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On normal representations of G-actions on projective varieties

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Introduction

Let G be a finite group. In this paper we examine a relation between G-actions on manifolds (in particular, on smooth projective varieties) and the polynomial invariants of G-representations.

Let $\Psi: G \to GL(V; \mathbb{C})$ be a G-representation in a complex n-dimensional vector space V with the property $V^G = \{\vec{0}\}$. Without loss of generality we assume that Ψ is faithful.

Given a finite set $\{\Psi_\gamma\}_{\gamma \in \Gamma}$ of such representations Ψ_γ of the same dimension n, one might ask the following question. For which finite sets Γ does there exist a smooth closed oriented G-manifold M^{2n} (G-action preserving the orientation) with the G-fixed point set M^G discrete and equal to Γ, and with the tangential G-representation at each $\gamma \in M^G \subseteq M$ SO-isomorphic to Ψ_γ? Here Ψ_γ is regarded as an orthogonal representation under the standard inclusion $U(n) \subset SO(2n)$ (thus, we are doing an oriented count of the fixed points in M). This question has a variety of versions which differ in the extra structures imposed on M (such as being stably-complex, algebraic, etc.) and in the restrictions on the isotropy groups of the G-action. If for a given set $\{\Psi_\gamma\}_{\gamma \in \Gamma}$ an appropriate G-manifold M exists, we say that Γ is *geometrically realizable* (in the given category).

In this paper we deal with a special case of the stated problem, namely, with the case where all Ψ_γ are pairwise isomorphic complex representations. Our main interest here is, for a given Ψ, to estimate the size of the subgroup of $\mathbb{Z}$, generated by the cardinalities $\#\Gamma$, Γ running over all geometrically realizable sets (it turns out that the "realizable" $\#\Gamma$ form a group using equivariant connected sums of G-manifolds and the effect of reversing orientations). We denote its additive generator by r_Ψ.

In this formulation of the problem the answer is often "trivial": $r_\Psi = 1$. For example, this is the case for any Ψ having no 1-dimensional G-invariant subspaces. In fact, for such Ψ its one copy is realizable on the complex projective space $\mathbb{P}(V \oplus \mathbb{C})$ with the G-action induced by $\Psi \oplus Id$. Another case, where the answer is often trivial (i.e., 1), but the argument is not, is the case where G is the cyclic group C_m of order m and Ψ is 1-dimensional. Then, by [5], $r_\Psi = 1$ if and only if m is not a power of a prime.

In the following we will work with a refined version of the problem, requiring further that any isotropy group G_x, $x \in M \backslash M^G$ (M realizing Γ) is contained in an appropriate isotropy group of a point of the unit sphere $S(V) \subset V$. This means that the G-action on $M \backslash M^G$ has to be not "less free" than the one on $S(V)$.

Let d_Ψ be the generator of the subgroup of $\mathbb{Z}$, generated by all $\#\Gamma$, where Γ is geometrically realizable in the aforementioned sense. Notice that $d_\Psi \mathbb{Z}$ is a subgroup of $r_\Psi \mathbb{Z}$ and thus r_Ψ divides d_Ψ. These numbers are not always equal. If $\Psi: C_{pq} \to U(1)$ is a faithful 1-dimensional representation, p, q distinct odd primes, then $r_\Psi = 1$ and $d_\Psi = pq$ [5].

If the G-action on $S(V)$ is free, the problem of computing d_Ψ is equivalent to the computation of the order of the element $[S(V)/G, f]$ in the oriented bordism group $\Omega_{2n-1}(BG)$ (or, for example, in the complex bordism group $U_{2n-1}(BG)$, depending on the structure imposed on G-manifolds). Here BG is the classifying space for G and $f: S(V)/G \to BG$ is the classifying map, produced by the G-action on $S(V)$. In particular, for $G \cong C_m$ one can ask how to compute d_Ψ in terms of the rotation numbers $j_1, \ldots, j_n$ characterizing the lense space $L_{2n-1}(m; j_1, \ldots, j_n) = S(V^n)/C_m$. By a spectral sequence argument in U-bordism it is easy to show that $[L_{2n-1}(m; j_1, \ldots, j_n), f]$, as an element of $U_{2n-1}(BC_m)$, can be represented by the formula

$$\sum_{l=0}^{n-1} \mathcal{S}_l(m; j_1, \ldots, j_n)[L_{2(n-l)-1}(m; 1, \ldots, 1), f_{n-l}], \tag{$*$}$$

where $\mathcal{S}_l(m; j_1, \ldots, j_n) \in U_{2l}(pt)$ and $f_{n-l}: L_{2(n-l)-1}(m; 1, \ldots, 1) \to BC_m$ denotes the classifying map. In fact, if one knows the classes $\{\mathcal{S}_l(m; j_1, \ldots, j_n)\}$, $\{\mathcal{S}_l(m; m+1, 1, \ldots, 1)\}$ and their images in $\Omega_{2l}(pt)$, then consecutively applying $(*)$ permits one, in principle, to describe all geometrically realizable Γ and, in particular, to compute d_Ψ. Therefore, the main difficulty is to compute $\{\mathcal{S}_l(m; j_1, \ldots, j_n)\}$. This was done in [7], [10] in terms of the Adams operations in U-bordism. Unfortunately, the Novikov–Kasparov methods do not give directly a precise numerical result (nevertheless, some nontrivial necessary conditions on the realizable Γ can be derived with their help). The best numerical result in this direction we know is the Conner–Floyd computation of the order of $[L_{2n-1}(p^k; 1, \ldots, 1), f_n]$ for an odd prime p [4]. It is equal to p^{k+a}, where a is defined by the condition $a(p-1) \leq n < (a+1)(p-1)$, and so grows exponentially with the dimension n.

If G does not act freely on $S(V)$, then one can still formulate the problem we are discussing in terms of stratified bordism of an appropriate (stratified) classifying space. However, the complexity of the stratified category and the lack of G-transversality makes it difficult to use this approach. We don't know any results of this sort for nonsemifree actions, besides those obtained in the present paper.

Let us describe the plan of the paper. In Section 1 we present an algorithm for constructing examples of smooth algebraic G-actions on complex projective varieties M with a prescribed normal representation $\Psi: G \to GL(V; \mathbb{C})$ at the set of isolated G-fixed points. This algorithm is the core of the proof of our main result – Theorem A. It exploits a new relationship between the geometrical problems stated above and the invariant theory of finite groups. We employ special sets of G-invariant polynomials on V, called projectively $\mathcal{K}$-free systems (see Definition 1), to construct such an M. Then the cardinality of M^G is expressed in terms of the degrees of these polynomials (see Theorem A). An important step in our construction uses Hironaka's Desingularization Theorem [6, Th. 7.1] which makes it possible to resolve equivariantly the singularities of a G-variety.

In this way we get some upper bounds for d_Ψ. Among other results we prove that, if $\Psi(G)$ acts on V as a group generated by pseudo-reflections [12], then d_Ψ divides the group order $|G|$ (See Corollary 2).

In Section 2 we illustrate the results of Section 1 by estimating and calculating d_Ψ for two special cases. In the first one we investigate the case of cyclic actions. Theorem A provides us with some upper bounds for d_Ψ. Combining this information with the lower bounds for d_Ψ, arising from simple considerations of bordism spectral sequences or from familiar number-theoretical considerations of the Atiyah–Bott G-signature theorem, we estimate d_Ψ and in some cases compute it entirely (see Propositions 2, 3). This should be compared with the only previously known results on semi-free cyclic actions of an odd prime power period [4], in which case our estimates may lead to a weaker result than the actual calculation of d_Ψ by Conner–Floyd, but the gap is not big and does not depend on the dimension of the representation Ψ.

The second case deals with the faithful 2-dimensional representations Ψ given by the inclusion of any subgroup of $SU(2)$. Since for those representations the structure of the polynomial invariants is classically known, Theorem A can be applied effectively. With some simple bordism considerations, this gives a calculation of d_Ψ for all but the cyclic subgroups of $SU(2)$ (in this case the ambiguity is of order 2) (See Proposition 3). These examples suggest tnat the categories of algebraic and, say, stably complex G-actions are perhaps different from the point of view of geometrically realizable Γ's. Such a result would be very interesting.

§1. Constructing smooth algebraic actions with a prescribed normal representation

To formulate the results we start with a few definitions. R_Ψ will denote the graded algebra of complex polynomials on the n-dimensional space V of the representation Ψ. Let $R^G_\Psi \subset R_\Psi$ be the subalgebra of G-invariant polynomials.

Denote by $\mathcal{S}(G)$ the set of all subgroups of G. For any subset $\mathcal{H} \subseteq \mathcal{S}(G)$ define its closure $\bar{\mathcal{H}}$ as $\{H \in \mathcal{S}(G) \mid H \subseteq H', H' \in \mathcal{H}\}$. $\mathcal{H}$ is *closed* if $\mathcal{H} = \bar{\mathcal{H}}$. $\mathcal{K} \subseteq \mathcal{S}(G)$ is *open* if its complement $\mathcal{S}(G)\backslash\mathcal{K}$ is closed.

Let $\mathbb{P}(V)$ be the complex projective space associated with V and carrying the natural linear G-action induced by Ψ.

DEFINITION 1. Let $\mathcal{K} \subseteq \mathcal{S}(G)$ be an open set. Homogeneous polynomials $P_1, \dots, P_s \in R_\Psi$ form a *projectively $\mathcal{K}$-free system* $\mathcal{P}$ if the intersection of the G-invariant subvariety $\mathcal{X}_\mathcal{P} \subseteq \mathbb{P}(V)$, defined by the equations $\{P_i = 0\}$, $1 \leq i \leq s$, with the H-fixed set $\mathbb{P}(V)^H$, is empty for any $H \in \mathcal{K}$.

Given $H \subseteq G$ and its one-dimensional character $\chi : H \to \mathbb{C}^*$, denote by V_χ the subspace of V, formed by the vectors $\vec{v}$ with the property $\Psi(h)(\vec{v}) = \chi(h)\vec{v}$ for any $h \in H$. Then Definition 1 means that for any $H \in \mathcal{K}$ and $\chi : H \to \mathbb{C}^*$ the equations $\{P_i = 0\}$, $1 \leq i \leq s$, restricted to V_χ, have only the trivial solution $\vec{0} \in V$. Consequently, the number of the equations s has to be at least $\max_{H \in \mathcal{K}, \chi}(\dim V_\chi)$. Thus, $s \geq \max_{H \in \mathcal{K}, \chi} \langle \mathrm{Res}_H \Psi, \chi \rangle$, where $\langle \mathrm{Res}_H \Psi, \chi \rangle$ is the multiplicity of χ in the representation $\mathrm{Res}_H \Psi$. Definition 1 implies that the H-action on $\mathcal{X}_\mathcal{P}$ is free for any H satisfying $H \cap H' = 1$ for every $H' \in \mathcal{S}(G)\backslash\mathcal{K}$. In [2] W. Browder and N. Katz proved that if a projective variety $\mathcal{X} \subset \mathbb{P}(V)$ is invariant under a linear G-action on $\mathbb{P}(V)$ and this action is free on $\mathcal{X}$, then the coefficients of the Hilbert polynomial of $\mathcal{X}$ are divisible by $|G|$. In particular, the degree $\deg \mathcal{X}$ is divisible by $|G|$. Therefore one gets

LEMMA 1. a) *For any projectively $\mathcal{K}$-free system $\mathcal{P}$ and any $H \in \mathcal{S}(G)$ satisfying $H \cap H' = 1$ for each $H' \in \mathcal{S}(G)\backslash\mathcal{K}$, the degree* $\deg \mathcal{X}_\mathcal{P}$ *is divisible by $|H|$. In particular, for such H, if $\mathcal{X}_\mathcal{P}$ is a complete intersection, then $\prod_{i=1}^s \deg P_i$ is divisible by $|H|$.*

b) *Moreover,* $\dim \mathcal{X}_\mathcal{P} \leq \dim V - \max_{H \in \mathcal{K}, \chi} \langle \mathrm{Res}_H \Psi, \chi \rangle$.

DEFINITION 2. The homogeneous polynomials $P_1, \dots, P_n \in R^G_\Psi$, $n = \dim V$, form a *regular system of parameters* if the system of equations $\{P_i = 0\}$, $1 \leq i \leq n$, has only the trivial solution $\vec{0} \in V$.

An alternative definition: R^G_Ψ is a finite $\mathbb{C}[P_1, \dots, P_n]$-module. Since for a finite G, R_Ψ is a finite R_Ψ^G-module [12, Th. 1.3] to verify the equivalence of the definitions it suffices to show that $\mathcal{P} = \{P_1, \ldots, P_n\}$ is a regular system of parameters (we view $\mathcal{P}$ also as a polynomial map $\mathcal{P}: V \to V$) iff the induced ring homomorphism $\mathcal{P}^*: R_\Psi \to R_\Psi$ introduces in R_Ψ a structure of a finitely generated R_Ψ-module. This happens iff the map $\mathcal{P}: V \to V$ is proper and with finite fibers. By [9, Lemma 3.11] $\mathcal{P}^{-1}(\vec{0}) = \vec{0}$ implies that locally, in a neighbourhood of $\vec{0}$, the map $\mathcal{P}$ is proper. By homogeneity of $\mathcal{P}$, it is proper everywhere. In particular, its fibers have to be finite. The inverse implication is obvious: if a nonzero vector belongs to $\mathcal{P}^{-1}(\vec{0})$, then the whole line, spanned by it, belongs to $\mathcal{P}^{-1}(\vec{0})$. Thus, $\mathcal{P}$ fails to be proper and $\mathcal{P}^*$ does not induce in R_Ψ a structure of a finite R_Ψ-module.

Since $\mathcal{Z}_{\mathcal{P}} \subset P(V)$ is empty for a regular system of parameters $\mathcal{P}$, such a $\mathcal{P}$ is a projectively $\mathcal{K}$-free system for any open $\mathcal{K} \subset \mathcal{S}(G)$.

For a regular system of parameters $\mathcal{P} = \{P_1, \ldots, P_n\}$ one has a useful formula: $\prod_{i=1}^n \deg P_i = |G| \dim_{\mathbb{C}} (R_\Psi^G/(P_1, \ldots, P_n))$, where $(P_1, \ldots, P_n)$ denotes the ideal of R_Ψ^G, generated by $\{P_1, \ldots, P_n\}$ (C.f. cor. 4.3 in [12]).

Let $\mathcal{H}_\Psi \subseteq \mathcal{S}(G)$ be defined as $\{G_x \in \mathcal{S}(G) \mid x \in S(V)\}$. Denote by $\mathcal{K}_\Psi$ the open set $\mathcal{S}(G) - \bar{\mathcal{H}}_\Psi$.

Now we are in a position to formulate the main result.

THEOREM A. *Let $\Psi: G \to GL(V; \mathbb{C})$ be a complex representation of a finite group G with the properties*: Ker $\Psi = 1$, $V^G = \{\vec{0}\}$. *Then, given any projectively $\mathcal{K}_\Psi$-free system $\mathcal{P} = \{P_1, \ldots, P_s\}$ of homogeneous G-invariant polynomials, there exists a smooth complex projective variety M with a smooth algebraic action on it, and such that M^G consists of $\prod_{i=1}^s \deg P_i$ isolated points. Moreover, the complex tangential G-representation at each point of M^G is isomorphic to the given Ψ. The G-action on $M \backslash M^G$ is not "less free" than the one on the unit sphere $S(V) \subset V$, i.e. for each $x \in M \backslash M^G$ the isotropy group G_x is contained in an appropriate isotropy group of some point of $S(V)$.*

Proof. Let W be an $(s+1)$-dimensional complex vector space. Consider the projective $(n+s)$-dimensional $(n = \dim V)$ space $\mathbb{P}(V \oplus W)$ with the G-action induced by the representation $\Psi \oplus Id: G \to GL(V \oplus W; \mathbb{C})$.

We are going to construct a G-invariant n-dimensional subvariety $M \subset \mathbb{P}(V \oplus W)$ employing a given projectively $\mathcal{K}_\Psi$-free system $\mathcal{P} = \{P_1, \ldots, P_s\}$, $P_i \in R_\Psi^G$. Define M by the system of homogeneous polynomial equations:

$$\{P_i(\vec{v}) + Q_i(\vec{w}) = 0,\ \vec{v} \in V,\ \vec{w} \in W;\ 1 \leq i \leq s\}. \tag{A}$$

Here the homogeneous polynomials $Q_i: W \to \mathbb{C}$ are chosen so as to satisfy the following conditions: 1) $\deg Q_i = \deg P_i$, 2) the equations $\{Q_i(\vec{w}) = 0\}$ define a 0-dimensional smooth complete intersection Z in $\mathbb{P}(W)$ ($\dim \mathbb{P}(W) = s$ and the differentials $\{dQ_i\}$, $1 \leq i \leq s$, are in a general position at points of Z). It is easy to check that such polynomials $\{Q_i\}$ exist.

Note that the variety M defined by (A) is invariant under the linear G-action on $\mathbb{P}(V \oplus W)$. Moreover, M is nonsingular in a Zariski neighborhood of $Z \subset \mathbb{P}(W) \subset \mathbb{P}(V \oplus W)$ because the differentials $\{dP_i(\vec{v}) + dQ_i(\vec{w})\}$ are in a general position at Z by the choice of $\{Q_i\}$.

Now we analyze closely the orbit-type stratification on $\mathbb{P}(V \oplus W)$ to understand the induced stratification on M. Let H be a subgroup of G. Consider the H-invariant decomposition $V^H \oplus V_{(H)}$ of V, where $V_{(H)}$ denotes the orthogonal complement of V^H in V with respect to a G-invariant hermitian metric. Then $\mathbb{P}(V \oplus W)^H = \mathbb{P}(V_{(H)})^H \amalg \mathbb{P}(V^H \oplus W)$. Therefore

$$M^H = M \cap \mathbb{P}(V \oplus W)^H = M \cap \mathbb{P}(V^H \oplus W) \coprod M \cap \mathbb{P}(V_{(H)})^H.$$

Since $\{P_1, \dots, P_s\}$ is projectively $\mathscr{K}_\Psi$-free system in the sense of Definition 1, $M \cap \mathbb{P}(V_{(H)})^H \subseteq M \cap \mathbb{P}(V)^H$ is empty for $H \in \mathscr{K}_\Psi$ (note that $P_i + Q_i$ coincides with P_i, restricted to $\mathbb{P}(V) \subset \mathbb{P}(V \oplus W)$). Consequently $M^H = M \cap \mathbb{P}(V^H \oplus W)$ for $H \in \mathscr{K}_\Psi$. Note also that M is smooth and transversal to $\mathbb{P}(W)$ in a Zariski neighborhood of Z by the choice of $\{Q_i\}$. This implies that for any H the variety M is transversal to $\mathbb{P}(V^H \oplus W) \supseteq \mathbb{P}(W)$ in a Zariski neighborhood U of Z. Thus, the normal bundle $\nu(\mathbb{P}(V^H \oplus W), \mathbb{P}(V \oplus W))$, restricted to $M^H \cap U$, is equivariantly isomorphic to $\nu(M^H, M) \mid M^H \cap U$. On the other hand, the fiber of the bundle $\nu(\mathbb{P}(W), \mathbb{P}(V \oplus W))$ is equivariantly V. Therefore, using the fact that $V^G = \{\vec{0}\}$ and that $G \in \mathscr{K}_\Psi$, one gets $M^G = Z$. Moreover, the tangential complex G-representations at the points of M^G are all isomorphic to Ψ.

Note that the isotropy groups (or even slice-types) of the manifold $\mathbb{P}(V \oplus W) \setminus [\mathbb{P}(V) \amalg \mathbb{P}(W)]$ are the same as the ones for the $\Psi(G)$-action on $S(V)$ (the space of $\nu(\mathbb{P}(W), \mathbb{P}(V \oplus W))$ is $\mathbb{P}(V \oplus W) \setminus \mathbb{P}(V)$). Therefore the points of $M \setminus [(M \cap \mathbb{P}(V)) \amalg M^G]$ do not contribute "new" isotropy groups in addition to the isotropy groups of $S(V)$. The only "new contribution", as we have seen, might come from $M \cap \mathbb{P}(V)$. But the last possibility is ruined by the choice of the projectively $\mathscr{K}_\Psi$-free system $P_1, \dots, P_s$ (since $M \cap \mathbb{P}(V)^H = \varnothing$ for every $H \in \mathscr{K}_\Psi$). Note that this argument implies also that $M \cap [\mathbb{P}(V^H \oplus W) \setminus \mathbb{P}(W)] = \varnothing$ for $H \in \mathscr{K}_\Psi$. Consequently, $M^H = M^G$ for $H \in \mathscr{K}_\Psi$.

Finally, note that $\# M^G = \prod_{i=1}^s \deg P_i$ for the complete intersection $Z = M^G$ in $\mathbb{P}(W)$.

This would be the end of the proof if M were nonsingular and the action on it were smooth. But our construction guarantees that this is the case only in some

Zariski neighborhood of $M^G \subset M$. To overcome this difficulty we will apply a quite remarkable principle (Proposition 1), which is an immediate corollary of Hironaka's Desingularization Theorem 7.1 [6]. This theorem implies that one can resolve the singularities of complex analytic (or algebraic, projective) G-varieties (G – a finite group) equivariantly. (In fact, Hironaka's theorem claims much more. There exists a nonsingular resolution, such that any automorphism of the original singular variety can be uniquely lifted to an automorphism of the resolution.)

Let $\mathcal{K} \subseteq \mathcal{S}(G)$ be an open subset, invariant under the conjugation by elements of G.

PROPOSITION 1. *Let G be a finite group, acting as a group of automorphisms of a complex analytic (algebraic, projective) variety M (which is countable at infinity). Assume that the singularities of M do not intersect the set $\bigcup_{H \in \mathcal{K}} M^H$ (i.e. M is nonsingular at the "$\mathcal{K}$-singularities" of the action). Then there exists a smooth analytic (algebraic, projective) G-variety $\tilde{M}$, such that some Zariski neighborhood of $\bigcup_{H \in \mathcal{K}} \tilde{M}^H$ in $\tilde{M}$ is equivariantly isomorphic to a Zariski neighborhood of $\bigcup_{H \in \mathcal{K}} M^H$ in M.* □

In particular, from the point of view of differential topology the following holds. If one can realize a collection of smooth equivariant normal bundles $\{\nu(M^H, M)\}$, $H \in \mathcal{K}$, on a complex analytic G-variety M (without boundary), then the same local data is realizable on a smooth (complex) G-manifold (without boundary).

Now to conclude the proof of the Theorem it is sufficient to describe the relations between orbit-types of M constructed above, and the equivariant resolution $\tilde{M}$ of M related to M by a "resolving" map $\pi : \tilde{M} \to M$. The continuity of π implies that $G_{\tilde{x}} \subseteq G_{\pi(\tilde{x})}$ for every $\tilde{x} \in \tilde{M}$. We have seen that M is nonsingular at least in some neighborhood of M^G in M^G in M. Recall that $G_x \in \bar{\mathcal{H}}_\Psi$ for every $x \in M \backslash M^G$. Consequently, $\tilde{M}^G$ in $\tilde{M}$ and M^G in M have G-isomorphic Zariski neighborhoods, moreover, since $\bar{\mathcal{H}}_\Psi$ is closed, $G_{\tilde{x}} \in \bar{\mathcal{H}}_\Psi$ for every $\tilde{x} \in \tilde{M} \backslash \tilde{M}^G$. □

One can derive a few corollaries from the Theorem.

COROLLARY 1. *There exists an oriented smooth closed G-manifold M, the G-action preserving the orientation and M^G a finite set, such that*

1) *the tangential G-representation at each point of M^G is SO-isomorphic to the given Ψ,*

2) *$\# M^G$ divides $\prod_{i=1}^s \deg P_i$ for any projectively $\mathcal{K}_\Psi$-free system $\{P_1, \ldots, P_s\}$.*

In particular, $\#M^G$ *divides* $|G| \cdot \dim_{\mathbb{C}}(R^G_\Psi/(P_1, \ldots, P_n))$ *for any regular system of parameters* $P_1, \ldots, P_n$.

3) $G_x \in \mathcal{H}_\Psi$ *for every* $x \in M \backslash M^G$.

Proof. Consider integral linear combinations of the G-actions constructed in the Theorem (the "negative" of a G-action means the change of the orientation for the corresponding G-manifold). Note that if the fixed point set M^G is "virtually" zero, then, by equivariantly attaching 1-handles to M at M^G, one gets a new smooth G-manifold with no G-fixed points. Of course, this operation will destroy the analytic (algebraic) structure of M. Now any subgroup of $\mathbb{Z}$, generated by some integers is, in fact, generated by finitely many of them, which completes the proof. □

COROLLARY 2. a) *For any regular system of parameters* $P_1, \ldots, P_n \in R^G_\Psi$ *there exists a smooth complex projective variety* M, *such that* $\#M^G = |G| \cdot \dim_{\mathbb{C}}(R^G_\Psi/(P_1, \ldots, P_n))$ (*which is also the degree of the polynomial map* $\mathcal{P}: V \to V$ *given by* $P_1, \ldots, P_n$). *Moreover, the complex tangential* G-*representation at each point of* M^G *is isomorphic to the given* Ψ, *and the isotropy groups* G_x *belongs to* $\mathcal{H}_\Psi$ *for any* $x \in M \backslash M^G$.

b) *In particular, if* $\Psi(G)$ *acts on* V *as a group generated by pseudo-reflections, one can construct such* M *with* $\#M^G = |G|$.

c) *If the variety* $\mathcal{X}_{\mathcal{P}} \subseteq \mathbb{P}(V)$, ($\mathcal{P}$ *projectively* $\mathcal{H}_\Psi$-*free*) *is a complete intersection*[1] *then one can construct an appropriate* M *with* $\#M^G = \prod_{i=1}^s \deg P_i = \deg \mathcal{X}_{\mathcal{P}}$. *This number* $\#M^G$ *is divisible by* $|H|$ *for each* $H \subseteq G$ *satisfying* $H \cap H' = 1$ *for any* $H' \in \mathcal{H}_\Psi$.

Proof. Since for any regular system of parameters $\mathcal{P} = \{P_1, \ldots, P_n\}$ the variety $\mathcal{X}_{\mathcal{P}} \subset \mathbb{P}(V)$ is empty, $\mathcal{P}$ is a $\mathcal{H}$-free system for any $\mathcal{H}$, and, in particular, for $\mathcal{H} = \mathcal{H}_\Psi$. It is known (c.f. Cor. 4.3 [12]) that for a regular system $\mathcal{P}$ we have $\prod_{i=1}^n \deg P_i = |G| \cdot \dim_{\mathbb{C}}(R^G_\Psi/(P_1, \ldots, P_n))$. It is also known that the last expression equals $|G|$ if and only if $\Psi(G)$ acts as a group generated by pseudo-reflections (c.f. Th. 4.1 and Cor. 4.4 in [12]). c) follows from Lemma 1. □

For any representation $\Psi: G \to GL(V, \mathbb{C})$ denote by A_Ψ the minimal positive integer such that the function $g \to A_\Psi \operatorname{Det}[(\Psi(g) + I)(\Psi(g) - I)^{-1}]$, well-defined for $g \in G \backslash \bigcup_{x \in S(V)} G_x$, extends to a character of a virtual G-representation.

COROLLARY 3. $A_{\operatorname{Res}_H \Psi}$ *divides* d_Ψ *for any* $H \subseteq G$. *Since* $d_\Psi \mid \prod_{i=1}^s \deg P_i$, $A_{\operatorname{Res}_H \Psi} \mid \prod_{i=1}^s \deg P_i$ *for any projectively* $\mathcal{H}_\Psi$-*free system* $\mathcal{P}$.

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[1] This happens iff $(P_1, \ldots, P_s)$ can be completed to a system of regular parameters for R^G_Ψ.

Proof. Note that for M, constructed with the help of $\mathscr{P}$ in the Theorem, $M^g = M^G$ for $g \in G \setminus \bigcup_{x \in S(V)} G_x$ (since $g \in \mathscr{K}_\Psi$ for such a g). In particular, it will be the case for $g \in H \setminus \bigcup_{x \in S(V)} H_x$, $H_x = G_x \cap H$. Now apply the Atiyah–Bott formula [1] to compute the g-signature Sign(g, M) of M. Use the fact that the contribution of each fixed point to Sign(g, M) (which is the value of the character Sign(G, M) at g) is $\mathrm{Det}\,[(\Psi(g) + I)(\Psi(g) - I)^{-1}]$ when M^g consists of isolated points [1]. □

§2. Calculating d_Ψ

Example 1 (*Cyclic groups*). Let $g \in C_m$ be a fixed generator. Let $\Psi: C_m \to GL(V; \mathbb{C})$ be an n-dimensional faithful representation. Correspond to the matrix $\Psi(g)$ its eigenvalues $\{\lambda^{j_s}\}$, $1 \le s \le n$, $0 < j_s < m$, where $\lambda = \exp(2\pi i/m)$, each λ^{j_s} counted according to its multiplicity. One can order the set $\{\lambda^{j_s}\}$ and associate with it the function $\omega: \{1, 2, \ldots, n\} \to \{1, 2, \ldots, m-1\}$ where $\omega(s) = j_s$.

Let $\omega_k: \{1, 2, \ldots, n\} \to \{0, 1, \ldots, m-1\}$, $1 \leqslant k \leqslant m$, be a new function defined by $\omega_k(s) = k\omega(s) (\mathrm{mod.}\, m)$. Note that $\dim_{\mathbb{C}} V^{\langle g^k \rangle} = \# \omega_k^{-1}(0)$. Thus, the isotropy groups of the sphere $S(V)$ are characterized in terms of ω as

$$\{\langle g^k \rangle : \omega_k^{-1}(0) \ne \varnothing,\ \#\omega_k^{-1}(0) > \#\omega_{k'}^{-1}(0)\ (k' \,|k|\, m,\ k' < k < m)\}.$$

Denote the set of such k's by $\mathscr{H}(\omega)$ and the set $\{k : k \mid m \text{ and } \omega_k^{-1}(0) \ne \varnothing\}$ by $\bar{\mathscr{H}}(\omega)$. Let $\mathscr{K}(\omega)$ be the complement of $\bar{\mathscr{H}}(\omega)$ in the set of all divisors of m. Note that with our previous notations $\{\langle g^k \rangle\}_{k \in \mathscr{K}(\omega)} = \mathscr{K}_\Psi$ and $\{\langle g^k \rangle\}_{k \in \bar{\mathscr{H}}(\omega)} = \bar{\mathscr{H}}_\Psi$. It is clear that $\dim_{\mathbb{C}} [\mathbb{P}(V)^{\langle g^k \rangle}] = \max_{0 \le j < m} \{\#\omega_k^{-1}(j)\} - 1$. Put $t(\omega) = \max_{k \in \mathscr{K}(\omega)}(\max_j \{\#\omega_k^{-1}(j)\})$, the maximal dimension of eigenspaces $V_{\lambda^j} \subseteq V$ for $\Psi(g^k)$, $k \in \mathscr{K}(\omega)$.

(a) We shall construct in two different ways a projectively $\mathscr{K}_\Psi$-free system $\mathscr{P}$ of $t(\omega)$ polynomials $P_1, \ldots, P_{t(\omega)} \in R_\Psi^{C_m}$.

(i) Choose each P_l, $1 \le l \le t(\omega)$, to be of the form $\sum_{s=1}^n a_{ls} z_s^m$, where $\{z_s\}_{s=1}^n$, the coordinate functions on the representation space V are compatible with the splitting of V into $\{V_{\lambda^j}\}$. It is easy to see that one can choose $\{a_{ls}\}$, $1 \le l \le t(\omega)$, $1 \le s \le n$, in such a way that the corresponding invariant subvariety $\mathscr{Z}_{\mathscr{P}}$ in $\mathbb{P}(V)$ will miss any given finite system of projective subspaces $\{\mathbb{P}(W_\alpha) \subseteq \mathbb{P}(V)\}_\alpha$, $\dim_{\mathbb{C}} W_\alpha \le t(\omega)$. In particular, it can miss the subspaces $\mathbb{P}(V)^{\langle g^k \rangle}$, $k \in \mathscr{K}(\omega)$. Thus, $\prod_{l=1}^{t(\omega)} \deg P_l = m^{t(\omega)}$, and by Theorem A, $m^{t(\omega)}$ copies of Ψ are realizable on a smooth projective C_m-variety. Consequently, $d_\Psi \mid m^{t(\omega)}$. For instance, if $\Psi(g)$ has an eigenvalue λ^{kr}, $(r, m/k) = 1$, for each proper divisor k of

m, then $\tilde{\mathcal{H}}(\omega)$ consists of all divisors of m, but 1, $\mathcal{H}(\omega)=\{1\}$, and $t(\omega)=t_\Psi$, the maximal multiplicity of eigenvalues of $\Psi(g)$. Thus for such Ψ, $d_\Psi \mid m^{t_\Psi}$.

(ii) Pick a number $l, 1 \leq l \leq t(\omega)$. We view ω as an element $\vec{\omega}=(\omega(1), \ldots, \omega(n))$ of the lattice $\mathbb{Z}_+^n \subset \mathbb{R}_+^n$, where $\mathbb{R}_+$ denotes the set of non-negative real numbers. We say that $d \in \mathbb{Z}_+$ is *(l, ω)-admissible* (in the strong sense) if for each $(l-1)$ subsimplex Δ_β^{l-1}, $1 \leq \beta \leq \binom{n}{l}$, of the simplex $\Delta^{n-1}(d)=\{a_1+\cdots+a_n=d\} \cap \mathbb{R}_+^n$ there exists a vector $\vec{a}_\beta \in [\operatorname{Int} \Delta_\beta^{l-1}(d)] \cap \mathbb{Z}_+^n$, such that $\langle \vec{a}_\beta, \vec{\omega} \rangle \equiv 0 \,(\text{mod.}\, m)$. Here $\langle \cdot, \cdot \rangle$ denotes the standard scalar product in $\mathbb{R}^n$.

One can weaken the requirements on d, by considering in the previous definition only the simplexes $\Delta_\beta^{l-1}(d)$, spanned by vertices $\{\vec{b}^{(i)}\}$, with the coordinates $(\vec{b}^{(i)})_s = d \cdot \delta_{is}$, $1 \leq s \leq n$, where δ_{is} is the Kronecker function and i runs over a subset of $\omega_k^{-1}(j)$ for some $k \in \tilde{\mathcal{H}}(\omega)$ and some $j \in \operatorname{Im}(\omega_k)$. Note that such a simplex $\Delta_\beta^{l-1}(d)$ is contained in $\bigoplus_{\{j': kj'=j\}} V_{\lambda^{j'}}$ and $\mathbb{P}(V)^{\langle g^k \rangle} = \coprod_{j \in \operatorname{Im}(\omega_k)} \mathbb{P}(\bigoplus_{\{j': kj'=j\}} V_{\lambda^{j'}})$. In fact, the second definition of an (l, ω)-admissible number suffices for the purposes of our construction, although the first one is more convenient and will be used subsequently. For example, d is $(1, \omega)$-admissible iff $d \cdot \omega(s) \equiv 0 \,(\text{mod.}\, m)$ for each $1 \leq s \leq n$ (i.e. $d \equiv 0 \,(\text{mod.}\, m)$, since $\Psi(g)$ is faithful); d is $(2, \omega)$-admissible iff for each piar s, s' of distinct integers between 1 and n there exists an integer $t, 0<t<d$, such that $t\omega(s)+(d-t)\omega(s') \equiv 0 \,(\text{mod.}\, m)$, and so forth.

It can be checked that an (l, ω)-admissible number always exists and is not less than l. Moreover, if d is (l, ω)-admissible, so is $d+m$.

Given an (l, ω)-admissible d, employing $\{\vec{a}_\beta\}$ one can construct an *invariant* homogeneous polynomial $\tilde{P}_l$ of degree $d_l = d$. Put $\tilde{P}_l = \sum_{\beta=1}^{\binom{n}{l}} z_1^{a_1(\beta)} \cdot \cdots \cdot z_n^{a_n(\beta)}$, where $\{a_s(\beta)\} = \vec{a}_\beta$ (here the coordinate functions $z_1, \ldots, z_n$ are consistent with the splitting of V by $\{V_{\lambda^j}\}$). Note that precisely l coordinates of $\vec{a}_\beta$ are nonzero. It can be verified inductively that such a system $\mathcal{P}=\{\tilde{P}_l\}$, $1 \leq l \leq t(\omega)$, is projectively $\mathcal{H}_\Psi$-free: $\mathfrak{X}_{\mathcal{P}}$ misses the projectivization of any $t(\omega)$-dimensional subspace $V_{\lambda^{k\omega(s)}}$, $k \in \tilde{\mathcal{H}}(\omega)$, spanned by the vectors of the preferred basis in V. Thus, there exists a smooth projective C_m-variety, realizing $\prod_{l=1}^{t(\omega)} d_l$ copies of Ψ, where d_l can be any (l, ω)-admissible number.

As an illustration, take a two-dimensional Ψ with $\vec{\omega}=(\eta, -\eta)$, $(\eta, m)=1$. For odd m, $z_1^m + z_2^m$ forms a projectively $\mathcal{H}_\Psi$-free system: the roots of $z_1^m + z_2^m$ in $\mathbb{P}(\mathbb{C}^2)$ differ from $[1:0]$ and $[0:1]$ (at the same time m is $(1, \omega)$-admissible). Thus, $d_\Psi \mid m$ (in fact, it will follow from our further discussion of the semi-free category that $d_\Psi = m$). For even m, $z_1 z_2$ and $z_1^m + z_2^m$ form a projectively $\mathcal{H}_\Psi$-free system. Note that $2=\deg(z_1 z_2)$ is $(2, \omega)$-admissible: $\langle (1,1), (\eta, -\eta) \rangle \equiv 0 \,(\text{mod.}\, m)$. Also, $m=\deg(z_1^m + z_2^m)$ is $(1, \omega)$-admissible: $\langle (m, 0), (\eta, -\eta) \rangle \equiv 0 \equiv \langle (0, m), (\eta, -\eta) \rangle (\text{mod.}\, m)$. Therefore, for even m, $d_\Psi \mid 2m$. Further arguments show that $m \mid d_{\Psi}$ and, moreover, the formula (4) of [7] implies that actually in the stably complex category of actions, $d_{\Psi} = 2m$.

(b) In line with Corollary 3 one can estimate d_{Ψ} from below. This is done by computing the minimal positive integer $A(\omega)$ for which all the numbers $A(\omega) \cdot \prod_{s=1}^{n} (\lambda^{kj_s} + 1)/(\lambda^{kj_s} - 1)$, $k \in \mathcal{K}(\omega)$, i.e. the "candidates" for Sign (g^k, M), are algebraic integers.

This computation is analogous to the one in [1], concerned with the case $m = p^l$, p – an odd prime, so, we omit the details.

To describe $A(\omega)$ we introduce some notation. Let $2^b \prod_{p|m} p^{c(p)}$ be the primary decomposition of m.

For any rational number $r = q/l$, $(q, l) = 1$, and any odd prime p define an auxiliary number

$$v(p, r) = \begin{cases} (-1)^{\varepsilon}/p^{e-1}(p-1) & \text{if } l \text{ is of the form } 2^{\varepsilon} \cdot p^{e}, \\ & \text{where } \varepsilon = 0,\ 1 \text{ and } e > 1 \\ 0 & \text{otherwise.} \end{cases}$$

For any $k \in \mathcal{K}(\omega)$ consider the number $a_k(p, \omega) = \lceil \sum_{s=1}^{n} v(p, k \cdot \omega(s)/m) \rceil_+$, where $\lceil x \rceil_+$ denotes the smallest non-negative integer greater or equal to x.

Then $A(\omega) = \prod_{p|m} p^{\max_{k \in \mathcal{K}(\omega)} \{a_k(p, \omega)\}}$. Note that if for any s, $\omega'(s) = q_s \cdot \omega(s)$, where $(q_s, m) = 1$, then $A(\omega) = A(\omega')$.

Summarizing (a) and (b), we have proved

PROPOSITION 2. (i) $\prod_{p|m} p^{\max_{k \in \mathcal{K}(\omega)} \{\lceil \sum_{s=1}^{n} v(p, k \cdot \omega(s)/m) \rceil_+\}}$ *divides* d_{Ψ}.

(ii) d_{Ψ} *divides* $m^{t(\omega)}$, *as well as* $\prod_{l=1}^{t(\omega)} d_l$, *where* $t(\omega) = \max_{k \in \mathcal{K}(\omega)} \{\max_j (\# \omega_k^{-1}(j))\}$ *and* d_l *can be any* (l, ω)*-admissible number.* □

For example, if $m = 15$, $n = 3$, and the eigenvalues of $\Psi(g)$ are λ, λ^3, λ^5, then $\omega: \{1, 2, 3\} \to \{1, 2, 5\}$. The computation of $\{\omega_k\}$, $k \mid 15$, shows that $\mathcal{K}(\omega) = \{1\}$, and $t(\omega) = 1$. Thus, $d_{\Psi} \mid 15$. On the other hand, $v(3, \frac{1}{15}) = 0$, $v(3, \frac{1}{5}) = 0$, $v(3, \frac{1}{3}) = \frac{1}{2}$; $v(5, \frac{1}{15}) = 0$, $v(5, \frac{1}{5}) = \frac{1}{4}$, $v(5, \frac{1}{3}) = 0$. Hence $A(\omega) = 3^{\lceil 1/2 \rceil_+} \cdot 5^{\lceil 1/4 \rceil_+} = 15$. Consequently, $d_{\Psi} = 15$.

The number $t(\omega)$ grows with the growth of the multiplicities. In this case the use of (l, ω)-admissible numbers becomes essential. For example, if $m = 15$, $n = 4$ and $\omega: \{1, 2, 3, 4\} \to \{1, 3, 3, 5\}$ (compare with the previous example), then still $\mathcal{K}(\omega) = \{1\}$, but $t(\omega) = 2$. Of course, 15 is $(1, \omega)$-admissible, and 5 is $(2, \omega)$-admissible (in the weak sense): $\langle (0, 2, 3, 0), (1, 3, 3, 5) \rangle \equiv 0 \,(\text{mod. } 15)$. Therefore, 15·5 copies of Ψ are realizable by a smooth algebraic action and hence, $d_{\Psi} \mid 75$. On the other hand, still $A(\omega) = 15$. Consequently, $15 \mid d_{\Psi} \mid 75$. It would be useful to find a general formula for d_l.

Now let us discuss the semi-free case in more detail. From the spectral sequence in stably complex bordism one can derive the following general fact [4]. Let X be a CW-complex. Assume that some singular manifolds $\{f_\alpha : M_\alpha \to X\}_\alpha$ generate the group $H_*(X;\mathbb{Z})$ by the natural homomorphism $\mu : U_*(X) \to H_*(X;\mathbb{Z})$, taking each (M_α, f_α) to $(f_\alpha)_*[M_\alpha] \in H_{\dim M_\alpha}(X, \mathbb{Z})$. Then $\{f_\alpha : M_\alpha \to X\}_\alpha$ generate $U_*(X)$ as a $U_*(pt)$-module.

Choose a sequence $j_1, \ldots, j_n$, where $j_s \in \{1, \ldots, m-1\}$ and $(j_s, m) = 1$; define $\omega_{[l]} : \{1, \ldots, l\} \to \{1, \ldots, m-1\}$ by $\omega_{[l]}(s) = j_s$, for each $1 \le l \le n$. If we denote $L_{2l-1}(\omega_{[l]}) = L_{2l-1}(m; j_1, \ldots, j_l)$, then $\tilde{U}_*(BC_m)$ is generated, as a $U_*(pt)$-module, in dimensions $\leqslant n$, by $\{(L_{2l-1}(\omega_{[l]}), f_l\}$, $1 \leqslant l \leqslant n$.

This implies the following formula, similar to (*) in the Introduction,

$$[L_{2n-1}(\omega), f] = \sum_{l=0}^{n-1} \mathscr{S}_l(\omega, \omega_{[n-l]}) \times [L_{2(n-l)-1}(\omega_{[n-l]}), f_{n-l}], \qquad (**)$$

where $\mathscr{S}_l(\omega, \omega_{[n-l]}) \in U_{2l}(pt)$ is defined by (**).

Applying the natural transformation $\varepsilon : U_*(\ ) \to \Omega_*(\ )$ to (**), we get a version of (**) in oriented bordism. In fact for odd m this formula in $\Omega_*(BC_m)$ will have only terms with even l's. By an easy transversality argument, analogous to the one used in [4, Prop. 34.7], one shows that the order of $L_{2(l+1)-1}(\omega_{[l+1]})$ in $\Omega_*(BC_m)$ is divisible by the order of $L_{2l-1}(\omega_{[l]})$. Thus, (**) implies that the order of $L_{2n-1}(\omega)$ in $\Omega_*(BC_m)$ divides the order of $L_{2n-1}(\omega_{[n]})$. Since $\omega_{[n]}$ was chosen arbitrarily, this means that the order of $L_{2n-1}(\omega)$ depends only on n.

Set $t(n) = \min\{t(\omega)\}$, where $\omega : \{1, \ldots, n\} \to \{1, \ldots, m-1\}$ runs over the set $\Theta(n)$ of maps satisfying $(\omega(s), m) = 1$ for $1 \leqslant s \leqslant n$. Thus, $d_\Psi \mid m^{t(n)}$ for any n-dimensional semi-free C_m-representation Ψ. Recall that, by definition

$$t(n) = \min_{\omega \in \Theta(n)} \Big(\max_{k \mid m, k \ne m} (\max_j \{\#\omega_k^{-1}(j)\}) \Big).$$

An elementary number theoretical argument shows that $t(n) = \lceil n/(p^* - 1) \rceil$, where p^* is the minimal prime dividing m ($\lceil a \rceil$ denotes the smallest integer greater or equal to a, and $\lfloor a \rfloor$ denotes the greatest integer less or equal to a). Thus, for semi-free actions, d_Ψ divides both $\prod_{l=1}^{\lceil n/(p^*-1) \rceil} d_l$ and $m^{\lceil n/(p^*-1) \rceil}$ for *any* $\omega \in \Theta(\omega)$ and (l, ω)-admissible number d_l.

On the other hand, using Proposition 2, the fact that $(\omega(s), m) = 1$ together with the definitions of $v(p, r)$, $a_k(p, \omega)$ and $A(\omega)$, we get that

$$\prod_{p \mid m} p^{\max_{k \mid m, k < m} \{ \lceil n v(p, k/m) \rceil_+ \}} = \prod_{p \mid m} p^{\lceil n/(p-1) \rceil}$$

has to divide d_Ψ. For a simple homological reason, $m \mid d_\Psi$. Consequently, $2^b \prod_{p|m} p^{\max(c(p), \lceil n/p-1 \rceil)}$ divides d_Ψ. This should be compared with the actual answer $d_\Psi = p^{c(p)+\lfloor n/(p-1) \rfloor}$ for the case $m = p^{c(p)}$ [4]. Hence, for odd m, $\prod_{p|m} p^{c(p)+\lfloor n/(p-1) \rfloor} \mid d_\Psi$ and $d_\Psi \mid$ g.c.d. $(\prod_{l=1}^{n} d_l, 2^{nb} \prod_{p|m} p^{nc(p)})$. Here d_l can be any (l, ω)-admissible number for an arbitrary chosen $\omega \in \Theta(n)$. By the way, we do not know a combinatorial proof of the following fact: $2^b \prod_{p|m} p^{c(p)+\lfloor n/(p-1) \rfloor} \mid \prod_{l=1}^{\lceil n/(p^*-1) \rceil} d_l$.

As we pointed out in the Introduction, it would be interesting to find out whether there is a difference between complex algebraic and stably complex actions from the point of view of geometrically realizable sets Γ. For instance, it might happen that for semi-free C_m-actions the lattice in $\mathbb{Z}$, generated by $\#\Gamma$, where Γ is algebraically realizable and consists of several copies of a Ψ, actually depends on Ψ, and not on n and m.

Example 2 (*finite subgroups* $SU(2)$). We consider the finite subgroups G in $SU(2)$, equipped with a 2-dimensional complex representation Ψ given by the inclusion $G \subset SU(2)$. It is well-known that these subgroups are: the cyclic group C_m of order m, the binary dihedral group $\mathbb{D}_m$ of order $4m$, the binary tetrahedral group $\mathbb{T}$ of order 24, the binary octahedral group $\mathbb{O}$ of order 48 and the binary icosahedral group $\mathbb{I}$ of order 120. It is also known that $\mathbb{C}[z_1, z_2]^G$, for these groups G, is an algebra generated by 3 invariants X, Y, Z with the single relation (cf. [10]). We present the information we need in the following table:

G	Relation	deg X	deg Y	deg Z	Regular systems $\mathcal{P}$	Product of degrees $d(\mathcal{P})$	d_Ψ divides	$\lvert G \rvert$
C_m	$X^m - YZ = 0$	2	m	m	$\{X, Y+Z\}$, $\{Y+Z, Y-Z\}$	$2m$ m^2	m if $m \equiv 1\ (2)$, $2m$ if $m \equiv 0\ (2)$	m
$\mathbb{D}_m$	$X^{m+1} + XY^2 + Z^2 = 0$	4	$2m$	$2m+2$	$\{Y, Z\}$, $\{X, Y\}$	$4m^2 + 4m$ $8m$	$4m$	$4m$
$\mathbb{T}$	$X^4 + Y^3 + Z^2 = 0$	6	8	12	$\{X, Y\}$, $\{X, Z\}$	48 72	24	24
$\mathbb{O}$	$X^3Y + Y^3 + Z^2 = 0$	8	12	18	$\{X, Y\}$, $\{X, Z\}$	96 144	48	48
$\mathbb{I}$	$X^5 + Y^3 + Z^2 = 0$	12	20	30	$\{X, Y\}$, $\{X, Z\}$	240 360	120	120

To prove that the polynomial pairs in the sixth column are regular systems for Ψ, we use the appropriate relation to verify that they have no common zeroes on $\mathbb{P}^1$. For a geometric description of the zeroes of X, Y, Z on $\mathbb{P}^1 \cong S^2$ for $G = \mathbb{T}, \mathbb{O}, \mathbb{I}$ see the classical work of Klein [8].

Since 1 is not an eigenvalue of a generic matrix $\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \in SU(2)$, G acts freely on $S^3 \subset \mathbb{C}^2$ for any $G \subseteq SU(2)$. In particular, any cyclic or generalized quaternion subgroup of a given G, from our list, acts freely.

Some of those subgroups are: the cyclic group of order 3 and the generalized quaternion group $\mathbb{D}_2$ for $\mathbb{T}$; the cyclic group of order 3 and $\mathbb{D}_4$ for $\mathbb{O}$; the cyclic groups of orders 3, 5 and $\mathbb{D}_2$ for $\mathbb{I}$.

Let H be a cyclic or a generalized quaternion group. A free H-action on S^3 induces a map $f: S^3/H \to BH$. If $d[S^3/H, f]$ is a boundary, then there is an obvious condition on d: it has to be divisible by the order of $f_*[S^3/H]$ in $H_3(BH; \mathbb{Z})$. This group is isomorphic to $\mathbb{Z}/m\mathbb{Z}$ for $H \cong C_m$ and to $\mathbb{Z}/4m\mathbb{Z}$ if H is a generalized quaternian group ([3], pp. 253–254).

It can be verified that $f_*[S^3/H]$ is a generator of $H_3(BH; \mathbb{Z})$ for any faithful Ψ. Consequently, $m \mid d_\Psi$ for $H \cong C_m$ and $4m \mid d_\Psi$ for $H \cong \mathbb{D}_m$. In particular, $8 \mid d_\Psi$ for $\mathbb{D}_2$ and $16 \mid d_\Psi$ for $\mathbb{D}_4$. Also, $3 \cdot 8 \mid d_\Psi$ for $\mathbb{T}$, $3 \cdot 16 \mid d_\Psi$ for $\mathbb{O}$, and $3 \cdot 5 \cdot 8 \mid d_\Psi$ for $\mathbb{I}$. Hence, using the table, we have proved

PROPOSITION 3. *Given $\Psi: G \to SU(2)$, where $G = C_m, \mathbb{D}_m, \mathbb{T}, \mathbb{O}, \mathbb{I}$, the number d_Ψ is: m if $m \equiv 1(2)$, m or $2m$ if $m \equiv 0(2)$ for C_m; $4m$ for $\mathbb{D}_m$; 24 for $\mathbb{T}$; 48 for $\mathbb{O}$; 120 for $\mathbb{I}$.*

In fact, $2m$ and m^2 copies of Ψ are realizable by semifree smooth algebraic C_m-actions on a projective complex nonsingular surface M. The corresponding numbers (realizable by smooth algebraic actions on surfaces) for other groups are: $8m$ and $4m^2 + 4m$ for $\mathbb{D}_m$, 48 and 72 for $\mathbb{T}$, 96 and 144 for $\mathbb{O}$, 240 and 360 for $\mathbb{I}$. □

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