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Autor: Knapp, K.

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Stably spherical classes in the K-homology of a finite group

K. KNAPP

§1. Introduction

Let G be a finite group and BG its classifying space. In this paper we study the Hurewicz map

$$h_K : \pi_n^s(BG) \to K_n(BG).$$

Here $\pi_n^s(X)$ denotes the stable homotopy and $K_n(X)$ the (reduced) K-homology of X; the homomorphism h_K is induced by the unit map from the sphere spectrum to the spectrum of complex K-theory. Since $K_n(BG) = 0$ for n even, we assume n to be odd. Somewhat surprisingly it turns out that for n sufficiently large the image of h_K is more accessible than for example $\operatorname{im}(h : \pi_n^s(BG) \to H_n(BG))$ or even $H_n(BG)$ itself. This simplicity in high dimensions is, of course, obscured by the fact that $\operatorname{im}(h_K)$ can be rather complicated for small n.

The major results of this paper are (i) a computation of the odd primary part of $\operatorname{im}(h_K : \pi_{2n-1}^s(BG) \to K_{2n-1}(BG))$ for all but finitely many n and (ii) a complete computation of the image for some small groups, e.g. $G = \mathbb{Z}/p$, $\mathbb{Z}/p^2$, $\mathbb{Z}/p \times \mathbb{Z}/p$, $(\mathbb{Z}/p \times \mathbb{Z}/p) \rtimes \mathbb{Z}/p$. To state (i) more precisely fix an odd prime p and choose l generating $(\mathbb{Z}/p^2)^*$. The Adams operation ψ^l defines a stable operation ψ_n^l on p-local K-homology $K_n(X)_{(p)}$ which acts as the identity on stably spherical classes. Therefore we have $\operatorname{im}(h_K) \subset \ker(\psi_n^l - 1)$. For $X = BG$ and n large it turns out that $\psi_n^l - 1$ gives the only restriction for an element to be stably spherical:

THEOREM 4.4. *Let G be a finite group and p an odd prime. Then there exists $n_0(G, p) \in \mathbb{N}$ such that for $n \geq n_0(G, p)$* $\operatorname{im}(h_K : \pi_{2n-1}^s(BG)_{(p)} \to K_{2n-1}(BG)_{(p)})$ *is the subgroup* $\ker(\psi_{2n-1}^l - 1)$.

There is a similar result for $p = 2$ with real K-theory.

One reason for being interested in $h_K : \pi_{2n-1}^s(BG) \to K_1(BG)$ is the following relation to equivariant topology. First of all $\pi_n^s(BG^+)$ can be identified with the equivariant bordism group $\Omega_n^{fr}(G)$ of free, equivariantly framed G-manifolds. By

Bott periodicity we have $K_{2n-1}(BG) = K_1(BG)$. For any finite group G there exists an embedding

$$\psi : K_1(BG) \to \bar{R}(G) \otimes (\mathbb{Q}/\mathbb{Z})$$

where $R(G)$ is the complex representation ring of G, $\bar{R}(G) = R(G)/(\text{reg})$ and reg is the regular representation of G (see §2). With respect to Pontrjagin duality ψ is dual to the well known map

$$\alpha : R(G) \to K^0(BG^+).$$

The Hurewicz map h_K composed with ψ has the following interpretation. Given x in $\pi^s_{2n-1}(BG^+)$, represent x by a free, equivariantly framed G-manifold M. Since $\pi^s_{2n-1}(BG^+)$ is finite, some multiple of M, say $m \cdot M$, bounds a free G-manifold W

$$m \cdot M = \partial W$$

For W, the G-signature $\operatorname{sign}(W, G) \in R(G)$ is defined. The value of $\alpha(M, G) := (1/m) \operatorname{sign}(W, G) \in \bar{R}(G) \otimes \mathbb{Q}$ depends only on M and is a bordism invariant in $\bar{R}(G) \otimes (\mathbb{Q}/\mathbb{Z})$. Then

$$(-2)^n \cdot \psi \circ h_K(x) = \frac{1}{m} \operatorname{sign}(W; G)$$

in $\bar{R}(G) \otimes \mathbb{Q}/\mathbb{Z}$. Thus a determination of $\operatorname{im}(h_K)$ shows which values the invariant $\alpha(M, G) \in \bar{R}(G) \otimes \mathbb{Q}/\mathbb{Z}$ can take on equivariantly framed free G-manifolds.

The paper is organized as follows. In §2 we collect the necessary facts about $K_*(BG)$ and its relation to $R(G)$ used in the rest of the paper. One main point is an investigation of the skeletal filtration of $K_1(BG)$.

In §3 we study the invariant $\ker(\psi^l_{2n-1} - 1)$. The fibre of the Adams operation $\psi^l_n - 1$ defines a p-local homology theory $\mathrm{Ad}_n(X)$ and for $X = BG$ we have $\ker(\psi^l_{2n-1} - 1) \cong \mathrm{Ad}_{2n-1}(BG)$. We show that for $n \neq 0$ $\mathrm{Ad}_{2n-1}(BG)$ is a finite group annihilated by $e \cdot n$ where e is the exponent of G. Because of the close relation to the representation ring, the structure of $\mathrm{Ad}_{2n-1}(BG)$ can be determined more easily than for example that of $H_*(BG)$. For example, if G is the symmetric group Σ_m then $\mathrm{Ad}_{2n-1}(B\Sigma_m)$ is isomorphic to $\bigoplus \mathbb{Z}/p^{1+v_p(n)}$ for $n \equiv 0(p-1)$ and 0 otherwise ($n \neq 0$). Here $v_p(n)$ denotes the exponent of p in the prime decomposition of n. At least for a p-group there is an explicit description of $\mathrm{Ad}_{2n-1}(BG)$ in terms of the conjugacy classes of G; see (3.16). A universal coefficient formula for Ad-theory relating Ad-homology to Ad-cohomology gives, in the special case $X = BG$ and $n \neq 0$, an isomorphism $\mathrm{Ad}^{1+2n}(BG) \cong \mathrm{Ad}_{1-2n}(BG)$, thus reducing the computation of $\mathrm{Ad}^*(BG)$ to that of $\mathrm{Ad}_*(BG)$.

In §4 we use a solution of the Adams conjecture to construct the elements necessary to generate $\mathrm{im}(h_k)$ and prove theorem (4.4). The method used to construct elements in $\mathrm{im}(h_K)$ relies on the fact that the skeletal filtration of the elements in question is small enough. This only works in dimensions larger than some $n_0(G) \in \mathbb{N}$ and seems to be a special property of the classifying space of a finite group. For G abelian we give an explicit estimate for the constant $n_0(G)$. In the case of some small groups the bound $n_0(G)$ turns out to be small enough for the computation of $\mathrm{im}(h_K)$ to be completed by other methods.

In §5 we deal with the case of a cyclic group C in the dimension range $n \leqslant n_0(C)$. We show how $\mathrm{im}(h_K)$ is related to the e-invariant on $\pi^s_{2n-1}(BS^1)_{(p)}$ (this study was already begun in [10]) and give examples where $\mathrm{im}(h_K)$ is acutally much smaller than $\ker(\psi^l_{2n-1} - 1)$. Results on $\pi^s_{2n-1}(BS^1)$ (see e.g. [13]) may then be used to show that $\mathrm{im}(h_K)$ can be rather complicated in the range $n \leq n_0(G)$ – even for cyclic groups. We close with the example $G = \mathbb{Z}/p^2$.

The case $G = \mathbb{Z}/p$, reproved in §5, is already known. For $p = 2$ $\mathrm{im}(h_K)$ was computed in [18], for $p \neq 2$ and $n \equiv 0(p-1)$ in [20] and for the other values of n in [10].

We shall use the convention that all (co-)homology theories which occur will be taken as reduced theories. Throughout the paper we shall work at an odd prime p, but with some additional work many of the results carry over for $p = 2$. The main result (4.4) is also true at $p = 2$ and this is contained in remark (4.14).

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§2. $K_*(BG)$ and the representation ring of G

Let EG denote a free contractible G-space with $EG/G = BG$ and $\mathrm{pr}: EG^+ \to S^0$ the canonical projection. The map $\alpha: R(G) \to K^0(BG^+)$ in Atiyah's theorem [2] may be viewed as the map induced by pr

$$R(G) = K^0_G(S^0) \xrightarrow{\mathrm{pr}^*} K^0_G(EG^+) \cong K^0(BG^+)$$

Here K^*_G is reduced G-equivariant K-theory and we have used $K^0_G(X^+) \cong K^0(X/G^+)$ for a free G-space X. Dual to this, with K^G_* G-equivariant

K-homology, we have a map

$$\psi_G: K_0(BG^+;\mathbb{Q}/\mathbb{Z}) \cong K_0^G(EG^+;\mathbb{Q}/\mathbb{Z}) \xrightarrow{\mathrm{pr}_*} K_0^G(S^0;\mathbb{Q}/\mathbb{Z}) = R(G)\otimes\mathbb{Q}/\mathbb{Z}. \quad (2.1)$$

To deduce the properties of ψ_G we relate ψ_G to α via Pontrjagin duality. The K-theory Kronecker pairing

$$\langle\ ,\ \rangle_K: K^i(X)\otimes K_i(X;\mathbb{Q}/\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$$

defines a linking form

$$L: K_i(X;\mathbb{Q}/\mathbb{Z}) \to \mathrm{Hom}\,(K^i(X);\mathbb{Q}/\mathbb{Z}). \quad (2.2)$$

Since $\mathbb{Q}/\mathbb{Z}$ is injective as a $\mathbb{Z}$-module, $\mathrm{Hom}\,(K^i(X);\mathbb{Q}/\mathbb{Z})$ is a (non-additive) homology theory in X and L a natural transformation of homology theories. Clearly L is an isomorphism on coefficients, hence L is an isomorphism for finite CW-complexes. Let $BG^{(m)}$ denote the m-skeleton for a CW-structure on BG, then L induces an isomorphism

$$L: K_i(BG;\mathbb{Q}/\mathbb{Z}) \cong \varinjlim_m \mathrm{Hom}\,(K^i(BG^{(m)});\mathbb{Q}/\mathbb{Z}) =: \mathrm{Hom}_c\,(K^i(BG);\mathbb{Q}/\mathbb{Z}).$$

The usual Kronecker pairing

$$\langle\ ,\ \rangle_R: R(G)\otimes R(G) \to \mathbb{Z}$$

defined by $\langle\lambda,\mu\rangle_R := \dim \mathrm{Hom}^G(\lambda,\mu)$ (e.g. see [19], §7) is nonsingular and induces an isomorphism

$$L_R: R(G)\otimes\mathbb{Q}/\mathbb{Z} \xrightarrow{\cong} \mathrm{Hom}\,(R(G);\mathbb{Q}/\mathbb{Z}).$$

The map $K_0(BG^+;\mathbb{Q}/\mathbb{Z}) \to R(G)\otimes\mathbb{Q}/\mathbb{Z}$ defined by the composition

$$K_0(BG^+;\mathbb{Q}/\mathbb{Z}) \xrightarrow{L}$$
$$\mathrm{Hom}\,(K^0(BG^+);\mathbb{Q}/\mathbb{Z}) \xrightarrow{\alpha^*} \mathrm{Hom}\,(R(G);\mathbb{Q}/\mathbb{Z}) \xleftarrow{L_R} R(G)\otimes\mathbb{Q}/\mathbb{Z} \quad (2.3)$$

is easily seen to be the same as ψ_G in (2.1). For simplicity we shall use (2.3) as definition for ψ_G and take (2.1) as motivation.

PROPOSITION 2.4.

(i) *ψ_G is injective.*

(ii) *For a p-group G, $\psi_G: K_0(BG^+; \mathbb{Q}/\mathbb{Z}_{(p)}) \to R(G) \otimes \mathbb{Q}/\mathbb{Z}_{(p)}$ is an isomorphism.*

(iii) *ψ_G is natural in G.*

Proof. Observe first that ψ_G factorizes through $\mathrm{Hom}_c(K^0(BG^+); \mathbb{Q}/\mathbb{Z})$. Since $\alpha: R(G) \to K^0(BG^+)$ is continuous, $R(G)$ having the I-adic and $K^0(BG^+)$ the skeletal topology, ([2], §7) we obtain a map $\mathrm{Hom}_c(K^0(BG^+); \mathbb{Q}/\mathbb{Z}) \xrightarrow{\alpha^*} \mathrm{Hom}_c(R(G); \mathbb{Q}/\mathbb{Z})$ which is easily seen to be injective. Therefore the composition

$$K_0(BG^+; \mathbb{Q}/\mathbb{Z}) \xrightarrow[\cong]{L} \mathrm{Hom}_c(K^0(BG^+); \mathbb{Q}/\mathbb{Z}) \xrightarrow{\alpha^*} \mathrm{Hom}_c(R(G); \mathbb{Q}/\mathbb{Z})$$

$$\subset \mathrm{Hom}(R(G); \mathbb{Q}/\mathbb{Z}) \xleftarrow[\cong]{L_R} R(G) \otimes \mathbb{Q}/\mathbb{Z}$$

which is simply ψ_G, is injective.

For a p-group G, I-adic completion is the same as p-adic completion. This implies that the map $\mathrm{Hom}_c(K^0(BG^+); \mathbb{Q}/\mathbb{Z}_{(p)}) \to \mathrm{Hom}(R(G); \mathbb{Q}/\mathbb{Z}_{(p)})$ is an isomorphism, hence ψ_G must be an isomorphism after localizing at (p).

The map ψ_G is characterised by the equation $\langle y, \psi_G(x)\rangle_R = \langle \alpha(y), x\rangle_K$, $y \in R(G)$. The naturality properties of ψ_G then follow simply from corresponding properties of α. Let $f: H \to G$ be a homomorphism of groups, $f^*: R(G) \to R(H)$ the induced map on representation rings and $\mathrm{ind}_H^G: R(H) \to R(G)$ the map generalizing the usual induction map for $H \to G$ an inclusion (see e.g. [19] 7.1.). Then

$$\begin{aligned}\langle y, \mathrm{ind}_H^G \psi_H(x)\rangle_R &= \langle f^*(y), \psi_H(x)\rangle_R = \langle \alpha(f^*(y)), x\rangle_K = \langle Bf^*(\alpha(y)), x\rangle_K \\ &= \langle y, \psi_G(Bf_*(x))\rangle_R\end{aligned}$$

implies $\psi_G(Bf_*(x)) = \mathrm{ind}_H^G(\psi_H(x))$.

Similarly, if f is an inclusion with tranfer $t: BG^+ \to BH^+$ we find $\psi_H(t(x)) = f^*\psi_G(x)$.

Remark 2.5. Let $i: * = B\{1\} \to BG$ denote the canonical map. Then $K_0(BG; \mathbb{Q}/\mathbb{Z}) \cong \mathrm{coker}\,(\mathrm{ind}_{\{1\}}^G: R(\{1\}) \otimes \mathbb{Q}/\mathbb{Z} \to R(G) \otimes \mathbb{Q}/\mathbb{Z})$ which is simply $\bar{R}(G) \otimes \mathbb{Q}/\mathbb{Z}$, where $\bar{R}(G) = R(G)/(\mathrm{reg})$ and $\mathrm{reg} = \mathrm{ind}_{\{1\}}^G(1)$. We obtain the embedding $\psi: K_1(BG) \cong K_0(BG; \mathbb{Q}/\mathbb{Z}) \to \bar{R}(G) \otimes \mathbb{Q}/\mathbb{Z}$ mentioned in the introduction. The relation of ψ to the G-signature is then proved in [11] (see also [24]), where also a different proof of (2.4) may be found).

Adams operations ψ^i on $R(G)$ may be defined by the character formula

$$\chi_{\psi^i(\rho)}(g) = \chi_\rho(g^i) \tag{2.6}$$

where $g \in G$, $\rho \in R(G)$ and χ_ρ is the character of ρ. Define for $i \not\equiv 0(p)$ $\psi^i_{2n}(\rho) := i^{-n}\psi^i$ on $R(G) \otimes \mathbb{Q}/\mathbb{Z}_{(p)}$. Then

LEMMA 2.7. *For $i \not\equiv 0(p)$, $(|G|, i) = 1$, the Adams operation ψ^i_{2n} commutes with $\psi_G : K_0(BG^+; \mathbb{Q}/\mathbb{Z}_{(p)}) \to R(G) \otimes \mathbb{Q}/\mathbb{Z}_{(p)}$.*

Proof. Let $c\psi^i_0$ be the operation which is adjoint to ψ^i_0 with respect to $\langle\ ,\ \rangle_K : \langle c\psi^i_0(x), y\rangle_K = \langle x, \psi^i_0(y)\rangle_K$. It is easy to see that $c\psi^i_0 = \psi_0^{1/i}$. The formula $\langle \psi^i_0(\rho), \psi^i_0(\mu)\rangle_R = \langle \rho, \mu\rangle_R$ in $R(G)$ ([19], 12.4) implies that $\psi_0^{1/i}$ is adjoint to ψ^i_0 with respect to $\langle\ ,\ \rangle_R$, $(i, |G|) = 1$. The well known property of Adams operations $\alpha \circ \psi^i = \psi^i \circ \alpha$ implies the result.

If G is not a p-group, then $\alpha : R(G) \to K^0(BG^+)$ has a non-trivial kernel. Dual to this ψ_G has a non-trivial cokernel. To describe im (ψ_G) we first introduce the following notation.

For a finite group G let G_p be a p-Sylow subgroup of G and res$: R(G) \to R(G_p)$, ind$: R(G_p) \to R(G)$ the restriction and induction map associated with $G_p \subset G$. Recall that every $x \in G$ may be written in a unique way $x = x_p \cdot x_p'$ where x_p has order a power of p ($=p$-element) and the order of x_p' is prime to p and x_p and x_p' commute. Choose $k \in \mathbb{N}$ such that the k-th power map $g \to g^k$ on G acts as the identity on p-elements and by $g^k = e$ for g with order prime to p. Using the decomposition $x = x_p \cdot x_p'$ above one easily sees that $\psi^k : R(G) \to R(G)$ is idempotent and res induces an isomorphism $\psi^k(R(G)) \to$ im (res).

Let $t : K_0(BG^+; \mathbb{Q}/\mathbb{Z}_{(p)}) \to K_0(BG_p^+; \mathbb{Q}/\mathbb{Z}_{(p)})$ denote the transfer associated to $G_p \subset G$. Since k is prime to p, t commutes with ψ^k. From the fact that $\psi^k = 1$ on $R(G_p)$ and the injectivity of t it follows that im $(\psi_G : K_0(BG^+; \mathbb{Q}/\mathbb{Z}_{(p)}) \to R(G) \otimes \mathbb{Q}/\mathbb{Z}_{(p)})$ is contained in $\psi^k(R(G)) \otimes \mathbb{Q}/\mathbb{Z}_{(p)}$. Let $N_G(G_p)$ be the normalizer of G_p in G and define $W := N_G(G_p)/G_p$. Then W acts on $R(G_p)$ and $K_1(BG)$ in the canonical way. Denote by $Kl(G)$ the $\mathbb{C}$-vector space of $\mathbb{C}$-valued class functions on G and by $Kl_p(G)$ the subspace of $Kl(G)$ consisting of class-functions vanishing outside the set of p-elements.

PROPOSITION 2.8. *With the notation as above we have*

$$K_0(BG^+; \mathbb{Q}/\mathbb{Z}_{(p)}) \cong \psi^k(R(G)) \otimes \mathbb{Q}/\mathbb{Z}_{(p)} \cong \frac{R(G)}{\ker(\text{res})} \otimes \mathbb{Q}/\mathbb{Z}_{(p)}$$
$$\cong \text{im}(\text{res}) \otimes \mathbb{Q}/\mathbb{Z}_{(p)}.$$

The last isomorphism describes $t(K_0(BG^+;\mathbb{Q}/\mathbb{Z}_{(p)}))$ as a direct summand in $K_0(BG_p^+;\mathbb{Q}/\mathbb{Z}_{(p)})\cong R(G_p)\otimes\mathbb{Q}/\mathbb{Z}_{(p)}$.

Proof. Consider the commutative diagram

$$\begin{tikzcd}
R(G)\otimes\mathbb{Z}/p^i \arrow[r, "\mathrm{res}"] \arrow[d] & \mathrm{im(res)} \arrow[r, "\mathrm{ind}"] \arrow[d, phantom, "\cap"] \arrow[dr] & R(G)\otimes\mathbb{Z}/p^i \arrow[d] \\
\dfrac{R(G)}{\ker(\mathrm{res})}\otimes\mathbb{Z}/p^i \arrow[ur] & R(G_p)\otimes\mathbb{Z}/p^i & \dfrac{R(G)}{\ker(\mathrm{res})}\otimes\mathbb{Z}/p^i
\end{tikzcd}$$

Then $\mathrm{ind}(\mathrm{res}(x))=x\cdot\mathrm{ind}(1)$. Now $\mathrm{ind}(1)=|G|/|G_p|+z$ with $z\in I(G)$ and $|G|/|G_p|\not\equiv 0(p)$.

Claim. In $(R(G)/\ker(\mathrm{res}))\otimes\mathbb{Z}/p^i$ we have $z^m=0$ for m large enough. To prove this claim, observe that it is enough to show $\mathrm{res}(z^m)=\mathrm{res}(z)^m=0$ in $R(G_p)\otimes\mathbb{Z}/p^i$. But $\mathrm{res}(z)^m\in I(G_p)^m$ and the elements in $I(G_p)^m$ are divisible by p^i, m large enough, because for a p-group I-adic and p-adic topology coincide. Therefore multiplication by $\mathrm{ind}(1)$ on $(R(G)/\ker(\mathrm{res}))\otimes\mathbb{Z}/p^i$ is an isomorphism. Since this is true for every i, it is true in the limit. The commutativity of the following diagram then implies the proposition.

$$\begin{tikzcd}
 & K_0(BG^+;\mathbb{Q}/\mathbb{Z}_{(p)}) \arrow[r, tail] \arrow[dl, "\psi_G"'] \arrow[d] & K_0(BG_p^+;\mathbb{Q}/\mathbb{Z}_{(p)}) \arrow[r, two heads] \arrow[d, "\cong"] & K_0(BG^+;\mathbb{Q}/\mathbb{Z}_{(p)}) \arrow[d] \\
\psi^k(R(G))\otimes\mathbb{Q}/\mathbb{Z}_{(p)} \arrow[r, phantom, "\cong"] & \dfrac{R(G)}{\ker(\mathrm{res})}\otimes\mathbb{Q}/\mathbb{Z}_{(p)} \arrow[r] & R(G_p)\otimes\mathbb{Q}/\mathbb{Z}_{(p)} \arrow[r] & \dfrac{R(G)}{\ker(\mathrm{res})}\otimes\mathbb{Q}/\mathbb{Z}_{(p)}
\end{tikzcd}$$

Observe that the discussion above also shows that $\mathrm{im}(\mathrm{res})_{(p)}\subset R(G_p)_{(p)}$ is a direct summand (that $\mathrm{im}(\mathrm{res})\subset R(G_p)$ is the inclusion of a direct summand was first proved in [15] by applying Brauer's theorem).

In some cases one can describe $\mathrm{im}(\psi_G)$ as the subgroup of elements invariant under $W=N_G(G_p)/G_p$.

PROPOSITION 2.9. *Let G_p be a p-Sylow subgroup of the finite group G and assume that the following condition is satisfied*: (c) *If $x, y\in G_p$ are conjugate in G, there exists $g\in N_G(G_p)$ with $gxg^{-1}=y$. Then*

(a) $\mathrm{im}(\mathrm{res})_{(p)}=R(G_p)^W_{(p)}$

(b) $K_1(BG)_{(p)}=K_1(BG_p)^W$.

(c) is satisfied, for example, if G_p is a normal subgroup of G or if G_p is abelian (see (2.5) IV in [7]).

Proof. (a) We trivially have $\operatorname{im}(\mathrm{res}) \subset R(G_p)^W$. As observed in the proof of (2.8) $\operatorname{im}(\mathrm{res})_{(p)}$ is a direct summand in $R(G_p)_{(p)}$, hence also in $R(G_p)^W_{(p)}$. To see $\operatorname{im}(\mathrm{res})_{(p)} = R(G_p)^W_{(p)}$ it is therefore enough to show that $\operatorname{im}(\mathrm{res})$ and $R(G_p)^W$ have the same rank.

We identify $R(G) \otimes \mathbb{C}$ with $Kl(G)$. The map res restricted to $Kl_p(G) \subset Kl(G)$ is still onto $\operatorname{im}(\mathrm{res} \otimes 1_{\mathbb{C}}) \subset R(G_p) \otimes \mathbb{C}$. It is injective since every conjugacy class of a p-element in G contains at least one element of G_p. We now show that the composition

$$\begin{aligned} Kl_p(G) &\cong \operatorname{im}(\mathrm{res} \otimes 1_{\mathbb{C}}) = \operatorname{im}(\mathrm{res}) \otimes \mathbb{C} \subset R(G_p)^W_{(p)} \otimes \mathbb{C} \\ &\subset (R(G_p) \otimes \mathbb{C})^W = Kl(G_p)^W \end{aligned}$$

is surjective. This then proves the assertion about the ranks of $\operatorname{im}(\mathrm{res})$ and $R(G_p)^W$ completing the proof of (a).

Let $f \in Kl(G_p)^W$ be a class function on G_p invariant under W. Define $F: G \to \mathbb{C}$ by $F(x) = 0$ if x is not a p-element and by $F(x) := f(gxg^{-1})$ for a p-element x where $g \in G$ is such that $gxg^{-1} \in G_p$. To see that F is well defined, assume that $h \in G$ also satisfies $hxh^{-1} \in G_p$. Since $hxh^{-1} \in G_p$ and $gxg^{-1} \in G_p$ are conjugate in G, they are conjugate in $N_G(G_p)$ by (c). Therefore f takes the same value on hxh^{-1} and gxg^{-1}, hence F is well defined. In the same way it follows that F is a class function, which clearly restricts to f.

Part (b) is an immediate consequence of (a) by observing that ψ_{G_p} carries the action of W on $K_0(BG_p^+; \mathbb{Q}/\mathbb{Z}_{(p)})$ to the action of W on $R(G_p) \otimes \mathbb{Q}/\mathbb{Z}_{(p)}$.

For reduction to cyclic subgroups we use

LEMMA 2.10. *In $K_0(BG^+; \mathbb{Q}/\mathbb{Z})$ we have*

$$x = \sum_{A \subset G} Bi_{A*}\left(\alpha(\theta_A) \cap t_A\left(\frac{x}{|G|}\right)\right)$$

where A runs through all cyclic subgroups of G, $i_A: A \hookrightarrow G$ is the inclusion with transfer $t_A: K_0(BG^+; \mathbb{Q}/\mathbb{Z}) \to K_0(BA^+; \mathbb{Q}/\mathbb{Z})$ and $\theta_A \in R(A)$ is the (virtual) representation whose character takes the value $|A|$ on group generators and is zero otherwise.

Proof. We deduce this from the corresponding formula in $R(G)$. In $R(G)$ we have $|G| \cdot y = \sum_{A \subset G} \operatorname{ind}_A^G(\theta_A \otimes \operatorname{res}_A(y))$ (e.g. see [19]). Applying $\alpha: R(G) \to$

$K^0(BG^+)$ to this gives

$$|G|\cdot y=\sum_{A\subset G} t_A(\alpha(\theta_A)\cup Bi_A^*(y))$$

for all $y\in \operatorname{im}(\alpha)$.

Since $\psi_G: K_0(BG^+;\mathbb{Q}/\mathbb{Z})\to R(G)\otimes\mathbb{Q}/\mathbb{Z}$ is injective, an element $x\in K_0(BG^+;\mathbb{Q}/\mathbb{Z})$ is determined by the values of $\langle y,x\rangle_K$ for all $y\in\operatorname{im}(\alpha)$. Hence

$$x=\sum_{A\subset G} Bi_{A*}\left(\alpha(\theta_A)\cap t_A\left(\frac{x}{|G|}\right)\right)$$

is true if $\langle y,x\rangle_K$ is equal to

$$\left\langle y,\sum_{A\subset G} Bi_{A*}\left(\alpha(\theta_A)\cap t_A\left(\frac{x}{|G|}\right)\right)\right\rangle_K$$

for all $y\in\operatorname{im}(\alpha)$. But this follows from the formula above.

A main ingredient for the proof of theorem (4.4) is some knowledge of the skeletal filtration on $K_1(BG)$. We first handle the case G cyclic and then reduce the general case to this using (2.10).

PROPOSITION 2.11. *Let p be a prime, $C=\mathbb{Z}/p^a$ and $x\in K_1(BC)$ be an element with $p^b\cdot x=0$. Then the skeletal filtration of x is less than $2(p-1)p^{a-1}(a+b)$.*

Proof. Well known properties of lens spaces translate by duality into

(a) $i_*: K_0(BC^{(2m)};\mathbb{Q}/\mathbb{Z}_{(p)})\to K_0(BC;\mathbb{Q}/\mathbb{Z}_{(p)})$ is injective and

(b) $i_*(K_0(BC^{(2m)};\mathbb{Q}/\mathbb{Z}_{(p)})=\{z\in K_0(BC;\mathbb{Q}/\mathbb{Z}_{(p)})\mid \langle (H-1)^s;z\rangle_K=0$ for $s>m\}$ (e.g. see [9]). Here $H\to BC$ is the Hopf line bundle, $H=\alpha(\lambda)$.

Now $K_0(BC^+;\mathbb{Q}/\mathbb{Z}_{(p)})\cong R(C)\otimes\mathbb{Q}/\mathbb{Z}_{(p)}$ is generated by elements x of the form $x=\lambda^{ip^r}/p^k$, $i\not\equiv 0(p)$, $r<a$ and every element x with $p^bx=0$ can be written as an integral linear combination of such elements with $k\le b$. Let $x=\lambda^{ip^r}/p^b$, then $\langle H^j,x\rangle_K=1/p^b$ if $j\equiv ip^r \bmod p^a$ and zero otherwise. Hence

$$\langle (H-1)^s,x\rangle_K=\sum_{j\ge 0}(-1)^{j+s}\binom{s}{j}\langle H^j,x\rangle_K=\sum_{j\ge 0}(-1)^{jp^a+ip^r+s}\binom{s}{ip^r+jp^a}\cdot p^{-b}.$$

If $f(t)=\sum a_k t^k$ is a power series in t and u, v integers with $0 \le u < v$ then

$$a_u t^u + a_{u+v} t^{u+v} + a_{u+2v} t^{u+2v} + \cdots = \frac{1}{v} \sum_{k=0}^{v-1} \omega^{-ku} f(\omega^k t)$$

where $\omega = e^{2\pi i/v}$. We apply this with $v = p^a$, $u = ip^r$, $i \not\equiv 0(p)$, $r < a$ and $f(t) = (1-t)^s$. The coefficient of $t^{ip^r + jp^a}$ in $f(t)$ is

$$(-1)^{jp^a + ip^r + s} \binom{s}{ip^r + jp^a}$$

hence with $t = 1$

$$\sum_{j \ge 0} (-1)^{jp^a + ip^r + s} \binom{s}{ip^r + jp^a} p^{-b} = p^{-b-a} \cdot \sum_{k=0}^{p^a - 1} \omega^{-kp^r \cdot i} (\omega^k - 1)^s.$$

But $(\omega^k - 1)^{p^{a-1}(p-1)}$ is divisible by p. This implies that for $s \ge p^{a-1}(p-1)(b+a)$ the Kronecker products $\langle (H-1)^s, \lambda^{ip^r}/p^b \rangle_K$ vanish.

COROLLARY 2.12. *Let G be a finite group with exponent $e = p^a \cdot m$, $(m, p) = 1$ and order $|G| = p^c \cdot m'$, $(m', p) = 1$. If $x \in K_0(BG; \mathbb{Q}/\mathbb{Z}_{(p)})$ is an element with $p^b \cdot x = 0$ then the skeletal filtration of x is less than $2(p-1)p^{a-1}(b + c + 1)$.*

Proof. Consider the element θ_A defined in (2.10). Let $A = \mathbb{Z}/p^r$, $B = \mathbb{Z}/p$ and $\mathrm{pr}: A \to B$ the reduction map. Then, as is easy to see, we have $\theta_A = \mathrm{pr}^* (p - \mathrm{reg}_A) \cdot p^{r-1}$ which implies that θ_A is divisible by p^{r-1}. Using this and (2.10) we can estimate the skeletal filtration of $\alpha(\theta_A) \cap t_A(x/|G|)$ by (2.11).

Remark 2.13. A slightly better estimate for the skeletal filtration of an element of order p^b may be obtained as follows:

$$\text{If } p^b \cdot x = 0 \text{ in } K_1(BC),\ C = \mathbb{Z}/p^a, \text{ then } x \in \operatorname{im}(K_1(BC^{(2m)}) \to K_1(BC)) \text{ for } m \ge (p-1)p^{a-1}(b+1). \tag{2.14}$$

Since we shall use this only in the examples we give only the main steps for the argument and refer to [11] for more details. The first step is to show that $(1 - \lambda^i)$ for $i \not\equiv 0(p)$ is invertible in $\bar{R}(C) \otimes \mathbb{Z}[1/p]$ where $\bar{R}(C) = R(C)/(\mathrm{reg}_C)$. Hence, for $a_i \not\equiv 0(p)$, $1/\prod_{i=1}^m (1 - \lambda^{a_i})$ is a well defined element in $K_1(BC) \cong \bar{R}(C) \otimes$

$\mathbb{Q}/\mathbb{Z}_{(p)}$. It is not hard to show that the skeletal filtration of such an element is at most $2m$. The formula

$$\theta_A \otimes \frac{x}{p^{a+c}} = (\theta_A \otimes x) \Big/ \left(\prod_{\substack{j=1 \\ j \neq 0(p)}}^{p^r - 1} (1 - \lambda^j)^{a+c} \right)$$

for $x \in I(A)$ and $A = \mathbb{Z}/p^r \subset C$ is proved by comparing characters on both sides. Since θ_A is divisible by p^{r-1}, this equation shows that the skeletal filtration of

$$x_A := \alpha(\theta_A / p^{r-1}) \cap t_A\left(\frac{x}{p^{a+b-r+1}}\right) \quad \text{for} \quad x \in I(C),$$

$A = \mathbb{Z}/p^r \subset C$ does not exceed $2(p-1)p^{r-1}(a+b-r+1)$. By (2.10) we have $x/p^a = \sum_A Bi_{A*}(x_A)$ with A running through the cyclic subgroups of C. Hence $x \in \operatorname{im}(K_0(BC^{(2m)}; \mathbb{Q}/\mathbb{Z}_{(p)}) \to K_0(BC; \mathbb{Q}/\mathbb{Z}_{(p)})$ for $m \geq \max\{(p-1)p^{r-1}(a+b-r+1) \mid 1 \leq r \leq a\} = (p-1)p^{a-1}(b+1)$.

If we use this bound in the proof for (2.12) we obtain $2(p-1)p^{a-1}(b+c-a+1)$ as an upper bound for the skeletal filtration of an element of order p^b in $K_0(BG; \mathbb{Q}/\mathbb{Z}_{(p)})$.

The last topic we have to discuss for the proof of theorem (4.4) is the kernel of the inclusion map $K_1(BG^{(m)}) \xrightarrow{i_*} K_1(BG)$. We need

PROPOSITION 2.15. *Let G be a finite group and $m \in \mathbb{N}$. Then there exists $r = r(m, G) \in \mathbb{N}$ such that $K_0(BG^{(m+r)}; \mathbb{Q}/\mathbb{Z}) \xrightarrow{i_*} K_0(BG; \mathbb{Q}/\mathbb{Z})$ is injective on $\operatorname{im}(K_0(BG^{(m)}; \mathbb{Q}/\mathbb{Z}) \xrightarrow{j_*} K_0(BG^{(m+r)}; \mathbb{Q}/\mathbb{Z}))$.*

This is essentially a statement about the Atiyah–Hirzebruch spectral sequence for the K-theory of BG. It is true for every CW-complex X with $H^i(X; \mathbb{Z})$ finite for all i. In the special case $X = BG$ much more is true, namely there exists $r(G) \in \mathbb{N}$ working for all m. This means simply that the Atiyah–Hirzebruch spectral sequence for the K-groups of BG collapses after finitely many steps, despite BG being an infinite complex. For $G = \mathbb{F}_p^n$, an elementary abelian group, this is stated in [2], there is only one non-zero differential d_{2p-1}. I learned the fact and the proof that the Atiyah–Hirzebruch spectral sequence for $K^0(BG)$ satisfies $d_r = 0$ for $r \geq r_0(G)$ from M. Hopkins. Later on I found out, that this result is already known [23]. We shall work with the weaker version (2.15).

To compute differentials in the Atiyah–Hirzebruch spectral sequence seems to be simpler in cohomology, so we turn to the dual statements. Using (2.2) we deduce (2.15) from the following two propositions.

PROPOSITION 2.16. *Let X be an CW-complex with $H^i(X;\mathbb{Z})$ finite for all i and $m \in \mathbb{N}$. Then there exists $r_0 = r_0(X, m) \in \mathbb{N}$ such that in the Atiyah–Hirzebruch spectral sequence for $K^n(X)$ we have $d_r^{p,q} = 0$ for $r \geq r_0$ and all $p \leq m$.*

Proof. This follows trivially from the fact that the E_2-term consists of finite groups.

PROPOSITION 2.17. *Suppose that in the Atiyah–Hirzebruch spectral sequence for $K^n(X)$ we have $d_r^{p,n-p} = 0$ for $r \geq r_0$ and all $p \leq m$. Then every $x \in K^n(X^{(m)})$ which lifts to $K^n(X^{(m+r_0-1)})$ lifts to $K^n(X)$.*

Proof. We use the exact couple

$$E_1^{s,q} = K^{s+q}(X^{(s)}, X^{(s-1)}) \qquad D_1^{s,q} = K^{s+q}(X^{(s)})$$

$$\begin{tikzcd}
K^*(X^{(s-1)}) \arrow[dr, dashed, "\delta"'] & & K^*(X^{(s)}) \arrow[ll, "i"'] \\
& K^*(X^{(s)}, X^{(s-1)}) \arrow[ur, "j"'] &
\end{tikzcd}$$

$$B_r^{s,q} = \delta(\ker i^{r-1}: K^{s+q-1}(X^{(s-1)}) \to K^{s+q-1}(X^{(s-r)}))$$

$$Z_r^{s,q} = j^{-1}(\operatorname{im} i^{r-1}: K^{s+q}(X^{(s+q-1)}) \to K^{q+s}(X^{(s)}))$$

Consider

$$\begin{tikzcd}
K^n(X^{(s-1)}) \arrow[d, dashed, "\delta"'] & K^n(X^{(s)}) \arrow[l] & K^n(X^{(m)}) \arrow[l, "i^{m-s}"'] & K^n(X^{(m+r)}) \arrow[l, "i^r"'] \arrow[d, dashed, "\delta"'] & \arrow[l, "i"'] \\
K^n(X^{(s)}, X^{(s-1)}) \arrow[ur, "j"'] & & & K^{n+1}(X^{(m+r+1)}, X^{(m+r)}) \arrow[ur, "j"'] &
\end{tikzcd}$$

We first show: If $x \in K^n(X^{(m)})$ has a lift to an element $x' \in K^n(X^{(m+r)})$, $r \geq r_0 - 1$, then it has a lift to $K^n(X^{(m+r+1)})$.

Let s be the exact filtration of x, i.e. $i^{m-s}(x) \neq 0$ in $K^*(X^{(s)})$ but $i^{m-s+1}(x) = 0$ in $K^n(X^{(s-1)})$. Then $i^{m-s}(x) = j(y)$ for some $y \in K^n(X^{(s)}, X^{(s-1)})$. If $x' \in K^n(X^{(m+r)})$ is not in $\operatorname{im}(i)$, then $\delta(x') = z \neq 0$ in $K^{n+1}(X^{(m+r+1)}, X^{(m+r)})$. By definition $z \in B_{m+r-s+2}^{m+r+1,n-m-r}$. Now $\operatorname{im}(d_k) = B_{k+1}/B_k$ and since $d_k^{p,n-q} = 0$ for $k \geq r_0$, $p \leq m$ and $m + r - s + 2 \geq r + 1$ we have $z \in B_{r+1}^{m+r+1,n-m-r}$. Hence there exists $x'' \in K^n(X^{(m+r)})$ with $\delta(x'') = z$ and $x'' \in \ker(i^r)$. Therefore $x' - x'' \in \ker(\delta) = \operatorname{im}(i)$. But $i^r(x' - x'') = i^r(x') = x$, hence $x \in \operatorname{im}(i^{r+1})$. We conclude: $x \in \varprojlim K^n(X^{(m+n)})$ and there exists an element $\bar{x} \in K^n(X)$ restricting to x.

§3. Upper bounds on $\mathrm{im}\,(h_K)$

Restrictions for elements in $K_*(X)$ to be stably spherical are given by Adams operations. The simplest way to handle these seems to be the following. Fix an odd prime p and choose l generating $(\mathbb{Z}/p^2)^*$. Recall that there exists a generalized homology theory Ad_* fitting into the long exact sequence

$$\to \mathrm{Ad}_i\,(X) \xrightarrow{D} K_i(X)_{(p)} \xrightarrow{\psi_i^l - 1} K_i(X)_{(p)} \xrightarrow{\Delta} \mathrm{Ad}_{i-1}(X) \to \tag{3.1}$$

Here ψ_i^l denotes the stable K-theory operation induced by the Adams operation ψ^l. We have a similar sequence in cohomology.

The K-theory Hurewicz map h_K factors through Ad-theory

$$h_K : \pi_n^s(X)_{(p)} \xrightarrow{h_A} \mathrm{Ad}_n\,(X) \xrightarrow{D} K_n(X)_{(p)}$$

and since $\psi_n^l - 1$ vanishes on $\mathrm{im}\,(h_K)$, $\ker(\psi_n^l - 1) = D(\mathrm{Ad}_n\,(X))$ is an upper bound for $\mathrm{im}\,(h_K)$. In the case $X = BG$, G a finite group, we have $K_0(BG) = 0$, so $\ker(\psi_{2n-1}^l - 1) = \mathrm{Ad}_{2n-1}(BG)$ and the Ad-group itself gives the upper bound.

Now stable homotopy is a connected homology theory, therefore the Ad-theory Hurewicz map will factor through the connected version $A_*(X)$ of Ad-theory. For X a CW-complex, connected Ad-theory may be defined by

$$A_n(X) := \mathrm{im}\,(\mathrm{Ad}_n\,(X^n) \to \mathrm{Ad}_n\,(X^{n+1})). \tag{3.2}$$

Let $d: A_n(X) \to \mathrm{Ad}_n\,(X)$ denote the canonical map. In general d is neither injective nor surjective and A-theory gives an even better bound for $\mathrm{im}\,(h_K)$. But usually $A_*(X)$ is much harder to calculate than $\mathrm{Ad}_*\,(X)$. The following result is the only one of this chapter which is needed in the proof of theorem (4.4).

PROPOSITION 3.3. *Let G be a finite group, e the exponent of G and $n \neq 0$. Then* (i) $\mathrm{Ad}_{2n}\,(BG) = 0$, (ii) $\mathrm{Ad}_{2n-1}\,(BG)$ *is finite and* (iii) $n \cdot e \cdot \mathrm{Ad}_{2n-1}\,(BG) = 0$.

Proof. First of all we may reduce to a p-group by the following standard transfer argument. Let t be the transfer associated with the inclusion $i: G_p \to G$ of a p-Sylow subgroup of G. Since $\mathrm{B}i_* \circ t$ induces multiplication with $|G|/|G_p|$ in ordinary homology, it follows that $\mathrm{B}i_* \circ t$ is an isomorphism for any p-local homology theory. Hence $\mathrm{Ad}_*\,(BG)$ is a direct summand in $\mathrm{Ad}_*\,(BG_p)$ and $\mathrm{B}i_* : \mathrm{Ad}_*\,(BG_p) \to \mathrm{Ad}_*\,(BG)$ is onto.

Let now G be a p-group, e the exponent of G and $v_p(e)$ the power of p in the prime factorization of e. Observe next that the Bockstein map $\beta: \mathrm{Ad}_{2n}(BG; \mathbb{Q}/\mathbb{Z}) \to \mathrm{Ad}_{2n-1}(BG)$ is bijective and that

$$\mathrm{Ad}_{2n}(BG; \mathbb{Q}/\mathbb{Z}) \cong \ker(\psi^l_{2n} - 1) \quad \text{on} \quad K_{2n}(BG; \mathbb{Q}/\mathbb{Z}_{(p)}).$$

This is the same as $\ker(l^{-n}\psi^l_0 - 1) = \ker(\psi^l_0 - l^n)$ on $K_0(BG; \mathbb{Q}/\mathbb{Z}_{(p)}) \cong R(G) \otimes \mathbb{Q}/\mathbb{Z}_{(p)}$. From (2.6) we see that ψ^s_0 is periodic in s with period e. Since l generates the p-adic units, it is easy to see that for $s \not\equiv 0(p)$ ψ^s_0 operates on $\ker(\psi^l_0 - l^n) \subset K_0(BG; \mathbb{Q}/\mathbb{Z}_{(p)})$ as multiplication by s^n. Let now $v = (p-1) \cdot e/p$, then $s^v \equiv 1 \bmod e$ and $\psi^{s^v}_0 = \psi^1_0 = \mathrm{id}$ on $K_0(BG; \mathbb{Q}/\mathbb{Z}_{(p)})$. Therefore

$$(s^{v \cdot n} - 1) \cdot \ker(\psi^l_0 - l^n) = 0.$$

But $\min\{v_p(s^{nv} - 1) \mid s \not\equiv 0(p)\} = v_p(n) + v_p(v) + 1 = v_p(e \cdot n)$. This proves (iii). Since $K_0(BG; \mathbb{Q}/\mathbb{Z}_{(p)})$ is a finite sum of copies of $\mathbb{Q}/\mathbb{Z}_{(p)}$ (ii) follows from (iii). The vanishing of $\mathrm{Ad}_{2n}(BG) \cong \mathrm{Ad}_{2n-1}(BG; \mathbb{Q}/\mathbb{Z})$ is then implied by (ii) using (3.1).

We first deal with $\mathrm{Ad}_*(BG)$ for a p-group G. Let G be a p-group with exponent e and ρ an irreducible representation of G. For l with $(l, |G|) = 1$, $\psi^l(\rho)$ is again irreducible (e.g. [19], 9.4). Hence ψ^l acts as a permutation on the set $\mathrm{Irr}(G)$ of irreducible representations of G. Denote by $V_i = \{\rho^{(i)}_1, \rho^{(i)}_2, \ldots, \rho^{(i)}_{s_i}\}$, $i = 0, \ldots, w$, the orbits of this action. By renumbering we may assume $\psi^l \rho^{(i)}_{(k)} = \rho^{(i)}_{(k+1)}$, $k < s_i$, and $\psi^l \rho^{(i)}_{s_i} = \rho^{(i)}_1$. We also assume $\rho^{(0)}_1 = 1$, the trivial representation. For an orbit $V = \{\rho_1, \ldots, \rho_s\}$ of the action of ψ^l on $\mathrm{Irr}(G)$ define

$$x_n(V) := \frac{1}{l^{n \cdot s} - 1} \cdot \sum_{j=0}^{s-1} \psi^{l^j}_{2n}(\rho_1) = \frac{1}{l^{ns} - 1} \sum_{j=0}^{s-1} l^{-n \cdot j} \cdot \rho_{j+1} \tag{3.4}$$

as an element in $R(G) \otimes \mathbb{Q}/\mathbb{Z}_{(p)} \cong K_{2n}(BG; \mathbb{Q}/\mathbb{Z}_{(p)})$.

LEMMA 3.5. $x_n(V) \in \ker(\psi^l_{2n} - 1)$

Proof. Because of $\psi^l_{2n} = l^{-n}\psi^l_0$ we have $\psi^l_{2n} \circ \psi^{l^j}_{2n} = \psi^{l^{j+1}}_{2n}$. Therefore

$$(\psi^l_{2n} - 1)(x_n(V)) = (l^{-sn}\rho_1 - \rho_1)/(l^{sn} - 1) \equiv 0 \bmod \mathbb{Z}_{(p)}.$$

Note that the denominator $l^{ns} - 1$ in $x_n(V)$ is maximal.

PROPOSITION 3.6. *Let G be a p-group. Denote by $V_i = \{\rho_1^{(i)}, \ldots, \rho_{s_i}^{(i)}\}$ $i = 1, \ldots, w$ the orbits of the action of ψ^l on the set of nontrivial irreducible representations of G. Then for $n \neq 0$*

$$\mathrm{Ad}_{2n-1}(BG) \cong \bigoplus_{i=1}^{w} (\mathbb{Z}_{(p)}/(l^{n \cdot s_i} - 1)) \cdot x_n(V_i)$$

Proof. Let $x = \sum a_{ij}\rho_j^{(i)}$ be an element in $\ker(\psi_{2n}^l - 1) \cong \mathrm{Ad}_{2n}(BG; \mathbb{Q}/\mathbb{Z})$. Then $\psi_{2n}^l x - x = \sum_{i,j}(a_{ij}\psi_{2n}^l(\rho_j^{(i)}) - a_{ij}\rho_j^{(i)}) = 0$ gives the congruences

$$a_{i,j}l^{-n} \equiv a_{i,j+1} \quad \text{for} \quad j < s_i \quad \text{and} \quad a_{i,s_i}l^{-n} \equiv a_{i,1}.$$

These imply $(l^{-sn} - 1)a_{1,j} \equiv 0 \bmod \mathbb{Z}_{(p)}$ showing that x can be written as an integral linear combination of the elements $x_n(V_i)$.

Next we give two reformulations of (3.6) and prove that $\mathrm{Ad}_{2n-1}(BG)$ has w cyclic summands (not depending on n).

The numbers w and s_i are related to the rational representation ring of G. To see this, recall the following facts from representation theory (e.g. see [19]). Let ξ be a primitive e-th root of unity. Then all representations of G may be realized over $\mathbb{Q}(\xi)$. The Galois group $\Gamma = \mathrm{Gal}(\mathbb{Q}(\xi):\mathbb{Q})$ operates in the usual way on $R(G)$ via its operation on the character values. If $\sigma \in \Gamma$ satisfies $\sigma(\xi) = \xi^t$ then $\sigma(\rho) = \psi^t(\rho)$, hence Galois action and action by Adams operation are the same. If G is a p-group with $p \neq 2$ then Γ is cyclic and we may take ψ^l as generator.

Denote by $R(G)^\Gamma$ the elements of $R(G)$ invariant under Γ. Then $R(G)^\Gamma$ consists of representations with rational characters and the rational representation ring $R_\mathbb{Q}(G)$ is contained in $R(G)^\Gamma$ as a subgroup of maximal rank. For a p-group, $p \neq 2$, they are actually the same since all Schur indices are 1 ([8], (10.14)). If $V = \{\rho_1, \ldots, \rho_s\}$ is an orbit of the action of $\langle\psi^l\rangle = \Gamma$, where G is now a p-group, then $\rho = \rho_1 + \rho_2 + \cdots + \rho_s$ is an irreducible rational representation. Thus $s = \langle\rho, \rho\rangle_R$ and w is the number of nontrivial irreducible rational representations. Hence

PROPOSITION 3.7. *Let G be a p-group. Then the number of orbits of the action of ψ^l on* $\mathrm{Irr}(G)$ *is equal to the rank of $R_\mathbb{Q}(G)$ and the number s of elements in an orbit $V = \{\rho_1, \ldots, \rho_s\}$ is $s = \langle\rho, \rho\rangle_R$ where $\rho = \rho_1 + \rho_2 + \cdots \rho_s$ is the rational representation corresponding to V.*

There is still another description of the orbit set of Γ. The Galois group Γ acts also on the set of conjugacy classes of G. If $g \in G$ and $\sigma \in \Gamma$ satisfies $\sigma(\xi) = \xi^t$, then $\sigma(g)=g^l$ is well defined. This induces an action of Γ on the set $C(G)=\{c_1,\ldots,c_m\}$ of conjugacy classes of G. Let $w_i=\{c_1^{(i)},\ldots,c_{h_i}^{(i)}\}$, $i=0,\ldots,r$, the orbits of this action, $c_j^{(i)}\in C(G)$, with the convention $c_1^{(0)}=\{1\}$. For a p-group G we take ψ^l as generator of Γ. We use this action for our final description of $\mathrm{Ad}_{2n-1}(G)$.

THEOREM 3.8. *Let G be a p-group with exponent e. Denote by $w_i=\{c_1^{(i)},\ldots,c_{h_i}^{(i)}\}$, $i=0,1,\ldots,r$, the orbits of the action of the cyclic group $\langle\psi^l\rangle$ on conjugacy classes $c_j^{(i)}$ of G induced by the l-th power map, $\psi^l(g)=g^l$ for $g\in c_j^{(i)}$. Let $c_1^{(0)}=\{1\}$ be the conjugacy class of 1. Then, for $n\neq 0$, $\mathrm{Ad}_{2n-1}(BG)$ has exactly r cyclic summands of order $p^{v_p(n)+v_p(h_i)}+1$, $i=1,\ldots,r$. In particular $h_i\equiv 0(p-1)$ for $i>0$, $v_p(h_i)\leq v_p(e)$ and $h_i=s_i$, $w=r$ in (3.6).*

Proof. Let $f_j^{(i)}$ be the class function on G which is 1 on $c_j^{(i)}$ and zero otherwise. Clearly the Γ set $X_1:=\{f_j^{(i)}\}$ is Γ-equivariantly isomorphic to the Γ set $C(G)$. The Γ set X_1 defines a complex permutation representation W^1 and the Γ set X_2 consisting of the irreducible representations of G defines another permutation representation W^2. Clearly $W^1=W^2=R(G)\otimes\mathbb{C}$ as complex vector spaces, but (2.6) shows that they are also the same as Γ-modules. To show that the two Γ sets X_1 and X_2 are isomorphic Γ sets, observe that it is enough to show that X_1 and X_2 define the same element in the Burnside ring $\Omega(\Gamma)$ of Γ. The map which associates to a Γ set its permutation representation defines a canonical map $\Omega(\Gamma)\to R(\Gamma)$. As long as Γ is cyclic this map is injective (e.g. [19]). Hence X_1 is isomprhic to X_2 as a Γ set, in particular $r=w$ and $h_i=s_i$.

It remains to show $h_i\equiv 0(p-1)$ for $i>0$. For then $v_p(l^{n\cdot s_i}-1)=v_p(n)+v_p(s_i)+1$ and the order of $x_n(w_i)$ is as stated. Let $w=\{c_1,\ldots,c_s\}$ be an orbit of the $\langle\psi^l\rangle$-action on $C(G)$ not containing 1, and arranged in such a way that $\psi^l c_i=c_{i+1}$, $i<s$, and $\psi^l c_s=c_1$. If $c_1=\{h^{-1}gh\mid h\in G\}$ then $c_i=\{h^{-1}g^{l^{i-1}}h\mid h\in G\}$. For some $h_0\in G$ we thus have $g^{l^s}=h_0^{-1}gh_0$. Induction shows that $h^{-1}gh=g^a$ implies $h^{-k}gh^k=g^{a^k}$. Let p^b be the order of h_0 and p^c the order of g, then $g^{(l^s)^{p^b}}=h^{-p^b}gh^{p^b}=g$ and therefore $l^{sp^b}\equiv 1 \bmod p^c$. By the choice of l this is only possible if $s\equiv 0(p-1)$.

COROLLARY 3.9. *Let G be an abelian p-group and I the family of cyclic subgroups of G different from $\{1\}$. Then*

$$\mathrm{Ad}_{2n-1}(BG)=\bigoplus_{A\in I}(\mathbb{Z}_{(p)}/|A|\cdot n).$$

Proof. Study the orbit set of $\langle\psi^l\rangle$ on $C(G)=G$.

EXAMPLE 3.10. (a) $G = \mathbb{Z}/p^a$, $R(\mathbb{Z}/p^a) = \mathbb{Z}[\lambda]/(\lambda^{p^a} = 1)$. The orbits of $\langle \psi^l \rangle$ are $v_i = \{\lambda^{p^i \cdot j} \mid j \neq 0(p)\}$, $i = 0, \ldots, a-1$, with $s_i = (p-1)p^{a-i-1}$. We have

$$x_n(v_i) = \left(\sum_{k=0}^{s_i - 1} l^{-k \cdot n} \lambda^{p^i l^k}\right) \Big/ p^{a-i+\nu_p(n)} = \left(\sum_{\substack{j=1 \\ j \neq 0(p)}}^{p^{a-1}} j^{-n} \lambda^{p^i \cdot j}\right) \Big/ p^{a-i+\nu_p(n)}$$

and

$$\mathrm{Ad}_{2n-1}(B\mathbb{Z}/p^a) = \bigoplus_{i=0}^{a-1} (\mathbb{Z}/p^{a-i+\nu_p(n)}) \cdot x_n(v_i).$$

(b) $G = \mathbb{Z}/p \times \mathbb{Z}/p$, $R(G) = R(\mathbb{Z}/p) \otimes R(\mathbb{Z}/p)$. The orbits are $v_i = \{\lambda^j \otimes \lambda^{j \cdot i} \mid j = 1, \ldots, p-1\}$, $i = 0, \ldots, p-1$ and $v_p = \{1 \otimes \lambda^j \mid j = 1, \ldots, p-1\}$. The elements

$$x_n(v_i) = \left(\sum_{j=1}^{p-1} j^{-n} \lambda^j \otimes \lambda^{j \cdot i}\right) \Big/ p^{1+\nu_p(n)}, \qquad i = 0, \ldots, p-1,$$

and

$$x_n(v_p) = \left(\sum_{j=1}^{p-1} j^{-n} \cdot 1 \otimes \lambda^j\right) \Big/ p^{1+\nu_p(n)} \quad \text{generate} \quad \mathrm{Ad}_{2n-1}(BG) \cong \bigoplus_{j=0}^{p} \mathbb{Z}/p^{1+\nu_p(n)}$$

(c) G a non abelian group of order p^3. For simplicity let $p = 3$. There are 2 non isomorphic groups $G_1 = (\mathbb{Z}/3 \times \mathbb{Z}/3) \rtimes \mathbb{Z}/3$ and $G_2 = \mathbb{Z}/9 \rtimes \mathbb{Z}/3$. In both cases we have 5 orbits of length $p - 1 = 2$. So

$$\mathrm{Ad}_{2n-1}(BG_i) = \bigoplus_{j=1}^{5} \mathbb{Z}/3^{1+\nu_3(n)} \quad \text{for} \quad i = 1, 2.$$

We now turn to an arbitrary finite group G with exponent m, $e = \nu_p(m) \geq 1$ and $s(e) := (p-1) \cdot p^{e-1}$.

To describe elements in $\mathrm{Ad}_{2n-1}(BG)$ we introduce the map

$$\tau : R(G) \to R(G) \otimes \mathbb{Q}/\mathbb{Z}_{(p)}$$

defined by

$$\tau(x) = \left(\sum_{j=0}^{s(e)-1} \psi_{2n}^{l^j}(x)\right) \Big/ (l^{n \cdot s(e)} - 1)$$

We assume l is chosen satisfying $(|G|, l) = 1$.

LEMMA 3.11. *If G is a p-group with exponent p^e then*

$$\operatorname{im}(\tau: R(G) \to R(G) \otimes \mathbb{Q}/\mathbb{Z}_{(p)}) \cong \mathrm{Ad}_{2n}(BG^+; \mathbb{Q}/\mathbb{Z}).$$

Proof. Let $v = \{\rho_1, \ldots, \rho_k\}$ be an orbit of the action of ψ^l on $\operatorname{Irr}(G) - \{1\}$. It suffices to show $\tau(\rho_1) = c \cdot x_n(v)$ for some $c \not\equiv 0(p)$. Let $r = s(e)/k$ then

$$\tau(\rho_1) = \left(\sum_{a=0}^{r-1} l^{-nka}(\rho_1 + l^{-n}\rho_2 + \cdots + l^{-n(k-1)}\rho_k)\right) \Big/ (l^{n \cdot s(e)} - 1)$$

$$= \frac{l^{n \cdot k} - 1}{l^{n \cdot s(e)} - 1} \cdot \left(\sum_{a=0}^{r-1} l^{-nka}\right) \cdot x_n(v) = \frac{l^{n \cdot k} - 1}{l^{n \cdot s(e)} - 1} \frac{1 - l^{-nkr}}{1 - l^{nk}} \cdot x_n(v)$$

Recall the notation introduced preceding (2.8). Let $G_p \subset G$ be a p-Sylow subgroup of G and $k \in \mathbb{N}$ as in (2.8). For $x \in R(G)$ the element $\tau(x) \in R(G) \otimes \mathbb{Q}/\mathbb{Z}_{(p)}$ is not necessarily in $\ker(\psi^l_{2n} - 1)$, but this is true for $x \in \operatorname{im}(\mathrm{ind}: R(G_p) \to R(G))$ or $x \in \psi^k(R(G))$ as we shall see now.

PROPOSITION 3.12. (a) $\tau(\operatorname{im}(\mathrm{ind})) \cong \psi_G(\mathrm{Ad}_{2n}(BG^+; \mathbb{Q}/\mathbb{Z}))$
(b) $\tau(\psi^k(R(G))) \cong \psi_G(\mathrm{Ad}_{2n}(BG^+; \mathbb{Q}/\mathbb{Z}))$ *in* $R(G) \otimes \mathbb{Q}/\mathbb{Z}_{(p)}$.

Proof. Consider the following commutative diagram

$$\begin{tikzcd}
R(G_p) \arrow[r, "\tau"] \arrow[d, "\mathrm{ind}"] & R(G_p) \otimes \mathbb{Q}/\mathbb{Z}_{(p)} \arrow[d, "\mathrm{ind}"] & K_0(BG_p^+; \mathbb{Q}/\mathbb{Z}_{(p)}) \arrow[l, "\psi_{G_p}"'] \arrow[d, "\mathrm{Bi}_*"] \\
R(G) \arrow[r, "\tau"] & R(G) \otimes \mathbb{Q}/\mathbb{Z}_{(p)} & K_0(BG^+; \mathbb{Q}/\mathbb{Z}_{(p)}) \arrow[l, tail, "\psi_G"'] \arrow[u, bend left, "t"]
\end{tikzcd} \tag{3.13}$$

Because of $(|G|, l) = 1$ $\mathrm{ind}^G_{G_p}$ and ψ^l will commute, so the left hand square in (3.13) is commutative. This shows $(\psi^l_{2n} - 1)(\tau(x)) = 0$ for $x \in \operatorname{im}(\mathrm{ind})$. The facts that Bi_* and ψ_{G_p} are onto in (3.13) then imply (a) by (3.11). Part (b) is proved similarly.

EXAMPLE 3.14. Let $G = \Sigma_m$ be the symmetric group on m symbols. Then $R(\Sigma_m) = R_{\mathbb{Q}}(\Sigma_m)$ ([19], 13.1) hence $\psi^l = 1$ on $R(\Sigma_m)$. Therefore

$$\tau(x) = x \cdot \left(\sum_{j=0}^{s(e)-1} l^{-nj}\right) \Big/ (l^{n \cdot s(e)} - 1) = x \cdot \frac{(1 - l^{-n \cdot s(e)})}{(1 - l^{-n}) \cdot (l^{n \cdot s(e)} - 1)}$$

and $\mathrm{Ad}_{2n-1}(B\Sigma_m)$ is zero if $n \not\equiv 0(p-1)$ and a direct sum of cyclic groups of order $p^{1+\nu_p(n)}$ if $n \equiv 0(p-1)$, $n \neq 0$.

The following proposition describes $\mathrm{Ad}_{2n-1}(BG^+)$ as a subgroup of $\mathrm{Ad}_{2n-1}(BG_p^+)$.

PROPOSITION 3.15. *The map* $\mathrm{Bi}_*: \mathrm{Ad}_{2n-1}(BG_p^+) \to \mathrm{Ad}_{2n-1}(BG^+)$ *restricted to* $\tau(\mathrm{im}(\mathrm{res})) \subset \mathrm{Ad}_{2n}(BG_p^+; \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Ad}_{2n-1}(BG_p^+)$ *is bijective.*

Proof. Let $x \in \mathrm{Ad}_{2n}(BG^+; \mathbb{Q}/\mathbb{Z})$ be given. By (3.12) (a) there exists x_1 with $\tau(\mathrm{ind}(x_1)) = x$. Choose i with $p^i \cdot \mathrm{Ad}_{2n-1}(BG_p^+) = 0$. The proof of (2.8) then shows that we can choose x_1 in $\mathrm{im}(\mathrm{res}) \bmod p^i$. Hence there is $x_2 \in \mathrm{im}(\mathrm{res})$ with $\tau(\mathrm{ind}(x_2)) = x$. But then $\mathrm{Bi}_*(\tau(x_2)) = x$ and Bi_* is onto.

Assume that we have $\tau(\mathrm{res}(z)) \in \ker(\mathrm{Bi}_*)$ for $z \in R(G)$. Since $\psi^k(z)$ and z (k as in (2.8)) have the same image under res, we may assume $z \in \psi^k(R(G))$. But then $\tau(z) \in R(G) \otimes \mathbb{Q}/\mathbb{Z}_{(p)}$ is in $\ker(\psi_{2n}^l - 1) = \mathrm{Ad}_{2n}(BG^+; \mathbb{Q}/\mathbb{Z})$ and $t(\tau(z)) = \tau(\mathrm{res}(z))$ in $\mathrm{Ad}_{2n}(BG_p^+; \mathbb{Q}/\mathbb{Z})$ by naturality. Since $\mathrm{Bi}_* \circ t$ is bijective it follows $\tau(\mathrm{res}(z)) = 0$ and Bi_* is injective.

We now turn to the problem of determining the group structure of $\mathrm{Ad}_{2n-1}(BG)$. Let $C_p(G)$ be the set of conjugacy classes of p-elements in the finite group G and Γ the finite cyclic group generated by the action of the l-th power map on $C_p(G)$. Denote by $B_i = \{C_1^{(i)}, C_2^{(i)}, \ldots, C_{h_i}^{(i)}\}$, $i = 0, 1, \ldots, r$, $C_j^{(i)} \in C_p(G)$, the orbits with respect to this action with the convention that B_0 is the orbit of the unit element of G.

The class functions dual to $C_j^{(i)}$ give a basis of $Kl_p(G)$, the vector space of class functions vanishing outside the p-elements of G. Since the action of ψ^l on $Kl_p(G)$ corresponds to the action of the l-th power map on $C_p(G)$ we see that $Kl_p(G)$ is a permutation representation of the finite cyclic group $\langle \psi^l \rangle$ generated by ψ^l. Consider now

$$P := \mathrm{im}(\mathrm{res})_{(p)} \subset R(G_p)_{(p)}.$$

A direct generalization of (3.8) from p-groups to arbitrary groups is possible if one knows that P is a $\langle \psi^l \rangle$-permutation representation. This is true at least rationally: In the proof of (2.9) we have seen that $P \otimes \mathbb{C}$ and $Kl_p(G)$ are isomorphic as $\langle \psi^l \rangle$-modules. Hence $P \otimes \mathbb{C}$ is a $\langle \psi^l \rangle$-permutation representation. Since $r: R_{\mathbb{Q}}(\langle \psi^l \rangle) \to R_{\mathbb{C}}(\langle \psi^l \rangle)$ is injective, $Kl_p(G)$, $P \otimes \mathbb{C}$ both are in $\mathrm{im}(r)$ and $Kl_p(G)$ may be realized as a permutation representation over $\mathbb{Z}$ it follows that $P \otimes \mathbb{Q}$ is a $\langle \psi^l \rangle$-permutation representation.

Whether or not P itself is a $\langle \psi^l \rangle$-permutation-representation seems to be unknown in general. In (3.17) we shall describe some cases where this is true.

PROPOSITION 3.16. *Let G be a finite group with p-Sylow subgroup G_p. Assume that $P = \operatorname{im}(\mathrm{res}: R(G) \to R(G_p))_{(p)}$ is a $\langle \psi^l \rangle$-permutation representation. Then*

$$\mathrm{Ad}_{2n-1}(BG) \cong \bigoplus_{i=1}^{r} \mathbb{Z}_{(p)}/(l^{n \cdot h_i} - 1).$$

Proof. By (3.15) we have $\mathrm{Ad}_{2n}(BG^+; \mathbb{Q}/\mathbb{Z}) \cong \tau(P)$ in $R(G_p) \otimes \mathbb{Q}/\mathbb{Z}_{(p)}$. Let S denote the $\langle \psi^l \rangle$-set defining the $\langle \psi^l \rangle$-permutation representation P. As in the case of a p-group we obtain a description of the group structure of $\tau(P)$ by number and length of the $\langle \psi^l \rangle$-orbits on S. The fact that the $\langle \psi^l \rangle$-permutation representation $P \otimes \mathbb{C}$ and $Kl_p(G)$ are $\langle \psi^l \rangle$-isomorphic implies that the $\langle \psi^l \rangle$-sets S and $C_p(G)$ have the same orbit structure (see proof of (3.8)).

The assumptions of (3.16) are clearly satisfied if the l-th power map acts trivially on $C_p(G)$, for then P has to be the trivial permutation representation. We may apply (3.16) also in the situation of (2.9):

PROPOSITION 3.17. *If $P = \operatorname{im}(\mathrm{res})_{(p)} = R(G_p)^W_{(p)}$ then P is a $\langle \psi^l \rangle$-permutation representation.*

Proof. The actions of $W = N_G(G_p)/G_p$ and $\langle \psi^l \rangle$ on $R(G_p)$ commute and we have an induced action of $\langle \psi^l \rangle$ on $R(G_p)^W_{(p)}$. Define $s: R(G_p)_{(p)} \to R(G_p)^W_{(p)}$ by

$$s(x) := \frac{1}{|W|} \sum_{\sigma \in W} \sigma(x).$$

Then s is $\langle \psi^l \rangle$-equivariant. It is easy to see that if $\lambda \in R(G_p)$ is irreducible and $\sigma \in W$ then $\sigma(\lambda)$ is irreducible too. Hence W operates on the set $\mathrm{Irr}(G_p)$. Under the map s the W-orbits on $\mathrm{Irr}(G_p)$ define a basis of $R(G_p)^W_{(p)}$ and since $\langle \psi^l \rangle$ maps W-orbits to W-orbits we see that $R(G_p)^W_{(p)}$ is a permutation representation of the group $\langle \psi^l \rangle$.

EXAMPLE 3.18. Let $G = D_{10} = \mathbb{Z}/10 \rtimes \mathbb{Z}/2$ be the dihedral group of order 20 and $p = 5$. There are two conjugacy classes of elements of order 5 in D_{10}, which are permutated by ψ^l. Hence we have one orbit of length 2, so

$$\mathrm{Ad}_{2n-1}(BD_{10}) = \mathbb{Z}_{(5)}/(l^{2n} - 1) = \begin{cases} \mathbb{Z}/5^{1+v_5(n)} & n \equiv 0(2) \\ 0 & n \equiv 1(2) \end{cases}.$$

Remark. The number of cyclic summands in $\mathrm{Ad}_{2n-1}(BG)$ depends only on the residue class of $n \bmod (p-1)$. This is easily seen by introducing mod p coefficients and observing that $\psi_{2n}^{l} \equiv \psi_{2m}^{l} \bmod p$ if $n \equiv m \bmod (p-1)$. Using this and the fact that $P \otimes \mathbb{Q}$ is a $\langle \psi^{l} \rangle$-permutation representation one can still prove: The groups

$$\mathrm{Ad}_{2n-1}(BG) \quad \text{and} \quad \bigoplus_{i=1}^{r} \mathbb{Z}_{(p)}/(l^{n \cdot h_i} - 1)$$

have the same order and the same number of cyclic summands. We omit the details.

Remark. Comparison of the exact sequences for $\mathrm{Ad}_n(BG; \mathbb{Q}/\mathbb{Z})$ and $\mathrm{Ad}_n^G(S^0; \mathbb{Q}/\mathbb{Z})$ (here Ad^G denotes G-equivariant Ad-theory), gives the relation between $\mathrm{Ad}_*(BG)$ and $\mathrm{Ad}_*^G(S^0)_{(p)}$

$$\begin{tikzcd}
\to \mathrm{Ad}_n^G(S^0; \mathbb{Q}/\mathbb{Z}_{(p)}) \arrow[r] & K_n^G(S^0; \mathbb{Q}/\mathbb{Z}_{(p)}) \arrow[r, "\psi_n^l - 1"] & K_n^G(S^0; \mathbb{Q}/\mathbb{Z}_{(p)}) \arrow[r] & \mathrm{Ad}_{n-1}^G(S^0; \mathbb{Q}/\mathbb{Z}_{(p)}) \to \\
\to \mathrm{Ad}_n(BG^+; \mathbb{Q}/\mathbb{Z}_{(p)}) \arrow[r] \arrow[u, "\bar{\psi}_G"] & K_n(BG^+; \mathbb{Q}/\mathbb{Z}_{(p)}) \arrow[r, "\psi_n^l - 1"] \arrow[u, "\psi_G"] & K_n(BG^+; \mathbb{Q}/\mathbb{Z}_{(p)}) \arrow[r] \arrow[u, "\psi_G"] & \mathrm{Ad}_{n-1}(BG^+; \mathbb{Q}/\mathbb{Z}_{(p)}) \arrow[u, "\psi_G"]
\end{tikzcd}$$

In particular, if G is a p-group, then $\bar{\psi}_G$ is an isomorphism. We close this chapter with a short discussion of $\mathrm{Ad}^*(BG)$.

For a finite group G, Rector [17] studied $K\mathbb{F}_q^*(BG)$, where $K\mathbb{F}_q^*$ is the algebraic K-theory associated with the finite field $\mathbb{F}_q$. If we choose the number l in the definition of A^* to be a prime power then $K\mathbb{F}_l^*(X)_{(p)} \cong A^*(X)$. Let $R_{\mathbb{F}_q}(G)$ denote the Grothendieck group of finitely generated $\mathbb{F}_q[G]$-modules, I the augmentation ideal of $R_{\mathbb{F}_q}(G)$ and $R_{\mathbb{F}_q}(G)^\wedge$ the I-adic completion of $R_{\mathbb{F}_q}(G)$. Rector proved an analogue of Atiyah's theorem [2], $K\mathbb{F}_q^0(BG^+) \cong R_{\mathbb{F}_q}(G)^\wedge$ and $K\mathbb{F}_q^{2i}(BG) = 0$ for $i < 0$. For $i < 0$ and $\mathbb{F}_s$ a splitting field of G he gave a description of $K\mathbb{F}_q^{2i+1}(BG^+)$ as the kernel of $\psi^q - 1$ on $R_{\mathbb{F}_s}(G)^\wedge \otimes K\mathbb{F}_s^{2i+1}(S^0)$.

Since we are mainly interested in $\mathrm{Ad}_*(BG)$ we have given a description of $\mathrm{Ad}_*(BG)$ which is independent of [17]. The two approaches may be related by the following universal coefficient formula for Ad-theory which is proved in [14].

PROPOSITION 3.19. *Let X be a CW-complex and R an abelian group. Then the following sequence is exact*

$$0 \to \mathrm{Ext}(\mathrm{Ad}_{i-1}(X); R_{(p)}) \to \mathrm{Ad}^{i+1}(X; R) \to \mathrm{Hom}(\mathrm{Ad}_i(X); R_{(p)}) \to 0.$$

COROLLARY 3.20. *Let G be a finite group and $n \neq 0$. Then*

$$\mathrm{Ad}^{1+2n}(BG) \cong \mathrm{Ad}_{2n-1}(BG).$$

§4. Lower bounds for im (h_K) and the proof of theorem (4.4)

We shall use an extension of the classical J-homomorphism to $A^0(X)$

$$j_A : A^0(X) \to \pi_s^0(X)_{(p)}$$

to construct elements in $\pi_*^s(BG)$. For other applications of this method see e.g. [4], [5]. The extension j_A of J results from a solution of the Adams conjecture and may be constructed as follows (in the complex case): Let BSF denote the classifying space for spherical fibrations, $\mathbf{J}$ the fibre of $\psi^l - 1$ and $J : U \to SF$ the classical (complex) J-map. Then $A^0(X) = [X, \mathbf{J}]$ and $[X, SF] \cong \pi_s^0(X)$ for connected X. A solution of the Adams conjecture provides us with a map α in the following commutative diagram of classifying spaces (localised at p):

$$\begin{tikzcd}
U \arrow[r, "\Delta"] \arrow[d, equal] & \mathbf{J} \arrow[r, "D"] \arrow[d, "j_A"] & BU \arrow[r, "\psi^l - 1"] \arrow[d, "\alpha"] & BU \arrow[d, equal] \\
U \arrow[r] & SF \arrow[r] & SF/U \arrow[r] & BU
\end{tikzcd}$$

Then j_A is the induced map between fibres and $J = j_A \circ \Delta$. It is then proved in [22] that $h_A \circ j_A$ is a bijection. Observe that j_A induces a map $A^0(X) \to \pi_s^0(X)_{(p)}$ only for X a space, not for a spectrum. So there is no corresponding map in A-homology.

Using S-duality we can still draw the following conclusion from the existence of j_A. For a CW-complex X with finite k-skeleton $X^{(k)}$ let $D(X^{(k)})$ be an S-dual of $X^{(k)}$.

PROPOSITION 4.1. *Suppose $x \in \mathrm{Ad}_n(X)$ is in* $\mathrm{im}(\mathrm{Ad}_n(X^{(k)}) \to \mathrm{Ad}_n(X))$ *and there exists a space Y such that $\Sigma^n D(X^{(k)})_{(p)}$ and $Y_{(p)}$ are homotopy equivalent. Then x is in* $\mathrm{im}(h_A : \pi_n^s(X)_{(p)} \to \mathrm{Ad}_n(X))$.

Proof. Consider the following commutative diagram

$$\begin{tikzcd}
\pi_n^s(X)_{(p)} \arrow[d] & \pi_n^s(X^{(k)})_{(p)} \arrow[l, "i_*"'] \arrow[d] \arrow[r, phantom, "\cong"] & \pi_s^0(\Sigma^n D(X^{(k)}))_{(p)} \arrow[d, "h_A"] \\
\mathrm{Ad}_n(X) & \mathrm{Ad}_n(X^{(k)}) \arrow[l, "i_*"'] \arrow[r, phantom, "\cong"] & \mathrm{Ad}^0(\Sigma^n D(X^{(k)}))
\end{tikzcd}$$

where the isomorphisms are defined by S-duality. Since $\Sigma^n D(X^{(k)})_{(p)} \simeq Y_{(p)}$ is a space and $\mathrm{Ad}^0(Y) = A^0(Y)$, the existence of j_A implies that $h_A : \pi_n^s(X^{(k)})_{(p)} \to \mathrm{Ad}_n(X^{(k)})$, is onto. Hence the result.

Theorem 1 of [6] states that a $2n-2$ connected spectrum of finite type (localized at $p \neq 2$) with dimension less than $2np - 1$ can be realized by a space. If we apply this to desuspend the stable dual $D(BG^{(2m)})$ we obtain

COROLLARY 4.2. *If* $x \in \mathrm{Ad}_{2n-1}(BG)$ *is in* $\mathrm{im}(\mathrm{Ad}_{2n-1}(BG^{(2m)}) \to \mathrm{Ad}_{2n-1}(BG))$ *with* $pm \leq n(p-1)$ *then* $x \in \mathrm{im}(h_A)$.

This is the lower bound for $\mathrm{im}(h_A)$ which we shall use. Examples show that usually this type of lower bound gives only a proper subgroup of $\mathrm{im}(h_A)$. This is also true in the case of BG in a certain dimension range. But as we shall see now, for n sufficiently large, the skeletal filtration of the elements in $\mathrm{Ad}_{2n-1}(BG)$ becomes small enough so that (4.2) applies. We first prove the main result in the special case of a cyclic group $G = \mathbb{Z}/p^a$. Let $c_0(a) \in \mathbb{N}$ be minimal with $p^c \geq p^a(2a + c)$ for $c \geq c_0(a)$.

THEOREM 4.3. *Let p be an odd prime, C be the cyclic group of order p^a and $n_0 = p^a(2a + c_0(a))$. Then, for $n \geq n_0$,*

$$h_A : \pi_{2n-1}^s(BC) \to \mathrm{Ad}_{2n-1}(BC) \text{ is onto.}$$

In particular, for $n \geq n_0$, $\mathrm{im}(h_K : \pi_{2n-1}^s(BC) \to K_1(BC))$ *is the subgroup*

$$\mathrm{Ad}_{2n-1}(BC) = \ker(\psi_{2n-1}^l - 1).$$

Proof. Let x be an element in $\mathrm{Ad}_{2n-1}(BC)$. Then $\nu_p(|D(x)|) \leq a + \nu_p(n)$ by (3.3) and this implies $D(x) = i_*(x_1)$ for some $x_1 \in K_1(BC^{(2m)})$ with $m \leq (p-1)p^{a-1}(2a + \nu_p(n))$. Since $i_* : K_1(BC^{(2m)}) \to K_1(BC)$ is injective we have $(\psi_{2n-1}^l - 1)(x_1) = 0$, hence there exists $x_2 \in \mathrm{Ad}_{2n-1}(BC^{(2m)})$ with $x_1 = D(x_2)$. If $pm \leq n(p-1)$ we have $x \in \mathrm{im}(h_A)$ by (4.2). This is satisfied if $p^a(2a + \nu_p(n)) \leq n$. Write $n = p^c \cdot n'$, $(n', p) = 1$. If $c \geq c_0(a)$ then $n = p^c n' \geq p^c \geq p^a(2a + c) = p^a(2a + \nu_p(n))$ by the definition of $c_0(a)$. If $c < c_0(a)$ then $n \geq n_0 = p^a(2a + c_0(a)) \geq p^a(2a + \nu_p(n))$ by the definition of n_0. Hence $n \geq n_0$ implies $p^a(2a + \nu_p(n)) \leq n$ finishing the proof.

Remarks. (a) Combining (2.14) with (5.4) gives a slight improvement for n_0: $c_0'(a) = \min\{c \mid p^c \geq p^a(a + c)\}$ and $n_0' = p^a(a + c_0'(a))$. (b) The exact skeletal filtration of the elements of order p in $\mathrm{Ad}_{2n-1}(BC)$ is known, see [13].

We now prove

THEOREM 4.4. *Let G be a finite group and p an odd prime. Then there exists $n_0(G) \in \mathbb{N}$ such that for $n \geq n_0(G)$ the image of the Hurewicz map*

$$h_K : \pi^s_{2n-1}(BG)_{(p)} \to K_1(BG)_{(p)}$$

is the subgroup $\mathrm{Ad}_{2n-1}(BG) = \ker(\psi^l_{2n-1} - 1)$.

Proof. Let $y \in \mathrm{Ad}_{2n}(BG; \mathbb{Q}/\mathbb{Z})$ be given. Then $x = D(y) \in \ker(\psi^l_{2n} - 1) \subset K_{2n}(BG; \mathbb{Q}/\mathbb{Z}_{(p)})$ satisfies $v_p(|x|) \leq c + v_p(n)$ for some $c \in \mathbb{N}$ by (3.3). Consider

$$x_A := \alpha(\theta_A) \cap t_A\left(\frac{x}{|G|}\right) \in K_{2n}(BA^+; \mathbb{Q}/\mathbb{Z}_{(p)})$$

for a cyclic subgroup $A = \mathbb{Z}/p^a$ of G. Then $v_p(|x_A|) \leq c' + v_p(n)$ and we have $x_A = i_*(x'_A)$ for some $x'_A \in K_{2n}(BA^{(2m)+}; \mathbb{Q}/\mathbb{Z}_{(p)})$ with $m = m(a) \leq (p-1)p^{a-1} \cdot (a + c' + v_p(n))$ by (2.11). Now x'_A may not be in $\ker(\psi^l_{2n} - 1)$. But $|G| \cdot x_A$ is, since $\psi^l_{2n}(\alpha(\theta_A) \cap t_A(x)) = \psi^l_0(\alpha(\theta_A)) \cap \psi^l_{2n}(t_A(x))) = \alpha(\theta_A) \cap t_A(x)$. Let $\bar{m} = \max\{m(a) \mid \mathbb{Z}/p^a < G\}$. Then $x' := \sum_A \mathrm{B}i_{A*}(x'_A) \in K_0(BG^{(2\bar{m})}; \mathbb{Q}/\mathbb{Z}_{(p)})$ maps to x under the inclusion $i : BG^{(2\bar{m})} \to BG$.

Since $i_* : K_0(BG^{(2\bar{m})}; \mathbb{Q}/\mathbb{Z}_{(p)}) \to K_0(BG; \mathbb{Q}/\mathbb{Z})$ is in general not injective we only know $(\psi^l_{2n} - 1)(x') \in \ker(i_*)$. But we know $(\psi^l_{2n} - 1)(x')$ is at most of order $|G|$ since $(\psi^l_{2n} - 1)(|G| \cdot x_A) = 0$ as observed above. The elements $z \in K_0(BG; \mathbb{Q}/\mathbb{Z}_{(p)})$ with $|G| \cdot z = 0$ are all in $\mathrm{im}(K_0(BG^{(m_1)}; \mathbb{Q}/\mathbb{Z}_{(p)}) \to K_0(BG; \mathbb{Q}/\mathbb{Z}_{(p)}))$ by (2.12), m_1 depending only on $|G|$. By (2.15) we can find $r = r(m_1, G)$ such that $i_* : K_0(BG^{(m_1+r)}; \mathbb{Q}/\mathbb{Z}_{(p)}) \to K_0(BG; \mathbb{Q}/\mathbb{Z}_{(p)})$ is injective on $\mathrm{im}(K_0(BG^{(m_1)}; \mathbb{Q}/\mathbb{Z}_{(p)}) \to K_0(BG^{(m_1+r)}; \mathbb{Q}/\mathbb{Z}_{(p)})$. We conclude that, letting $m_2 := \max\{2\bar{m}, m_1 + r\}$, the element $i_*(x') \in K_0(BG^{(m_2)}; \mathbb{Q}/\mathbb{Z}_{(p)})$ is in $\ker(\psi^l_{2n} - 1)$. Hence $y \in \mathrm{im}(\mathrm{Ad}_{2n}(BG^{(m_2)}; \mathbb{Q}/\mathbb{Z}) \to \mathrm{Ad}_{2n}(BG; \mathbb{Q}/\mathbb{Z}))$.

Now m_2 depends linearly on $v_p(n)$, so we may argue as in the proof of (4.3) that there exists $n_0(G)$ such that for $n \geq n_0(G)$ the element y is in $\mathrm{im}(h_A)$ by (4.2), proving the theorem.

Remark 4.5. (a) The only constant going into $n_0(G)$ in (4.4) which is not known in general is the constant $r(m_1, G)$ coming in by using (2.15).

(b) For $G = (\mathbb{Z}/p)^k$, an elementary abelian group, we know $r_0(G) \leq 2p$ by [2].

(c) for $G = (\mathbb{Z}/p^b)^k$, $k \geq 2$, the length of the non-trivial differentials in the Atiyah–Hirzebruch spectral sequence for $K^0(BG)$ grows with b. It can be made arbitrarily large by increasing b.

(d) The case $(\mathbb{Z}/p^b)^2$ is special, since there are no non-trivial differentials starting in $E_2^{p,-p}$ with $p=0(2)$, hence $r(m_1, G)=1$ in this case.

(e) Closely related to this is the case of a p-group with p-rank at most 2 (the p-rank of a p-group is the rank of a maximal elementary abelian subgroup). If $p \geq 5$, it is known [21] that $H^{\text{even}}(BG;\mathbb{Z})$ is generated by Chern classes of representations, implying that $H^{\text{even}}(BG;\mathbb{Z})$ consists of permanent cycles. So we have $r(m_1, G)=1$ for those groups too. The groups of example (3.10c) belong to this class of groups (even for $p=3$).

(f) By using the fact that the Atiyah–Hirzebruch spectral sequence for $K_*(BG)$ collapses after finitely many steps [23], one can easily prove that there exists a constant $n_1(G) \geq n_0(G)$ such that for $n \geq n_1(G)$ $h_A: \pi^s_{2n-1}(BG) \to \mathrm{Ad}_{2n-1}(BG)$ is a split surjection.

For G abelian we can avoid the use of (2.15) and thus obtain an explicit estimate for $n_0(G)$ as follows.

We first prepare a lemma true for arbitrary finite G. Recall the situation in (3.12). Let $G_p \xrightarrow{i} G$ be a p-Sylow subgroup of G, $R_p = \mathrm{im}\,(\mathrm{ind}: R(G_p) \to R(G))$, $E=$ exponent of G, $e=v_p(E)$ and $s(e)=(p-1)p^{e-1}$, then $\psi^{l^{s(e)}}=1$ in $R(G_p)$ and therefore in $K_1(BG)$ (2.8). By (3.12) every element in $\mathrm{Ad}_{2n}(BG;\mathbb{Q}/\mathbb{Z})$ is given by $\tau(x)=\sum_{i=0}^{s(e)-1} \psi^{l^i}_{2n}(x)/(l^{ns(e)}-1)$ for some $x \in R_p$. Denote by $i: Y(m) \to BG$ the inclusion of a subcomplex contained in $BG^{(2m)}$.

LEMMA 4.6. *With the notations above, let $x' \in K_0(Y(m);\mathbb{Q}/\mathbb{Z}_{(p)})$ be an element with $i_*(x')=x/(l^{n\cdot s(e)}-1)$ for $x \in R_p$. Assume that the following conditions are satisfied*:

(a) $mp \leq n(p-1)$

(b) $\psi_0^{l^{s(e)}}=1$ *on* $K_0(Y(2m);\mathbb{Q}/\mathbb{Z}_{(p)})$

(c) $(l^{n\cdot s(e)}-1)x'=0$

Then $\tau(x) \in \mathrm{im}\,(h_A)$.

Proof. We show that $x'' := \sum_{i=0}^{s(e)-1} \psi^{l^i}_{2n}(x')$ is in $\ker(\psi^l_{2n}-1)$. We have $(\psi^l_{2n}-1)(x'')=x'-\psi^{l^{s(e)}}_{2n}(x')=x'(1-l^{-ns(e)})=0$ by (b) and (c). Then $\tau(x)=i_*(x'')$ is in $\mathrm{im}\,(h_A)$ by (a) and (4.2).

Next we show how to satisfy (b) and (c) for G abelian. Let G be an abelian p-group.

LEMMA 4.7. *Let $G=\prod_{i=1}^c \mathbb{Z}/p^{a_i}$ be an abelian p-group, $p \neq 2$, and set $e=\max\{a_i\}$, $s(e)=(p-1)p^{e-1}$, $X_i = B\mathbb{Z}/p^{a_i}$, $Y=\bigwedge_{i=1}^c X_i$ and $Y(m):= \bigwedge_{i=1}^c X_i^{(2m)}$ with inclusion $i_Y: Y(m) \to Y$. Then*

(a) $\ker(i_{Y*}) \subset K_1(Y(m))$ *is a direct summand and*

(b) $\psi_0^{l^{s(e)}}=1$ *on* $K_0(Y(m);\mathbb{Q}/\mathbb{Z})$

Proof. (a) we use the Künneth sequence [3] to compute the K-theory of $Y(m)$. From [3] it follows that there exists a natural short exact sequence

$$0 \to K_*(X) \otimes K_*(Y) \xrightarrow{\mu} K_*(X \wedge Y) \xrightarrow{\gamma} K_*(X) * K_*(Y) \to 0$$

with γ of degree -1. Moreover, this sequence splits (unnaturally), e.g. see [16]. We now use induction on c. Consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & K_1(X^{(2m)}) \otimes K_0(Y(m)) \arrow[r, "\mu"] & K_1(X^{(2m)} \wedge Y(m)) \arrow[r, "\gamma"] \arrow[d, "i_{X \wedge Y^*}"] & K_1(X^{(2m)}) * K_1(Y(m)) \arrow[r] \arrow[d, "(i_{X^*}) * (i_{Y^*})"] & 0 \\
& 0 \arrow[r] & K_1(X \wedge Y) \arrow[r, "\gamma"] & K_1(X) * K_1(Y(m)) &
\end{tikzcd}$$

with $X = B\mathbb{Z}/p^a$ and we have used $K_0(X^{(2m)}) = 0$. The case $c = 1, 2$ are trivially true since i_{Y^*} is injective, starting the induction. By induction assumption the kernel of i_{Y^*} splits off, hence the kernel of $(i_{X^*}) * (i_{Y^*})$. The splitting of the Künneth sequence then proves the induction step.

To prove (b) we prove the dual statement and use the following fact, proved in [16] p. 267

$K^*(X \wedge Y)$ is generated by elements of the form $x_0 \wedge y_0$, $\beta_k(x_k \wedge y_k)$ for $k \geq 1$, where $x_0 \in K^*(X)$, $y_0 \in K^*(Y)$, $x_k \in K^*(X; \mathbb{Z}/k)$, $y_k \in K^*(Y; \mathbb{Z}/k)$, $\beta_k : K^*(X \wedge Y; \mathbb{Z}/k) \to K^*(X \wedge Y)$ is the Bockstein map and $\wedge$, $\wedge_k$ denote the product maps for $K^*(\ )$ and $K^*(-; \mathbb{Z}/k)$. (4.8)

We again use induction on c and the Künneth sequence. The exact sequence

$$0 \to K^*(X^{(2m)}; \mathbb{Z}/p^k) \to K^*(X^{(2m)}) \xrightarrow{p^k} K^*(X^{(2m)}) \to K^*(X^{(2m)}; \mathbb{Z}/p^k) \to 0$$

shows $\psi^{l^{s(e)}} = 1$ to be true on $K^*(X^{(2m)}; \mathbb{Z}/p^k)$ since it is true on $K^*(X)$ and $i_X^* : K^*(X) \to K^*(X^{(2m)})$ is onto. Let M_r denote the Moore spectrum for $\mathbb{Z}/p^r$. Then $K^i(X; \mathbb{Z}/p^r) = K^{i+1}(X \wedge M_r)$ and $K^i(X \wedge M_k; \mathbb{Z}/p^r) = K^{i+1}(X \wedge M_k \wedge M_r)$. Since $p \neq 2$ we have a splitting $M_k \wedge M_r \simeq M_t \vee \Sigma M_t$ where $t = \min(k, r)$. This shows $\psi^{l^{s(e)}} = 1$ on $K^*(X \wedge M_p; \mathbb{Z}/p^r)$. Inductively we get the statement for

$K^*(Y(m))$ and $K^*(Y(m); \mathbb{Z}/p^r)$ using (4.8) and the facts that β_k and $\wedge_k$ commute with ψ^l.

It is now rather clear how (4.6) and (4.7) give an estimate for $n_0(G)$, G abelian.

COROLLARY 4.9. $G = \prod_{i=1}^{c} \mathbb{Z}/p^{a_i}$, $e = \max\{a_i\}$, $b = \nu_p(|G|)$, $d_0 = \min\{d \mid p^d \geq p^e \cdot c(e+b+1+d)\}$, $n_1(G) := p^e \cdot c(e+b+1+d_0)$. *Then, for* $n \geq n_1(G)$, $h_A : \pi_{2n-1}^s(BG) \to \mathrm{Ad}_{2n-1}(BG)$ *is onto.*

Proof. Let $x \in R(G)$ be given. Then $x/p^{e+\nu_p(n)}$ comes from the $2m$-skeleton with $m \leq (p-1)p^{e-1}(e+\nu_p(n)+b+1)$ by (2.12). We have

$$BG^{(2m)} \subset \bigwedge_{i=1}^{c} (B\mathbb{Z}/p^{a_i+})^{(2m)} = Y(m) \subset BG^{(2mc)}.$$

By (4.6) $\tau(x) \in \mathrm{im}(h_A)$ provided $m \cdot cp \leq (p-1)n$. As in the proof for (4.3) we find that this is the case for $n \geq n_1$ with n_1 as above.

Remark. (4.8) gives only a rough estimate for $n_0(G)$. It can be improved using splittings of BG and the better estimates (2.13).

EXAMPLES 4.10. (a) As first example we treat $G = \mathbb{Z}/p \times \mathbb{Z}/p$, p an odd prime. Recall the description of $\mathrm{Ad}_*(BG)$ from (3.10b). For $G = \mathbb{Z}/p \times \mathbb{Z}/p$, $n \neq 0$, $\mathrm{Ad}_{2n-1}(BG) = \bigoplus_{i=0}^{p} \mathbb{Z}/p^{1+\nu_p(n)}$ is generated by

$$x_n(v_i) = \left(\sum_{j=0}^{p-1} j^{-n} \lambda^j \otimes \lambda^{j \cdot i}\right) \Big/ p^{1+\nu_p(n)}, \qquad i = 0, \ldots, p-1,$$

and

$$x_n(v_p) = \left(\sum_{j=0}^{p-1} j^{-n} 1 \otimes \lambda^j\right) \Big/ p^{1+\nu_p(n)}.$$

We need to know the skeletal filtration on $\mathrm{Ad}_{2n-1}(BG)$, which we abreviate by SF. Observe first that if we know $\mathrm{SF}(x_n(v_1))$ we know $\mathrm{SF}(x_n(v_i))$ for $i = 1, \ldots, p-1$. The reason is that the map induced by multiplication by i in the second factor maps $x_n(v_1)$ to $x_n(v_i)$. Therefore if $x_n(v_1)$ is in $\mathrm{im}(h_A)$, then h_A is onto. Of course, for a given n, appropriate linear combinations of $x_n(v_i)$'s have a smaller skeletal filtration and may be in $\mathrm{im}(h_A)$ whereas $x_n(v_1)$ may not.

We use (2.10) to estimate SF $(x_n(v_1))$. Let C denote a cyclic subgroup of G and i_c the inclusion. Then

$$x_c := \alpha(\theta_c) \cap t_c(x_n(v_1))/p^2 \in K_0(BC^+; \mathbb{Q}/\mathbb{Z})$$

has order at most $p^{3+v_p(n)}$. In the case $C \cong \mathbb{Z}/p$ the skeletal filtration of an element of order p^t does not exceed $2(p-1)\cdot t$. Since SF $(\theta_c) = 2(p-1)$ (for $C \cong \mathbb{Z}/p$), we find SF $(x_c) \le 2(p-1)(2+v_p(n))$ and thus SF $(x) \le 2(p-1)(2+v_p(n))$.

Also, $K_0(BG^{(2m+1)}; \mathbb{Q}/\mathbb{Z}) \to K_0(BG; \mathbb{Q}/\mathbb{Z})$ is injective on $\operatorname{im}(K_0(BG^{(2m)}; \mathbb{Q}/\mathbb{Z}) \to K_0(BG^{(2m+1)}; \mathbb{Q}/\mathbb{Z})$ or simpler $K_1(BG^{(2m)}) \to K_1(BG)$ is injective in this special case. By (4.2) we know $x \in \operatorname{im}(h_A)$ provided $n \ge p(2+v_p(n))$. This is the case for $n \ge 2p+1$ if $p \ge 5 (n \ge 3p+1$ if $p=3)$. On the other hand, since $h_A: \pi_k^s(S^0)_{(p)} \to A_k(S^0)$ in bijective for $k < 2(p^2-p-1)$, we have $h_A: \pi_{2n-1}^s(BG) \to A_{2n-1}(BG)$ is onto for $n \le p^2 - p$. For $p \ge 5$ all dimensions are covered; for $p=3$ one needs an extra check by hand for $n = 3p$, which is not covered by (4.2). We have proved:

PROPOSITION 4.11. *Let p be an odd prime and $G = \mathbb{Z}/p \times \mathbb{Z}/p$. Then for $n > 2p$* $\operatorname{im}(h_K: \pi_{2n-1}^s(BG) \to K_1(BG))$ *is isomorphic to* $\operatorname{Ad}_{2n-1}(BG) = \ker(\psi^l - 1)$. *For $n \le 2p$* $\operatorname{im}(h_K)$ *is isomorphic to $d(A_{2n-1}(BG))$.*

(b) Since the range where $h_A: \pi_n^s(X)_{(p)} \to A_n(X)$ is bijective depends on p, the argument above easily extends to $G = (\mathbb{Z}/p)^b$, $b \le p-1$. Only the case $n = p^2$, $b = p-1$, is not covered by (3.2) and has to be handled by a different method.

PROPOSITION 4.12. *Let $p > 3$ be a prime and $G = (\mathbb{Z}/p)^b$, $b \le (p-1)$. Then for $n > p^2 - 2p$* $\operatorname{im}(h_K: \pi_{2n-1}^s(BG) \to K_1(BG))$ *is isomorphic to* $\operatorname{Ad}_{2n-1}(BG)$ *and for $n \le p^2 - 2p$ to $d(A_{2n-1}(BG))$.*

(c) Note that the evaluation of $(H-1)^{p-1} \otimes (H-1)^{p-1}$ on $x_n(v_1)$ (for $v_p(n) = 0$, $n \equiv 0(p-1)$) shows SF $(x_n(v_1)) \ge 2(p-1)\cdot 2$ (in $\operatorname{Ad}_{2n}(B(\mathbb{Z}/p \times \mathbb{Z}/p); \mathbb{Q}/\mathbb{Z})$, if $n = 2(p-1)$ the skeletal filtration of $\beta x_n(v_1) \in \operatorname{Ad}_{2n-1}(B(\mathbb{Z}/p \times \mathbb{Z}/p))$ may be smaller). This shows that $d(A_{2n-1}(BG))$ and $\operatorname{Ad}_{2n-1}(BG)$ actually may differ in low dimensions.

EXAMPLE. $p = 3$, $G = \mathbb{Z}/3 \times \mathbb{Z}/3$, $\operatorname{coker}(d: A_{2n-1}(BG) \to \operatorname{Ad}_{2n-1}(BG)) \cong \mathbb{Z}/3 \oplus \mathbb{Z}/3$ for $n = 1, 3$ and $\mathbb{Z}/3$ for $n = 2$.

The skeletal filtration of the element

$$x_n(V_1^{(b)}) = \left(\sum_{j=1}^{p-1} j^{-n}\lambda_1^j \otimes \lambda_2^j \otimes \cdots \otimes \lambda_b^j\right) \Big/ p^{1+\nu_p(n)}$$

in $\mathrm{Ad}_{2n-1}(B(\mathbb{Z}/p)^b)$ increases with b; for example

$$\mathrm{SF}(x_n(V_1^{(b)})) = 2 \cdot b(p-1) \quad (n \equiv 0(p-1),\ \nu_p(n) = 0,\ n \neq b(p-1)).$$

So to cover the dimension range left open by (4.4) for $b \geq p$ requires more sophisticated arguments than the simple one used above.

(d) To treat a non-abelian example, we consider the extra special p-group

$$\begin{aligned} Q = (\mathbb{Z}/p \times \mathbb{Z}/p) \rtimes \mathbb{Z}/p = \{A, B, C \mid A^p = B^p = C^p \\ = [A, C] = [B, C] = 1, [B, A] = C\}. \end{aligned}$$

Since Q has $H^{\mathrm{ev}}(BQ; \mathbb{Z})$ generated by Chern classes there is no problem with (2.15), see (4.5). We may proceed exactly as in the case $G = \mathbb{Z}/p \times \mathbb{Z}/p$. The skeletal filtration in $\mathrm{Ad}_{2n-1}(BQ) = \bigoplus_{i=1}^{p+2} \mathbb{Z}/p^{1+\nu_p(n)}$ is bounded by $2(p-1)(3 + \nu_p(n))$. Therefore $\mathrm{Ad}_{2n-1}(BQ) = \mathrm{im}(h_A)$ if $n > 3p$ by (4.2). If $n \leq 3p$ we use the fact that $h_A : \pi_{2n-1}^s(BQ) \to A_{2n-1}(BQ)$ is onto for $n \leq (p-1)p$. Hence if $p \geq 5$ all dimensions are covered and we have

PROPOSITION 4.13. *Let $p \geq 5$ be a prime and $Q = (\mathbb{Z}/p \times \mathbb{Z}/p) \rtimes \mathbb{Z}/p$. Then*

$$\mathrm{im}(h_K : \pi_{2n-1}^s(BQ) \to K_1(BQ)) \cong \begin{cases} \mathrm{Ad}_{2n-1}(BQ) & n > 3p \\ d(A_{2n-1}(BQ)) & n \leqslant 3p. \end{cases}$$

Remark 4.14. Theorem 4.4 is also true at $p = 2$. Only the technical details are more involved. The Hurewicz map h_K for complex K-theory factors through the KO-Hurewicz map $h_{\mathrm{KO}} : \pi_{2n-1}^s(BG)_{(2)} \to \mathrm{KO}_{2n-1}(BG)_{(2)}$. Since h_{KO} determines h_K we consider h_{KO}. Then the 2-primary version of (4.4) is

THEOREM 4.15. *Let G be a finite group. Then there exists $n_0(G)$ such that for $n \geq n_0(G)$ the image of the Hurewicz map*

$$h_{\mathrm{KO}} : \pi_{2n-1}^s(BG)_{(2)} \to \mathrm{KO}_{2n-1}(BG)_{(2)}$$

is the subgroup $\ker(\psi^3 - 1)$.

We indicate only the main changes in the proof: Denote by Ad_* the fibre theory belonging to $\psi^3-1:\mathrm{KO}_*(\ )_{(2)}\to\mathrm{KO}_*(\ )_{(2)}$ and by A_* the homology theory fitting into the exact sequence

$$\to A_i(X)\to \mathrm{ko}_i(X)_{(2)}\xrightarrow{\psi^3-1}\mathrm{kspin}_i(X)_{(2)}\to A_{i-1}(X)\to$$

where ko_* is (-1)-connected KO-theory and kspin_* its 2-connected cover. There is a J-map $j_A:A^0(X)\to\pi_s^0(X)_{(2)}$ such that the composition of j_A with the Hurewicz map $\pi_s^0(X)_{(2)}\to A^0(X)$ is a bijection. If X is simply connected, then $A^0(X)=\mathrm{Ad}^0(X)$ (e.g. see [4]). Hence we have a statement corresponding to (4.2) for $p=2$: If the skeletal filtration of $x\in\mathrm{Ad}_{2n-1}(X)$ is small enough it will be in $\mathrm{im}(h_A)$.

Next one has to compute $\mathrm{KO}_*(BG)_{(2)}$. This may be done as in the complex case. One finds that $\mathrm{KO}_*(BG)_{(2)}$ consists of finitely many summands $\mathbb{Q}/\mathbb{Z}_{(2)}$ and $\mathbb{Z}/2\mathbb{Z}$ depending on the congruence class of n mod 4.

Elements of order 2 in $\ker(\psi^3-1)$ not divisible by 2 are most easily handled using Adams periodicity operators, e.g. see [4]. If $x\in\ker(\psi^3-1)$ is divisible by 2 then $y:=c(x/2)\in\mathrm{KU}_{2n-1}(BG)$ is an element with $r(y)=x$ where $c:\mathrm{KO}\to\mathrm{KU}$ and $r:\mathrm{KU}\to\mathrm{KO}$ are the canonical maps. Since $2\cdot y$ is in $\ker(\psi^3-1)$, the exponent of 2 in the order of y is of the form $a+v_2(n)$. Arguing as for $p\neq 2$ we find that for n large enough y comes from $\mathrm{KU}_{2n-1}(BG^{(2m)})$ and $(\psi^3-1)(y)$ from $\mathrm{KU}_{2n-1}(BG^{(2d)})$ with m sufficiently small and d depending only on G. Then the same is true for $x=r(y)$. Using the statement analogous to (2.15) the proof is finished as in the odd primary case.

§5. Im (h_K) for cyclic groups in low dimensions

Let C be the cyclic group of order p^c, $p\neq 2$. For the image of $h_K:\pi^s_{2n-1}(BC)\to K_1(BC)$ there are two ranges for the values of n and c where we can find some sort of stability. The case of c fixed and n large is the one discussed above. The other extreme case is where c is large and n is fixed. In this case $B\mathbb{Z}/p^c$ approximates the complex projective space $P_\infty\mathbb{C}$. For the rest of this section we fix the notation $\xi=H^{p^c}$ where H is the universal line bundle over $P_\infty\mathbb{C}$. Then the sphere bundle $S(\xi)$ is a model for $B\mathbb{Z}/p^c$ and the cofibre sequence

$$\to S(\xi)^+\xrightarrow{\pi}P_\infty\mathbb{C}^+\xrightarrow{j}P_\infty\mathbb{C}^\xi\xrightarrow{\delta}\Sigma S(\xi)^+\to \tag{5.1}$$

relates $B\mathbb{Z}/p^c$ to $P_\infty\mathbb{C}$.

We recall the following facts about $A_*(P_\infty\mathbb{C})$ from [13]: $A_{2n}(P_\infty\mathbb{C}) = \mathrm{Ad}_{2n}(P_\infty\mathbb{C}) = \mathbb{Z}_{(p)}$ is generated by $b_1^n \in K_0(P_\infty\mathbb{C})_{(p)}$ where $\{b_j \mid j \geq 1\}$ is the usual basis of $K_0(P_\infty\mathbb{C})$ dual to the powers of $(H-1)$ and b_1^n is the n-th power of b_1 with respect to the Pontrajagin product. The index of the canonical map $T: A_{2n}(P_\infty\mathbb{C}) \to H_{2n}(P_\infty\mathbb{C})_{(p)}$ is the p-part of $n!$. Write $n = t(p-1)+s$ with $0 < s \leq (p-1)$ and let

$$m = \sum_{i=1}^{t} (1 + v_p(i)) - v_p(n!), \qquad r = \left[\log\left(\frac{n+1}{s+1}\right) \Big/ \log(p)\right].$$

Then $A_{2n-1}(P_\infty\mathbb{C})$ is a group of order p^m with r cyclic summands. The kernel of the Bockstein map $\beta: A_{2n}(P_\infty\mathbb{C}; \mathbb{Q}/\mathbb{Z}) \to A_{2n-1}(P_\infty\mathbb{C})$ is given by the multiples of b_1^n.

The long exact sequence in A-theory induced by (5.1) reduces to a six term sequence

$$\begin{aligned} &0 \to A_{2n}(P_\infty\mathbb{C}^+) \xrightarrow{j_*} A_{2n}(P_\infty\mathbb{C}^\xi) \xrightarrow{\delta} A_{2n-1}(B\mathbb{C}^+) \xrightarrow{\pi_*} \\ &A_{2n-1}(P_\infty\mathbb{C}^+) \xrightarrow{j_*} A_{2n-1}(P_\infty\mathbb{C}^\xi) \xrightarrow{\delta} A_{2n-2}(B\mathbb{C}^+) \to 0. \end{aligned} \tag{5.2}$$

The description of $\mathrm{Ad}_*(BC)$ was given in (3.10) and the relation between $\mathrm{Ad}_*(BC)$ and $A_*(BC)$ is as follows. First of all, the exact sequence of $(BC^{(2n)}, BC^{(2n-1)})$ shows

$$A_{2n-1}(BC) = \mathrm{Ad}_{2n-1}(BC^{(2n)}).$$

Since BC, $BC/BC^{(2m)}$ and $BC^{(2m)}$ have only non-zero homology in odd degrees, we find $K_0(BC) = 0$, $K_0(BC/BC^{(2m)}) = 0$ and $K_0(BC^{(2m)}) = 0$. This implies that the maps $i_*: K_1(BC^{(2m)}) \to K_1(BC)$, $D: \mathrm{Ad}_{2n-1}(BC) \to K_1(BC)$ and $D: \mathrm{Ad}_{2n-1}(BC^{(2m)}) \to K_1(BC^{(2m)})$ are all injective. Therefore $i_*: \mathrm{Ad}_{2n-1}(BC^{(2n)}) \to \mathrm{Ad}_{2n-1}(BC)$ and thus $d: A_{2n-1}(BC) \to \mathrm{Ad}_{2n-1}(BC)$ is injective. We shall describe elements in $A_{2n-1}(BC)$ by their images in $\mathrm{Ad}_{2n-1}(BC)$.

The first step in an investigation of (5.2) is the determination of $\ker(\pi_*)$.

LEMMA 5.3. $\ker(\pi_*) \cong \mathbb{Z}/p^{c+v_p(n)} \subset A_{2n-1}(BC)$ *is generated by*

$$z = x_n(v_0) + p^{n-1}x_n(v_1) + p^{2n-2}x_n(v_2) + \cdots$$

Proof. The commutative diagram

$$\begin{tikzcd}
& A_{2n-1}(BC) \arrow[r, "\pi_*"] & A_{2n-1}(P_\infty\mathbb{C}) \\
& A_{2n}(BC;\mathbb{Q}/\mathbb{Z}) \arrow[u, "\beta", "\cong"'] \arrow[r, "\pi_*"] \arrow[d, "D"] & A_{2n}(P_\infty\mathbb{C};\mathbb{Q}/\mathbb{Z}) \arrow[u, "\beta"] \arrow[d, "D"] \\
0 \arrow[r] & K_{2n}(BC;\mathbb{Q}/\mathbb{Z}_{(p)}) \arrow[r, "\pi_*"] & K_{2n}(P_\infty\mathbb{C};\mathbb{Q}/\mathbb{Z}_{(p)})
\end{tikzcd}$$

shows that $\ker(\pi_*:A_{2n-1}(BC)\to A_{2n-1}(P_\infty\mathbb{C}))$ in contained in $\ker(\beta)=\mathbb{Q}/\mathbb{Z}_{(p)}\cdot b_1^n$. The equation

$$D\left(\pi_*\left(\sum_{i=0}^{c-1} x_n(v_i)\cdot p^{i(n-1)}\right)\right)=\frac{1}{p^{c+\nu_p(n)}}\cdot b_1^n$$

is proved by evaluating $H^k-1\in K^0(P_\infty\mathbb{C})$, $k\geq 1$, on both sides.

To see that $\ker(\pi_*)$ is always in $\operatorname{im}(h_K)$ we use the transfer map $t:A_{2n-2}(P_\infty\mathbb{C}^+)\to A_{2n-1}(BC^+)$. Denote by $\sigma\in\pi_2^s(P_\infty\mathbb{C}^+)$ a generator with $h_K(\sigma)=b_1$. Then σ^{n-1} generates $\pi_{2n-2}^s(P_\infty\mathbb{C}^+)/\mathrm{tor}$ and $h_A(\sigma^{n-1})$ generates $A_{2n-2}(P_\infty\mathbb{C}^+)$.

LEMMA 5.4.

$$h_A(t(\sigma^{n-1}))=z=\sum_{i\geq 0} x_n(v_i)\cdot p^{i(n-1)} \quad \textit{in} \quad A_{2n-1}(BC).$$

Proof. We prove here only $h_A t(\sigma^{n-1})=\lambda\cdot z$ with $\lambda\not\equiv 0(p)$.

Since $\pi_*\circ t=0$ we have $\operatorname{im}(t)\subset\ker(\pi_*)$, so it is enough to compute the order of $\operatorname{im}(t)$. For this we use that the transfer map t appears as boundary map in the cofibre sequence

$$\to S(\xi)^{-\xi}\to P_m\mathbb{C}^{-\xi}\xrightarrow{j} P_m\mathbb{C}^+\xrightarrow{t}\Sigma S(\xi)^{-\xi}\to.$$

Hence the order of $\operatorname{im}(t)$ is given by the degree of the map

$$j_*:A_{2n-2}(P_m\mathbb{C}^{-\xi})\to A_{2n-2}(P_m\mathbb{C}^+),\quad m>n.$$

Claim. The index of $T:A_{2n-2}(P_m\mathbb{C}^{-\xi})\to H_{2n-2}(P_m\mathbb{C}^{-\xi};\mathbb{Z}_{(p)})$ is the p-part of

$n!$ (independently of c, except for $p=2$, $c=1$). Then by comparing the degree of j_* in A-theory with the degree of j_* in homology gives: $\deg(j_*)=n\cdot p^c$ in A-theory. This shows $\operatorname{im}(t)=\ker(\pi_*)$ in $A_{2n-1}(BC)$ since both subgroups have the same order.

To prove the claim let $U_K(E)$ be the standard Thom class of a complex vector bundle E in K-theory with the conventions $\operatorname{ch}(U_K(E))=\operatorname{Todd}(E)\cup U_H(E)$ and $\operatorname{Todd}(L)=(\exp(c_1(L))-1)/c_1(L)$ for a line bundle L. Let $\operatorname{Bh}(E)=\operatorname{ch}^{-1}(\operatorname{Todd}(E))\in K^0(X^+;\mathbb{Q})$ and $\phi_K:K_*(X^+)\to K_*(X^E)$ be the Thom isomorphism associated with $U_K(E)$. It is an easy exercise to show that $w=\phi_K^{-1}(b_1^n\cap\operatorname{Bh}(-\xi))\in K_{2n-2}(P_n\mathbb{C}^{-\xi};\mathbb{Q})$ is a p-integral class which is not divisible by any positive power of p. Since the Chern character of this class has only components in $H_{2n-2}(P_n\mathbb{C}^{-\xi};\mathbb{Q})$ it must be in $\ker(\psi^l_{2n-2}-1)$. Now $\ker(\psi^l_{2n-2}-1)=\operatorname{Ad}_{2n-2}(P_n\mathbb{C}^{-\xi})=A_{2n-2}(P_n\mathbb{C}^{-\xi})$ is a direct summand in $K_{2n-2}(P_n\mathbb{C}^{-\xi})_{(p)}$. Hence w is a generator of $\ker(\psi^l_{2n-2}-1)$. It's image in $H_{2n-2}(P_n\mathbb{C}^{-\xi};\mathbb{Z}_{(p)})\cong H_{2n-2}(P_n\mathbb{C};\mathbb{Z}_{(p)})$ is easily seen to be $n!$ times a generator.

Remarks. (i) A more direct computation of $t:A_{2n-2}(P_\infty\mathbb{C}^+)\to A_{2n-1}(BC^+)$ may be given as follows. It is proved in [12] that the composition

$$K_0(P_\infty\mathbb{C})\xrightarrow{t}K_1(BC)\xleftarrow[\cong]{\beta}K_0(BC;\mathbb{Q}/\mathbb{Z})\xrightarrow{\pi_*}K_0(P_\infty\mathbb{C};\mathbb{Q}/\mathbb{Z})$$

is given by $y\to y\cap e(\xi)^{-1}$ where $e(\xi)=H^{p^c}-1$ is the K-theory Euler class of ξ. (For $y\in K_0(P_\infty\mathbb{C})$, the cap product $y\cap e(\xi)^{-1}$ in $K_0(P_\infty\mathbb{C};\mathbb{Q})$ makes sense, since $y=y'\cap(H-1)$ and $e(\xi)/(H-1)$ is invertible in $K^0(P_\infty\mathbb{C};\mathbb{Q})$). The equation $h_A(t(\sigma^{n-1}))=z$ then follows by evaluating H^k-1, $k\geq 1$, on both sides. This resolves the factor λ.

(ii) For $p\neq 2$ (5.3) and (5.4) reprove:

$$\operatorname{im}(h_K:\pi^s_{2n-1}(B\mathbb{Z}/p)\to K_1(B/\mathbb{Z}/p))=\mathbb{Z}/p^{1+\nu_p(n)}\text{ is generated by }h_K(t(\sigma^{n-1})).$$

The sequence (5.2) can now be written as

$$0\to\mathbb{Z}/p^{c+\nu_p(n)}\to A_{2n-1}(BC^+)\xrightarrow{\pi_*}A_{2n-1}(P_\infty\mathbb{C}^+)$$
$$\xrightarrow{j_*}A_{2n-1}(P_\infty\mathbb{C}^\xi)\to A_{2n-2}(BC^+)\to 0.\qquad(5.5)$$

Fix now n. If c is large enough, the map j_* must be the zero map in A_*-theory as well as in stable homotopy. This may be seen as follows. We can write j as a composition of maps $P_\infty\mathbb{C}^{H^{p^i}} \to P_\infty\mathbb{C}^{H^{p^{i+1}}}$ of Adams filtration >0. By increasing c we can increase the Adams filtration of j. If the Adams filtration of j is large enough, j_* is the zero map. Or: for c large the bundle $\xi \to P_n\mathbb{C}$ is stably fibre homotopy trivial at (p). So ξ is orientable for $\pi_s^*(\ )_{(p)}$ and A^* and (5.5) becomes the usual Gysin sequence of the line bundle ξ. Then j_* is given by the cap-product with the Euler class of ξ. If c is large enough j_* must be zero since $e(\xi)$ is divisible by a large power of p. We have proved

PROPOSITION 5.6. *For n fixed c large enough we have*

$$A_{2n-1}(B\mathbb{Z}/p^c) \cong A_{2n-1}(P_\infty\mathbb{C}) \oplus \mathbb{Z}/p^{c+\nu_p(n)}$$

and $\operatorname{cok}(h_A)$ *in* $A_{2n-1}(B\mathbb{Z}/p^c)$ *is isomorphic to* $\operatorname{cok}(h_A)$ *in* $A_{2n-1}(P_\infty\mathbb{C})$.

Notice that this means that $A_{2n-1}(B\mathbb{Z}/p^c)/\operatorname{im}(\delta)$ is independent of c for c large.

Some elements in $\operatorname{cok}(h_A : \pi_{2n-1}^s(P_\infty\mathbb{C}) \to A_{2n-1}(P_\infty\mathbb{C}))$ are determined for example in [13]. Except for very low values of n, h_A is never onto and $\operatorname{cok}(h_A)$ becomes arbitrary large for increasing n, see [13] or [14].

The lowest dimensional example where $h_A : \pi_{2n-1}^s(B\mathbb{Z}/p^c) \to A_{2n-1}(B\mathbb{Z}/p^c)$ is not onto is the following

EXAMPLE 5.7. Let $n = p^2 - 1$, $p \neq 2$, then

$A_{2n-1}(P_\infty\mathbb{C}) \cong \mathbb{Z}/p^2$ generated by $\pi_*(x_n(v_1))$, $\quad \operatorname{im}(h_A) \cong \mathbb{Z}/p$

$\mathrm{Ad}_{2n-1}(B\mathbb{Z}/p^3) = \mathbb{Z}/p^3 \oplus \mathbb{Z}/p^2 \oplus \mathbb{Z}/p$

$A_{2n-1}(B\mathbb{Z}/p^3) = \mathbb{Z}/p^3 \oplus \mathbb{Z}/p^2$

$\operatorname{im}(h_A) = \mathbb{Z}/p^3 \oplus \mathbb{Z}/p$ generated by $x_n(v_0)$ and $p \cdot x_n(v_1)$

If $\pi_*(x_n(v_1))$ is not stably spherical, then the same must be true for $x_n(v_1)$. To show that $\pi_*(x_1(v_1))$ is not stably spherical is particularly easy. The following method usually detects few elements in $\operatorname{cok}(h_A)$ but works in this case.

Let ch_m denote the m-th term of the Chern character, $q = 2p - 2$ and k connected p-local K-theory. Then it is well known that $p^m \cdot \mathrm{ch}_{qm}$ is integral at p and gives a class in $H^{q \cdot m}(k; \mathbf{Z}_{(p)})$. This class defines natural transformations

$$Q_m : k_i(X) \to H_{i-mq}(X; \mathbb{Z}_{(p)}) \quad \text{and} \quad Q_m : k_i(X; \mathbb{Q}/\mathbb{Z}) \to H_{i-mq}(X; \mathbb{Q}/\mathbb{Z}_{(p)}).$$

The composition

$$A_{2n}(X;\mathbb{Q}/\mathbb{Z}) \xrightarrow{D} k_{2n}(X;\mathbb{Q}/\mathbb{Z}) \xrightarrow{Q_m} H_{2n-qm}(X;\mathbb{Q}/\mathbb{Z}_{(p)})$$

clearly vanishes on stably spherical classes ($m>0$). For $m=p$ it can be used to show $\pi_*(x_n(v_1)) \notin \operatorname{im}(h_A)$, $n=p^2-1$. We omit the easy calculation; see e.g. [14].

Notice also that this case is already stable in the sense of (5.6). For $n=p^2-1$ and all $c\geq 3$ we have $A_{2n-1}(B\mathbb{Z}/p^c)=\mathbb{Z}/p^c\oplus\mathbb{Z}/p^2$ and $\operatorname{im}(h_A)=\mathbb{Z}/p^c\oplus\mathbb{Z}/p$ by the same argument as above. For $c=1$ we have $\mathrm{Ad}_{2n-1}(B\mathbb{Z}/p)=A_{2n-1}(B\mathbb{Z}/p)$ for all $n\geq 2$ and h_A is onto by (5.4). The case $c=2$ is also exceptional. It turns out that the bound $n_0(\mathbb{Z}/p^2)$ beyond which h_A is onto by (4.4) is small enough to allow a direct computation for the rest of the dimensions.

PROPOSITION 5.8. *Let p be an odd prime, then*

$h_A:\pi^s_{2n-1}(B\mathbb{Z}/p^2)\rightarrow A_{2n-1}(B\mathbb{Z}/p^2)$ *is onto.*

Proof. We only sketch the argument. Only the elements $x_n(v_1)$ of order $p^{1+\nu_p(n)}$ have to be considered. In the first step one works out that the skeletal filtration of $x_n(v_1)$ is less than $2(p-1)p(2+\nu_p(n))$ by (2.14). Except for $p=3$ this implies: if $\nu_p(n)\geq 3$ then $x_n(v_1)\in\operatorname{im}(h_A)$ for all n and if $\nu_p(n)\leq 2$ then $x_n(v_1)\in\operatorname{im}(h_A)$ if $n>3p^2$.

The remaining cases are checked by hand. Only for $n\geq(p-1)p$ there is something to show; for smaller values of n $h_A:\pi^s_{2n-1}(X)\rightarrow A_{2n-1}(X)$ is bijective since $\operatorname{cok}(J)=0$. Next one works out which multiple $p^s\cdot x_n(v_1)$ is in $A_{2n-1}(B\mathbb{Z}/p^2)$. The elements in $\pi^s_{2n-1}(B\mathbb{Z}/p^2)$ needed to generate $\operatorname{im}(h_A)$ are first of all constructed in $\pi^s_{2n-1}(P_\infty\mathbb{C})$ using the transfer map $t:\pi^s_{2n-2}(P_\infty\mathbb{C}\wedge P_\infty\mathbb{C})\rightarrow\pi^s_{2n-1}(P_\infty\mathbb{C})$ (see e.g. [13], [14]) and then by considering the exact sequence in stable homotopy induced by (5.1). To see that the elements in question indeed come from $\pi^s_{2n-1}(B\mathbb{Z}/p^2)$ one shows that they map to zero in $\pi^s_{2n-1}(P_\infty\mathbb{C}^{Hp^2})$. This is done by proving that in those dimension h_A is injective, allowing to do the calculation in A-theory. Adams periodicity (e.g. see [4]) reduces the number of cases where one has to construct an element mapping to

$$x_n(v_1) \quad\text{or}\quad p\cdot x_1(v_1).$$

EXAMPLE 5.9. Let $p=3$. $A_{2n-1}(B\mathbb{Z}/9)$ is cyclic for $n=1,2,3,4,6,$ the index of $A_{2n-1}(B\mathbb{Z}/9)$ in $\mathrm{Ad}_{2n-1}(B\mathbb{Z}/9)$ is 3 for $n=12$, 18 and 9 for $n=9$. In the other dimensions we have $A_{2n-1}(B\mathbb{Z}/9)=\mathrm{Ad}_{2n-1}(B\mathbb{Z}/9)$.

REFERENCES

[1] J. F. ADAMS, *On the groups $J(X)-IV$*, Topology *5* (1966) 21–71.
[2] M. F. ATIYAH, *Characters and cohomology of finite groups*, Publ. math. Inst. ht. Etud. scient. *9* (1961) 23–64.
[3] ——, *Vector bundles and the Künneth formula*, Topology *1* (1962) 245–248.
[4] M. C. CRABB and K. KNAPP, *Adams periodicity in stable homotopy*, Topology *24* (1985) 475–486.
[5] —— and K. KNAPP, *The Hurewicz map on stunted complex projective spaces* (Preprint).
[6] B. GRAY, *Desuspension at an odd prime, in Algebraic Topology Aarhus* 1982, Lecture Notes in Math. *1051* (1984) 360–370.
[7] B. HUPPERT, *Endliche Gruppen I*, Springer Verlag 1967.
[8] I. M. ISAACS, *Character Theory of finite groups*, Academic Press 1976.
[9] T. KAMBE, *The structure of K_Λ-rings of the lens space and their applications*, J. Math. Soc. Japan *18* (1966) 135–146.
[10] K. KNAPP, *Das Bild des Hurewicz-Homomorphismus $h: \pi_*^s(B\mathbb{Z}_p) \to K_1(B\mathbb{Z}_p)$*, Math. Ann. *223* (1976) 119–138.
[11] ——, *On the K-homology of classifying spaces*, Math. Ann. *233* (1978) 103–124.
[12] ——, *On odd-primary components of Lie groups*, Proc. A.M.S. *79* (1980) 147–152.
[13] ——, *Some applications of K-theory to framed bordism*, Habilitationsschrift, Bonn 1979.
[14] ——, revised version of *13* (in preparation).
[15] N. KUHN, *The* mod p *K-theory of classifying spaces of finite groups* (Preprint 1985).
[16] V. PUPPE, *Bocksteinspektrum und multiplikative Struktur in der komplexen K-Theorie von Produkträumen*, Arch. Math. *22* (1971) 260–270.
[17] D. L. RECTOR, *Modular characters and K-theory with coefficients in a finite field*, J. of Pure and Appl. Algebra *4* (1974) 137–158.
[18] F. W. ROUSH, Transfer in generalized cohomology theories, Thesis Princeton 1871.
[19] J. P. SERRE, *Linear representations of finite groups*, Springer Verlag (1977).
[20] L. SMITH, *On* Im $\{\Omega_*^{fr}(B\mathbb{Z}_p) \to \Omega_*^U(B\mathbb{Z}_p)\}$, J. appl. algebra *6* (1975) 223–234.
[21] C. B. THOMAS, *Filtration on the representation ring of a finite group*, in Contemporary Mathematics Vol. *19* (1983) 397–405.
[22] J. TORNEHAVE, *The splitting of spherical fibrations at odd primes*, Preprint series Aarhus 1972/73 No. 28.
[23] E. A. WEISS, *Kohomologiering und Darstellungsring endlicher Gruppen*, Bonner Mathematische Schriften Nr. *36* (1969).
[24] G. WILSON, *K-theory invariants for unitary G-Bordism*, Quart. J. Math. Oxford *24* (1973) 499–526.

Bergische Universität
Gesamthochschule Wuppertal
Fachbereich Mathematik
Gaußstraße 20
D 5600 *Wuppertal* 1

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