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**Autor:** Garnett, John / Trubowitz, Eugene  
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# Gaps and bands of one dimensional periodic Schrödinger operators, II

JOHN GARNETT and EUGENE TRUBOWITZ

## §1. Introduction

Let  $q(x) \in L^2_{\mathbf{R}}[0, 1]$ , the Hilbert space of square integrable real valued functions on the unit interval. Extend  $q(x)$  to the whole line  $\mathbf{R}$  by  $q(x + 1) = q(x)$ . The spectrum of the Schrödinger operator  $-d^2/dx^2 + q(x)$ , acting on  $L^2(\mathbf{R})$ , is the set of  $\lambda$  such that

$$-y'' + q(x)y = \lambda y \tag{1.1}$$

has a nontrivial solution bounded on  $\mathbf{R}$ . The spectrum is contained in  $\mathbf{R}$  and it is the union of a sequence of closed intervals  $[\lambda_{2n-2}, \lambda_{2n-1}]$ , where  $\lambda_n = \lambda_n(q)$ ,  $n \geq 0$ , satisfies

$$\lambda_0 < \lambda_1 \leq \lambda_2 < \lambda_3 \leq \lambda_4 < \dots$$

These intervals are called *bands* and the intervening, possibly void, open intervals are called *gaps*. The possible arrangements of gaps and bands were investigated in [1]. This paper continues that study and includes some applications and simplifications.

Let  $\gamma_n(q) = \lambda_{2n}(q) - \lambda_{2n-1}(q)$  be the  $n$ -th gap length. It is well known that  $\gamma_n(q) \in (l^2)^+$ , the space of nonnegative sequences with  $\sum \gamma_n^2 < \infty$ . Two of the three main results of [1] are:

(a) *Whenever  $\gamma_n \in (l^2)^+$ , there exists  $q \in L^2_{\mathbf{R}}([0, 1])$  such that  $\gamma_n(q) = \gamma_n$ ,  $n = 1, 2, \dots$ . Moreover,  $q$  can be chosen from the even subspace  $E$  of  $q \in L^2_{\mathbf{R}}[0, 1]$  such that*

$$q(1 - x) = q(x)$$

(b) *the spectrum is determined, up to a translation, by the gap lengths  $\gamma_n(q)$ .*

Let  $\mu_n(q)$ ,  $n \geq 1$ , be the Dirichlet spectrum of  $q$ , that is, the spectrum of (1.1) for the boundary condition

$$y(0) = y(1) = 0,$$

and let  $\nu_n(q)$ ,  $n \geq 0$ , be it's Neumann spectrum, i.e. the spectrum of (1.1) with boundary condition

$$y'(0) = y'(1) = 0.$$

Then  $q \in E$  if and only if  $\{\mu_n(q), \nu_n(q)\} = \{\lambda_{2n-1}(q), \lambda_{2n}(q)\}$ , so that for  $q$  even,

$$\gamma_n(q) = |\mu_n(q) - \nu_n(q)|.$$

As functions on  $L^2_{\mathbf{R}}[0, 1]$ ,  $\mu_n(q)$  and  $\nu_n(q)$  are real analytic (while  $\lambda_{2n}$  is not analytic at a  $q$  for which  $\lambda_{2n}(q) = \lambda_{2n-1}(q)$ ) and hence the *signed gap length*  $\sigma_n(q) = \mu_n(q) - \nu_n(q)$ ,  $n \geq 1$ , is real analytic in  $q$ . Furthermore, the map  $\sigma: L^2_{\mathbf{R}}[0, 1] \rightarrow \ell^2$  defined by  $\sigma(q) = (\sigma_n(q))$ ,  $n \geq 1$ , is a real analytic mapping from the Hilbert space  $L^2_{\mathbf{R}}[0, 1]$  to the Hilbert space  $\ell^2$ . The third main result of [1] is:

(c) *Let  $E_0$  be the space of even potentials in  $L^2_{\mathbf{R}}[0, 1]$  satisfying  $\int_0^1 q(x) dx = 0$ . Then the map*

$$E_0 \ni q \rightarrow \sigma(q) = (\sigma_1(q), \sigma_2(q), \dots)$$

*is a real analytic isomorphism between  $E_0$  and  $\ell^2$ , that is,  $\sigma$  is one-to-one and onto and both  $\sigma$  and  $\sigma^{-1}$  are real analytic maps of Hilbert space.*

Of course, since  $\gamma_n(q) = |\sigma_n(q)|$ ,  $q \in E_0$ , result (c) included result (a).

The proof of (a), (b) and (c) in [1] applied harmonic measure arguments to the identification, due to Marčenko and Ostrovskii [3], of band configurations with certain slit quarter planes. In Section 2 we give a direct proof, using analysis in Hilbert space, that the Jacobian

$$d_q \sigma: E_0 \rightarrow \ell^2$$

is invertible. From this it follows easily that  $\sigma$  is one-to-one, and that, if  $\sigma$  is onto, than by the Inverse Function Theorem,  $\sigma^{-1}$  is real analytic. Consequently, result (c) can be proved without the intricate Section 6 of [1]. We cannot prove  $\sigma$  is onto  $\ell^2$  using only the method of Section 2 without a still unknown estimate of  $\|q\|_2$  in terms of  $\|\sigma(q)\|_{\ell^2}$ . However, in Sobolev space such an estimate is

available and thus we show in Section 2 that  $\sigma$  is an isomorphism from  $E_0 \cap H^k = \{q \in E_0 : q \text{ has } k \text{ derivatives periodic and in } L^2([0, 1])\}$  onto  $\mathcal{L}_k^2 = \{(\sigma_n) : \sum n^{2k} \sigma_n^2 < \infty\}$ .

In Section 3 result (c) is used to prove Marčenko's theorem [2] that the finite band potentials (those  $q$  with  $\gamma_n(q) = 0$  for large  $n$ ) are norm dense in  $L^2$ , and that  $q$  has primitive period  $1/k$  if and only if  $\gamma_n(q) = 0$  when  $k$  does not divide  $n$ .

In Section 4 we give some inequalities that band lengths must satisfy and we show that for real analytic potentials the band lengths determine the spectrum up to a translation. Here the harmonic measure methods of [1] reappear.

## §2. Signed gap lengths

We need a general interpolation lemma.

LEMMA 2.1. *Suppose  $\phi(\lambda)$  is an entire function satisfying*

$$\sup_{|\lambda|=(n+1/2)^2\pi^2} \left| \phi(\lambda) / \frac{\sin \sqrt{\lambda}}{\sqrt{\lambda}} \right| = o(1)$$

as  $n \rightarrow \infty$ . Then

$$\phi(\lambda) = \sum_{n \geq 1} \phi(\xi_n) \prod_{\substack{m \geq 1 \\ m \neq n}} \frac{\xi_m - \lambda}{\xi_m - \xi_n}$$

for any sequence  $\xi_n$ ,  $n \geq 1$ , of distinct complex numbers satisfying  $\xi_n = n^2\pi^2 + o(1)$ .

*Proof.* If  $\xi_m$ ,  $m \geq 1$ , is a distinct sequence with  $\xi_m = m^2\pi^2 + o(1)$ , then

$$\prod \frac{\xi_m - z}{m^2\pi^2} = \frac{\sin \sqrt{z}}{\sqrt{z}} \left( 1 + o\left(\frac{\log n}{n}\right) \right)$$

uniformly on the circles  $\Gamma_n = \{|z| = (n + 1/2)^2\pi^2\}$ . Hence the meromorphic function

$$f(z) = \frac{\phi(z)}{z - \lambda} \prod_{m \geq 1} \frac{m^2\pi^2}{\xi_m - z}$$

satisfies  $\sup_{\Gamma_n} |f(z)| = o(n^{-2})$ ,  $n \rightarrow \infty$ ; and the sum of its residues inside  $\Gamma_n$  has



limit 0 as  $n \rightarrow \infty$ . But  $f(z)$  has simple poles of  $\lambda$  and at  $\xi_n$ ,  $n \geq 1$ , and  $f(z)$  is regular elsewhere. Summing the residues, we obtain

$$0 = \phi(\lambda) \prod_{m \geq 1} \frac{m^2 \pi^2}{\xi_m - \lambda} - \sum_{n=1}^{\infty} \phi(\xi_n) \frac{n^2 \pi^2}{\lambda_n - \lambda} \prod_{m \neq n} \frac{m^2 \pi^2}{\xi_m - \xi_n},$$

which is the assertion of the lemma.  $\square$

We turn to the main result of this section.

**THEOREM 2.2.** *For all  $q \in E_0$ , the Jacobian  $d_q \sigma: E_0 \rightarrow \ell^2$  is an isomorphism onto  $\ell^2$ .*

*Proof.* See Chapter 2 of [6] for the facts used in this proof.

The components of  $d_q \sigma$  are

$$d_q \sigma_n = d_q \mu_n - d_q \nu_n = g_n^2 - h_n^2,$$

where

$$g_n^2(t) = 2 \sin^2 n\pi t + o\left(\frac{1}{n}\right)$$

and

$$h_n^2(t) = 2 \cos^2 n\pi t + o\left(\frac{1}{n}\right)$$

are the respective squares of the  $n$ -th Dirichlet and Neumann eigenfunctions. Hence the operator  $d_q \sigma$  is the sum of the isomorphic Fourier series operator

$$E_0 \ni f \rightarrow (-2 \langle \cos 2n\pi t, f \rangle, n \geq 1)$$

and the compact operator

$$E_0 \ni f \rightarrow \left( \left\langle o\left(\frac{1}{n}\right), f \right\rangle, n \geq 1 \right),$$

and  $d_q \sigma: E_0 \rightarrow \ell^2$  is a Fredholm operator.

When  $q$  is even the vectors  $g_m^2 - 1$ ,  $m \geq 1$ , form a basis for  $E_0$  with dual basis  $-2a'_m(x)$ , where

$$a_m(x) = y_1(x, \mu_m) y_2(x, \mu_m)$$

and where  $y_1(x, \mu_m), y_2(x, \mu_m)$  are the fundamental solutions of (1.1) for  $\lambda = \mu_m$  with

$$\begin{pmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

That is, if  $f, g \in E_0$ , then

$$\begin{aligned} \langle f, -2a'_m \rangle &\in \ell^2 \\ \langle g_m^2 - 1, g \rangle &\in \ell^2 \end{aligned}$$

and

$$\int fg \, dx = \sum_{m=1}^{\infty} \langle f, -2a'_m \rangle \langle g_m^2 - 1, g \rangle.$$

Therefore it is sufficient to prove that the matrix

$$a_{n,m} = \langle g_n^2 - h_n^2, -2a'_m \rangle$$

is invertible in  $B(\ell^2, \ell^2)$ .

We have  $\langle g_n^2, -2a'_m \rangle = \delta_{n,m}$ , and because  $a_m(0) = a_m(1) = 0$ ,

$$\begin{aligned} \langle -h_n^2, -2a'_m \rangle &= 2 \int h_n^2 a'_m \, dx = -4 \int h'_n h_n y_1(x, \mu_m) y_2(x, \mu_m) \, dx \\ &= \int y_1 h_n [h_n, y_2] + y_2 h_n [h_n, y_1] \, dx \end{aligned}$$

where  $[f, g] = fg' - f'g$ . But by (1.1),

$$\frac{d}{dx} [h_n, y_j] = (v_n - \mu_m) h_n y_j.$$

So if  $v_n \neq \mu_m$ , then

$$\begin{aligned} \langle -h_n^2, -2a'_m \rangle &= \frac{1}{v_n - \mu_m} ([h_n, y_1][h_n, y_2]) \Big|_0^1 \\ &= \frac{1}{v_n - \mu_m} h_n^2(1) y_1'(1) y_2'(1) = \frac{(-1)^m}{v_n - \mu_m} h_n^2(1) y_1'(1, \mu_m) \end{aligned}$$

since  $h'_n(0) = h'_n(1) = 0$ , since  $y'_1(0) = 0$  and since, when  $q$  is even,  $y'_2(1, \mu_m) = (-1)^m$ . Also

$$h_n^2(1) = \frac{y_1^2(1, \nu_n)}{\|y_1(\cdot, \nu_n)\|_2^2} = \frac{(-1)^{n+1}}{y'_1(1, \nu_n)}$$

where  $\dot{y} = \partial y / \partial \lambda$ , because  $y_1(1, \nu_n) = (-1)^n$  when  $q$  is even and because  $\|y_1(\cdot, \nu_n)\|_2^2 = -\dot{y}'_1(1, \nu_n)y_1(1, \nu_n)$ . From the product formulas

$$y'_1(1, \mu_m) = (\nu_0 - \mu_m) \prod_{k \geq 1} \frac{\nu_k - \mu_m}{k^2 \pi^2}$$

$$\dot{y}'_1(1, \nu_n) = \frac{-(\nu_0 - \nu_n)}{n^2 \pi^2} \prod_{1 \leq k \neq n} \frac{\nu_k - \nu_n}{k^2 \pi^2}$$

we conclude that

$$\langle -h_n^2, -2a'_m \rangle = (-1)^{n+m} \prod_{0 \leq k \neq n} \frac{\nu_k - \mu_m}{\nu_k - \nu_n} \quad (2.1)$$

when  $\nu_n \neq \mu_m$ . If  $\nu_n = \mu_m$ , then  $n = m$  and  $[h_n, y_j] = \|y_1\|_2^{-1} \delta_{j,2}$  because the Wronskian  $[y_1, y_2] = 1$ . Consequently  $\langle -h_n^2, -2a'_m \rangle = 1$  and (2.1) also holds when  $\nu_n = \mu_m$ . Thus our matrix is

$$a_{n,m} = \delta_{n,m} + (-1)^{n+m} \prod_{0 \leq k \neq n} \frac{\nu_k - \mu_m}{\nu_k - \nu_n},$$

and  $(a_{n,m})$  is Fredholm because  $d_q \sigma$  is a Fredholm operator.

By the Fredholm alternative,  $d_q \sigma$  is an isomorphism of  $E_0$  onto  $\ell^2$  if the transpose  $(a_{m,n})$  is one-to-one. Now suppose  $\tau = (\tau_n, n \geq 1) \in \ell^2$  lies in the kernel of  $(a_{m,n})$ . Then

$$0 = (-1)^n \frac{\tau_n}{\nu_0 - \mu_n} + \sum_{m \geq 1} (-1)^m \frac{\tau_m}{\nu_0 - \nu_m} \prod_{1 \leq k \neq m} \frac{\nu_k - \mu_n}{\nu_k - \nu_m}.$$

Consider the function

$$\phi(\lambda) = \sum_{m \geq 1} (-1)^m \frac{\tau_m}{\nu_0 - \nu_m} \prod_{1 \leq k \neq m} \frac{\nu_k - \lambda}{\nu_k - \nu_m}.$$

We will show in a moment that  $\phi(\lambda)$  is an entire function of  $\lambda$  satisfying

$$\left| \phi(\lambda) / \frac{\sin \sqrt{\lambda}}{\sqrt{\lambda}} \right| = o(1) \quad (2.2)$$

uniformly on the circles  $|\lambda|(n + 1/2)^2\pi^2$  as  $n \rightarrow \infty$ . But since

$$\phi(\mu_n) = (-1)^{n+1} \frac{\tau_n}{\nu_0 - \mu_n}, \quad \phi(\nu_n) = (-1)^n \frac{\tau_n}{\nu_0 - \nu_n},$$

$\phi(\xi_n) = 0$  at some point  $\xi_n$  in the  $n$ -th gap. Consequently  $\phi \equiv 0$  by Lemma 2.1 and  $\tau_n = 0$ ,  $n \geq 1$ . That means the transpose  $(a_{m,n})$  is one-to-one and  $d_q\sigma$  is an isomorphism.

It remains to prove (2.2). Since  $\nu_n - n^2\pi^2 \in \ell^2$ ,

$$\begin{aligned} \prod_{1 \leq k \neq m} \frac{\nu_n - \lambda}{\nu_k - \nu_m} &= \left( \prod_{1 \leq k \neq m} \frac{k^2\pi^2 - \lambda}{k^2\pi^2 - m^2\pi^2} \right) \left( 1 + o\left(\frac{\log n}{n}\right) \right) \\ &= \frac{m^2\pi^2}{m^2\pi^2 - \lambda} \left( \prod_{1 \leq k \neq m} \frac{k^2}{k^2 - m^2} \right) \frac{\sin \sqrt{\lambda}}{\sqrt{\lambda}} \left( 1 + o\left(\frac{\log n}{n}\right) \right) \\ &= 2(-1)^{m+1} \frac{m^2\pi^2}{m^2\pi^2 - \lambda} \frac{\sin \sqrt{\lambda}}{\sqrt{\lambda}} \left( 1 + o\left(\frac{\log n}{n}\right) \right) \end{aligned}$$

on  $|\lambda| = (n - 1/2)^2\pi^2$ . Hence for such  $\lambda$ ,

$$\left| \phi(\lambda) / \frac{\sin \sqrt{\lambda}}{\sqrt{\lambda}} \right| \leq \text{Const.} \sum_m \frac{\tau_m}{|m^2 - (n + 1/2)^2|} = o(1). \quad \square$$

It will be convenient to replace  $E_0$  by

$$\mathcal{E}_0 = \{q \in E : \lambda_0(q) = 0\}.$$

Since even potentials are determined by their Dirichlet spectra and since  $\mu_n(q + c) = \mu_n(q) + c$ , and  $\nu_n(q + c) = \nu_n(q) + c$ , the map  $q \rightarrow q - [q]$ , where  $[q] = \int_0^1 q(x) dx$ , is an isomorphism from  $\mathcal{E}_0$  to  $E_0$  preserving signed gap lengths. Let

$$\mathcal{E}^n = \{q \in \mathcal{E}_0 : \gamma_m(q) = \sigma_m(q) = 0, m > n\}.$$

Because, by Theorem 2.2,  $\sigma$  is local analytic isomorphism on  $\mathcal{E}_0$ ,  $\mathcal{E}^n$  is a real analytic submanifold of  $\mathcal{E}_0$  of dimension  $n$ .

**COROLLARY 2.3.** *For each  $n \geq 1$ , the signed gap length map is a real analytic isomorphism of  $\mathcal{E}^n$  onto  $\mathbf{R}^n$ .*

*Proof.* The image  $\sigma(\mathcal{E}^n)$  is an open subset  $\mathbf{R}^n$  because  $\sigma: \mathcal{E}_0 \rightarrow \ell^2$  is a local homeomorphism and  $\mathcal{E}^n = \sigma^{-1}(\mathbf{R}^n) \cap \mathcal{E}_0$ . We next show  $\sigma(\mathcal{E}^n)$  is closed. The identity from [7],

$$q(t) = \lambda_0 + \sum_{m \geq 1} \{\lambda_{2m} + \lambda_{2m-1} - 2\mu_m(T_t q)\},$$

where  $T_t q(x) = q(x + t)$ , yields

$$|q(t)| \leq \sum_{m=1}^n \gamma_m(q), \quad q \in \mathcal{E}^n \tag{2.3}$$

Hence the preimage in  $\mathcal{E}^n$  of any compact subset of  $\mathbf{R}^n$  is bounded in  $L^2$ . It is also weakly closed because the functions  $\sigma_m(q) = \mu_m(q) - \nu_m(q)$  are weakly continuous. Thus the preimage of a compact subset of  $\mathbf{R}^n$  is a weakly compact subset of  $\mathcal{E}^n$ , and it follows that the map  $\sigma: \mathcal{E}^n \rightarrow \mathbf{R}^n$  is proper and that  $\sigma(\mathcal{E}^n)$  is a nonempty, closed subset of  $\mathbf{R}^n$ . Therefore  $\sigma$  maps  $\mathcal{E}^n$  onto  $\mathbf{R}^n$ .

Now let  $M$  be the set of points in  $\mathbf{R}^n$  having more than one preimage. Then  $M$  is open because  $\sigma$  is a local homeomorphism. But  $M$  is also closed. Indeed, if there are distinct points  $q_j$  and  $p_j$  in  $\mathcal{E}^n$  such that  $\sigma(p_j) = \sigma(q_j) \rightarrow \sigma \in \mathbf{R}^n$ , then because the map is proper there are subsequences such that  $p_j \rightarrow p \in \mathcal{E}^n$  and  $q_j \rightarrow q \in \mathcal{E}^n$ . If  $p = q$  then  $p_j = q_j$  for  $j$  large because the map  $\sigma$  is homeomorphic on a neighborhood of  $p$ . So  $p \neq q$  and  $M$  is closed. But  $0 \notin M$  by (2.3). Thus  $M \neq \emptyset$  and the mapping is one-to-one.

The map  $\sigma: \mathcal{E}^n \rightarrow \mathbf{R}^n$  is real analytic because  $\mu_m$  and  $\nu_m$  are real analytic on  $L^2_{\mathbf{R}}[0, 1]$ . The inverse map is real analytic because  $d_q \sigma$  is invertible.  $\square$

It is now easy to show that the map  $\sigma$  is one-to-one on  $\mathcal{E}_0$  (and hence on  $E_0$ ).

**COROLLARY 2.4.** *The signed gap length map is one-to-one on  $\mathcal{E}_0$ .*

*Proof.* Suppose not. Then some point  $\tau \in \ell^2$  has at least two preimages. Since  $\sigma$  is a local homeomorphism, the same is true for each point in some neighborhood of  $\tau$ , so it is also true at

$$\tau^{(N)} = (\tau_1, \dots, \tau_N, 0, 0, \dots)$$

for  $N$  sufficiently large. But that contradicts Corollary 2.3.  $\square$

Write  $\ell_k^2$  for the space of sequences  $(a_n)$  with  $\sum n^{2k} |a_n|^2 < \infty$ . From the asymptotics for  $y_2(1, \lambda, q)$  and  $y_1'(1, \lambda, q)$  we have

$$\begin{aligned}\mu_n(q) &= n^2\pi^2 + [q] - \langle \cos 2n\pi x, q \rangle + \ell_1^2 \\ \nu_n(q) &= n^2\pi^2 + [q] + \langle \cos 2n\pi x, q \rangle + \ell_1^2.\end{aligned}$$

Hence for  $q \in E_0$ ,  $\sigma_n(q) \in \ell_1^2$  if and only if  $\langle \cos 2n\pi x, q \rangle + \ell_1^2$ ; i.e. if and only if  $q$  is in the Sobolev space

$$H^1 = \{q \in L_{\mathbf{R}}^2[0, 1] : q' \in L_{\mathbf{R}}^2[0, 1]\}.$$

**THEOREM 2.5.** *The signed gap length map from  $E_0 \cap H^1$  to  $\ell_1^2$  is one-to-one and onto.*

*Proof.* By Corollary 2.4  $\sigma$  is one-to-one. To prove it is onto fix  $\tau \in \ell_1^2$  and let  $\tau^{(N)} = (\tau_1, \tau_2, \dots, \tau_N, 0, 0, \dots)$ . By Corollary 2.3 there is  $q_N \in \varepsilon^N$  such that  $\sigma(q_N) = \tau^{(N)}$ , and by (2.3)

$$|q_N(t)| \leq \sum_{n=1}^N |\tau_n| \leq \left( \sum_{n=1}^N n^2 \tau_n^2 \right)^{1/2} \left( \sum_{n=1}^{\infty} n^{-2} \right)^{1/2},$$

so that  $\|q_N\|_2 \leq \text{Const.} \|\tau\|_{\ell_1^2}$ . Let  $q \in \varepsilon_0$  be a weak limit of the sequence  $\{q_N\}$ . Then

$$\sigma_n(q - [q]) = \tau_n$$

for all  $n$ , and  $q - [q] \in H^1 \cap E_0$  since  $\tau \in \ell_1^2$ .  $\square$

*Remark 2.6.* We are unable to prove the full result that  $\sigma$  maps  $E_0$  onto  $\ell^2$  by this method. What is needed is an estimate of  $\|q\|_2$  in terms of  $\gamma_n(q)$  more powerful than (2.3). Such an estimate should be useful for other problems.

*Remark 2.7.* It is possible, by refining the proof of Theorem 2.2, to show that  $\sigma : E_0 \cap H^1 \rightarrow \ell_1^2$  is an analytic isomorphism. We omit the details.

*Remark 2.8.* It is known [3, p. 534] that  $\gamma_n(q) \in \ell_k^2$  if and only if  $q \in H^k$ , i.e. if and only if  $q$  has  $k$  derivatives which are periodic and lie in  $L_{\mathbf{R}}^2[0, 1]$ . Thus the proof of Theorem 2.5 shows that

$$\sigma : E_0 \cap H^k \rightarrow \ell_k^2$$

is one-to-one and onto. We have not verified the likely statement that this map is bianalytic.

### §3. Two applications

The potential  $q \in L^2_{\mathbf{R}}[0, 1]$  is called a *finite band* potential if  $\gamma_n(q) = 0$  for all but finitely many  $n$ . Marčenko [2, p. 258] proved that the set of finite band potentials is norm dense in  $L^2_{\mathbf{R}}[0, 1]$ . Here we derive that Theorem from result (c), stated in the introduction.

**THEOREM 3.1 (Marčenko).** *The set of finite band potentials is norm dense in  $L^2_{\mathbf{R}}[0, 1]$ .*

For  $q \in E$ , Theorem 3.1 is immediate from results (c). To prove it for arbitrary  $q$  we need two additional theorems. Define

$$\kappa_n(q) = \log((-1)^n y'_2(1, \mu_u, q)).$$

In [6] it is proved that  $\kappa_n(q) \in \ell^2_1$ , i.e. that  $\sum n^2 \kappa_n^2(q) < \infty$ , and that the correspondence

$$q \rightarrow (\mu_n(q) - [q], \kappa_n(q))$$

is a homeomorphism from  $L^2_{\mathbf{R}}[0, 1]$  onto  $\ell^2 \times \ell^2_1$ . That is the first theorem.

The second theorem is the description of the isospectral manifold

$$L(q) = \{p \in L^2_{\mathbf{R}}[0, 1] : \lambda_n(p) = \lambda_n(q), \text{ all } n\}$$

given in [4]. The parameters

$$\mu_n(p) \in [\lambda_{2n-1}, \lambda_{2n}]$$

and

$$\text{sign } \kappa_n(p)$$

uniquely determine  $p \in L(q)$ . Although true generally, this theorem will only be used for finite band potentials, and such potentials satisfy the smoothness assumptions of [4].

*Proof of Theorem 3.1.* Fix  $q \in L_{\mathbf{R}}^2[0, 1]$ . Since  $\lambda_n(q + c) = \lambda_n(q)$ , we may suppose  $\lambda_0(q) = 0$ . By result (c) there exist, for  $N = 1, 2, \dots$ ,  $e_N \in \mathcal{E}_0$  such that

$$\gamma_n(e_0) = \begin{cases} \gamma_n(q) & n \leq N \\ 0 & n > N \end{cases}$$

and

$$\mu_n(e_N) = \lambda_{2n-1}(e_N), \quad n = 1, 2, \dots$$

Since  $\mu_n(q) \in [\lambda_{2n-1}(q), \lambda_{2n}(q)]$  there exists  $t_n \in [0, 1]$  such that

$$\mu_n(q) = t_n \lambda_{2n}(q) + (1 - t_n) \lambda_{2n-1}(q),$$

and by the second theorem just cited there exists  $q_N \in L(e_N)$  such that for all  $n$ ,

$$\mu_n(q_N) = t_n \lambda_{2n}(e_N) + (1 - t_n) \lambda_{2n-1}(e_N)$$

and

$$\text{sign } \kappa_n(q_N) = \text{sign } \kappa_n(q).$$

By the first cited theorem  $\|q_N - q\|_2 \rightarrow 0$  if

$$\|\mu(q_N) - \mu(q)\|_{\ell^2} \rightarrow 0 \tag{3.1}$$

and

$$\|\kappa_n(q_N) - \kappa_n(q)\|_{\ell^1} \rightarrow 0. \tag{3.2}$$

By the second theorem there exists  $e \in \mathcal{E}_0$  such that for all  $n$ ,

$$\lambda_n(e) = \lambda_n(q)$$

$$\mu_n(e) = \lambda_{2n-1}(q).$$

Then  $\|\gamma_n(e_N) - \gamma_n(e)\|_{\ell^2} \rightarrow 0$  and  $\sigma_n(e_N)$  and  $\sigma_n(e)$  have the same sign, so that

$$\|\sigma_n(e_N) - \sigma_n(e)\|_{\ell^2} \rightarrow 0. \tag{3.3}$$

Hence by result (c),  $\|e_N - e\|_2 \rightarrow 0$  and by the first theorem

$$\|\mu_n(e_N) - \mu_n(e)\|_{\ell^2} \rightarrow 0. \tag{3.4}$$



But then by the choices of  $q_N$ ,  $e_N$  and  $e$ ,

$$\mu_n(q_N) - \mu_n(q) = -t_n(\sigma_n(e_N) - \sigma_n(e)) + \mu_n(e_N) - \mu_n(e),$$

and (3.3) and (3.4) imply (3.1).

To prove (3.2) we use the identity

$$2 \cosh \kappa_n(q) = (-1)^n \Delta(\mu_n(q), q),$$

where  $\Delta(\lambda, q)$  is the discriminant function

$$\Delta(\lambda, q) = y_1(1, \lambda, q) + y_2'(1, \lambda, q)$$

and the inequality

$$|x - y|^2 \leq 2 |\cosh x - \cosh y|,$$

valid when  $x$  and  $y$  have the same sign. They give

$$\begin{aligned} n^2 |\kappa_n(q_N) - \kappa_n(q)|^2 &\leq n^2 |\Delta(\mu_n(q_N), q_N) - \Delta(\mu_n(q), q)| \\ &= n^2 |\Delta(\mu_n(q_N), e_N) - \Delta(\mu_n(q), e)|. \end{aligned}$$

Since  $\|e_N - e\|_2 \rightarrow 0$ ,  $\Delta(\lambda, e_N) \rightarrow \Delta(\lambda, e)$  uniformly on compact sets. Thus by (3.1)

$$|\Delta(\mu_n(q_N), e_N) - \Delta(\mu_n(q), e)| \rightarrow 0$$

for each  $n$ . Moreover,

$$\begin{aligned} &|\Delta(\mu_n(q_N), e_N) - \Delta(\mu_n(q), e)| \\ &\leq |\Delta(\mu_n(q_N), e_N) - 2(-1)^n| + |\Delta(\mu_n(q), e) - 2(-1)^n| \end{aligned}$$

since  $\Delta(\lambda_{2n-1}, e_N) = \Delta(\lambda_{2n-1}, e) = 2(-1)^n$ . Because  $\dot{\Delta}(\lambda) = \partial\Delta/\partial\lambda$  is an entire function of order  $\frac{1}{2}$ , having one zero  $\dot{\lambda}_n$  in each gap  $[\lambda_{2n-1} \leq \lambda \leq \lambda_{2n}]$  and no other zeros, the product representation

$$\dot{\Delta}(\lambda) = \prod_{n \geq 1} \frac{\dot{\lambda}_n - \lambda}{n^2 \pi^2}$$

shows that

$$\sup_{\lambda_{2n-1} \leq \lambda \leq \lambda_{2n}} |\dot{\Delta}(\lambda, q)| \leq c \frac{\gamma_n(q)}{n^2}.$$

Hence by (3.5)

$$n^2 |\Delta(\mu_n(q_N), e_N) - \Delta(\mu_n(q), e)| \leq c\gamma_n^2(q)$$

and by dominated convergence

$$\lim_{N \rightarrow \infty} \sum_n n^2 |\kappa_n(q_N) - \kappa_n(q)|^2 = 0. \quad \square$$

Our second application concerns the subspace  $L_k^2 \subset L_R^2[0, 1]$  of all functions whose periodic extensions have primitive period  $1/k$ ,  $k = 1, 2, \dots$ . First of all, if  $q \in L_k^2$ ,  $\gamma_n(q) = 0$  whenever  $k \nmid n$ . To see this, recall from [7] that  $\mu_n(t) = \mu_n(T_t q)$ , where  $T_t q(x) = q(x - t)$ , then as  $t$  runs from 0 to 1,  $\mu_n(t)$  makes  $n$  complete trips between  $\lambda_{2n-1}(q)$  and  $\lambda_{2n}(q)$ , when  $\gamma_n(q) \neq 0$ . By assumption,  $\mu_n(t + 1/k) = \mu_n(t)$ . Therefore,  $n$  is equal to  $k$  times the number of complete trips in time  $1/k$ , and so  $k$  divides  $n$ ,  $k \mid n$ .

Now let  $E_0^k = E_0 \cap L_k^2$ . It follows from the observation above that the restriction of  $\sigma$  to  $E_0^k$  maps into  $\mathcal{I}^2(k) = \{\sigma \in \mathcal{I}^2 : \sigma_n = 0 \text{ whenever } k \nmid n\}$ . Without any change in the argument of [1] or that of Section 2, one can show that  $\sigma$  is a real analytic isomorphism between  $E_0^k$  and  $\mathcal{I}^2(k)$  or between  $E_0^k \cap H^1$  and  $\mathcal{I}_1^2(k)$ .

Suppose  $q \in E$  and  $\sigma(q) \in \mathcal{I}^2(k)$ . Then there is a  $p \in E_0^k$  such that  $\sigma(p) = \sigma(q)$ . However,  $\sigma$  is globally one-to-one on  $E_0$  so that  $p = q - [q]$ . In other words,  $q \in E_0$  has primitive period  $1/k$  if and only if  $\gamma_n(q) = 0$  whenever  $k \nmid n$ .

It is easy to extend this observation to all of  $L_R^2[0, 1]$ . Let  $q \in L_R^2[0, 1]$  and  $L(q) = \{r \in L_R^2[0, 1] \mid \lambda_i(r) = \lambda_i(q) i \geq 0\}$ , i.e., the isospectral set of  $q$ . It is not hard to see that  $L(q) \cap E \neq \emptyset$  and that all points in  $L(q)$  have the same primitive period. See [4]. Thus we have proved

**THEOREM 3.2.** *The potential  $q$  has primitive period  $1/k$  if and only if  $\gamma_n(q) = 0$  whenever  $k$  does not divide  $n$ .*

#### §4. Band lengths

Let  $\alpha_n(q) = \alpha_n = \lambda_{2n-1} - \lambda_{2n}$  be the length of the  $n$ -th band. It is well known that

$$\alpha_n(q) - (2n - 1)\pi^2 \in \mathcal{I}^2 \tag{4.1}$$

and in [1] and [5] it was shown that

$$\tilde{\alpha}_n = (2n - 1)\pi^2 - \alpha_n \geq 0, \quad (4.2)$$

with equality holding for some  $n$  if and only if  $q$  is constant.

**THEOREM 4.1.** *For all  $n$  and all  $q$ ,*

$$\alpha_n - \alpha_{n-1} + \alpha_{n-2} \mp \cdots > 0, \quad (4.3)$$

and

$$\beta_n = \tilde{\alpha}_n - \tilde{\alpha}_{n-1} + \tilde{\alpha}_{n-2} \mp \cdots \geq 0. \quad (4.4)$$

Moreover, if  $\beta_n = 0$  for some  $n$ , then  $q$  is constant and  $\beta_k = 0$  for all  $k$ .

Note that by (4.1), (4.3) has content only for small  $n$ . By (4.4)  $\beta_n \leq \tilde{\alpha}_n$ , so that by (4.1)

$$\beta_n \in \mathcal{L}_+^2$$

and by (4.4) and (4.3),

$$0 \leq \beta_n < n\pi^2. \quad (4.5)$$

We shall show that (4.5) is sharp for every  $n$  and that, properly interpreted, the Jacobian  $d_g \beta_n : E_0 \rightarrow \mathcal{L}^2$  is invertible at  $q = 0$ . A simple characterization of band lengths thus seems unlikely.

*Proof.* Recall from [1] that there exists  $h_n = h_n(q) \geq 0$ , such that  $\sum n^2 h_n^2 < \infty$ , and such that

$$\delta(\lambda, q) = \cos^{-1} \left( \frac{\Delta(\lambda, q)}{2} \right)$$

is a conformal mapping from the half plane  $\{\text{Im } \lambda > 0\}$  onto the slit quarter plane

$$\Omega(h) = \{x > 0, y > 0\} \setminus \bigcup_{n=1}^{\infty} T_n$$

where

$$T_n = \{n\pi + iy : 0 < y \leq h_n\}.$$

Under  $\delta(\lambda, q)$  the  $n$ -th band is mapped onto the segment

$$B_n = [(n-1)\pi-, n\pi+] \subset \partial\Omega(h),$$

and if

$$u_n(z, h) = \omega(z, B_n, \Omega(h)) = u_n(z)$$

is the harmonic measure of  $B_n$  in  $\Omega(h)$ , then

$$\alpha_n = \lim_{x \rightarrow \infty} 2\pi x^2 u_n(x + ix, h). \quad (4.6)$$

Let  $k \leq n$  and let  $z = x + ix$  with  $x > n\pi$ . Then  $u_k(z)$  is the probability that a Brownian path starting at  $z$  makes its first exit from  $\Omega(h)$  through  $B_k$ . Letting  $S_k$  be the set of such paths, we write

$$u_k(z) = P_z(S_k)$$

Brownian paths can be assumed continuous. Thus every path in  $S_k$  must cross the half line  $J_k = \{x = k\pi, y > 0\}$  before it leaves  $\Omega(h)$ . Let  $R_k$  be those paths in  $S_k$  which, before leaving  $\Omega(h)$ , last meet  $J_k \cup J_{k-1}$  in  $J_k$ , and let  $L_k$  be those whose last contact with  $J_k \cup J_{k-1}$ , before departing from  $\Omega(h)$ , is in  $J_{k-1}$ . Then  $R_k$  and  $L_k$  are  $P_z$  measurable,  $R_k \cap L_k = \phi$  and

$$P_z(S_k) = P_z(R_k) + P_z(L_k).$$

But

$$P_z(L_k) = P_z(R_{k-1})$$

by a reflection. Since  $L_1 = \emptyset$ , we conclude that

$$u_n(z) - u_{n-1}(z) \pm \cdots = P_z(R_n) > 0$$

which by (4.6) yields (4.3). To prove (4.4), let

$$V_n(z) = u_n(z, 0) - u_n(z, h).$$

Then

$$\tilde{\alpha}_n = \lim_{x \rightarrow \infty} 2\pi x^2 V_n(x + ix).$$

On  $\partial\Omega(h)$ ,

$$V_n(\xi) = \sum_{k=1}^{\infty} u_n(\xi, 0) \chi_{T_k}(\xi), \quad (4.7)$$

and the argument above shows

$$\sum_{k=1}^n (-1)^{n-k} u_k(\xi, 0) > 0$$

on  $\bigcup T_k$ . Hence for  $x$  large

$$\sum_{k=t}^n (-1)^{n-k} V_k(x + ix) \geq 0$$

and (4.4) holds. If equality holds in (4.4) then by (4.7),  $\bigcup T_k$  has zero harmonic measure in  $\Omega(h)$ . That means all gap lengths are zero and  $q$  is constant.  $\square$

To see that (4.5) is sharp, note that  $q \rightarrow h_n(q)$  maps onto  $\ell_1^2$  and that by (4.6),

$$\lim_{h_n \rightarrow \infty} \alpha_k = 0, \quad 1 \leq k \leq n.$$

For  $q \in E$ , define

$$a_n(q) = \mu_n(q) - \nu_{n-1}(q), \quad n \geq 1$$

and

$$b_n(q) = \sum_{k=1}^n (-1)^{n-k} ((2n-1)\pi^2 - a_n(q)) = n\pi^2 - \sum_{k=1}^n (-1)^{n-k} a_n(q).$$

Then for  $q \in E$

$$b_n(q) = \beta_n(q) + \text{Max}(\sigma_n(q), 0),$$

and for each potential  $q \in L^2$  there is  $q^+ \in E$  with  $\mu_n(q^+) \leq \nu_n(q^+)$  and  $\lambda_n(q^+) = \lambda_n(q)$ , so that  $b_n(q^+) = \beta_n(q)$ .

**THEOREM 4.2.** *At  $q=0$  the Jacobian  $d_q(b_n): E_0 \rightarrow \ell^2$  is an isomorphism onto  $\ell^2$ .*

*Proof.* At  $q = 0$ ,  $f \in E_0$ ,

$$\langle d_q a_n, f \rangle = \langle 2 \sin^2 n\pi t - 2 \cos^2 (n-1)\pi t, f \rangle$$

and

$$\langle d_q b_n, f \rangle = -2 \langle \sin^2 n\pi t, f \rangle = \langle \cos 2n\pi t, f \rangle,$$

and  $(\cos 2n\pi t)_{n \geq 1}$  is a complete orthonormal system in  $E_0$ .  $\square$

**THEOREM 4.3.** (a) *If  $q$  and  $\bar{q}$  are finite band potentials and if  $\alpha_n(q) = \alpha_n(\bar{q})$  for infinitely many  $n$ , then the periodic spectra of  $q$  and  $\bar{q}$  agree up to a translation.*

(b) *If  $q$  and  $\bar{q}$  are real analytic, and if  $\alpha_n(q) = \alpha_n(\bar{q})$  for all large  $n$ , then  $q$  and  $\bar{q}$  have the same periodic spectrum up to a translation.*

*Proof.* Let  $\phi(z, q)$  be the inverse of the mapping  $\delta(\lambda, q)$ . If  $q$  is a finite band potential then  $h_n(q) = 0$ ,  $n > N$  and  $\phi(z, q)$  reflects to be analytic in the complement of the finite union of vertical slits  $\{|x| = n\pi, |y| \leq h_n(q), 1 \leq n \leq N\}$ . For  $z$  large we have

$$\phi(z, q) = z^2 + o\left(\frac{1}{z}\right).$$

By the hypothesis of (a),

$$\phi(z + \pi, q) - \phi(z, q) = \phi(z + \pi, \bar{q}) - \phi(z, \bar{q}) \quad (4.8)$$

holds for an infinite sequence of integers tending to  $\infty$ . Hence (4.8) holds for all  $z$ , and  $\phi(z, q)$  and  $\phi(z, \bar{q})$  have the same singularities. Therefore  $h_n(q) = h_n(\bar{q})$  for all  $n$ , which means the spectra of  $q$  and  $\bar{q}$  differ by at most a translation.

To prove (b), set  $f(z) = \phi(z, q) - \phi(z, \bar{q})$ . By reflection  $f(z)$  is analytic in

$$\Omega^* = \mathbf{C} / \bigcup_{n=1}^{\infty} (S_n \cup S_{-n})$$

where  $S_n = \{x = n\pi, |y| \leq \text{Max}(h_{|n|}(q), h_{|n|}(\bar{q}))\}$ , and by the asymptotics for  $\Delta(\lambda, q)$ ,  $f(z)$  is bounded on  $\Omega^*$ . Since  $q$  and  $\bar{q}$  are real analytic, we have by [7],

$$\begin{aligned} \text{Max}(h_n(q), h_n(\bar{q})) &\leq C \text{Max}(\gamma_n(q), \gamma_n(\bar{q})) \\ &\leq C e^{-an} \end{aligned}$$

for constants  $a$  and  $C$ . Viewing  $S_n$  as two-sided, we see that  $f(z)$  has continuous boundary values on  $S_n$  and that for  $n \geq 1$ ,

$$\sup_{z \in \bar{S}_n} |f(z) - f(n\pi+)| \leq \gamma_n(q) + \gamma_n(\bar{q}) \leq Ce^{-an}.$$

By hypothesis there is  $N$  so that

$$f((n+1)\pi-) - f(n\pi+) = \alpha_n(q) - \alpha_n(\bar{q}) = 0$$

for  $n \geq N$ , and hence

$$\sup_{z \in \bar{S}_n} |f(z)| \leq Ce^{-an}, \quad n \geq N. \quad (4.9)$$

Set  $h^* = \sup_n \{h_n(q), h_n(\bar{q})\}$ . We shall prove

$$|f(x + Ch^*)| \leq Ce^{-a'x}, \quad x > x_0 \quad (4.10)$$

First assume (4.10). Then because  $f(z)$  is bounded and analytic in  $\{y > h^*\}$ ,

$$\log |f(z)| \leq C_1 + C_2 \int_{x>x_0} \frac{-a'x}{1+x^2} dx = -\infty$$

on  $|z - i(h^* + 1)| < \frac{1}{2}$ . Therefore  $f = 0$  and (b) is proved.

We turn to the proof of (4.10). Let  $\Delta_n$  be the disc  $\{|z - n\pi| < 2Ae^{-a|n|}\}$  with  $A$  so large that  $\text{dist}(S_n, \partial\Delta_n) \geq Ae^{-an}$  for all  $n \geq 1$ . Then by (4.9) and the three circles theorem,

$$\sup_{\partial\Delta_n} |f(z)| \leq C_2 e^{-an} \quad (4.11)$$

Now let  $\Omega = \mathbf{C} \setminus \bigcup_{n=1}^{\infty} (\bar{\Delta}_n \cup \bar{\Delta}_{-n})$  and for  $\delta > 0$  fixed and  $x$  large, set

$$E_x = \bigcup \{\Delta_n : |n\pi - x| < \delta x\}.$$

LEMMA 4.4. *There is  $C(h^*, a)$  such that for  $x$  large,*

$$\omega(x + ih^*, E_x, \Omega) \geq C(h^*, a)$$

Note that by the subharmonicity of  $\log |f|$ , this lemma and (4.11) imply (4.10) and hence the theorem.

*Proof of Lemma 4.9.* Fix  $\delta_1$ ,  $0 < \delta_1 < \delta$ , to be determined later, and let  $N_x \sim 2\delta_1 x / \pi$  be the number of  $n$  such that  $|n\pi - x| < \delta_1 x$ . Set

$$u(z) = \frac{1}{N_x} \sum_{|n\pi - x| < \delta_1 x} \log \frac{1}{|z - n\pi|}.$$

Then  $u(z)$  is harmonic and bounded above in  $\Omega$ , and

$$\sup_{z \in \partial\Omega \setminus E_x} u(z) \leq \log \frac{1}{(\delta - \delta_1)x} + c = \alpha.$$

But if  $z \in E_x$  then

$$\begin{aligned} u(z) &\leq \frac{1}{N_x} \log (Ae^{-an}) + \frac{1}{N_x} \sum_{i=2}^{N_x/2} \log \frac{1}{v\pi} \\ &\leq \log \frac{1}{\delta_1 x} + c'(a) = \beta, \end{aligned}$$

and by a similar calculation,

$$u(x + ih^*) \geq \beta + c(h^*, a).$$

We choose  $\delta_1$  so that  $\beta - \alpha = c'' > 1$ . Then by the maximum principle,

$$\omega(z, E_x, \Omega) \geq \frac{u(z) - \alpha}{\beta - \alpha}$$

and

$$\omega(x + ih^*, E_x, \Omega) \geq \frac{c(h^*, a)}{c''} \quad \square.$$

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*University of California, Los Angeles*  
*and*  
*E.T.H. Zurich, Switzerland*

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