Zeitschrift: Commentarii Mathematici Helvetici

Herausgeber: Schweizerische Mathematische Gesellschaft

Band: 63 (1988)

Artikel: Properties of the scattering map II.

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DOI: https://doi.org/10.5169/seals-48200

Comment. Math. Helvetici 63 (1988) 150–167 0010-2571/88/010150-18$01.50 + 0.20/0

Properties of the scattering map II

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1. Introduction

This paper continues the investigation of the scattering map as it was started in [3].

Again we consider the Schrödinger equation

$$-\frac{d^2}{dx^2}y(x)+q(x)y(x)=k^2y(x) \tag{1.1}$$

on the whole line where q is a real valued potential in the weighted Sobolev space $H_{N,N}$ ($N\geq 3$). The aim of this paper is to extend the study of the map SD, associating to a potential q its scattering data for the case where q has bound states. To be more precise let us denote by $f_1(x,k):=f_1(x,k,q)$ and $f_2(x,k):=f_2(x,k,q)$ the Jost functions of (1.1) and by $W(k):=W(k,q)$ and $S(k):=S(k,q)$ the Wronskians $W(k):=W[f_2(x,k),f_1(x,k)]$ ($\operatorname{Im}k\geq 0$) and $S(k):=W[f_1(x,-k),f_2(x,k)]$ ($\operatorname{Im}k=0$).

Let us denote by $Q_{N,n}(\mathbb{R})$ the set of all real valued potentials q in $H_{N,N}$ such that $W(k,q)\neq 0$ for $\operatorname{Im}k=0$ and such that $W(k,q)$ has exactly n zeroes in $\operatorname{Im}k>0$. Due to the fact that q is real valued all zeroes of $W(k,q)$ are situated on the imaginary axis. Let us denote them by $i\kappa_1(q),\ldots,i\kappa_n(q)$ where $\kappa_j=\kappa_j(q)>0$ and $\kappa_1<\cdots<\kappa_n$. Clearly for $k=i\kappa_j$ the Jost functions $f_1(x,k)$ and $f_2(x,k)$ are linearly dependent. So there exist real numbers $d_1(q),\ldots,d_n(q)$ such that $f_2(x,i\kappa_j)=d_j(q)f_1(x,i\kappa_j)$ (x in $\mathbb{R}$) where $d_j=d_j(q)\neq 0$ ($1\leq j\leq n$). Define $\eta_j(q):=\log d_j(q)^2$ and the norming constants $c_j=c_j(q):=(\int_{-\infty}^{\infty}f_1^2(x,i\kappa_j)\,dx)^{-1}$. Then $S(\cdot,q),\kappa_1(q),\ldots,\kappa_n(q),\eta_1(q),\ldots,\eta_n(q)$ is called the scattering data of q. We define the following map

$$\begin{aligned}SD:Q_{N,n}(\mathbb{R})&\to \mathcal{S}_{N,n}\times E_+^n\times\mathbb{R}^n\\ q&\mapsto (S(\cdot,q),\kappa_1(q),\ldots,\kappa_n(q),\eta_1(q),\ldots,\eta_n(q))\end{aligned}$$

where $E_+^n:=\{(x_1,\ldots,x_n)\in\mathbb{R}^n:0<x_1<\cdots<x_n\}$ and $\mathcal{S}_{N,n}:=\{\sigma\in H^{\#}_{N-1,N}:\sigma(-k)=\sigma(k)^*,\ (-1)^n\sigma(0)>0\}$. $H^{\#}_{N-1,N}$ denotes a weighted Sobolev space as introduced in [3] and * denotes as usual complex conjugation. For convenience we denote by $\cdot$ the derivative with respect to k.

Pointing out that $Q_{N,n}(\mathbb{R})$ is open in $H_{N,N}(\mathbb{R}, \mathbb{R})$ as will be proved in section 2 we state the following

THEOREM. *If* $N \geq 3$ *then*

(1) $SD: Q_{N,n}(\mathbb{R}) \to \mathcal{S}_{N,n} \times E_+^n \times \mathbb{R}^n$ *is a real analytic isomorphism.*

(2) *In particular at every* q *in* $Q_{N,n}(\mathbb{R})$ *the Jacobian*

$$d_qSD = d_qS \times \left(\bigtimes_{j=1}^{n} d_q\kappa_j\right) \times \left(\bigtimes_{j=1}^{n} d_q\eta_j\right)$$

is boundedly invertible.

(3) *The Jacobians* d_qS, $d_q\kappa_j$, $d_q\eta_j$ $(1 \leq j \leq n)$ *are integral operators given by* $(v \in H_{N,N}(\mathbb{R}, \mathbb{R}))$

$$d_qS[v](k) = \int_{-\infty}^{\infty} f_1(x, -k, q) f_2(x, k, q) v(x)\, dx$$

$$d_q\kappa_j[v] = -\frac{c_j}{2\kappa_j} \int_{-\infty}^{\infty} f_1^2(x, i\kappa_j) v(x)\, dx$$

$$d_q\eta_j[v] = \frac{ic_j}{\kappa_j d_j} \int_{-\infty}^{\infty} (\dot{f}_1 f_2 - \dot{f}_2 f_1)(x, i\kappa_j) v(x)\, dx$$

(4) *The inverse* $(d_qSD)^{-1}$ *of* d_qSD *is also an integral operator given by*

$$\begin{aligned}
&(d_qSD)^{-1}(\sigma, \alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n)(x) \\
&= \frac{1}{\pi} \int_{-\infty}^{\infty} dk\, \sigma(k) \frac{2ik}{W(-k)W(k)} \frac{\partial}{\partial x} (f_1(x, k) f_2(x, -k)) \\
&\quad + \sum_{j=1}^{n} \alpha_j \frac{2ic_j}{d_j} \frac{\partial}{\partial x} (\dot{f}_1 f_2 - f_1 \dot{f}_2)(x, i\kappa_j) \\
&\quad + \sum_{j=1}^{n} \beta_j c_j \frac{\partial}{\partial x} f_1^2(x, i\kappa_j)
\end{aligned}$$

where $\sigma \in H_{N-1,N}^{\#r}$, $(\alpha_1, \ldots, \alpha_n) \in E_+^n$ *and* $(\beta_1, \ldots, \beta_n) \in \mathbb{R}^n$.

From Theorem 4.4 in [3] and a well-known result in [2] it follows that SD is 1–1 and onto. Only the fact that for q in $Q_{N,n}(\mathbb{R})$ one has $(-1)^n S(0) > 0$ needs a remark. Let us recall that $S(0) = -W(0)$ and that $W(i\kappa)$ is a real valued function for $\kappa \geq 0$ with exactly n zeroes in $\kappa > 0$, all of them simple. But asymptotically $W(i\kappa) \sim -2\kappa$ ($\kappa \to +\infty$), thus $(-1)^n W(0) < 0$. From [3] we already know that S is real analytic on $Q_{N,n}(\mathbb{R})$. In section 2 we will show that $\kappa_j(q)$ and $\eta_j(q)$ are real analytic ($1 \leq j \leq N$). Then (1) follows from the inverse function theorem provided one can show that $d_q SD$ is boundedly invertible. This will be shown in section 3. (3) and (4) are proved in section 2 and 3 respectively.

In section 4 we will discuss the set of potentials with resonance, i.e. where the Wronskian $W(k)$ does vanish at $k = 0$. We then indicate how all the sets $Q_{N,n}(\mathbb{R})$ lie in $H_{N,N}(\mathbb{R}, \mathbb{R})$. The notation is the same as used in [3].

2. Investigation of the functions $\kappa_j(q)$ and $\eta_j(q)$

Let us first look at the functions $\kappa_j(q)$ ($1 \leq j \leq n$) and recall that $i\kappa_j(q)$ are the zeroes of $W(k, q)$. From Lemma 2.7 in [3] we know that

$$H_{N,N} \to C^0_+(\mathbb{R}, L^2(\mathbb{R}^+)), \qquad q \mapsto \partial_x^j B_1(x, y, q)$$

is holomorphic for $0 \leq j \leq N+1$ where $B_1(x, y, q)$ is given by $f_1(x, k) = e^{ikx}(1 + \int_0^\infty B_1(x, y) e^{2iky}\, dy)$. We conclude that

$$\{\operatorname{Im} k > 0\} \times H_{N,N} \to C^0_+(\mathbb{R}) \cap L^2_+(\mathbb{R}), \; (k, q) \mapsto \partial_x^j f_1(\cdot, k, q)$$

is holomorphic in q and k for $0 \leq j \leq N+1$. Similar results hold for $f_2(x, k, q)$. Thus we have proved

LEMMA 2.1. *If $N \geq 3$ then $W: \{\operatorname{Im} k > 0\} \times H_{N,N} \to \mathbb{C}$ is holomorphic.*

Let q be in $Q_{N,n}(\mathbb{R})$ with $N \geq 3$ and fix $1 \leq j \leq n$. As all the zeroes $W(k, q)$ are simple we conclude that $(\partial/\partial k) W(i\kappa_j, q) \neq 0$. By the implicit function theorem there exists an open neighborhood V_j of q in $H_{N,N}(\mathbb{R}, \mathbb{R})$ and a real analytic function $\hat{\kappa}_j$ defined on V_j such that $W(i\hat{\kappa}_j(p), p) = 0$ (p in V_j). For $\varepsilon > 0$, $c := \inf \{|W(i\kappa, q)| : \kappa > 0,\ |\kappa - \kappa_j(q)| \geq \varepsilon,\ 1 \leq j \leq n\} > 0$. So there exists an open neighborhood $U_\varepsilon \subseteq \bigcap_{j=1}^n V_j$ in $H_{N,N}(\mathbb{R}, \mathbb{R})$ such that $\inf \{|W(i\kappa, p)| : \kappa > 0,\ |\kappa - \kappa_j(q)| \geq \varepsilon,\ 1 \leq j \leq n\} \geq c/2$ for all p in U_ε. For sufficiently small $\varepsilon > 0$ $(\partial/\partial k) W(k, p) \neq 0$ for $p \in U_\varepsilon$ and k with $|k - i\kappa_j(q)| < \varepsilon$ ($1 \leq j \leq n$) and thus $U_\varepsilon \subseteq Q_{N,n}(\mathbb{R})$. To summarize we have got

PROPOSITION 2.2. *Let $N \geq 3$ and n in $\mathbb{N}$. Then*
(1) $Q_{N,n}(\mathbb{R})$ is an open subset of $H_{N,N}(\mathbb{R}, \mathbb{R})$.
(2) $\kappa_j : Q_{N,n}(\mathbb{R}) \to \mathbb{R}^+$ is real analytic.

Now let us turn to $\eta_j(q)$. Recall from [3] that for $a < b$

$$\{\operatorname{Im} k > 0\} \times H_{N,N} \to \mathbb{C},\ (k, q) \mapsto \int_a^\infty f_1^2(x, k)\, dx \quad \text{and}$$

$$\{\operatorname{Im} k > 0\} \times H_{N,N} \to \mathbb{C},\ (k, q) \mapsto \int_a^b f_1(x, k) f_2(x, k)\, dx$$

are holomorphic.

Using the chain rule one then concludes that

$$\int_a^\infty f_1^2(x, i\kappa_j(q), q)\, dx \quad \text{and} \quad \int_a^b (f_1 f_2)(x, i\kappa_j(q), q)\, dx$$

are real analytic on $Q_{N,n}(\mathbb{R})$. Further let us recall that $f_2(x, i\kappa_j(q)) = d_j(q) f_1(x, i\kappa_j(q))$ and thus

$$d_j(q)^2 = \frac{f_2(0, i\kappa_j)^2 + f_2'(0, i\kappa_j)^2}{f_1(0, i\kappa_j)^2 + f_1'(0, i\kappa_j)^2}$$

is locally bounded on $Q_{N,n}(\mathbb{R})$. This implies that $\int_{-\infty}^{\infty} dx f_1^2(x, i\kappa_j(q))$ and $\int_{-\infty}^{\infty} dx (f_1 f_2)x, i\kappa_j(q))$ are real analytic on $Q_{N,n}(\mathbb{R})$ where we used the fact that the limit of a sequence of real analytic functions converging locally uniformly is again real analytic. So we have proved

PROPOSITION 2.3. *If $N \geq 3$ then η_j is a real analytic function on $Q_{N,n}(\mathbb{R})$ with values in $\mathbb{R}$ $(1 \leq j \leq n)$.*

Let us now derive the formulae for the derivatives of $\kappa_j(q)$ and $\eta_j(q)$ as stated in the theorem of section 1. We need the following

LEMMA 2.4. *If $N \geq 3$, q in $Q_{N,n}(\mathbb{R})$ and v in $H_{N,N}(\mathbb{R}, \mathbb{R})$ then*

$$d_q f_1[v](x, i\kappa_j) = \frac{1}{2i\kappa_j} \frac{c_j}{d_j} \int_x^\infty dt\, v(t) f_1(t, i\kappa_j)$$
$$\times \frac{\partial}{\partial k} (f_2(x, k) f_1(t, k) - f_2(t, k) f_1(x, k))\big|_{k = i\kappa_j}.$$

Remark. Similarly on can show that

$$d_q f_2[v](x, i\kappa_j) = \frac{1}{2i\kappa_j} \frac{c_j}{d_j} \int_{-\infty}^{x} dt v(t) f_2(t, i\kappa_j)$$
$$\times \frac{\partial}{\partial k} (f_1(x, k) f_2(t, k) - f_2(x, k) f_1(t, k))\big|_{k = i\kappa_j}.$$

Proof (of Lemma 2.4). As in [3] one checks that for $k \neq i\kappa_\alpha$ $(1 \leq \alpha \leq n)$, $1 \leq j \leq n$ and v in $H_{N,N}(\mathbb{R}, \mathbb{R})$

$$d_q f_1[v](x, k) = \frac{k - i\kappa_j}{W(k)} \int_x^{\infty} dt\, v(t) f_1(t, k) \frac{f_2(x, k) f_1(t, k) - f_2(x, i\kappa_j) f_1(t, i\kappa_j)}{k - i\kappa_j}$$
$$- \frac{k - i\kappa_j}{W(k)} \int_x^{\infty} dt\, v(t) f_1(t, k) \frac{f_2(t, k) f_1(x, k) - f_2(t, i\kappa_j) f_1(x, i\kappa_j)}{k - i\kappa_j}$$

where we used that $f_2(x, i\kappa_j) f_1(t, i\kappa_j) = f_1(x, i\kappa_j) f_2(t, i\kappa_j)$. Clearly $d_q f_1[v](x, i\kappa_j) = \lim_{k \to i\kappa_j} d_q f_1[v](x, k)$. By [2] one sees that

$$\lim_{k \to i\kappa_j} \frac{k - i\kappa_j}{W(k)} = \frac{1}{2i\kappa_j} \frac{c_j}{d_j}.$$

Recall from [3] that $m_1(x, k) := e^{-ikx} f_1(x, k)$ and $m_2(x, k) := e^{ikx} f_2(x, k)$. Taking into account that $\dfrac{m_2(\cdot, k) - m_2(\cdot, i\kappa_j)}{k - i\kappa_j}$ is bounded in $L^\infty(\mathbb{R})$ uniformly for $\operatorname{Im} k \geq \kappa > 0$ $(k \neq i\kappa_j)$ as will be shown in the next lemma one can apply Lebesgue's convergence theorem to get the claimed result.

LEMMA 2.5. *For* $k \neq k'$ *with* $\operatorname{Im} k$, $\operatorname{Im} k' \geq \kappa > 0$ *we have*

$$\left\| \frac{m_i(\cdot, k) - m_i(\cdot, k')}{k - k'} \right\|_{L^\infty(\mathbb{R})} \leq M(\kappa) \quad (i = 1, 2)$$

Remark. As

$$\lim_{k' \to k} \frac{m_i(x, k) - m_i(x, k')}{k - k'} = m_i^{\cdot}(x, k)$$

pointwise for x in $\mathbb{R}$ it follows from Lemma 2.5 that $\|m_i^{\cdot}(\cdot, k)\|_{L^\infty(\mathbb{R})} \leq M(\kappa)$ $(i = 1, 2)$.

Proof (of Lemma 2.5). It suffices to prove the statement for $i = 1$. For convenience we write $m(x, k)$ for $m_1(x, k)$. For $k \neq k'$ the difference $m(x, k) - m(x, k')$ can be written as

$$\frac{m(x,k) - m(x,k')}{k - k'} = \int_x^\infty D_k(t-x) \frac{m(t,k) - m(t,k')}{k - k'} q(t)\, dt + \int_x^\infty \frac{D_k(t-x) - D_{k'}(t-x)}{k - k'} q(t) m(t, k')\, dt \tag{2.1}$$

where $D_k(y) = \int_0^y e^{2ikz}\, dz$. Without loss of any generality assume that $\operatorname{Im}(k' - k) \geq 0$. Then

$$\left| \frac{D_k(t-x) - D_{k'}(t-x)}{k - k'} \right| = \left| \int_0^{t-x} dz e^{2ikz} \frac{e^{2i(k'-k)z} - 1}{k' - k} \right| \leq K(\kappa)$$

for a suitably chosen $K(\kappa) > 0$ and all k with $\operatorname{Im} k \geq \kappa$. From [1] we learn that $|m(t, k)| \leq c(\kappa)$ for all t in $\mathbb{R}$ and for $\operatorname{Im} k \geq \kappa$ and c sufficiently big. (2.1) is an integral equation of Volterra type and thus we get for $\operatorname{Im} k$, $\operatorname{Im} k' \geq \kappa > 0$, $k \neq k'$

$$\left| \frac{m(x,k) - m(x,k')}{k - k'} \right| \leq \exp\left\{ \frac{1}{\kappa} \int_{-\infty}^\infty |q(x)|\, dx \right\} c(\kappa) K(\kappa)$$

and Lemma 2.5 follows.

Let us point out two corollaries of Lemma 2.6 which will be needed later.

COROLLARY 2.6. *If $N \geq 3$ and q in $Q_{N,n}(\mathbb{R})$ then the limit $h_j := \lim_{x \to -\infty} f_1^{\cdot}(x, i\kappa_j) e^{\kappa_j x}$ exists for $1 \leq j \leq n$.*

Remark. A similar result holds for f_2: For $1 \leq j \leq n$ the limit $g_j := \lim_{x \to +\infty} f_2^{\cdot}(x, i\kappa_j) e^{-\kappa_j x}$ exists.

COROLLARY 2.7. *If $N \geq 3$ and q in $Q_{N,n}(\mathbb{R})$ then*

$$\int_{-\infty}^\infty f_1^2(x, i\kappa_j)\, dx = i \frac{h_j}{d_j} \quad (1 \leq j \leq n).$$

Remark 1. A similar result holds for f_2:

$$\int_{-\infty}^{\infty} f_2^2(x, i\kappa_j)\,dx = id_j g_j \quad (1 \le j \le n).$$

Remark 2. It follows from Corollary 2.7 that $h_j = g_j$ $(1 \le j \le n)$.

Proof (of Corollary 2.7). We have

$$\int_{-\infty}^{\infty} f_1^2(x, i\kappa_j)\,dx = i\frac{1}{d_j}\cdot\frac{\partial}{\partial k}\frac{W(k)}{2ik}\bigg|_{k=i\kappa_j},$$

so it is to show that

$$\frac{\partial}{\partial k}\frac{W(k)}{2ik}\bigg|_{k=i\kappa_j} = h_j.$$

Now

$$\begin{aligned}\frac{\partial}{\partial k}\frac{W(k)}{2ik}\bigg|_{k=i\kappa_j} &= -\frac{1}{2\kappa_j}W^{\cdot}(i\kappa_j)\\ &= -\frac{1}{2\kappa_j}\{f_2^{\cdot}(x, i\kappa_j)f_1'(x, i\kappa_j) - f_2^{\cdot\prime}(x, i\kappa_j)f_1(x, i\kappa_j)\\ &\quad + f_2(x, i\kappa_j)f_1^{\cdot\prime}(x, i\kappa_j) - f_2'(x, i\kappa_j)f_1^{\cdot}(x, i\kappa_j)\} \quad (x \text{ in } \mathbb{R}).\end{aligned}$$

As this last equality is true for all x in $\mathbb{R}$ it remains true also in the limit as $x \to -\infty$. Using Corollary 2.6 and deducing from Lemma 2.12 the fact that $\lim_{x\to-\infty} m_1^{\cdot\prime}(x, i\kappa_k) = 0$ the claimed result follows.

PROPOSITION 2.8. *If* $N \ge 3$, q *in* $Q_{N,n}(\mathbb{R})$ *and* $1 \le j \le n$ *then* (v *in* $H_{N,N}(\mathbb{R}, \mathbb{R})$)

$$d_q\kappa_j[v] = -\frac{c_j}{2\kappa_j}\int_{-\infty}^{\infty} f_1(x, i\kappa_j)^2 v(x)\,dx$$

Proof. Clearly $W(i\kappa_j(q), q) = 0$ and thus by the chain rule

$$0 = d_q W[v](i\kappa_j, q) + W^{\cdot}(i\kappa_j, q)\,i d_q\kappa_j[v].$$

From [3] recall that $d_qW[v](k, q) = -\int_{-\infty}^{\infty} (f_2 f_1)(x, k)v(x)\, dx$. From [2] we know that

$$W^{\cdot}(i\kappa_j, q) = 2i\kappa_j \int_{-\infty}^{\infty} f_1(x, i\kappa_j) f_2(x, i\kappa_j)\, dx = 2i\kappa_j \frac{d_j}{c_j}.$$

So we get

$$d_q\kappa_j[v] = \frac{-c_j}{2\kappa_j d_j} d_j \int_{-\infty}^{\infty} f_1^2(x, k)v(x)\, dx.$$

PROPOSITION 2.9 *If* $N \geq 3$, q *in* $Q_{N,n}(\mathbb{R})$ *and* $1 \leq j \leq n$ *then* (v *in* $H_{N,N}(\mathbb{R}, \mathbb{R})$)

$$d_q\eta_j[v] = \frac{ic_j}{\kappa_j d_j} \int_{-\infty}^{\infty} (f_1^{\cdot} f_2 - f_1 f_2^{\cdot})(x, i\kappa_j)v(x)\, dx.$$

Proof. Let us introduce for $m \geq 1$ the functions

$$\eta_{jm}(q) := \ln \left(\frac{\int_{-m}^{m} f_1 f_2(x, i\kappa_j)\, dx}{\int_{-m}^{m} f_1^2(x, i\kappa_j)\, dx} \right)^2.$$

Then it follows as in Proposition 2.3 that $\eta_{jm}(q)$ is real analytic on $Q_{N,n}(\mathbb{R})$. Using well-known properties of f_1 and f_2 one obtains

$$\begin{aligned} d_q\eta_{jm}[v] &= 2\frac{\int_{-m}^{m} f_1 d_q f_2[v](x, i\kappa_j)\, dx}{\int_{-m}^{m} f_1 f_2(x, i\kappa_j)\, dx} - 2\frac{\int_{-m}^{m} f_1 d_q f_1[v](x, i\kappa_j)\, dx}{\int_{-m}^{m} f_1^2(x, i\kappa_j)\, dx} \\ &= \frac{2c_{jm}}{d_j}(I + II) \end{aligned}$$

where $c_{jm} := \int_{-m}^{m} f_1 f_2(x, i\kappa_j)\, dx$,

$$I = \int_{-m}^{m} dx (f_1 f_2^{\cdot} - f_2 f_1^{\cdot})(x, i\kappa_j) i d_q\kappa_j[v]$$

and

$$II = \int_{-m}^{m} dx\{f_1(x, i\kappa_j)d_q f_2[v](x, i\kappa_j) - f_2(x, i\kappa_j)d_q f_1[v](x, i\kappa_j)\}.$$

Using Lemma 2.4 one can write II as a sum $III + IV$ where

$$III = \frac{ic_j}{2\kappa_j d_j}\int_{-m}^{m} dx \int_{-\infty}^{\infty} dt\, v(t) f_2(t, i\kappa_j)^2\left(\frac{1}{d_j} f_1 f_2^{\cdot} - \frac{1}{d_j} f_2 f_1^{\cdot}\right)(x, i\kappa_j),$$

$$IV = \frac{ic_j}{2\kappa_j d_j}\int_{-m}^{m} dx \int_{-\infty}^{\infty} dt\, v(t) f_2(t, i\kappa_j) f_1(x, i\kappa_j)$$
$$\times \{f_2(x, i\kappa_j) f_1^{\cdot}(t, i\kappa_j) - f_2^{\cdot}(t, i\kappa_j) f_1(x, i\kappa_j)\}.$$

Using Proposition 2.8 one sees that $I + III = 0$. So in all we get

$$d_q\eta_{jm}[v] = \frac{2c_{jm}}{d_j} IV = \frac{ic_{jm}}{\kappa_j d_j} c_j \int_{-m}^{m} dx f_1^2(x, i\kappa_j)\int_{-\infty}^{\infty} dt\, v(t)(f_2 f_1^{\cdot} - f_2^{\cdot} f_1)(t, i\kappa_j).$$

It is easy to see that $\eta_j(q) = \lim_{m\to\infty} \eta_{jm}(q)$ $(1 \le j \le n)$ locally uniformly on $Q_{N,n}(\mathbb{R})$. As η_j and η_{jm} are real analytic on $Q_{N,n}(\mathbb{R})$ we conclude that $\lim_{m\to\infty} d_q\eta_{jm}[v] = d_q\eta_j[v]$. But clearly $\lim_{m\to\infty} c_j \int_{-m}^{m} dx f_1^2(x, i\kappa_j) = 1$ and the claimed result follows.

To finish this section we present two results which will be needed later.

PROPOSITION 2.10. *If $N \ge 3$ and q in $Q_{N,n}(\mathbb{R})$ then $(\partial/\partial x) f_1^2(x, i\kappa_j)$ is in $H_{N,N}(\mathbb{R}, \mathbb{R})$ $(1 \le j \le n)$.*

Proof. Recall that $f_1(x, i\kappa_j) = e^{-\kappa_j x} m_1(x, i\kappa_j) = e^{-\kappa_j x}(1 + \int_0^\infty B_1(x, y) e^{-2\kappa_j y}\, dy)$. From the considerations before Lemma 2.1 we know that $\partial_x^\alpha(m_1(x, i\kappa_j) - 1) = \int_0^\infty \partial_x^\alpha B_1(x, y) e^{-2\kappa_j y}\, dy$ is in $C_+^0(\mathbb{R})$ for $0 \le \alpha \le N+1$ and $1 \le j \le n$. So it follows that $x^m \partial_x^\alpha f_1^2(x, i\kappa_j) \in L_+^2(\mathbb{R})$ $(0 \le \alpha \le N+1,\ m \ge 0)$. Similarly one proves that $x^m \partial_x^\alpha f_2^2(x, i\kappa_j) \in L_-^2(\mathbb{R})$ $(0 \le \alpha \le N+1,\ m \ge 0)$. Because $f_2(x, i\kappa_j) = d_j f_1(x, i\kappa_j)$ the proposition follows.

PROPOSITION 2.11. *If $N \ge 3$ and q in $Q_{N,n}(\mathbb{R})$ then $(\partial/\partial x)(f_1^{\cdot} f_2 - f_1 f_2^{\cdot})(x, i\kappa_j)$ is in $H_{N,N}(\mathbb{R}, \mathbb{R})$ $(1 \le j \le n)$.*

Proof. It suffices to show that $\partial_x f_1^{\cdot} f_2(x, i\kappa_j)$ is in $H_{N,N}(\mathbb{R}, \mathbb{R})$.

As $f_2(x, i\kappa_j) = d_j f_1(x, i\kappa_j)$ we conclude that

$$\partial_x f_1^{\cdot} f_2 = -2\kappa_j d_j e^{-2\kappa_j x}(m_1^{\cdot}(x, i\kappa_j)m_1(x, i\kappa_j) + (ix)m_1^2(x, i\kappa_j))$$
$$+ d_j e^{-2\kappa_j x}\partial_x(m_1^{\cdot}(x, i\kappa_j)m_1(x, i\kappa_j) + (ix)m_1^2(x, i\kappa_j)).$$

It is not hard to see that $\partial_x^\alpha(m_1(x, i\kappa_j) - 1)$ and $\partial_x^\alpha m_1^{\cdot}(x, i\kappa_j)$ are in $C_+^0(\mathbb{R})$ for $0 \le \alpha \le N+1$ and it follows that $x^m\partial_x^{\alpha+1} f_1^{\cdot} f_2(x, i\kappa_j) \in L_+^2(\mathbb{R})$ $(0 \le m, \alpha \le N)$.

It remains to show that $x^m\partial_x^{\alpha+1} f_1^{\cdot} f_2(x, i\kappa_j) \in L_-^2(\mathbb{R})$. Using $f_1(x, i\kappa_j) = (1/d_j)f_2(x, i\kappa_j)$ we get

$$\partial_x(f_1^{\cdot} f_2)(x, i\kappa_j) = \frac{\partial}{\partial x}\left(ix\frac{1}{d_j}f_2^2(x, i\kappa_j)\right) + m_2'(x, i\kappa_j)m_1^{\cdot}(x, i\kappa_j)$$
$$+ m_2(x, i\kappa_j)m_1^{\cdot\prime}(x, i\kappa_j).$$

Again it is not hard to see that $\partial_x(ixf_2^2(x, i\kappa_j)) \in H_{N,N}(\mathbb{R}, \mathbb{R})$. Taking Lemma 2.6 into account where it is proved that $m_1^{\cdot}(x, i\kappa_j)$ is in $L^\infty(\mathbb{R})$ it suffices to show

(i) $x^m\partial_x^\alpha(m_1(x, i\kappa_j) - 1) \in L_+^2(\mathbb{R}) \qquad 0 \le \alpha \le N+1, \qquad 0 \le m \le N$

(ii) $x^m\partial_x^{\alpha+1} m_1^{\cdot}(x, i\kappa_j) \in L_-^2(\mathbb{R}) \qquad 0 \le \alpha \le N, \qquad 0 \le m \le N.$

(ii) will be proved in the following lemma. Towards (i) recall that

$$\partial_x m_1(x, i\kappa_j) = -\int_x^\infty dt e^{-2\kappa_j(t-x)}\, q(t)m_1(t, i\kappa_j) \quad \text{and for} \quad 2 \le \alpha \le N+1$$

$$\partial_x^\alpha m_1(x, i\kappa_j) = \sum_{\beta=0}^{\alpha-2} (2\kappa_j)^\beta \partial_x^{\alpha-2-\beta}(m_1(x, i\kappa_j)q(x))$$
$$- (2\kappa_j)^{\alpha-1}\int_x^\infty dt e^{-2\kappa_j(t-x)}\, q(t)m_1(t, i\kappa_j).$$

As $q(x)$ is in $H_{N,N}(\mathbb{R}, \mathbb{R})$ and $\partial_x^\alpha m_1(x, i\kappa_j) \in C_+^0(\mathbb{R})$ for $0 \le \alpha \le N+1$ it suffices to prove that $\int_x^\infty dt e^{-2\kappa_j(t-x)}\, q(t)m_1(t, i\kappa_j)$ is in $L_N^2(\mathbb{R}^+)$. For $0 \le \alpha \le N$ it is easy to see that

$$\int_0^\infty dx\left(\int_x^\infty dt e^{-2\kappa_j(t-x)} q(t)m_1(t, i\kappa_j)\right)^2 x^{2\alpha}$$
$$\le \left(\int_0^\infty dt e^{-2\kappa_j t}\right)^2 \int_0^\infty dx\, x^{2\alpha}\, |q(x)m_1(x, i\kappa_j)|^2 < \infty.$$

LEMMA 2.12. *If* $N \geq 3$ *and* q *in* $Q_{N,n}(\mathbb{R})$ *then* $\partial_x m_\alpha^{\cdot}(x, i\kappa_j)$ *is in* $H_{N,N}(\mathbb{R}, \mathbb{R})$ *for* $\alpha = 1, 2$ *and* $1 \leq j \leq n$.

Proof. Clearly it suffices to prove the statement for $\alpha = 1$. Recall that $\partial_x m_1^{\cdot}(x, i\kappa_j)$ satisfies

$$\partial_x m_1^{\cdot}(x, i\kappa_j) = -2ie^{\kappa_j x} I - II$$

where

$$I := \int_x^\infty (t-x) e^{-\kappa_j(t-x)} q(t) f_1(t, i\kappa_j)\, dt$$

and

$$II := \int_x^\infty e^{-2\kappa_j(t-x)} q(t) m_1^{\cdot}(t, i\kappa_j)\, dt.$$

As concerns I it suffices to show that $\partial_x^\alpha I$ is in $L_-^\infty(\mathbb{R})$ for $0 \leq \alpha \leq N$. This follows easily from the following expressions for the derivatives ($1 \leq j \leq n$)

$$\begin{aligned}\partial_x I = \kappa_j &\int_x^\infty dt (t-x) e^{-\kappa_j(t-x)} q(t) f_1(t, i\kappa_j) \\ &- \int_x^\infty dt e^{-\kappa_j(t-x)} q(t) f_1(t, i\kappa_j)\end{aligned}$$

and for $2 \leq \alpha \leq N+1$

$$\begin{aligned}\partial_x^\alpha I = \kappa_j^\alpha &\int_x^\infty dt e^{-\kappa_j(t-x)} (t-x) q(t) f_1(t, i\kappa_j) - \alpha \kappa_j^{(\alpha-1)} \int_x^\infty dt e^{-\kappa_j(t-x)} q(t) f_1(t, i\kappa_j) \\ &+ \sum_{\beta=0}^{\alpha-2} (\alpha - 1 - \beta) \kappa_j^{\alpha-2-\beta} \partial_x^\beta (q(x) f_1(x, i\kappa_j)).\end{aligned}$$

Towards II we have to show that $\partial_x^\alpha II \in L_N^2(-\infty)$. For the derivatives of II we get the following expressions

$$\partial_x II = -q(x) m_1^{\cdot}(x, i\kappa_j) + 2\kappa_j \int_x^\infty e^{-2\kappa_j(t-x)} q(t) m_1^{\cdot}(t, i\kappa_j)\, dt$$

and for $2 \leq \alpha \leq N$

$$\partial_x^\alpha II = (2\kappa_j)^\alpha \int_x^\infty e^{-2\kappa_j(t-x)} q(t) m_1^{\cdot}(t, i\kappa_j)\, dt - \sum_{\beta=0}^{\alpha-1} (2\kappa_j)^{\alpha-1-\beta} \partial_x^\beta (q(x) m_1^{\cdot}(x, i\kappa_j)).$$

Thus it suffices to prove, due to Lemma 2.6

(i) $\partial_x^\alpha m_1^{\cdot}(x, i\kappa_j) \in L^\infty(\mathbb{R}) \qquad 0 \leq \alpha \leq N-2$

(ii) $\int_x^\infty e^{-2\kappa_j(t-x)} q(t)\, dt \in L_N^2(-\infty)$.

(i) is proved by using an easy induction argument on the expressions for $\partial_x^\alpha I$ and $\partial_x^\alpha II$ together with Lemma 2.5.

Towards (ii) let us split the integral: First prove that $\int_{x/2}^\infty e^{-2\kappa_j(t-x)} q(t)\, dt$ is in $L_N^2(\mathbb{R}^-)$. Observe that for $t \geq x/2$ it follows that $t - x \geq |x|/2$ for $x < 0$. So we get

$$\left| \int_{x/2}^\infty e^{-2\kappa_j(t-x)} q(t)\, dt \right| \leq e^{-\kappa_j|x|} \int_{x/2}^\infty dt\, |q(t)|.$$

Clearly $\int_{x/2}^\infty dt\, |q(t)|$ is in $L^\infty(\mathbb{R})$ and thus it follows that $\int_{x/2}^\infty e^{-2\kappa_j(t-x)} q(t) m_1^{\cdot}(t, i\kappa_j)\, dt$ is in $L_N^2(\mathbb{R}^-)$. Further we observe that for $0 \leq \alpha \leq N$, $x < 0$

$$\left|\frac{x}{2}\right|^\alpha \int_x^{x/2} dt e^{-2\kappa_j(t-x)}\, |q(t)| \leqq \int_0^{-x/2} ds e^{-2\kappa_j s}\, |s + x|^\alpha\, |q(s + x)|.$$

Thus it suffices to prove that for $0 \leq \alpha \leq N$, $\int_0^{x/2} ds e^{-2\kappa_j s}\, |s - x|^\alpha\, |q(s - x)|$ is in $L^2(\mathbb{R}^+)$. We have by a change of the order of integration $(0 \leq \alpha \leq N)$

$$\begin{aligned}\int_0^\infty dx \left(\int_0^{x/2} ds e^{-2\kappa_j s}\, |s - x|^\alpha\, |q(s - x)| \right)^2 &\leq \int_0^\infty dt e^{-2\kappa_j t} \int_{2t}^\infty dx\, |t - x|^\alpha\, |q(t - x)| \\ &\quad \times \int_0^{x/2} ds\, |s - x|^\alpha\, |q(s - x)| e^{-2\kappa_j s} \\ &= III + IV\end{aligned}$$

where

$$III = \int_0^\infty dt e^{-2\kappa_j t} \int_{2t}^\infty ds e^{-2\kappa_j s} \int_{2s}^\infty dx\, |q(t - x)|\, |t - x|^\alpha\, |q(s - x)|\, |s - x|^\alpha$$

and

$$IV = \int_0^\infty dte^{-2\kappa_j t}\int_0^{2t} dse^{-2\kappa_j s}\int_{2t}^\infty dx\ |q(t-x)|\ |t-x|^\alpha\ |q(s-x)|\ |s-x|^\alpha.$$

Now it is easy to see that

$$III \le \int_0^\infty dte^{-2\kappa_j t}\int_{2t}^\infty dse^{-2\kappa_j s}\left(\int_{-\infty}^{-s} dy\ |q(y)|^2\ |y|^{2\alpha}\right)^{1/2}\left(\int_{-\infty}^{t-2s} dz\ |q(z)|^2\ |z|^{2\alpha}\right)^{1/2}$$
$$\le \left(\int_0^\infty dte^{-2\kappa_j t}\right)^2\int_{-\infty}^0 dy\ |q(y)|^2\ |y|^{2\alpha} < \infty$$

and

$$IV \le \int_0^\infty dte^{-2\kappa_j t}\int_0^{2t} dse^{-2\kappa_j s}\left(\int_{-\infty}^{-t} dyy^{2\alpha}\ |q(y)|^2\right)^{1/2}\left(\int_{-\infty}^{s-2t} dzz^{2\alpha}\ |q(z)|^2\right)^{1/2}$$
$$\le \left(\int_0^\infty dte^{-2\kappa_j t}\right)^2\int_{-\infty}^0 dyy^{2\alpha}\ |q(y)|^2.$$

This ends the proof of Lemma 2.12.

3. The Jacobian of the scattering data map and its inverse

In this section we will prove that the Jacobian of SD is boundedly invertible. In order to do so we have to derive certain orthogonality relations. We omit their proofs because they can be shown in very similar way as the orthogonality relation derived in Proposition 3.6 of [3] using Lemma 2.5, Corollary 2.6 and Lemma 2.12. Recall that we have introduced $h_j = \lim_{x\to-\infty} e^{\kappa_j x} f_1^\cdot(x, i\kappa_j)$ and that we have proved that $h_j = \lim_{x\to+\infty} e^{-\kappa_j x} f_2^\cdot(x, i\kappa_j)$ (Corollary 2.7).

LEMMA 3.1. *If $N \ge 3$, q in $Q_{N,n}(\mathbb{R})$ and $1 \le j \le n$ then*

(1) $0 = \int_{-\infty}^\infty f_1(x, -k) f_2(x, k)\partial_x(f_1 f_2^\cdot - f_1^\cdot f_2)(x, i\kappa_j)\, dx \quad (k \text{ in } \mathbb{R})$

(2) $0 = \int_{-\infty}^\infty f_1(x, -k) f_2(x, k)\partial_x f_1^2(x, i\kappa_j)\, dx \quad (k \text{ in } \mathbb{R}).$

LEMMA 3.2. *If $N \geq 3$, q in $Q_{N,n}(\mathbb{R})$ and $1 \leq j, j' \leq n$ then*

$$(1)\ 0 = \int_{-\infty}^{\infty} f_1^2(x, i\kappa_j)\partial_x f_1^2(x, i\kappa_{j'})\, dx$$

$$(2)\ -i\kappa_j \frac{h_j^2}{d_j}\delta_{jj'} = \int_{-\infty}^{\infty} f_1^2(x, i\kappa_j)\partial_x(f_1^{\cdot}f_2 - f_1f_2^{\cdot})(x, i\kappa_{j'})$$

where $\delta_{jj'} = 1$ if $j = j'$ and 0 otherwise.

LEMMA 3.3. *If $N \geq 3$, q in $Q_{N,n}(\mathbb{R})$ and $1 \leq j, j' \leq n$ then*

$$(1)\ 0 = \int_{-\infty}^{\infty} (f_1^{\cdot}f_2 - f_1f_2^{\cdot})(x, i\kappa_j)\partial_x(f_1^{\cdot}f_2 - f_1f_2^{\cdot})(x, i\kappa_{j'})\, dx$$

$$(2)\ i\kappa_j \frac{h_j^2}{d_j}\delta_{jj'} = \int_{-\infty}^{\infty} (f_1^{\cdot}f_2 - f_1f_2^{\cdot})(x, i\kappa_j)\partial_x f_1^2(x, i\kappa_j)\, dx.$$

Now we are in a position to prove the following

THEOREM 3.4. *If $N \geq 3$ and q in $Q_{N,n}(\mathbb{R})$ then d_qSD is boundedly invertible.*

Proof. We start proving that d_qSD is onto. First observe that $d_qS : H_{N,N}(\mathbb{R}, \mathbb{R}) \to H^{\#r}_{N-1,N}$ is onto for q in $Q_{N,n}(\mathbb{R})$ convincing oneself that the proof of Proposition 3.6 in [3] holds also for q in $Q_{N,n}(\mathbb{R})$.

Let us denote by C the map defined on $H^{\#r}_{N-1,N} \times E^n_+ \times \mathbb{R}^n$

$$C(\sigma, \alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n) = \frac{1}{\pi}\int_{-\infty}^{\infty} dk\, \sigma(k)\frac{2i\kappa}{W(-k)W(k)}\partial_x(f_1(x, k)f_2(x, -k))$$
$$+ \sum_{j=1}^{n} \alpha_j \frac{2ic_j}{d_j}\partial_x(f_1^{\cdot}f_2 - f_1f_2^{\cdot})(x, i\kappa_j)$$
$$+ \sum_{j=1}^{n} \beta_j c_j \partial_x f_1^2(x, i\kappa_j).$$

Apply Proposition 3.4 of [3] and use Proposition 2.10 and 2.11 to conclude that $C(\sigma, \alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n)$ is an element in $H_{N,N}(\mathbb{R}, \mathbb{R})$. Using Proposition 3.6 in [3] and Lemma 3.1, 3.2 and 3.3 we get that $d_qSD \circ C = Id$ on $H^{\#r}_{N-1,N} \times E^n_+ \times \mathbb{R}^n$ so it follows that d_qSD is onto.

To see that d_qSD is 1–1 let us introduce the subspace V of $H_{N,N}(\mathbb{R}, \mathbb{R})$ generated by the functions $\partial_x \dot{f}_1^2(x, i\kappa_j)$ and $\partial_x(\dot{f}_1 f_2 - f_1 \dot{f}_2)(x, i\kappa_j)$ $(1 \le j \le n)$. From Lemma 3.2 and 3.3 we conclude that V is of dimension $2n$ and that $\kappa_1 \times \cdots \times \kappa_n \times \eta_1 \times \cdots \times \eta_n$ is 1–1 on V where Lemma 3.1 tells us that $d_qS\,|_V = 0$ i.e. $V \subseteq \operatorname{Ker} d_qS$. As in [3] we can use the fact that C is a right inverse of d_qSD to show that d_qSD is $1-1$ provided one can show that d_qSD is 1–1 for every q in $Q_{N,n}(\mathbb{R})$ with compact support. So let q be in $Q_{N,n}(\mathbb{R})$ with compact support. Combining Lemma 3.6 with $V \subseteq \operatorname{Ker} d_qS$ and $\dim V = 2n$ we conclude that $V = \operatorname{Ker} d_qS$. But then it follows that d_qSD is 1–1.

Remark. Let us point out that it follows from the proof of Theorem 3.4 that C is the inverse of d_qSD as stated in the theorem of section 1.

LEMMA 3.6. *If* $N \ge 3$ *and* q *in* $Q_{N,n}(\mathbb{R})$ *with compact support then* $\dim \operatorname{Ker} d_qS \le 2n$.

Proof. As in [3] let us denote the natural extension of d_qS to all of $H_{1,1}(\mathbb{R}, \mathbb{R})$ by A. We will show that $\dim \operatorname{Ker} A \le 2n$. Take q in $Q_{N,n}(\mathbb{R})$ with $\operatorname{supp} q \subseteq [-b, b]$ for a certain b in $\mathbb{R}$. Introduce the subspace G of $H_{1,1}$ defined by

$$G := \left\{ g \in H_{1,1}(\mathbb{R}, \mathbb{R}) : \int_{-\infty}^{\infty} g(x) f_1(x, -i\kappa_j) f_2(x, i\kappa_j)\, dx = 0 \quad \text{and} \right.$$

$$\left. \int_{-\infty}^{\infty} g(x) f_1(x, i\kappa_j) f_2(x, -i\kappa_j)\, dx = 0, \qquad 1 \le j \le n \right\}.$$

Applying the proof of Lemma 3.10 in [3] one concludes that for $a > b$ $d_qS|_{H_{1,1}(a) \cap G}$ is 1–1 where $H_{1,1}(a)$ is defined as in [3]. Clearly the codimension of G is at most $2n$. Moreover one convinces oneself that every element v in G can be approximated by a sequence $(v_n)_{n \in N}$ with $v_n \in H_{1,1}(n) \cap G$. Combining these three results Lemma 3.6 follows.

4. The set of potentials with resonance

Let us denote by $\mathcal{R}_N$ the set of all potentials q in $H_{N,N}(\mathbb{R}, \mathbb{R})$ with $W(0, q) = 0$.

PROPOSITION 4.1.. *Let* $N \ge 3$ *then* $\mathcal{R}_N$ *is a real analytic submanifold of codimension* 1 *in* $H_{N,N}(\mathbb{R}, \mathbb{R})$.

Proof. From Theorem 2.17 in [3] we conclude that $W(0, q)$ is a real analytic function of q. From Theorem 3.1 in [3] we know that $\partial W(0, q)/\partial q(x) = -f_1(x, 0, q)f_2(x, 0, q)$ and so $\partial W(0, q)/\partial q(x) \not\equiv 0$ in x. Thus Proposition 4.1 follows from the implicit function theorem. Now let us introduce for $n = 0, 1, \ldots$

$$\mathcal{R}_{N,n} := \{q \in \mathcal{R}_N : W(i\kappa, q) \text{ has exactly } n \text{ zeroes in } 0 < \kappa < \infty\}.$$

Then all the $\mathcal{R}_{N,n}$ are pairwise disjoint and $\mathcal{R}_N = \bigcup_{n=0}^{\infty} \mathcal{R}_{N,n}$. Let us remark that $Q_{N,n}(\mathbb{R})$ is an open connected and unbounded subset of $H_{N,N}(\mathbb{R}, \mathbb{R})$ $(n = 0, 1, \ldots)$.

It is not difficult to see that between $Q_{N,n}(\mathbb{R})$ and $Q_{N,n+1}(\mathbb{R})$ sits the set $\mathcal{R}_{N,n}$. To prove it let us recall the well known fact that for q in $H_{N,N}(\mathbb{R}, \mathbb{R})$ all the zeroes of $W(i\kappa)$, $\kappa > 0$, are simple. We will show in Lemma 4.3 below that if q is in $H_{N,N}(\mathbb{R}, \mathbb{R})$ with $W(0, q) = 0$ then $W^{\cdot}(0, q) \neq 0$. Thus all zeroes of $W(i\kappa, q)$ on $0 \leq \kappa < \infty$ are simple.

Therefore there exists for q in $\mathcal{R}_{N,n}$ an open neighborhood U of q in $H_{N,N}(\mathbb{R}, \mathbb{R})$ such that for all p in U, $W(i\kappa, p)$ has exactly n or $n + 1$ roots in $0 \leq \kappa < \infty$.

Then let us define

$$\begin{aligned}
U_0 &:= \{p \in U : W(0, p) = 0\} = U \cap \mathcal{R}_N \\
U_1 &:= \{p \in U : (-1)^n W(0, p) < 0\} \\
U_2 &:= \{p \in U : (-1)^{n+1} W(0, p) < 0\}.
\end{aligned}$$

Clearly U_0 is a neighborhood of q in $\mathcal{R}_N$ which is contained completely in $\mathcal{R}_{N,n}$ using the fact that $W(0, p) = -S(0, p)$ one follows that $U_1 \subseteq Q_{N,n}(\mathbb{R})$ and $U_2 \subseteq Q_{N,n+1}(\mathbb{R})$ where we used that $W(i\kappa) \sim -2k$ as $\kappa \to +\infty$ and that all the zeroes of $W(i\kappa)$ on $0 \leq \kappa < \infty$ are simple. Observe that this shows also that for $n = 0, 1, \ldots$ $\mathcal{R}_{N,n}$ is a real analytic submanifold of codimension 1. We summarize our results in the following

PROPOSITION 4.2. *Let $N \geq 3$. For $n = 0, 1, \ldots$ $\mathcal{R}_{N,n}$ is a real analytic submanifold of codimension* 1 *which sits in between $Q_{N,n}(\mathbb{R})$ and $Q_{N,n+1}(\mathbb{R})$.*

LEMMA 4.3. *Let $N \geq 3$ and q be in $H_{N,N}(\mathbb{R})$ such that $W(0) = 0$. Then $W^{\cdot}(0) \neq 0$.*

Proof. Let k be in $\mathbb{R}\setminus\{0\}$. Then $f_2(x, k)$ can be written as

$$f_2(x, k) = S(k)\frac{f_1(x, k) - f_1(x, -k)}{2ik} + \frac{W(k) + S(k)}{2ik} f_1(x, -k).$$

So for all x in $\mathbb{R}$

$$f_2(x, 0) = \lim_{k\to 0} f_2(x, k) = \left(\frac{W^{\cdot}(0)}{2i} + \frac{S^{\cdot}(0)}{2i}\right) f_1(x, 0).$$

Similarly one gets (k in $\mathbb{R}\setminus\{0\}$)

$$\begin{aligned} f_1(x, k) &= \frac{S(-k)}{2ik} f_2(x, k) + \frac{W(k)}{2ik} f_2(x, -k) \\ &= S(-k)\frac{f_2(x, k) - f_2(x, -k)}{2ik} + \frac{W(k) + S(-k)}{2ik} f_2(x, -k) \end{aligned}$$

and

$$f_1(x, 0) = \lim_{k\to 0} f_1(x, k) = \left(\frac{W^{\cdot}(0)}{2i} - \frac{S^{\cdot}(0)}{2i}\right) f_2(x, 0).$$

Using the fact that $f_1(x, 0) \neq 0$ and $f_2(x, 0) \neq 0$ one gets combining the two results

$$4 - S^{\cdot}(0)^2 = -W^{\cdot}(0)^2.$$

From the fact that $W(k)^* = W(-k)$ and $S(k)^* = S(-k)$ (k in $\mathbb{R}$) we conclude that $W^{\cdot}(0)$ and $S^{\cdot}(0)$ must be purely imaginary. The above equation can thus be written as

$$|W^{\cdot}(0)|^2 = 4 + |S^{\cdot}(0)|^2$$

and Lemma 4.3 follows.

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Received July 10, 1986

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Ph. Cassou-Noguès and M. J. Taylor, **Elliptic Functions and Rings of Integers,** (Progress in mathematics 66) Birkhäuser Verlag Boston – Basel – Stuttgart, 1987 198 pp. SFr. 58.––.

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