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## Every rational surface is separably split

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Let  $F$  be a field with algebraic closure  $\bar{F}$ . A surface  $X$  is called rational over  $F$  if  $\bar{X} = X \times_F \bar{F}$  is birational to  $\mathbb{P}^2$ . An extension  $E/F$  is said to split  $X$  if the surface  $X_E$  is birational with  $\mathbb{P}_E^2$  by a sequence of monoidal transformations centered at  $E$ -points.

When Bloch wrote the paper [1] which renewed the study of zero cycles on rational surfaces, he made an unpleasant technical assumption. His techniques only worked for separably split rational surfaces; i.e., surfaces admitting a splitting field  $E/F$  which is a separable extension of  $F$ . Those who followed needed the same assumption [2, 3, 10].

This unfortunate circumstance arose because one did not yet know the validity of

**THEOREM 1.** *Every rational surface is separably split.*

This note will rectify matters. Consequently, the results of [1, 2, 3, 10] actually hold for all rational surfaces. For instance, the exact sequence of [1, Thm. 0.1] actually exists for every rational surface, and Thms. 0.3 and 0.4 (*op. cit.*) hold for all conic bundle surfaces. Similarly, one can generalize [10, Thm. 6.1] by removing the restriction to prime-to- $p$  torsion.

**COROLLARY** [10, Thm. 6.1]. *Let  $X$  be a rational surface over either a local field or a field of cohomological dimension  $\leq 1$ . Then the group  $A_0(X)$ , of zero cycles of degree zero modulo rational equivalence, is finite.*

Not surprisingly, the principal tool used is the work of Iskovskih [7] classifying minimal rational surfaces over any field. However, his argument relies on the “Adjunction Lemma” [7, p. 20, Lem. 2], whose proof has only appeared in print in the case when the ground field is perfect [Manin, 8]. In the first section of this paper, the Adjunction Lemma is proved for separably closed fields. Iskovskih’s results are then used in the second section to prove Theorem 1.

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## §1. The adjunction lemma

Throughout this section, let  $X$  be a smooth projective surface over a separably closed field  $F$ . Recall that  $X$  is said to be  $F$ -minimal if the only  $F$ -morphisms  $X \rightarrow Y$  which are birational equivalences to a smooth projective  $F$ -surface  $Y$  are actually isomorphisms.

**THEOREM 2** (The Adjunction Lemma). *Let  $X$  be an  $F$ -minimal surface with  $q = P_2 = 0$ . For any invertible sheaf  $\mathcal{L}$  on  $X$ , there exists an integer  $n_0 \geq 0$  such that  $n \geq n_0$  implies*

$$H^0(X, \mathcal{L} \otimes \omega_X^n) = 0.$$

*Proof.* Because  $X$  is smooth and proper over  $F$ , the language of divisors can be used in place of the language of invertible sheaves. Thus, it is necessary to show, for any divisor  $D$  and for sufficiently large  $n$ , that there are no effective representatives of the divisor class of  $D + nK$ .

Since Fulton's [4] version of the Hirzebruch–Riemann–Roch (HRR) Theorem holds for a projective variety over any field, one has

$$\chi(\mathcal{O}_X) = 1 - q + p_g = 1 + p_g > 0.$$

Because  $X$  is smooth over  $F$ , Serre duality [5] holds with dualizing sheaf  $\omega_X = \Omega_{X/F}^2 = \mathcal{O}_X(-K)$ . Therefore,

$$h^2(-K) = h^0(2K) = P_2 = 0.$$

So, applying HRR to the anticanonical divisor gives

$$h^0(-K) - h^1(-K) = \chi(\mathcal{O}_X(-K)) = \frac{1}{2}(-K) \cdot (-2K) + \chi(\mathcal{O}_X) = K^2 + 1 + p_g.$$

If  $K^2 \geq 0$ , then the last chain of equations implies  $h^0(-K) > 0$ . So,  $-K$  has an effective representative, which is not zero because  $P_2 = 0$ . Thus, there exists a hyperplane section  $H$  with  $K \cdot H < 0$ . Furthermore, if  $n$  is large, then  $(D + nK) \cdot H = D \cdot H + nK \cdot H < 0$ . Thus,  $D + nK$  cannot be linearly equivalent to an effective divisor.

So, assume  $K^2 < 0$ . For  $n$  large,  $(D + nK) \cdot K < 0$ . If  $D + nK$  is linearly equivalent to an effective divisor for arbitrarily large  $n$ , then at some point  $n_1$  we have both  $(D + n_1K) \cdot K < 0$  and  $D + n_1K \sim \sum m_i C_i$  with all  $m_i \geq 0$ . In particular, there is a component  $C = C_i$  such that  $C \cdot K < 0$ . Then  $n$  large implies  $C \cdot (D + nK) < 0$ . But if  $C^2 \geq 0$  and  $D + nK$  has an effective representative, then  $C \cdot (D + nK) \geq 0$ . Since this is impossible, the proof of the theorem has been reduced to:

**LEMMA 3.** *Let  $X$  be a smooth projective surface over a separably closed field  $F$ . Let  $C$  be an integral curve on  $X$  with  $C^2 < 0$  and  $C \cdot K < 0$ . Then  $C$  is an exceptional curve of the first kind defined over  $F$ .*

*Proof.* Define  $\omega_C = \mathcal{O}_C(C + K)$ . By definition, there are exact sequences

$$\begin{aligned} 0 \rightarrow \mathcal{O}_X(-C) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_C \rightarrow 0 \\ 0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X(K + C) \rightarrow \omega_C \rightarrow 0. \end{aligned}$$

By Serre duality on  $X$  and the additivity of the Euler characteristic, one has

$$\chi(\omega_C) = \chi(\mathcal{O}_X(K + C)) - \chi(\mathcal{O}_X) = \chi(\mathcal{O}_X(-C)) - \chi(\mathcal{O}_X) = -\chi(\mathcal{O}_C).$$

By HRR [4, Ex. 18.3.4] on the projective curve  $C$  over  $F$ , one has

$$\deg(\omega_C) = \chi(\omega_C) - \chi(\mathcal{O}_C) = -2\chi(\mathcal{O}_C).$$

But by [4, Ex. 2.4.9], one knows  $\deg(\omega_C) = C \cdot (C + K)$ .

Now consider the natural morphism  $\pi: \bar{X} \rightarrow X$ . Because  $F$  is separably closed, the pullback  $\pi^*(C) = qD$ , where  $q$  is a power of the characteristic of  $F$  and  $D$  is an integral curve on  $\bar{X}$ . Since the formation of relative differentials is stable under base change [6], one also knows that  $\pi^*(K) = \bar{K}$  is a canonical divisor on  $\bar{X}$ . Therefore,

$$C^2 = q^2 D^2 \quad \text{and} \quad C \cdot K = qD \cdot \bar{K}.$$

In particular, both  $D^2$  and  $D \cdot \bar{K}$  are negative. By the Adjunction Formula over the algebraically closed field  $\bar{F}$ ,  $D$  is an exceptional curve of the first kind. Hence

$$C^2 = -q^2 \quad \text{and} \quad C \cdot K = -q.$$

Let  $E$  be the algebraic closure of  $F$  in the function field  $F(C)$ . Then  $E$  is a

purely inseparable extension of  $F$  of degree  $q$  and  $D$  is already defined and isomorphic to  $\mathbb{P}^1$  over  $E$ . In fact,  $F(C) \approx E(t)$  is a rational function field in one variable, and  $D \rightarrow C$  is the normalization. Thus, there is an exact sequence

$$0 \rightarrow \mathcal{O}_C \rightarrow \phi_*(\mathcal{O}_D) \rightarrow \text{Torsion} \rightarrow 0.$$

Therefore,

$$\begin{aligned} q^2 + q &= -C \cdot (C + K) = -\deg(\omega_C) = 2\chi(\mathcal{O}_C) \\ &\leq 2h^0(\mathcal{O}_C) \leq 2h^0(\mathcal{O}_D) = 2[E : F] = 2q. \end{aligned}$$

In other words,  $q^2 \leq q$  or  $q \leq 1$ . But this is only possible if  $q = 1$  and therefore  $E = F$ .

## §2. Proof of Theorem 1

The proof will proceed through a series of reductions. First, Theorem 1 is clearly equivalent to

**THEOREM 4.** *Let  $F$  be a separably closed field. Every rational surface over  $F$  is split by  $F$ .*

It is in this form that the theorem will be proved. Assume hereafter that  $F$  is separably closed.

**PROPOSITION 5.** *Let  $f: X \rightarrow Y$  be a birational morphism of smooth projective surfaces over  $F$ . Then  $f$  factors as a sequence of monoidal transformations centered at  $F$ -points.*

*Proof.* Because  $f$  factors over  $\bar{F}$  as a sequence of blowups of points, it must factor over  $F$  as a sequence of blowups of closed points. It suffices to show that the blowup of a closed point whose residue field is a nontrivial purely inseparable extension of  $F$  can never give rise to a smooth surface.

Let  $Q$  be a closed point of  $Y$  with residue field  $E$  a purely inseparable extension of degree  $p^n > 1$ . Since smoothness is local, one may replace  $Y$  by an open affine  $\text{Spec } A$  where  $Q$  is defined by a maximal ideal  $\mathfrak{m} = (x, y)$ . Let  $R = A_{\mathfrak{m}}$  be the (regular) local ring of  $Q$  on  $Y$ . The Second Exact Sequence of

[Matsumura, 9]

$$\begin{array}{c} m/m^2 \rightarrow \Omega_{R/F} \otimes E \rightarrow \Omega_{E/F} \rightarrow 0 \\ \parallel \quad \cup \\ E \oplus E \quad E \end{array}$$

shows that there is at least one linear relation between  $dx$  and  $dy$  over  $E \approx A/m$ . Without loss of generality, one may assume there exists  $t \in A$  such that  $dx \equiv t dy \pmod{m}$ .

By shrinking the affine  $Y$  if necessary, the blowup can be defined locally by  $X = \text{Spec } B$  where  $B = A[T]/(xT - y)$ . The module of differentials of this ring is

$$\Omega_{B/F} = (B dT \otimes (\Omega_{A/F} \otimes B))/(T dx + x dT - dy).$$

At the point defined by  $\mathfrak{n} = (x, y, T - t)$ , one has

$$T dx + x dT - dy \equiv t dx - dy \equiv 0 \pmod{\mathfrak{n}}.$$

Therefore,  $\Omega_{B/F} \otimes B/\mathfrak{n}$  is three-dimensional and  $\mathfrak{n}$  is not a smooth point on the blowup  $X$ .

*Remark.* Let  $E = F(\alpha)$  be a purely inseparable extension of degree  $p$ . Let  $X$  be the blowup of  $\mathbb{P}^2$  at the  $F$ -scheme defined by the  $E$ -point  $(\alpha : 0 : 1)$ . Although the proposition shows that  $X$  is not a smooth  $F$ -surface, it is a regular projective  $F$ -surface which becomes birationally equivalent to  $\mathbb{P}^2$  over the algebraic closure. It is not separably split.

By Prop. 5, one may assume that  $X$  is an  $F$ -minimal rational surface. By Theorem 2, one may use Iskovskih's classification of such surfaces [7]. One property that all such surfaces share is that the rank of the Picard group is small.

**LEMMA 6.** *Let  $X$  be a rational surface over a separably closed field  $F$ . Then  $\text{Pic}(X)$  is a subgroup of finite index in  $\text{Pic}(\bar{X})$ .*

*Proof.* Let  $C_1, \dots, C_r$  be a finite set of curves generating  $\text{Pic}(\bar{X})$ . Each  $C_i$  is defined over some purely inseparable extension of  $F$  of finite degree. So, for some  $n \geq 0$ , each  $p^n C_i$  is an  $F$ -rational divisor. Therefore

$$p^n \text{Pic}(\bar{X}) \subset \text{Pic}(X) \subset \text{Pic}(\bar{X}).$$

**PROPOSITION 7.** *The only possible minimal rational surfaces over  $F$  are*

- (I) *Severi–Brauer surfaces*
- (II) *smooth conic bundle surfaces  $X \rightarrow \mathbb{P}^1$ .*

*Proof.* Iskovskih [7] has shown that the minimal surfaces are either (i) del Pezzo surfaces of Picard rank 1 over  $F$  or (ii) generically smooth conic bundles of Picard rank 2, possibly with singular fibres, over a smooth genus zero curve.

By Lem. 6, the rank of the Picard group is unchanged by passage to the algebraic closure. The only del Pezzo surfaces with Picard rank 1 are the Severi–Brauer surfaces (del Pezzo surfaces of degree 9). Since smooth projective curves of genus zero always have points over a separably closed field, they are all isomorphic to  $\mathbb{P}^1$ . The only conic bundles over  $\mathbb{P}^1$  with Picard rank 2 are the smooth conic bundles.

Now it is well-known that every Severi–Brauer surface has points over a separably closed field, and is therefore separably split. See, for instance [11]. So, it only remains to consider conic bundles.

**PROPOSITION 8.** *All smooth conic bundles  $X \rightarrow \mathbb{P}^1$  are separably split.*

*Proof.* By [7, Thm. 3], every smooth conic bundle is associated to a rank 2 vector bundle. But vector bundles on  $\mathbb{P}^1$  split as a sum of line bundles, and their projectivization is unchanged by twisting by line bundles [6]. So, it is enough to consider the surfaces

$$X_n = \mathbb{P}(\mathcal{O} \otimes \mathcal{O}(n)) \quad n \geq 0$$

over  $\mathbb{P}^1$ .

When  $n = 0$ ,  $X_0 = \mathbb{P}^1 \times \mathbb{P}^1$  is clearly separably split. When  $n > 0$ , the surface  $X_n$  contains a unique  $F$ -curve  $B_n$  of self intersection  $-n$ . Now  $X_n$  can be split over  $F$  by choosing  $n - 1$  points

$$t_1, \dots, t_{n-1} \in \mathbb{P}^1(F),$$

writing  $Q_i = f^{-1}(t_i) \cap B_n$ , and then blowing up all the  $Q_i$  and blowing down the proper transform of  $B_n$ .

This completes the proof of Theorems 1 and 4.

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