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Every rational surface is separably split

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Let F be a field with algebraic closure $\bar{F}$. A surface X is called rational over F if $\bar{X} = X \times_F \bar{F}$ is birational to $\mathbb{P}^2$. An extension E/F is said to split X if the surface X_E is birational with $\mathbb{P}^2_E$ by a sequence of monoidal transformations centered at E-points.

When Bloch wrote the paper [1] which renewed the study of zero cycles on rational surfaces, he made an unpleasant technical assumption. His techniques only worked for separably split rational surfaces; i.e., surfaces admitting a splitting field E/F which is a separable extension of F. Those who followed needed the same assumption [2, 3, 10].

This unfortunate circumstance arose because one did not yet know the validity of

THEOREM 1. *Every rational surface is separably split.*

This note will rectify matters. Consequently, the results of [1, 2, 3, 10] actually hold for all rational surfaces. For instance, the exact sequence of [1, Thm. 0.1] actually exists for every rational surface, and Thms. 0.3 and 0.4 (*op. cit.*) hold for all conic bundle surfaces. Similarly, one can generalize [10, Thm. 6.1] by removing the restriction to prime-to-p torsion.

COROLLARY [10, Thm. 6.1]. *Let X be a rational surface over either a local field or a field of cohomological dimension ≤ 1. Then the group $A_0(X)$, of zero cycles of degree zero modulo rational equivalence, is finite.*

Not surprisingly, the principal tool used is the work of Iskovskih [7] classifying minimal rational surfaces over any field. However, his argument relies on the "Adjunction Lemma" [7, p. 20, Lem. 2], whose proof has only appeared in print in the case when the ground field is perfect [Manin, 8]. In the first section of this paper, the Adjunction Lemma is proved for separably closed fields. Iskovskih's results are then used in the second section to prove Theorem 1.

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§1. The adjunction lemma

Throughout this section, let X be a smooth projective surface over a separably closed field F. Recall that X is said to be F-minimal if the only F-morphisms $X \to Y$ which are birational equivalences to a smooth projective F-surface Y are actually isomorphisms.

THEOREM 2 (The Adjunction Lemma). *Let X be an F-minimal surface with $q = P_2 = 0$. For any invertible sheaf $\mathscr{L}$ on X, there exists an integer $n_0 \geq 0$ such that $n \geq n_0$ implies*

$$H^0(X, \mathscr{L} \otimes \omega_X^n) = 0.$$

Proof. Because X is smooth and proper over F, the language of divisors can be used in place of the language of invertible sheaves. Thus, it is necessary to show, for any divisor D and for sufficiently large n, that there are no effective representatives of the divisor class of $D + nK$.

Since Fulton's [4] version of the Hirzebruch–Riemann–Roch (HRR) Theorem holds for a projective variety over any field, one has

$$\chi(\mathcal{O}_X) = 1 - q + p_g = 1 + p_g > 0.$$

Because X is smooth over F, Serre duality [5] holds with dualizing sheaf $\omega_X = \Omega^2_{X/F} = \mathcal{O}_X(-K)$. Therefore,

$$h^2(-K) = h^0(2K) = P_2 = 0.$$

So, applying HRR to the anticanonical divisor gives

$$h^0(-K) - h^1(-K) = \chi(\mathcal{O}_X(-K)) = \tfrac{1}{2}(-K) \cdot (-2K) + \chi(\mathcal{O}_X) = K^2 + 1 + p_g.$$

If $K^2 \geq 0$, then the last chain of equations implies $h^0(-K) > 0$. So, $-K$ has an effective representative, which is not zero because $P_2 = 0$. Thus, there exists a hyperplane section H with $K \cdot H < 0$. Furthermore, if n is large, then $(D + nK) \cdot H = D \cdot H + nK \cdot H < 0$. Thus, $D + nK$ cannot be linearly equivalent to an effective divisor.

So, assume $K^2 < 0$. For n large, $(D + nK) \cdot K < 0$. If $D + nK$ is linearly equivalent to an effective divisor for arbitrarily large n, then at some point n_1 we have both $(D + n_1K) \cdot K < 0$ and $D + n_1K \sim \sum m_iC_i$ with all $m_i \geq 0$. In particular, there is a component $C = C_i$ such that $C \cdot K < 0$. Then n large implies $C \cdot (D + nK) < 0$. But if $C^2 \geq 0$ and $D + nK$ has an effective representative, then $C \cdot (D + nK) \geq 0$. Since this is impossible, the proof of the theorem has been reduced to:

LEMMA 3. *Let X be a smooth projective surface over a separably closed field F. Let C be an integral curve on X with $C^2 < 0$ and $C \cdot K < 0$. Then C is an exceptional curve of the first kind defined over F.*

Proof. Define $\omega_C = \mathcal{O}_C(C + K)$. By definition, there are exact sequences

$$0 \to \mathcal{O}_X(-C) \to \mathcal{O}_X \to \mathcal{O}_C \to 0$$
$$0 \to \mathcal{O}_X \to \mathcal{O}_X(K + C) \to \omega_C \to 0.$$

By Serre duality on X and the additivity of the Euler characteristic, one has

$$\chi(\omega_C) = \chi(\mathcal{O}_X(K + C)) - \chi(\mathcal{O}_X) = \chi(\mathcal{O}_X(-C)) - \chi(\mathcal{O}_X) = -\chi(\mathcal{O}_C).$$

By HRR [4, Ex. 18.3.4] on the projective curve C over F, one has

$$\deg(\omega_C) = \chi(\omega_C) - \chi(\mathcal{O}_C) = -2\chi(\mathcal{O}_C).$$

But by [4, Ex. 2.4.9], one knows $\deg(\omega_C) = C \cdot (C + K)$.

Now consider the natural morphism $\pi : \bar{X} \to X$. Because F is separably closed, the pullback $\pi^*(C) = qD$, where q is a power of the characteristic of F and D is an integral curve on $\bar{X}$. Since the formation of relative differentials is stable under base change [6], one also knows that $\pi^*(K) = \bar{K}$ is a canonical divisor on $\bar{X}$. Therefore,

$$C^2 = q^2D^2 \quad \text{and} \quad C \cdot K = qD \cdot \bar{K}.$$

In particular, both D^2 and $D \cdot \bar{K}$ are negative. By the Adjunction Formula over the algebraically closed field $\bar{F}$, D is an exceptional curve of the first kind. Hence

$$C^2 = -q^2 \quad \text{and} \quad C \cdot K = -q.$$

Let E be the algebraic closure of F in the function field $F(C)$. Then E is a purely inseparable extension of F of degree q and D is already defined and isomphic to $\mathbb{P}^1$ over E. In fact, $F(C) \approx E(t)$ is a rational function field in one variable, and $D \to C$ is the normalization. Thus, there is an exact sequence

$$0 \to \mathcal{O}_C \to \phi_*(\mathcal{O}_D) \to \text{Torsion} \to 0.$$

Therefore,

$$\begin{aligned} q^2 + q = -C \cdot (C + K) = -\deg(\omega_C) = 2\chi(\mathcal{O}_C) \\ \leq 2h^0(\mathcal{O}_C) \leq 2h^0(\mathcal{O}_D) = 2[E:F] = 2q. \end{aligned}$$

In other words, $q^2 \leq q$ or $q \leq 1$. But this is only possible if $q = 1$ and therefore $E = F$.

§2. Proof of Theorem 1

The proof will proceed through a series of reductions. First, Theorem 1 is clearly equivalent to

THEOREM 4. *Let F be a separably closed field. Every rational surface over F is split by F.*

It is in this form that the theorem will be proved. Assume hereafter that F is separably closed.

PROPOSITION 5. *Let $f: X \to Y$ be a birational morphism of smooth projective surfaces over F. Then f factors as a sequence of monoidal transformations centered at F-points.*

Proof. Because f factors over $\bar{F}$ as a sequence of blowups of points, it must factor over F as a sequence of blowups of closed points. It suffices to show that the blowup of a closed point whose residue field is a nontrivial purely inseparable extension of F can never give rise to a smooth surface.

Let Q be a closed point of Y with residue field E a purely inseparable extension of degree $p^n > 1$. Since smoothness is local, one may replace Y by an open affine $\operatorname{Spec} A$ where Q is defined by a maximal ideal $\mathfrak{m} = (x, y)$. Let $R = A_{\mathfrak{m}}$ be the (regular) local ring of Q on Y. The Second Exact Sequence of

[Matsumura, 9]

$$\begin{array}{ccccc} \mathfrak{m}/\mathfrak{m}^2 \to \Omega_{R/F} \otimes E \to & \Omega_{E/F} \to 0 \\ \wr\!\wr & \cup \\ E \oplus E & E \end{array}$$

shows that there is at least one linear relation between dx and dy over $E \approx A/\mathfrak{m}$. Without loss of generality, one may assume there exists $t \in A$ such that $dx \equiv t\, dy \bmod \mathfrak{m}$.

By shrinking the affine Y if necessary, the blowup can be defined locally by $X = \operatorname{Spec} B$ where $B = A[T]/(xT - y)$. The module of differentials of this ring is

$$\Omega_{B/F} = (B\, dT \otimes (\Omega_{A/F} \otimes B))/(T\, dx + x\, dT - dy).$$

At the point defined by $\mathfrak{n} = (x, y, T - t)$, one has

$$T\, dx + x\, dT - dy \equiv t\, dx - dy \equiv 0 \bmod \mathfrak{n}.$$

Therefore, $\Omega_{B/F} \otimes B/\mathfrak{n}$ is three-dimensional and $\mathfrak{n}$ is not a smooth point on the blowup X.

Remark. Let $E = F(\alpha)$ be a purely inseparable extension of degree p. Let X be the blowup of $\mathbb{P}^2$ at the F-scheme defined by the E-point $(\alpha : 0 : 1)$. Although the proposition shows that X is not a smooth F-surface, it is a regular projective F-surface which becomes birationally equivalent to $\mathbb{P}^2$ over the algebraic closure. It is not separably split.

By Prop. 5, one may assume that X is an F-minimal rational surface. By Theorem 2, one may use Iskovskih's classification of such surfaces [7]. One property that all such surfaces share is that the rank of the Picard group is small.

LEMMA 6. *Let X be a rational surface over a separably closed field F. Then* $\operatorname{Pic}(X)$ *is a subgroup of finite index in* $\operatorname{Pic}(\bar{X})$.

Proof. Let $C_1, \dots, C_r$ be a finite set of curves generating $\operatorname{Pic}(\bar{X})$. Each C_i is defined over some purely inseparable extension of F of finite degree. So, for some $n \geq 0$, each $p^n C_i$ is an F-rational divisor. Therefore

$$p^n \operatorname{Pic}(\bar{X}) \subset \operatorname{Pic}(X) \subset \operatorname{Pic}(\bar{X}).$$

PROPOSITION 7. *The only possible minimal rational surfaces over F are*
(I) *Severi–Brauer surfaces*
(II) *smooth conic bundle surfaces $X \to \mathbb{P}^1$.*

Proof. Iskovskih [7] has shown that the minimal surfaces are either (i) del Pezzo surfaces of Picard rank 1 over F or (ii) generically smooth conic bundles of Picard rank 2, possibly with singular fibres, over a smooth genus zero curve.

By Lem. 6, the rank of the Picard group is unchanged by passage to the algebraic closure. The only del Pezzo surfaces with Picard rank 1 are the Severi–Brauer surfaces (del Pezzo surfaces of degree 9). Since smooth projective curves of genus zero always have points over a separably closed field, they are all isomorphic to $\mathbb{P}^1$. The only conic bundles over $\mathbb{P}^1$ with Picard rank 2 are the smooth conic bundles.

Now it is well-known that every Severi–Brauer surface has points over a separably closed field, and is therefore separably split. See, for instance [11]. So, it only remains to consider conic bundles.

PROPOSITION 8. *All smooth conic bundles $X \to \mathbb{P}^1$ are separably split.*

Proof. By [7, Thm. 3], every smooth conic bundle is associated to a rank 2 vector bundle. But vector bundles on $\mathbb{P}^1$ split as a sum of line bundles, and their projectivization is unchanged by twisting by line bundles [6]. So, it is enough to consider the surfaces

$$X_n = \mathbb{P}(\mathcal{O} \otimes \mathcal{O}(n)) \qquad n \geq 0$$

over $\mathbb{P}^1$.

When $n = 0$, $X_0 = \mathbb{P}^1 \times \mathbb{P}^1$ is clearly separably split. When $n > 0$, the surface X_n contains a unique F-curve B_n of self intersection $-n$. Now X_n can be split over F by choosing $n - 1$ points

$$t_1, \ldots, t_{n-1} \in \mathbb{P}^1(F),$$

writing $Q_i = f^{-1}(t_i) \cap B_n$, and then blowing up all the Q_i and blowing down the proper transform of B_n.

This completes the proof of Theorems 1 and 4.

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