}_b which are incomparable with $\{a\}_1$ are $\{b\}_0$ and $\{b, c\}_0$, where $a \asymp c \asymp b$. The elements V of \hat{Q}_a which are incomparable with $\{b\}_0$ are $\{a\}_1$, $\{a, c\}_0$ where $a \asymp c \asymp b$ and $\{a, c, s\}_1$ where $c \asymp a \asymp s \asymp c \asymp b > s$.*

Proof. It is clear that the listed indecomposables have the required properties. And the elements of \hat{Q}_b which are incomparable with $\{a\}_1$ are the listed ones, because \hat{Q}_b consists of $\{b\}_0$, $\{b\}_1$ and indecomposables of the form $\{b, c\}_0$ where $b \asymp c$.

It remains for us to examine the indecomposables $V \in \hat{Q}_a$ whose support Σ does not have the form (1) or (1, 1) of Fig. 10. The existence of d implies that a has multiplicity ≥ 3 in Σ (2.3) and excludes the posets (1, 2, 3) and (1, 3 \leftarrow 3) of Fig. 10. We shall consider the 4 remaining cases separately.

If $\Sigma = \{a, c, s\}$ has the form (1, 1, 1), V equals $\{a, c, s\}_1$ or $\{a, c, s\}_0 = (d, M)$ where $d = [1111]$ and $M = [111]$. The first eventuality is “accepted” by our lemma. In the second one, c or s is comparable with b (2.4(1)), say $s < b$. But then $V(a) = [1] = V(s)$, and we have $V < \{b\}_0$ by 3.1.

If $\Sigma = \{x_1 \rightarrow x_2, y, z\}$ has the form (1, 1, 2), V has the dimension-vector $d = [21111]$ and the matrix $M = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$. Then three cases are possible. (1) In case $a \in \{y, z\}$, say $a = y$, we must have $x_1 < x_2 < b$ or $x_1 < b > z$ because of 2.4(1) and of the dual of 2.6(2). Accordingly, $T(a)$ is a linear combination of $T(x_1), T(x_2)$ in the first subcase, of $T(x_1), T(z)$ in the second. (2) In case $a = x_1$, we have $x_2 \in \dot{P}$ by 2.5 and $x_2 \not< b$ by 0.4. By 2.4(1) this implies $y < b$ and $z < b$. The associated columns $T(y)$ and $T(z)$ generate $T(a)$. (3) In case $a = x_2$, b is comparable with y or z , say $y < b$ (2.4(1)). Then $T(a)$ is a linear combination of $T(x_1)$ and $T(y)$.

If $\Sigma = \{x_1 \rightarrow x_2, y, z_1 \rightarrow z_2\}$ has the form (1, 2, 2), two cases are to be considered (2.3): (1) In case $a = x_2$, we have $z_1 < z_2 < b$ or $z_1 < b > y$ (2.4(1) and 2.6(2)). If we let T run through the 3 faithful representations with support Σ [3], it remains to check that $T(a)$ is a linear combination of $T(x_1), T(z_1), T(z_2)$ in the first subcase, of $T(x_1), T(z_1), T(y)$ in the second. (2) In case $a = x_1$, we have $x_2 \in \dot{P}$ by 2.5 and $x_2 \not< b$ by 0.4. Since b is normal, we have $z_1 < z_2 < b > y$, and $T(a)$ is a linear combination of $T(y), T(z_1), T(z_2)$ by 3.3.

Finally, if $\Sigma = \{x_1 \rightarrow x_2 \leftarrow y_1 \rightarrow y_2, z_1 \rightarrow z_2\}$, a equals x_2 or y_1 (Fig. 10). The two cases are treated like case 1) and 2) of (1, 2, 2).

3.5. By \hat{Q}_{ab} we denote the full subposet of \hat{Q} formed by the representations $V \in \hat{Q}_a \cup \hat{Q}_b$ which are incomparable with $\{b\}_0$ or with $\{a\}_1$. By P_{ab} we denote the union of their supports equipped with the order induced by P .

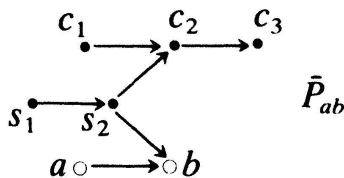


Figure 11

LEMMA. Under the assumptions of Lemma 3.4, P_{ab} is equal to $\{a, b\}$ or isomorphic to a full subposet of \bar{P}_{ab} (Fig. 11) containing $\{a, b, c_1\}$. The poset \hat{Q}_{ab}

is identified with the full subposet of \bar{Q}_{ab} (Fig. 12) formed by the vertices which involve only points of P_{ab} .

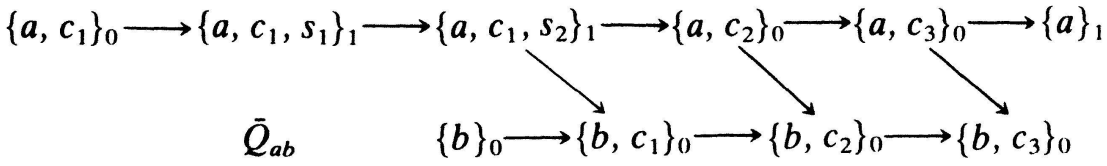


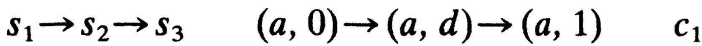
Figure 12

Proof. By 2.4(1), the points $c \in P$ such that $a \times c \times b$ form a linearly ordered set $c_1 \rightarrow c_2 \rightarrow \dots \rightarrow c_k$. If k was ≥ 4 , \hat{P} would contain the full subposet



If there was an $s \in P$ such that $a \times s \times c_i$ for some $i \geq 2$, we would have $c_1 \rightarrow s \rightarrow b$ by (2.4)(1) and the dual of 2.6(2).

Finally, the points $s \in P$ such that $a \times s \times c_1$ form a linearly ordered set $s_1 \rightarrow s_2 \rightarrow \dots \rightarrow s_l$ (0.3). If l was ≥ 3 , \hat{P} would contain the full subposet



The rest should be clear.

3.6. LEMMA. Assume that $a \in \hat{P}$ is normal and that there is a $d \in \hat{Q}$ such that $a' \times d \times b'$. Then P_{ab} is equal to $\{a, b\}$ or isomorphic to a full subposet of \underline{P}_{ab} (Fig. 13) containing $\{a, b, c_3\}$. The poset \hat{Q}_{ab} is identified with the full subposet of \underline{Q}_{ab} (Fig. 14) formed by the vertices which involve only points of P_{ab} .

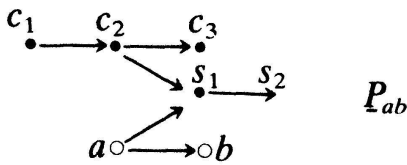


Figure 13

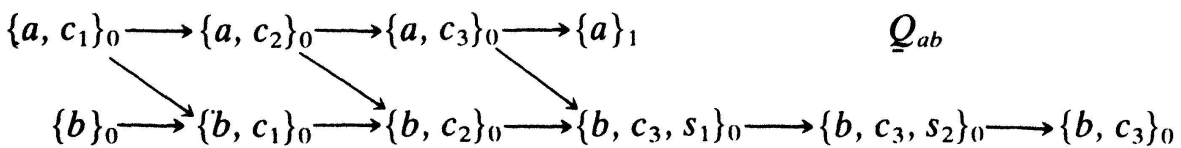


Figure 14

Proof. This is “the dual of the quasi-dual” of lemma 3.5. Since duality theory is screened by the use of matrices, we sketch the essentials: For each representation $V = (d, M)$ of P , we choose a matrix $K \in k^{|\bar{d}| \times e_0}$ such that $MK = 0$ and $\text{rank } K = e_0 = |\bar{d}| - \text{rank } M$. Setting $e = [e_0 \bar{d}] \in \mathbb{N}^{n+1}$, we then interpret the pair $\mathcal{D}V = (e, K^T)$ as a representation of the opposite poset P^0 , and we assemble a contravariant functor $\mathcal{D}: \text{rep } P \rightarrow \text{rep}(P^0)$ by piecing out the map $V \mapsto \mathcal{D}V$ as follows: First we notice that each morphism $B \in \text{Hom}(\bar{d}, \bar{d}')$ of P_k (0.2) produces a morphism $B^T \in \text{Hom}(\bar{d}', \bar{d})$ of $(P^0)_k$. Our second observation is that, for each morphism $(A, B): (d, M) \rightarrow (d', M')$ of $\text{rep } P$, there is a unique matrix $C \in k^{e_0 \times e'_0}$ such that $KC = B^T K'$, where $\mathcal{D}(d', M') = (e', K')$. This means that $(C^T, B^T) = \mathcal{D}(A, B)$ is a morphism of $\text{rep}(P^0)$ from (e', K') to (e, K) . The contravariant functor thus defined induces an antiequivalence from $\text{rep}_0 P$ (the full subcategory of $\text{rep } P$ formed by the (d, M) such that $d_0 = \text{rank } M$) to $\text{rep}_0(P^0)$. For instance, we have $\mathcal{D}\{a\}_0 = \{a\}_1$, $\mathcal{D}\{a, c\}_0 = \{a, c\}_0$, $\mathcal{D}\{a, c, s\}_1 = \mathcal{D}\{a, c, s\}_0 \cdots$

3.7. LEMMA. *Assume that there is a $d \in \hat{Q}$ satisfying $a' \times d \times b'$ and that a or b is normal. Let further $z \in \hat{P}$ be such that $b < z$ and $(b, z) \sim (b', z')$. Then $\hat{Q}_{ab} \cap \hat{Q}_{bz}$ consists of the representations $\{b, c\}_0$ where c is incomparable with a, b and z . If there is only one such c , then $\{a\}_1$ is the only element of \hat{Q}_a which is incomparable with $\{b, c\}_0$.*

Proof. The first statement directly follows from 3.5 if b is normal. If a is normal, we must prove that $\{b, c_3, s_i\}_0 \notin \hat{Q}_{ab} \cap \hat{Q}_{bz}$ (3.6). But this follows from the validity of $c_3 < z$ or of $s_i < z$ (2.4(1)).

Now, the points incomparable with a and b form a chain $c_1 \rightarrow c_2 \rightarrow \cdots \rightarrow c_l$. If there is only one c as above, we must have $c = c_1$. Our second statement therefore follows from Fig. 12 or Fig. 14.

Remark. By duality and quasi-duality, the first statement of the lemma is also true under the assumption that there is a q satisfying $b' \times q \times z'$ and that b or z is normal. If, moreover, there is only one c , then $\{z\}_0$ is the unique element of \hat{Q}_z such that $\{z\}_0 \times \{b, a\}_0$.

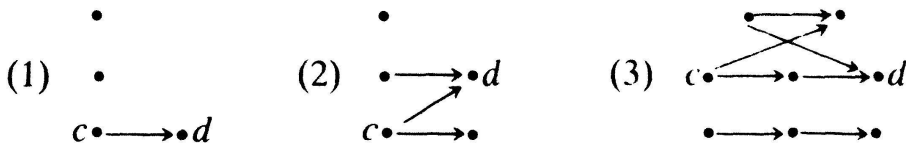
4. Mixed edges

From now onwards, we suppose that \tilde{S} admits a faithful indecomposable representation $U = (d, M)$ (0.6). We denote by Σ_U the support of the associated representation $U_Q = (d_Q, M_Q)$ of \hat{Q} (1.4, 2.6). We investigate Σ_U under the following assumption, valid throughout section 4: $a \in \hat{P}$ is thick, $b \in \hat{P}$ is

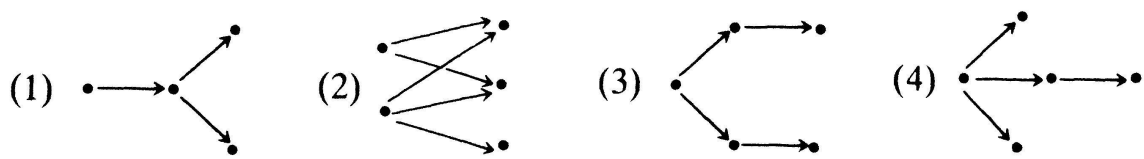
subsequent to a , $a' \in \hat{Q}$ is normal and $b' \in \hat{Q}$ is not. By \bar{a} and \bar{b} we denote a maximal and a minimal element of $\hat{Q}_a \cap \Sigma_U$ and $\hat{Q}_b \cap \Sigma_U$ respectively.

The lemmas 4.1–4.5 are preliminary and follow directly from 2.1, Fig. 6. As in 2.1, we denote by Σ the support of an indecomposable representation of a representation-finite poset T .

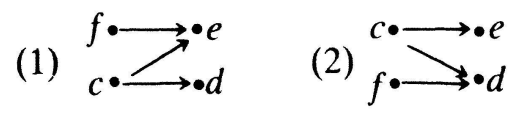
4.1. LEMMA. *Suppose that Σ has at least three points. Then, for any two points c and d (comparable or not), there is an x such that $c \not\asymp x \not\asymp d$. In case $c < d$, $\{c, d\}$ is contained in a full subposet of Σ having one of the following three forms.*



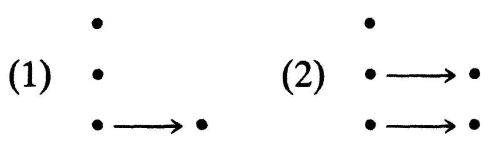
4.2. LEMMA. *Σ contains no full subposet which is isomorphic or dual to one of the following posets.*



4.3. LEMMA. *If $d \in \Sigma$ and $e \in \Sigma$ are subsequent to $c \in \Sigma$ and satisfy $d \not\asymp e$, then Σ contains a full subposet of one of the following two forms. Moreover, we have $g \not\asymp f$ whenever $g \in \Sigma$ is incomparable with c, d and e .*

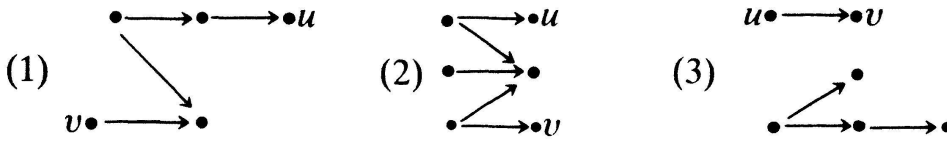


4.4. LEMMA. *A proper full subposet Σ of the form (1) below is contained in a full subposet of Σ isomorphic to (2).*



4.5. In case $v \in T$, we call *duplicate of v in T* an element $w \in T$ which is comparable with v and such that, for any $t \in T \setminus \{v, w\}$, the inequality $v < t$ is equivalent to $w < t$ and $t < v$ to $t < w$.

LEMMA. If X is a full subposet of Σ (resp. of Σ^0) of one of the three forms below, then $\Sigma \setminus X$ (resp. $\Sigma^0 \setminus X$) contains a duplicate of u in Σ or of v .



4.6. LEMMA. \hat{Q} contains a point q subsequent to a' and a point d such that $b' \not\asymp q \not\asymp d$ and $a' \not\asymp d \not\asymp b'$.

Proof. Consider the full subposet $Q(b', \cdot)$ of Q formed by the points $s \in Q$ which can be incorporated into a triad $\{b', c, s\}$ of three pairwise incomparable elements of Q . This subposet contains at least two minimal elements, say q_1 and q_2 . If q_1 is incomparable with a' , we can set $d = q_1$ and $q = q_2$: Indeed, 2.4(1) implies $a' < q_2$; if q_2 was not subsequent to a' , each element q_3 such that $a' < q_3 < q_2$ should be incomparable with b' (which is subsequent to a');

accordingly, $q_3 < q_2$ would imply $q_3 < q_1$ and Σ_U would contain $\bar{a} \longrightarrow q_3 \begin{matrix} \nearrow q_1 \\ \searrow q_2 \end{matrix}$ in contradiction with 4.2(1).

In case $a' < q_1$ and $a' < q_2$, the same argument shows that q_1 and q_2 are both subsequent to a' . By lemma 4.3, Σ_U contains a full subposet of the form, say $\bar{a} \longrightarrow q_1, x \longrightarrow q_2$, which satisfies $x \not\asymp \bar{b}$ if $\bar{a} \not\asymp \bar{b}$. We claim that $x \in \hat{Q}$, because $y \in \hat{Q}, x \in \hat{Q}_y$, and $x < q_2$ would imply $y \cong a$, hence $x < q_1$. Furthermore, we have $x < \bar{b}$, because q_2 is minimal in $Q(b', \cdot)$. We infer that $\bar{a} < \bar{b}$ and that Σ_U contains the full subposet of Fig. 15 in contradiction with 4.2(2).

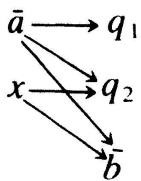


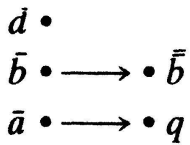
Figure 15

4.7. LEMMA. Let P contain a point which is incomparable with all points of \hat{P} . Then $\bar{a} \in \hat{Q}_a \cap \Sigma_U$ and $\bar{b} \in \hat{Q}_b \cap \Sigma_U$ can be chosen so that $\bar{a} \not\asymp \bar{b}$.

Proof. Otherwise, we can apply 4.3 to the subset $\begin{matrix} d \bullet & \nearrow & \bullet q \\ \bar{a} \bullet & \longrightarrow & \bullet \bar{b} \end{matrix}$ of Σ_U and find an $x \in \Sigma_U$ such that $\bar{a} \not\asymp x \not\asymp d$ and that either $q \not\asymp x < \bar{b}$ or $\bar{b} \not\asymp x < q$. In both cases

we have $x \notin \hat{Q}$ since a' is normal. So let x be in \hat{Q}_y , $y \in \hat{P}$, and first suppose that $y \leq a$: Then $x < q$, $x \not\asymp \bar{b}$ and $y < a$ since $x \in \hat{Q}_a \cap \Sigma_U$ is supposed to imply $x < \bar{b}$; it follows that $(y, b) \sim (y', b')$, and we obtain a contradiction with 2.6(6), which reduces our proof to the case $y \geq b$. But then we have $q \not\asymp x < \bar{b}$, hence $y = b$ and the contradiction $\bar{a} < x$ (since $x \in \hat{Q}_b \cap \Sigma_U$ is supposed to imply $\bar{a} < x$).

4.8. LEMMA. *Let \hat{P} contain a point which is not normal and P a point which is incomparable with all points of \hat{P} . If Σ_U has at least 5 points, it contains a full subposet of the following form, where $\bar{a} \in \hat{Q}_a$, $\{\bar{b}, \bar{b}'\} \subset \hat{Q}_b$ and $\{q, d\} \subset \hat{Q}$.*



Proof. We apply 4.4 to the full subposet $\begin{array}{c} d \bullet \quad \bullet \bar{b}' \\ \bar{a} \bullet \longrightarrow \bullet q \end{array}$ of Σ_U which is provided by 4.6 and 4.7. By 4.4 there is an $x \in \Sigma_U$ such that $\bar{a} \not\asymp x \not\asymp q$ and that $d \not\asymp x$ or $\bar{b} \not\asymp x$. If x was in \hat{Q} , it would be comparable with d (because a' is normal) and provide a contradiction to 2.6(2).

Therefore, we have $x = \bar{y} \in \hat{Q}_y$ for some $y \geq b$ (because $x \not\asymp q$). In case $y = b$ the proof is perfect. So it remains for us to exclude the case $y > b$. In this case, 2.6(6) implies $d < y'$, and S, Σ_U contain the full subposets of Fig. 16, where $(a, y) \sim (a', y')$. By 2.6(1), b is normal; by 2.6(6), y is subsequent to b ; by 2.6(5), there is at most one $c \in \hat{P}$ such that $a \not\asymp c \not\asymp y$; by 3.7 and the assumptions of the lemma, we have $\bar{b} = \{b, c\}_0$, $\bar{a} = \{a\}_1$ and $\bar{y} = \{y\}_0$, where $c \not\asymp z$ for all $z \in P$.

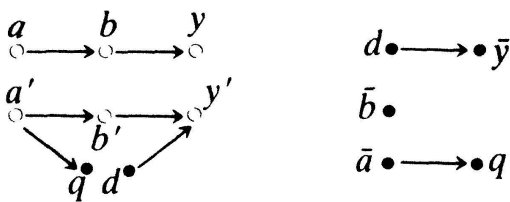


Figure 16

By assumption, \hat{P} contains a point $e \not\asymp x$. Since U is faithful and b normal, e belongs to the support of some $\bar{t} \in \hat{Q}_t \cap \Sigma_U$, where $t \neq b$. Up to duality and quasi-duality, we may assume that $t < b$. Let us then compare \bar{t} with $\bar{a}, q, \bar{b}, d, \bar{y}$: Obviously, $\bar{t} < \bar{a} = \{a\}_1$. It follows that $\bar{t} \not\asymp d$, because $\bar{t} < d$ would contradict 4.2(3). We claim that $\bar{t} < \bar{b}$: Indeed, this follows from 3.7 if $t = a$; and the case $t < a$, $\bar{t} \not\asymp \bar{b}$ is excluded by 2.6(6) (since c is incomparable with t, a, b and d with t', a', b'). Finally, we have $\bar{t} \not\asymp \bar{y}$ because $\bar{t} < \bar{y}$ would contradict 4.2(4).

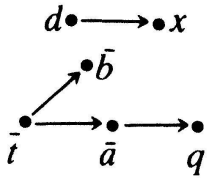


Figure 17

Now, by 4.5(3) d or x has a duplicate z in Σ_U . If $z \in \hat{Q}$, \tilde{S} contains the stable subset of Fig. 18 in contradiction to 2.6(2). If $z = \bar{r} \in \hat{Q}_r$, $q \times z$ implies $b \leq r$, and $b' \times d$ implies $y \leq r$. Then the dual of the quasi-dual of the argument applied to \bar{t} above yields the contradiction $\bar{b} < \bar{r}$.

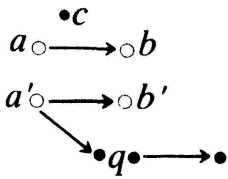


Figure 18

Remark. Our proof involves quasi-duality. This may need some explanation since we formally use the fact that the “dual of the quasi-dual” $\tilde{S}^{*0} = P^0 \triangleleft Q^0$ also admits a faithful indecomposable representation. In fact, the antiequivalence $\mathcal{D} : \text{rep } P \rightarrow \text{rep } (P^0)$ of 3.6 induces an isomorphism of $(\hat{Q})^0$ onto the poset $(Q^0)^\wedge$ attached to $\tilde{S}^{*0} = P^0 \triangleleft Q^0$. The needed faithful indecomposable representation V of \tilde{S}^{*0} is defined by $V_{Q^0} \cong \mathcal{D}(U_Q)$.

Using similar arguments, one shows that the dual $\tilde{S}^0 = Q^0 \triangleleft P^0$ and the quasi-dual $\tilde{S}^* = Q \triangleleft P$ admit faithful indecomposable representations.

5. Proof of theorem 3

As in section 4, we denote by U a faithful indecomposable representation of the bipartite completed poset $\tilde{S} = P \triangleleft Q$. We suppose that \hat{P} and \hat{Q} contain non-normal points.

5.1. *We first prove theorem 3 under the assumption that \hat{P} has cardinality 2.* Using quasi-duality and 2.6(1), we may assume that $\hat{P} = \{a < b\}$ and $\hat{Q} = \{a' < b'\}$, where a', b are normal and a, b' not. By lemma 4.6 and its dual, S then contains the full subposet of Fig. 19. If there is any other point which is incomparable with a and b or with a' and b' , we may by duality assume that it lies in Q , hence that c is the unique point which is incomparable with a and b (2.6(3)).

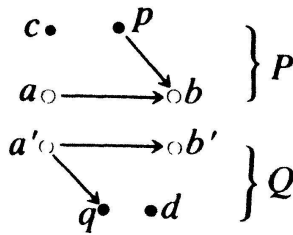


Figure 19

Let us now assume that Q contains more points and find the contradiction. The dual argument will then show that P also has 4 points, and our proof will be complete.

By lemma 4.8, Σ_U contains a full subposet of the form $\begin{matrix} \bar{a} \bullet \rightarrow \bullet q \\ \bar{b} \bullet \rightarrow \bullet \bar{b} \bullet d \end{matrix}$, where $\bar{a} \in \hat{Q}_a$ and $\{\bar{b}, \bar{b}\} \subset \hat{Q}_b$. By lemma 3.5, $\bar{a} = \{a\}_1$, $\bar{b} = \{b\}_0$ and $\bar{b} = \{b, c\}_0$. Accordingly, since U has dimension ≥ 1 at p , Σ_U must contain some other $\bar{a} \in \hat{Q}_a$. Lemma 3.5 implies $\bar{a} < \bar{b}$ and lemma 4.2(3) $\bar{a} \times \bar{b}$ (Fig. 20).

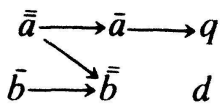


Figure 20

We now apply 4.5: A duplicate of \bar{b} in Σ_U cannot belong to \hat{Q} because a' is normal, to \hat{Q}_b because of 3.5, to \hat{Q}_a because $b \times q$. Accordingly, only q can admit a duplicate in Σ_U . Therefore, Σ_U contains the full subposet of Fig. 21, where q is one of the non-specified points. By 2.1, Fig. 6 it follows that Σ_U has the form of Fig. 22. But y cannot belong to \hat{Q} because $\bar{b} \times y < \bar{b}$. And y cannot belong to $\hat{Q}_a \cup \hat{Q}_b$, i.e. in fact to \hat{Q}_{ab} , because of 3.5.

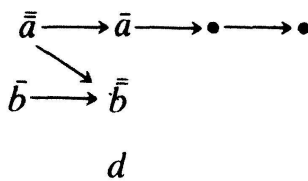


Figure 21

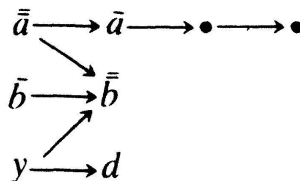


Figure 22

5.2. LEMMA. *Suppose that \hat{P} contains points $a < b < c$ and \hat{P} a point d such that $(a, c) \sim (a', c')$ and $a \times d \times c$. Then b is normal.*

Proof. Otherwise, \hat{P} contains two points d_1, d_2 such that $d_1 \times b \times d_2 \times d_1$. Since $a < d_i < c$ is excluded by 0.4, each point d_i satisfies $a \times d_i$ or $d_i \times c$. By

2.4(1) it follows that we have, say $a \times d_1 < c$ and $a < d_2 \times c$. By 2.6(1), d_1 and d_2 are both comparable with d . So we obtain $d_1 < d$ ($d < d_1$ would imply $d < c!$), $d < d_2$ and the contradiction $d_1 < d_2$.

5.3. LEMMA. \hat{P} contains a point s incomparable with all $u \in \hat{P}$.

Proof. By the dual of 4.6 we can assume that \hat{P} has cardinality ≥ 3 .

Let a be the minimal and c the maximal point of \hat{P} . Choose \bar{a} in $\hat{Q}_a \cap \Sigma_U$, \bar{c} in $\hat{Q}_c \cap \Sigma_U$. By 4.1, Σ_U contains a point x such that $\bar{a} \times x \times \bar{c}$. If $x \in \hat{Q}$, it follows that $a \times x \times c$, and we can set $s = x$. Hence we may suppose that $x \in \hat{Q}_b$ for some $b \in \hat{P}$; if $b \neq c$, \hat{P} contains an element incomparable with b and c , since the contrary would imply $x = \{b\}_1 \cong \{a\}_1 \cong \bar{a}$ (3.2). Similarly, if $a \neq b$, \hat{P} contains an element incomparable with a and b .

This solves our problem in case $a = b$ or $b = c$. In general, it implies that, whenever $v \in \hat{P}$ is subsequent to $w \in \hat{P}$, there is a point incomparable with v and w . By duality, the same statement holds for \hat{Q} .

Now suppose that $a < b < c$. Let $v \in \hat{P}$ be subsequent to b , and b to $u \in \hat{P}$. We claim that one of the points u, b, v is normal: Indeed, by 5.2 u is normal if $a \neq u$, and v is if $c \neq v$; if neither u nor v is normal, we have $a = u, c = v$, and b must be normal (otherwise, all points a', b', c' of \hat{Q} would be normal by 2.6(1)).

So we can apply 3.7 to u, b, w . Since $x \in \hat{Q}_b$ satisfies $\{a\}_1 \times x \times \{c\}_0$, it satisfies $\{u\}_1 \times x \times \{v\}_0$ and has the form $x = \{b, s\}_0$. But $\{a\}_1 \times \{b, s\}_0 \times \{c\}_0$ implies $a \times s \times c$.

5.4. LEMMA. Let $b \in \hat{P}$ and $p \in \hat{P}$ be such that b is normal and $p < b$. Then there is a point $x \in \hat{P}$ such that $x < b$ and a representation $t \in \hat{Q}_x \cap \Sigma_U$ with dimension ≥ 1 at p .

Proof. Since U is faithful, there is an $x \in \hat{P}$ and a $t \in \hat{Q}_x \cap \Sigma_U$ with dimension ≥ 1 at p . Since b is normal, \hat{Q}_b consists of $\{b\}_0, \{b\}_1$ and of representations $\{b, c\}_0$ with $b \times c$. We infer that $x \neq b$. Suppose that $x > b$. By 4.1, the support Σ of t contains a full subposet X or Y as shown in Fig. 23 (the case (3) of 4.1 is excluded because x has multiplicity ≥ 2). In case $\Sigma \supset X$, the inequalities $p < b < x$

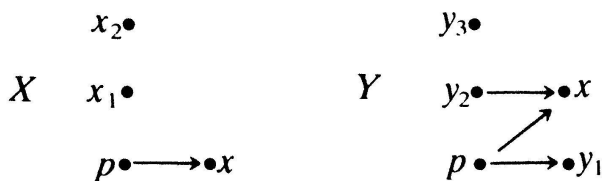
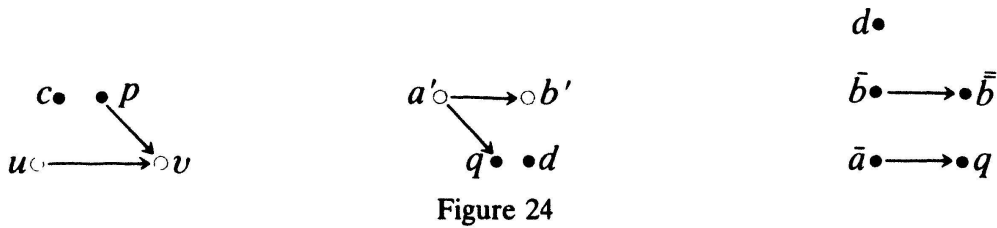


Figure 23

imply $x_1 \asymp b \asymp x_2$ in contradiction to the normality of b . In case $\Sigma \supset Y$, this normality and the condition $p \asymp y_3 \asymp x$ imply that b is comparable with y_1 and y_2 , hence that $b < y_1$ ($x \asymp y_1$ implies $b \not\asymp y_1$) and $y_2 < b$. This leads us to the contradiction $y_2 < y_1$.

5.5. *Proof of theorem 3.* Applying 2.6(6), 5.3 and the dual of 5.3, we first observe that y must be subsequent to x if $(x, y) \sim (x', y')$ and $x \neq y$.

By lemma 2.6(1), out of two equivalent points at least one is normal. We choose two equivalence classes $\{u \sim u'\}$, $\{b \sim b'\}$ such that $u, b \in \hat{P}$ and $u', b' \in \hat{Q}$, that u', b are normal and that u, b' are not. Furthermore, we suppose that all points of \hat{P} between u and b (if there are any) are normal, as well as all points of \hat{Q} between u' and b' . Up to quasi-duality, we may also suppose that $u < b$. We then denote by $v \in \hat{P}$ the point subsequent to u , by $a \in \hat{P}$ the point to which b is subsequent ($u \leq a, v \leq b$). To the pairs (a, b) and (u, v) thus constructed we apply the lemmas 4.6, 4.8 and their duals, which provide us with the full subsets of P, Q and \hat{Q} described in Fig. 24.



By 2.6(4), there are at most two points incomparable with a and b . Using 3.5, we infer that $\bar{a} = \{a\}_1$ or $\bar{a} = \{a, c'\}_0$ where $a \asymp c' \asymp b$. Accordingly, \bar{a} has dimension 0 at p , and is distinct from the point $t \in \hat{Q}_x, x \leq a$, constructed in 5.4 and obviously subjected to $t < q$.

Let us suppose that $t < \bar{b}$. Then we have $\bar{a} \asymp t \asymp d$ by 4.2((3) and (4)) and can apply 4.5(1) to $\begin{matrix} t \rightarrow \bar{b} \rightarrow \bar{\bar{b}} \\ \bar{a} \rightarrow q \end{matrix}$. But there is no way of obtaining a duplicate z of $\bar{\bar{b}}$ in Σ_U from \hat{Q} because a' is normal; from \hat{Q}_b because $\bar{b}, \bar{\bar{b}}$ exhaust the elements of \hat{Q}_b incomparable with \bar{a} ($\bar{a} \asymp t$ implies $\bar{a} = \{a, c_2\}_0, \bar{\bar{b}} = \{b, c_1\}_0, \bar{b} = \{b\}_0$ in 3.5); from $\hat{Q}_y, y > b$, because $a < b < y$ implies $(a, y) \not\sim (a', y')$ and $\bar{a} < z$. Nor can we obtain a duplicate z of \bar{a} from \hat{Q} (q is subsequent to a' and $z < \bar{a}$ would imply $z < \bar{b}$); from \hat{Q}_a by 3.5; from $\hat{Q}_y, y < a$, because $y < a < b$ implies $z < \bar{b}$.

So we are reduced to the case $t \asymp \bar{b}$, hence $x = a$. By 3.5, t is comparable with \bar{a} . It is $< \bar{a}$ because \bar{a} is supposed to be maximal in $\hat{Q}_a \cap \Sigma_U$. The case $\bar{a} = \{a\}_1, t = \{a, c_2\}_0, \bar{\bar{b}} = \{b, c_1\}_0, \bar{b} = \{b\}_0$ is excluded by the assumption that t has dimension ≥ 1 at p . By 3.5 this implies that $t < \bar{\bar{b}}$. In this case, we obtain Fig. 20

and can repeat the argument produced in the last paragraph of 5.1. Theorem 3 is proved.

6. Appendix

Our objective in this section is to expose a more synthetical point of view for the reduction used in section 1. The following is due to P. Gabriel.

6.1. Let $k^m = k^{1 \times m}$ be the space of m -rows and $\text{mod } k$ the category of finite dimensional vector spaces. The category \tilde{S}_k (0.2) is naturally equipped with a functor $F: \tilde{S}_k \rightarrow \text{mod } k$, $v \mapsto k^{|v|}$ which maps the morphism $B \in \text{Hom}(u, v)$ onto $x \mapsto xB^T$. Using F , we can interpret a representation (d, M) as a pair (\bar{d}, f) consisting of an object $\bar{d} \in \tilde{S}_k$ and a linear map $f: k^{d_0} \rightarrow F\bar{d}$, $y \mapsto yM$. In this way, we obtain an *equivalence between $\text{rep } \tilde{S}$ and the following F -subspace category* sub F [5]: An object of sub F is an “ F -subspace”, i.e. a pair (v, f) formed by an object $v \in \tilde{S}_k$ and a morphism $f: V \rightarrow Fv$ of $\text{mod } k$. A morphism $(u, e) \rightarrow (v, f)$ is given by a pair of morphisms $B \in \text{Hom}(u, v)$ and $A \in \text{Hom}(U, V)$ such that $fA = (FB)e$.

The natural decompositions of the rows $v \in \tilde{S}_k$ and $x \in Fv$ into “blocks” $v_p = [v_1 \cdots v_{|p|}]$, $v_Q = [v_{|p|+1} \cdots v_n]$ and $x_p = [x_1 \cdots x_{|v_p|}]$, $x_Q = [x_{|v_p|+1} \cdots x_{|v|}]$ yield an exact sequence of functors

$$0 \longrightarrow F_Q \xrightarrow{\iota} F \xrightarrow{\pi} F_P \longrightarrow 0,$$

where $F_P v = k^{|v_p|}$, $\pi v: x \mapsto x_p$, $F_Q v = k^{|v_Q|}$ and $(\iota v)[y_1 \cdots y_{|v_Q|}] = [0 \cdots 0 y_1 \cdots y_{|v_Q|}]$. The residue-functor F_P gives rise to an F_P -subspace category sub F_P , which is defined like sub F and contains the full subcategory $\text{sub}_0 F_P$ formed by the *proper F_P -subspaces*, i.e. by the pairs (v, g) such that g is injective. The subcategory $\text{sub}_0 F_P$ finally provides us with the wanted *reduction-functor*

$$\mathcal{R}: \text{sub } F \longrightarrow \text{sub } F'_Q$$

$$(v, V \xrightarrow{f} Fv) \mapsto ((v, \text{Im}(\pi v)f \xrightarrow{f_p} F_P v), \text{Ker}(\pi v)f \xrightarrow{f_Q} F_Q v)$$

where f_p, f_Q are induced by f and

$$F'_Q: \text{sub}_0 F_P \rightarrow \text{mod } k$$

maps $(v, U \xrightarrow{g} F_P v)$ onto $F_Q v$.

PROPOSITION. *The reduction-functor $\mathcal{R} : \text{sub } F \rightarrow \text{sub } F'_Q$ induces a bijection between the isomorphism classes of $\text{sub } F$ and of $\text{sub } F'_Q$.*

It follows that \mathcal{R} also induces a bijection between the isomorphism classes of indecomposables.

Proof. It is easy to show that \mathcal{R} hits each isomorphism class of $\text{sub } F'_Q$. Indeed, each object $((v, U \xrightarrow{g} F_P v), W \xrightarrow{h} F_Q v)$ of $\text{sub } F'_Q$ is isomorphic to the image of the object $(v, U \oplus W \xrightarrow{[sg \ h]} Fv)$ of $\text{sub } F$, where $s : F_P v \rightarrow Fv$ denotes an arbitrary linear section of πv .

To prove the injectivity of the map induced by \mathcal{R} , we first remark that, for each linear map $e : F_P u \rightarrow F_Q v$, there is a morphism $E : u \rightarrow v$ of \tilde{S}_k such that $FE = (\iota v)e(\pi u)$ (if u and v are indecomposable in \tilde{S}_k , this immediately follows from (0.4b)). We then consider two objects $(v, V \xrightarrow{f} Fv)$ and $(v, V' \xrightarrow{f'} Fv)$ of $\text{sub } F$ having isomorphic images in $\text{sub } F'_Q$. This means that there are isomorphisms B, C and D which make commutative the first two squares of Fig. 7. We extend D to an isomorphism $A : V \cong V'$ which induces C . Then $f'A - (FB)f$ vanishes on $\text{Ker}(\pi v)f$ and factors through $\iota v : F_P v \rightarrow Fv$. In other words, $f'A - (fB)f$ can be written as a composition

$$V \xrightarrow{(\pi v)f} F_P v \xrightarrow{e} F_Q v \xrightarrow{\iota v} Fv$$

We infer that $f'A - (FB)f = (\iota v)e(\pi v)f = (fE)f$ for some $E : v \rightarrow v$ such that $(fE)^2 = 0$, hence $E^2 = 0$. So we finally obtain the isomorphism $(A, B + E) : (v, f) \cong (v, f')$.

$$\begin{array}{ccc} \text{Ker}(\pi v)f \xrightarrow{f_Q} F_Q v & & \text{Im}(\pi v)f \xrightarrow{f_p} F_P v \\ \downarrow D \wr & & \downarrow C \wr \\ \text{Ker}(\pi v)f' \xrightarrow{f'_q} F_Q v & & \text{Im}(\pi v)f' \xrightarrow{f'_p} F_P v \end{array}$$

$$\begin{array}{ccc} V \xrightarrow{f} Fv & & \\ A \downarrow \wr & & \downarrow FB \\ V' \xrightarrow{f'} Fv & & \end{array}$$

Fig. 7

6.2. The construction of the subspace category $\text{sub } F'_Q$ considered in 6.1 is based on the category $\text{sub}_0 F_Q$ which, in general, does not have the form \tilde{T}_k . We therefore insert some remarks about general subspace categories [5] [9].

Let K be a k -linear category such that the dimensions of the morphism spaces are finite and that each object is a finite direct sum of indecomposables with local endomorphism algebras. If $\Phi: K \rightarrow \text{mod } k$ is a k -linear functor, $\text{sub } \Phi$ is related in a simple way to the category of representations of a poset: Let U_1, \dots, U_s be pairwise non-isomorphic indecomposables such that $\dim \Phi U_i = 1$. Define a partial order on the set $\mathcal{V} = \{U_1, \dots, U_s\}$ by setting $U_i \leq U_j$ if $\Phi \mu \neq 0$ for some $\mu: U_i \rightarrow U_j$. Denote by $\bar{\Phi}: K/\text{Ker } \Phi = \bar{K} \rightarrow \text{mod } k$ the functor induced by Φ , where $\text{Ker } \Phi$ denotes the ideal of K formed by the morphisms v such that $\Phi v = 0$. We then have the following *comparison diagram*

$$\text{sub } \Phi \xrightarrow{\gamma} \text{sub } \bar{\Phi} \xleftarrow{\varepsilon} \text{rep } \mathcal{V},$$

where γ is the functor $(N, f) \mapsto (N, f)$ and ε is determined by the choice of a basis vector in each ΦU_i . The functor γ induces a bijection between the “isoclasses” of indecomposables of $\text{sub } \bar{\Phi}$ and the isoclasses of indecomposables of $\text{sub } \Phi$ which are not of the form $(N, 0)$ with $\Phi N = 0$. The functor ε is fully faithful; it is an equivalence if U_1, \dots, U_s exhaust the indecomposables of K . This takes place for instance in case $K = \text{sub}_0 F_P$ and $\Phi = F'_P$, when $\tilde{S} = P \triangleleft Q$ is representation-finite (2.5).

6.3. With proposition 6.1 we can also prove that, if $\tilde{S} = P \triangleleft Q$ is faithful, the subsets P and Q are uniquely determined by \tilde{S} . Indeed, suppose that $\tilde{S} = P \triangleleft Q = P' \triangleleft Q'$ and that, say, $P \cap Q' \neq \emptyset$. The (trivially completed) poset P then has the form $P = (P \setminus Q') \triangleleft (P \cap Q')$. From 6.1 we infer that an indecomposable representation of P has its support in $P \setminus Q'$ or in $P \cap Q'$. In particular, there is no indecomposable of $\text{rep } P$ whose support intersects $\dot{P} \subset P \setminus Q'$ and $P \cap Q'$. From 6.1 it then follows that there is no indecomposable of \tilde{S} whose support intersects \dot{P} and $P \cap Q'$.

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