Zeitschrift: Commentarii Mathematici Helvetici

Herausgeber: Schweizerische Mathematische Gesellschaft

Band: 63 (1988)

Artikel: Fixed points and homotopy fixed points.

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DOI: https://doi.org/10.5169/seals-48209

Comment. Math. Helvetici 63 (1988) 286–295 0010-2571/88/020286-10$01.50 + 0.20/0

Fixed points and homotopy fixed points

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Introduction

In this note we consider G-maps $C \to X$ from a contractible G-space C into a finite G-simplicial complex X. The group G is almost always taken to be finite.

We consider the relation between the existence of G-fixed points in X (i.e. $\lim_G X \overset{\text{def}}{=} X^G \neq \varnothing$) and the existence of homotopy G-fixed points, namely maps $EG \to X$ (i.e. $\operatorname{holim}_G X = \operatorname{map}_G (EG, X) \neq \varnothing$) where EG is a free, contractible G – C.W.-complex.

A compact topological group G is said to have the *homotopy fixed point property* (HFPP) if for every finite G-simplicial complex X one has $X^G = \varnothing$ *if and only if* $X^{hG} = \varnothing$; where X^{hG} denotes the space of homotopy fixed points $X^{hG} \equiv \hom_G (EG, X)$. In other words G has HFPP if fixed point free finite G-complexes do not admit G-maps from EG.

THEOREM A. *A finite group has HFPP if and only if G is a p-group for some prime p.*

If G has no HFPP, i.e. there exist a finite G-complex with $X^G = \varnothing$ and a map $EG \to X$ we say that G is *compressible*.

Since any contractible G-space X admits map $EG \to X$, it follows that if for a given group G there exists a contractible G-simplical complex K, on which G acts without fixed points then G must be compressible. Now we use heavily the following theorem of Oliver:

THEOREM [Oliver] p. 156). *Let $\mathcal{G}$ be the class of all finite groups G having the following subnormal decomposition: $P \lhd H \lhd G$ where P is a p-group H/P is cyclic and G/H is a q group, p and q are (not necessarily distinct) primes. Then the following are equivalent:*

(1) $G \notin \mathcal{G}$

(2) *G acts simplicially and without a fixed point on a contractible finite simplicial complex.*

It follows that all groups not in $\mathcal{G}$ are compressible and our study can be restricted to groups in $\mathcal{G}$.

This is done using the following steps.

Applying methods similar to [Tom Dieck, 7.1, 7.3] we show that p-tori (or real tori T^n) have HFPP.

THEOREM B. *Let G be an elementary abelian p-group $G \cong \oplus \mathbb{Z}/p\mathbb{Z}$, K a finite simplicial G-complex. Then the following are equivalent:*

(1) $K^G \neq \emptyset$

(2) $\mathrm{map}_G(EG, K) \neq \emptyset$

(3) *The classifying* map $\chi : EG \times_G K \to BG$ *induces a monomorphism on* mod-p *cohomology.*

Furthermore, the same holds for the torus $T^n = \mathbb{R}^n/\mathbb{Z}^n$, where rational cohomology is used in (3).

To get from elementary abelian groups to general p-groups we use the fact that the former are *Frattini factors* (see 2, below) of the latter. Using H. Miller's version of the "Sullivan conjecture" [Miller] one shows:

THEOREM C. *A finite group G is compressible if and only if its frattini factor is compressible.*

From which one gets immediately

COROLLARY D. *Finite p-groups have HFPP.*

This last result has been proved independently by Haeberly and Jackowsky [6, 7] using Carlsson's version of the "Segal conjecture."

THEOREM E. *Let p, q be two distinct primes, G_p, G_q a p-group and a q-group. Then a semi-direct product $G = G_p \rtimes G_q$ is compressible.*

Since (Lemma 3.4 below) any group G in the Oliver class $\mathcal{G}$ that is not a p-group has a factor group of the form $G_p \rtimes G_q$ for $p \neq q$ it follows from Theorem E and the results of Oliver mentioned above:

COROLLARY F. *A finite group that is not a p-group is compressible.*

Theorem A is a combination of D and F.

Remark. Theorem B is evidently related to the general Sullivan conjecture [Miller] [Sullivan]. That conjecture can be formulated by saying: *If K is a finite G-simplicial complex with G a p-group then the natural* map $X^G \to \hom_G(EG, X)$ *is a* mod-p *homology isomorphism.*

In the special case where $X^G = \varnothing$ it say that $\mathrm{map}_G(EG, X) = \varnothing$. But this is the main content of Theorem B. Notice also that if X is a finite contractible G-complex, it follows from Smith theory [Bredon] that $H_*(X^G, Z_p) \cong H_*(pt, Z_p)$ in conformity with the conjecture for that case.

The "generalized Sullivan conjecture" has been proven independently by J. Lannes, H. Miller, and G. Carlsson. Carlsson's most recent version claims the general case, independent of fundamental groups of fixed point sets.

The rest of this note is organized as follows: In the first section we prove Theorem B. In the second and third we prove C, E and F.

Part of this work was done while the second author was visiting the Centre de Recerca Mathematics Institut Destudis Catalans in Barcelona. We would like to express our gratitude for their kind hospitality.

NOTE. This paper was completed while the late Professor Alexander Zabrodsky was still alive. He was killed in a car accident on November 20th, 1986. His death, at the prime of his life while in the midst of a vigorous mathematical activity, has been a terrible loss to his many friends, students and colleagues as well as to the ongoing research work in algebraic topology. (E. D. F.)

1. Proof of Theorem B

We show $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (1)$. The first one is immediate since every fixed point $x \in K^G$ gives a G-map $EG \to x \hookrightarrow K^G \hookrightarrow K$. To show $(2) \Rightarrow (3)$ let $h : EG \to K$ be a G-map from a contractible free G-simplicial space EG. Consider the factorization of the identity.

$$EG \xrightarrow{1 \times h} EG \times K \xrightarrow{pr_1} EG.$$

Take G orbit spaces, the composite gives the identity $BG \to BG$, so that $pr_1/G = \chi$ is the right inverse of $1 \times h/G$, $BG \to EG \times_G K$, hence the monomorphism on cohomology.

The implication (3)$\Leftrightarrow$(1) is given by Borel see [Hsiang p. 45, Cor. 1]. We outline a direct, elementary proof: One uses an open covering of $K \times_G EG$ by $\{K_v\}$ indexed by the vertices v of K/G, with K_v homotopy equivalent to $EG \times_G G/G_v$ where G/G_v is the orbit over v. Now if $K^G = \phi$ we get that no orbit is free, so for every v one can choose a polynomial generator U_v in $H^2(BG_v, Z/pZ)$, that goes to zero in $H^2(BG_v, Z/pZ)$, this is possible for a p-torus. Thus one gets $\prod U_v \neq 0$, but since $\{K_v\}$ is a cover this product pulls back to zero in $EG \times_G K$ contrary to (3). Thus $K^G \neq \phi$ as needed.

1.1. Remark on function complexes

Although in the formulation of the theorems above the space of maps or G-maps from X to Y is not used, it is used heavily in the proofs. For any two such spaces we need a model for the space of all maps map (X, Y) and all equivariant maps $\mathrm{map}_G\,(X, Y)$. Some care is needed to avoid point-set problems. The basic property of this function spaces that we need is the exponential law: That is we need a canonical identification $\mathrm{map}_G\,(X, \mathrm{map}\,(Y, Z)) \xrightarrow{=} \mathrm{map}_G\,(X \times Y, Z)$.

This we can guarantee either by working in the Steenrod category of compactly generated spaces or by taking map (X, Y) to be the simplicial set with a discrete group G action, where the n-simplices are topological maps $X \times \Delta_n \to Y$. The simplicial subset of G-fixed point will then be the equivariant function complex $\mathrm{map}_G\,(X, Y)$.

2. Compressibility of Frattini factors

Recall that for a group G, the Frattini subgroup ΦG is the intersection of all maximal subgroups of G; one gets a normal subgroup and the factor group $G/\Phi G = G_0$ is called the Frattini factor group of G. See [Gorenstein, p. 173]. In the present section we establish Theorem C, namely prove that a finite group G is compressible if its Frattini factor is compressible.

We start with a consequence of [Miller]:

LEMMA 2.1. *Let $\alpha: G \to G_0$ be a surjection of groups with a finite kernel. Let K be any finite G_0-simplicial complex. Then the natural composition*

$$\mathrm{map}_{G_0}\,(EG_0, K) \overset{=}{\leftrightarrow} \mathrm{map}_G\,(EG_0, K) \to \mathrm{map}_G\,(EG, K)$$

is a weak equivalence of spaces. (The second map is induced by an G-map $EG \to EG_0$.)

Proof. Let $H \subset G$ denote $\ker \alpha$. Then

$$\mathrm{map}_G(EG, K) = \mathrm{map}_G(EG/H, K) = \mathrm{map}(BH, K)$$

H has a trivial action on K, and EG is a free contractible H-space.

Now consider the diagram

$$\begin{array}{ccc} \mathrm{map}_G(EG, K) & \xrightarrow{\simeq} & \mathrm{map}_G(EG/H, K) \\ \uparrow & & \downarrow \\ \mathrm{map}_{G_0}(EG_0, K) & \longrightarrow & \mathrm{map}_G(BH, K) \end{array}$$

In order to show that the map on the left is a weak equivalence it is sufficient to show that the bottom arrow is a weak equivalence. But taking the full function spaces, the map on the bottom becomes $\mathrm{map}(EG_0, K) \to \mathrm{map}(BH, K)$. By [Miller] the right hand side is homotopy equivalent to K since K is finite complex and H is a finite group. Thus this last map is a G-map that, ignoring the G-action, is a weak homotopy equivalence. Therefore, the map induces equivalence on the homotopy fixed points [Bousfield–Kan]. However, both EG_0 and $BH = EG/H$ are free G_0-spaces. Thus the above map must be a weak G_0-equivalence by the following Lemma 2.2, and therefore their fixed point are weakly equivalent as needed.

LEMMA 2.2. *Let F be any free G-simplicial complex and X any G-space, consider the G-space map (F, X) with the diagonal action by G, then the natural map* $\mathrm{map}_G(F, X) \equiv (\mathrm{map}(F, X))^G \to \mathrm{map}_G(EG, \mathrm{map}(F, X))$ *is a homotopy equivalence.*

Proof. We use the exponential law 1.2. For a free G-simplicial complex F the projection $EG \times F \to F$ is actually a G-homotopy equivalence by Bredon's Theorem. Therefore $\mathrm{map}_G(F, X) \to \mathrm{map}_G(EG \times F, X)$ is a homotopy equivalence. By 1.2 this is the same as the map in the lemma.

Proof of Theorem C. First note that if G_0 is any factor group of a group G and G_0 is compressible so is G. (Every G_0 space is a G space). Thus Theorem C in one direction is obvious.

Let K be a finite G-simplicial complex with $K^G = \varnothing$. Let $K_0 = K/\Phi G$. Then K_0 is a $G_0 = G/\Phi G$-simplicial complex. ΦG has the following property. If $H \subsetneq G$ then $(\Phi G, H) \subsetneq G$ [5].

Hence, the G_0 isotropy groups of K_0 are proper subgroups of G_0 and $K^{G_0}=\emptyset$. If G_0 is incompressible $\mathrm{map}_{G_0}(EG_0, K_0)=\emptyset$. By 2.1 $\mathrm{map}_{G_0}(EG_0, K)\approx (EG, K_0)=\emptyset$. But as one has a map $\mathrm{map}_G(EG, K)\to \mathrm{map}_G(EG, K_0)$, $\mathrm{map}_G(EG, K)=\emptyset$.

As the Frattini factors of p-groups are elementary abelian the following is an immediate consequence of Theorems B and C:

COROLLARY D. *p-groups have HFPP.*

The following is a simple application of Corollary D:

EXAMPLE D1. Let G be a compact Lie group, G_0 – a closed subgroup of G. $i: G_0\to G$. Given a homomorphism $\varphi:\pi\to G$ (where π is either a p-group or a torus) then $\varphi(\pi)$ is conjugate in G to a subgroup in G_0 if and only if B_φ ;$B\pi\to BG$ lifts (up to homotopy) to $B\pi\to BG_0$.

Proof. If $g\varphi(\pi)g^{-1}\subset G_0$ for some $g\in G$ consider the composed homomorphism $\varphi_0:\pi\to\varphi(\pi)\to g\varphi(\pi)g^{-1}\overset{\subseteq}{\to} G_0$. If $a_g: G\to G$ is the conjugation by g then $i\circ\varphi_0=a_g\circ\varphi$, hence $B_i\circ B\varphi_0=Ba_g\circ B\varphi$. But $Ba_g\sim 1$ implies that $B\varphi_0$ is the desired homotopy lifting.

Conversely, choose the following models for $B\pi$, BG, BG_0, B_i and $B\varphi$: Consider EG and G/G_0 as π-spaces via φ. Then $B_i: BG_0\to BG$ and $B\varphi: B\pi\to BG$ could be chosen to be the fibration $BG_0=G/G_0\times_G EG\to EG/G=BG$ and the map $B\pi=EG/\pi\to EG/G=BG$. Using $B\varphi$ to pull the fibration B_i over $B\pi$ one obtains the fibration $h: W=G/G_0\times_\pi EG\to EG/\pi=B\pi$ and a homotopy lifting of $B\varphi$ corresponds to a section of j, hence, to an element in $\mathrm{map}_\pi(EG, G/G_0)$. As G/G_0 is a finite π-complex [1] by Theorem B (for π – a torus) and Corollary B (for π – a p-group) $(G/G_0)^\pi\neq\emptyset$. But this is equivalent to the existence of g with $g\pi g^{-1}\subset G_0$.

3. Proof of Theorem E

Let p, q be distinct primes and let G_p and G_q be nontrivial p-group and q-group respectively. Let $G=G_p\rtimes G_q$. Embed $G_q\subset U(n_1)$, $G\subset U(n)$ where $U(n_1)$, $U(n)$ are any compact connected Lie groups. Then $U(n_1)$ and the homogenous space $U(n)/G_q$ are fixed point free G spaces and so in their joint $W=U(n_1)*U(n)/G_q$. Theorem E follows from the following:

PROPOSITION 3.3. *There exists a G* map *$EG\to W$ and $W^G=\emptyset$.*

Proof. Fix a point $u \in U(n_1) \subset W$ and let u denote the constant map $u: EG \to W$ ($u(EG) = u$). Then u is a G_p-map as $W^{G_p} = U(n_1)$. Since this set is a connected G-subspace of W – the path component of u in $\operatorname{map}_{G_p}(EG, W)$, which we denote $\operatorname{map}_{G_p}(EG, W)_u$ is a G subspace of the G-space $\operatorname{map}(EG, W)$. Consider the inclusion $i: \operatorname{map}_{G_p}(EG, W)_u \to \operatorname{map}(EG, W)$ as G_q map (by restriction). Note that the G-action on $\operatorname{map}_{G_p}(EG, W)$ factors through a G_q action which coincides with the restriction of the G action to G_q. As $W^{G_q} \neq \emptyset$ there exists a G_q map $v: EG_q \to \operatorname{map}(EG, W)$ where v is a constant map. To complete the proof of 3.1 suffices to show that v can be deformed into a G_q map $\hat{v}: EG_q \to \operatorname{map}_{G_p}(EG, W)_u$ for then the adjoint $\hat{v}_{\#}: EG_q \times EG \to W$ will be a G map from the contractible free G space $EG_q \times EG$ into W.

The fact that v could be deformed into a G_q map $\hat{v}$ is a consequence of the following two lemmas.

LEMMA 3.2. *Let G_0 be a discrete group and $f: X \to Y$ a G_0* map *between the G_0 spaces X and Y. Let $h: EG_0 \to Y$ be a G_0* map *and choose $u \in EG_0$, $x_0 \in X_0$ with $h(n) = f(x_0) = g_0$. If F_f – the homotopy fiber of f over y_0 – is path connected, then the obstructions to deforming $(EG_0, u) \to (Y, y_0)$ to a G_0 map $(EG_0, u) \to (X, x_0)$ are given as follows: The first obstruction is an element of $H^2(G_0, Z(\pi_1 F_f)$, where $Z(\ )$ denotes the center of a group. If this vanishes one can define a G_0 action on $\pi_n(F_f)$, $n > 1$ and the subsequent obstructions are in $H^{n+1}(G_0, \pi_n(F_f))$, $n \geq 2$. (F_f has a natural base point once x_0, y_0 are given).*

LEMMA 3.3. *The homotopy fiber of the map $i: \operatorname{map}_{G_p}(EG, W)_u \to \operatorname{map}(EG, W)$ (say over n) is path connected and its homotopy groups are p-profinite groups.*

3.2 and 3.3 imply the solution of the deformation $v \to \hat{v}$ in the proof of 3.1 and $H^i(G_q, M) = 0$ for $i > 0$, M a p-profinite group.

Proof. For any G_0 space M $\operatorname{map}_{G_0}(EG_0, M)$ is homeomorphic to the space of sections $EG_0 \times_{G_0} M \to BG_0$, thus deforming h is equivalent to the ordinary homotopy lifting problem

$$
\begin{array}{ccc}
* & \longrightarrow & EG_0 \times_{G_0} X \\
\downarrow & & \downarrow{\scriptstyle 1 \times_{G_0} f} \\
BG_0 & \xrightarrow{h'} & EG_0 \times_{G_0} Y
\end{array}
$$

This is equivalent to finding a section to $\tilde{f}: E \to BG_0$ where E is the homotopy pullback of $f \times_{G_0} EG_0$ and h'. The homotopy fiber of $\tilde{f}$ is homotopy equivalent to the homotopy fiber of f. As $F_f \approx F_{\tilde{f}}$ is path connected one has a short exact sequence

$$1 \to \pi_1(F_f) \to \pi_1(E) \to \pi_1(BG_0) = G_0 \to 1$$

and the first obstruction to obtain a section is the splitting of that exact sequence. This is an element in $H^2(G_0, Z(\pi_1 F_f))$ ([MacLane] Theorem IV 8.8). The vanishing of this obstruction yields a lifting of the second skeleton of BG_0:

$$\begin{tikzcd} & E \arrow[d] \\ B^2G_0 \arrow[r] \arrow[ur] & BG_0 \end{tikzcd}$$

and a homomorphism $G_0 \to \pi_1(E)$ which induces an action of G_0 on $\pi_n(E) \approx \pi_n(F_f)$, $n > 1$. The remaining obstructions are the classical ones for an ordinary extension problem

$$\begin{tikzcd} B^2G_0 \arrow[r, "\subset"] \arrow[d] & BG_0 \arrow[dl] \\ E & \end{tikzcd}$$

hence elements in $H^{n+1}(BG_0, \pi_n(E)) = H^{n+1}(G_0, \pi_n(F_f))$.

Proof of 3.3. As map(EG, W) is connected all homotopy fibers of i are homotopy equivalent. The homotopy fibre over u consists of all diagrams

$$\begin{tikzcd} EG \arrow[r, "h"] \arrow[d] & W \arrow[d, equal] \\ CEG \arrow[r] & W \arrow[d, equal] \\ * \arrow[u] \arrow[r, "u"] & W \end{tikzcd}$$

where h is a G_p-map that is G_p-homotopic to the constant map u. This is homeomorphic to a path component the space of pointed G_p-maps $M_{G_p}EG \to W$ where $M_{G_p}EG$ is the G_p-Hopf–Milnor construction of EG, i.e. the pushout of $EG \xrightarrow{\text{action}} G_p \times EG \to G_p \times CEG$. $(M_{G_p}EG \approx_{G_p} EG)$ the base point in $M_{G_p}EG$ is taken to be $1_{G_p} \times (\text{vertex } CEG)$ and the path component in $\mathrm{map}_{*G_p}(M_{G_p}EG, W)$ is of the constant map u. This space is homotopy equivalent to the path component of $\mathrm{map}_*(BG_p, EG \times_{G_p} W)_{u_0}$, $u_0: BG_p \xrightarrow{=} EG_p \times_{G_p} u = BG_p \subset EG \times_{G_p} U$. Now, the action of G_p on W can be extended to a $U(n)$ action by $U(n)$ acting trivially on $U(n_1)$ and naturally on $U(n)/G_q$ (this action does not extend to the

G action!). Hence the G_p action on W is simple (every $g \in G$ is homotopic to the identity as a map $W \to W$) and consequently as W is simply connected $EG \times_{G_p} W$ is nilpotent (n-simple for $n > 1$). It follows that $\pi_n(\mathrm{map}_* (BG_p, EG \times_{G_p} W), u_0)$ is an inverse limit of $\pi_n(\mathrm{map}_* (BG_p, (EG \times_{G_p} W)_r), u_0, r)$ $(EG \times_{G_p} W)_r$ – the r-th Postnikov section. These are finite p-groups as $H^r(BG_p, \pi_s(EG \times_{G_p} W))$ (simple coefficients) are such and 3.3 follows.

Remark 3.3.1. Note that $\pi_i(\mathrm{map}_*(BG_p, X), f)$ are not necessarily p profinite if X is not nilpotent and f is not null-homotopic. For example take $X = BG_p \vee BG_q$ and $f: BG_p \to BG_p \vee BG_q$ – the inclusion. If $\pi_1(\mathrm{map}_*(BG_p, BG_p \vee BG_q), f)$ is a p-profinite group one can lift the inclusion $BG_q^2 \to BG_p \vee BG_q$ of the second skeleton of BG_q to the unpointed function space $\mathrm{map}(BG_p, BG_p \vee BG_q)_f$. By adjointing this yields a map $BG_p \times BG_q^{(2)} \to BG_p \vee BG_q$. Now on the fundamental group one gets a splitting $G_p \times G_q \to G_p * G_g \to G_p \times G_q$ which is impossible.

Now Corollary F will follow from Theorem E if one shows:

LEMMA 3.4. *A group $G \in \mathscr{G}$ which is not a p-group for any p has a factor group of the form $G_k \rtimes G_l$ for some primes $k \neq l$.*

Proof. $G \in \mathscr{G}$ it has a normal decomposition $P \lhd H \lhd G$ where P is a p-group, H/P cyclic and G/H a q-group for some primes p, q. If G/P is a q-group we are done by a lemma of Schur that says that any extension of a p-group by a q-group for $p \neq q$ splits. If not there must be a prime $l \neq q$ such that $l \mid |H/P|$. Let S be the intersection of all the normal subgroups $T \subseteq H$ such that H/T is an l-group. Then $S \neq H$ and S is characteristic in H hence normal in G. The group H/S must itself be an l-group because it is embedded in a product of l-groups. Thus G/S has the required property for the primes l, q.

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Received January 13, 1986/May 20, 1987