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Autor: Dyer, Michael N.
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Localization of group rings and applications to 2-complexes

MICHAEL N. DYER*

In this paper we use recent results of S. Rosset [R] on the localization of group rings to give applications to the theory of 2-dimensional CW-complexes and related fields. If G is a group, we let $\mathbb{Z}G$ denote the integral group ring of G. If A is a non-trivial normal abelian torsion-free subgroup of G (In this case we say that G is a **Rosset group** or just an ***R*-group**, for short), we let S denote the multiplicatively closed subset $\mathbb{Z}A$-0 and localize $\mathbb{Z}G \to \mathbb{Z}G_S$ so as to invert the elements of S.

The first application is concerned with extending the Kaplansky rank κP (see [DV]) for finitely generated projective $\mathbb{Z}G$-modules P to projective $\mathbb{Z}G_S$-modules. This extension has a number of interesting applications because many $\mathbb{Z}G$-modules (such as the second homotopy module of a 2-complex) become projective upon localization.

The second application generalizes a theorem of Hillman [H]. If X is a connected 2-complex with fundamental group isomorphic to G (in this case X is called a **$[G, 2]$-complex**) and L is a subgroup of G, let X_L denote the covering of X corresponding to L. We say that X is ***L*-Cockcroft** if the Hurewicz map $\pi_2X \to H_2X_L$ is trivial. Let H_iL denote the ith-homology group of L with coefficients in the trivial $\mathbb{Z}G$-module $\mathbb{Z}$. If L is a normal subgroup of a group G, the **weight of L in G** (denoted by wt_GL) is the minimal number of elements whose normal closure in G is L.

THEOREM 1. *Suppose $L \rightarrowtail G \twoheadrightarrow H$ is an exact sequence of groups with H a Rosset group, G finitely presented, H_1L finitely generated as an abelian group and wt_GL finite. Let X be any $[G, 2]$-complex. Then the Euler characteristic $\chi X \geq 0$, with $\chi X = 0$ iff X is L-Cockcroft and $H_2L = 0$.*

COROLLARY 2. *In addition to the above hypotheses, if either*
(a) *H_1L is torsion-free and L has no perfect subgroups*

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or

(b) *L is locally indicable,*
then the $[G, 2]$-complex X is aspherical iff $\chi X = 0$ and $H_2 L = 0$.

Let G be a finitely presented group with $H = H_1 G$ infinite and $L = G'$, the derived group of G. Furthermore, assume that $H_1 G'$ is finitely generated as an abelian group. By theorem 1, G always has deficiency ≤ 1. If, in addition, G has no perfect subgroups (e.g., if G is residually nilpotent) and $H_1 G'$ is torsion-free, then it follows from corollary 2 that G has deficiency 1 iff it has geometric dimension 2 and $H_2 L = 0$.

The outline of the paper is as follows. In section 1 we describe the localization results of S. Rosset and in section 2 we give the extension of Kaplansky's invariant to projectives over localized rings. In section 3 we apply the earlier results to shed new light on the aspherical question of J. H. C. Whitehead: is every connected subcomplex of an aspherical 2-complex aspherical? Section 4 contains the proof of theorem 1 while in section 5 we derive an algebraic analog of theorem 1.

1. Localization of certain group rings

In this section we describe recent results of S. Rosset [R]. Let G be a group and let A be a non-trivial torsion-free abelian normal subgroup of G. If a group G has such a normal subgroup, we will say that G is an ***R*-group.** Then the set $S = \mathbb{Z}A - 0$ is a multiplicatively closed subset of the integral group ring $\mathbb{Z}G$ and satisfies the Ore conditions [P, page 146]. Thus there exists a left ring of fractions

$$\mathbb{Z}G_S = \{\beta^{-1}\alpha \mid \alpha \in \mathbb{Z}G, \beta \in S\}$$

and a canonical injection $i: \mathbb{Z}G \to \mathbb{Z}G_S$ given by carrying $\alpha \to 1^{-1}\alpha$.

This localization has the following properties:

(L1) The right $\mathbb{Z}G$-module $\mathbb{Z}G_S$ is flat.

(L2) If M is any left $\mathbb{Z}G$-module, then the localization M_S of M is given by $M_S = \mathbb{Z}G_S \otimes_{\mathbb{Z}G} M$. If the underlying abelian group M^0 of M is finitely generated or consists only of elements of finite order, then $M_S = 0$.

(L3) The ring $\mathbb{Z}G_S$ has rank invariance for finitely generated free modules; i.e., if $\mathbb{Z}G_S^m \approx \mathbb{Z}G_S^n$, then $m = n$.

The property (L3) is proved via the stronger Kaplansky property:

(L4) Let $\varphi: \mathbb{Z}G_S^m \to \mathbb{Z}G_S^m$ be any *surjection* from a free $\mathbb{Z}G_S$-module of rank m to itself. Then φ is an isomorphism, as well (see [R], theorem F).

Using properties L1–L3 S. Rosset [R] gives the following remarkable generalization of a theorem of D. Gottlieb ([G], [S]):

THEOREM 1.0. *If X is a finite aspherical complex whose fundamental group $\pi_1 X$ an R-group, then the Euler characteristic $\chi(X)=0$.*

In another paper [D_1] we show the following generalization of Rosset's theorem. This has also been discovered independently by L. Fornera in her Ph.D. thesis at ETH.

THEOREM 1.1. *Let X be a finite aspherical complex with $\pi_1 X = G$. Let $L \rightarrowtail G \twoheadrightarrow H$ be an exact sequence of groups with $H_* L$ finitely generated as an abelian group and H an R-group. Then $\chi(X)=0$.*

DEFINITION 1.1. Let m be an integer $\geqslant 2$. A $[G, m]$-complex is a connected CW-complex whose dimension is $\leq m$, whose fundamental group $\pi_1 X$ is isomorphic to G, and whose universal cover $\tilde{X}$ is $(m-1)$-connected. For example, any connected 2-complex is a $[\pi_1 X, 2]$-complex.

Combining the results of [R] and [H], we have the following.

THEOREM 1.2 (Hillman–Rosset). *Let X be a finite $[G, m]$-complex whose fundamental group is an R-group. Then the Euler characteristic $\chi(X) \geq 0$. The Euler characteristic of X is zero iff X is aspherical.*

Before giving the proof, we give the following:

LEMMA 1.3. *Let M be a submodule of a free $\mathbb{Z}G$-module F. Then $M_S = 0$ iff $M = 0$.*

Proof. The exact sequence $M \rightarrowtail F \twoheadrightarrow Q = F/M$ localizes to the exact sequence $M_S \rightarrowtail F_S \twoheadrightarrow Q_S$. The inclusion $F \to F_S$ induces an inclusion $M \to M_S$. The result follows. ■

Proof of the theorem. Let $C_*\tilde{X} \twoheadrightarrow \mathbb{Z}$ denote the augmented cellular chain complex of the universal cover $\tilde{X}$, considered as a sequence of finitely generated free $\mathbb{Z}G$-modules. Let $K = \ker[d_m : C_m \to C_{m-1}]$ be the mth-homotopy group of X. Localize the exact sequence $K \to C_*\tilde{X} \twoheadrightarrow \mathbb{Z}$ to obtain the exact sequence of stably-free projectives $K_S \to C_*\tilde{X}_S \to 0$. Thus the rank of K_S as a stably-free $\mathbb{Z}G_S$-module is $\chi(X)$, which must necessarily be ≥ 0. If $\chi(X)=0$, then rank $K_S = 0$. It follows from L4 that $K_S = 0$ and from the lemma that $K = 0$. ■

This theorem has two very lovely corollaries, the first of which was noted in [H]. We say that a finitely presented group G has **(finite) geometric dimension ≤ 2** if G admits a (finite) aspherical $[G, 2]$-complex.

COROLLARY 1.4. *If G is a finitely presented R-group, then the deficiency of G is ≤ 1. The deficiency of G is equal to 1 iff G has finite geometric dimension 2.* ■

COROLLARY 1.5. *If H is any finitely presented group, then the deficiency of the cartesian product $\mathbb{Z} \times H$ is ≤ 1. The deficiency of $\mathbb{Z} \times H = 1$ iff H is free.*

Proof. By the previous corollary, we need only show that the geometric dimension $\mathbb{Z} \times H \leq 2$ iff H is free. First, if H is finitely generated and free, then the obvious presentation of $\mathbb{Z} \times H$ of deficiency 1 may be realized as an aspherical $[\mathbb{Z} \times H, 2]$-complex. In order to see the converse, we apply the Lyndon–Hochschield–Serre spectral sequence to the split exact sequence $\mathbb{Z} \rightarrowtail \mathbb{Z} \times H \twoheadrightarrow H$. If M is any $\mathbb{Z}H$-module, then we obtain the split exact sequence

$$H^3(H; M) \rightarrowtail H^3(\mathbb{Z} \times H; M) \twoheadrightarrow H^2(H; M).$$

Thus $H^3(\mathbb{Z} \times H, M) = 0$ implies that $H^2(H; M) = 0$. This says that H has cohomological dimension ≤ 1. That H has cohomological dimension 1 follows because H is torsion free. Now H is free by the famous result of J. Stallings [S_2, p. 58]. ■

2. Extending the Kaplansky invariant

In this section we show how to use the results of [D_2] to extend the invariant of I. Kaplansky (see [DV]) to localized group rings. We assume that the group G has a non-trivial normal abelian torsion-free subgroup A (we call such an A an **NATF-subgroup**). Let $S = \mathbb{Z}A - 0$ and localize $\mathbb{Z}G \rightarrow \mathbb{Z}G_S$. References for this section include [S], [D_2], [DV], and [P].

For any ring R, a **trace function on R** is a linear map $T: R \rightarrow B$, where B is an abelian group such that, for each $r, s \in R$, $T(rs) = T(sr)$. If we define the set $[R, R]$ to be the subgroup generated by the Lie brackets $[r, s] = rs - sr$, then the **universal trace function** is given by $T_u: R \rightarrow \tau R = R/[R, R]$. Any trace function T on R may be extended in the usual way to any $n \times n$-matrix $M = [m_{ij}]$ over R via the formula $T(M) = \sum T(m_{ii})$. Any trace function T has the properties (a) $T(M + N) = T(M) + T(N)$ and (b) $T(PQ) = T(QP)$, where M, N are

$n \times n$-matrices, P is an $m \times n$-matrix, and Q is an $n \times m$-matrix over R. Also, $T(1_n) = n \cdot T(1)$, provided R has a multiplicative identity 1, and 1_n is the identity $n \times n$-matrix over R.

If G is a group and $\mathbb{Z}G$ is the integral group ring, then the universal trace group $\tau\mathbb{Z}G$ is easy to describe. Let CG denote the set of conjugacy classes of G. Then the group $\tau\mathbb{Z}G$ is equal to the free abelian group $\mathbb{Z}CG$ generated by the set CG. For an element $x \in G$, let $\langle x \rangle \in CG$ denote the conjugacy class of the element x.

The trace function $T_1 : \mathbb{Z}G \to \mathbb{Z}$ is given in either of two (equivalent) ways. First, for any $v \in \mathbb{Z}G$, let $T_1(v)$ be the coefficient of 1 in v. Secondly it can be described as the coefficient of $\langle 1 \rangle$ in $T_u(v)$.

Following [S] we extend the trace T to any endomorphism $f : R^n \to R^n$ by choosing a basis for R^n and defining $T(f)$ to be the trace of the matrix M of f with respect to this basis. This is independent of the choice of basis. Further, if P is any finitely generated projective R-module, choose an integer $n \geq 0$ and an idempotent endomorphism $e : R^n \to R^n$ whose image is isomorphic to P. Define the **rank of P** with respect to T to be $T(e)$. See [S] for the proof that this is well defined. We denote this rank by $\rho_T P$. If $R = \mathbb{Z}G$ and $T = T_1$, we denote this rank as κP. This is the **Kaplansky rank** (it is called iP in [DV]). The rank (**Hattori–Stallings**) for the universal trace function $T_u : \mathbb{Z}G \to \tau\mathbb{Z}G$ is usually denoted by $r_G P$.

The Kaplansky rank is known to have the following properties (see [DV]).

K(a) κP is an integer $\geqq 0$.

K(b) If P and Q are finitely generated projective $\mathbb{Z}G$-modules, then $\kappa(P \oplus Q) = \kappa P + \kappa Q$.

K(c) If $n(P)$ is the minimum number of generators of P as a $\mathbb{Z}G$-module, then $\kappa P \leqq n(P)$.

K(d) $\kappa P = 0$ iff $P = 0$.

K(e) $\kappa P = n(P)$ iff $P \approx \mathbb{Z}G^{n[P]}$.

Now let $S = \mathbb{Z}A - 0$ and localize $\mathbb{Z}G$ to $\mathbb{Z}G_S$ via the inclusion map i. Let H be the quotient G/A and $\pi : G \to H$ be the natural surjection. For any element $h \in H$ and $a \in A$, let $h * a$ denote the action induced by conjugation by any preimage of h under π (that is, if $\pi g = h$, then $h * a = g \cdot a \cdot g^{-1}$). This makes A into a $\mathbb{Z}H$-module. In this case, we will give a complete description of a direct summand $\mathcal{F}^\wedge$ of $\tau\mathbb{Z}G_S$. The proofs for this description are given in [DF].

First, let $\mathcal{F}$ denote the quotient field of $\mathbb{Z}A$. It is easy to see that, by choosing a set E of right coset generators for H in G (let $1 \in E$), the ring $\mathbb{Z}G_S$ is an $\mathcal{F}$-module and that it is $\mathcal{F}$-isomorphic to the vector space $\mathcal{F}(E)$ with natural basis E. Consider the projection $T : \mathbb{Z}G_S \to \mathcal{F} \cdot 1 = \mathcal{F}$ of the ring onto the coordinate corresponding to $1 \in E$. Note that $\mathbb{Z}G_S$ and $L = [\mathbb{Z}G_S, \mathbb{Z}G_S]$ are $\mathbb{Q}$-vector spaces, where $\mathbb{Q}$ is the rational numbers. Factoring out by the image of L under T defines the vector space $\mathcal{F}^\wedge$. (It is shown in [DF] that $T(L)$ is precisely the $\mathbb{Q}$-subspace $\mathbb{H} \cdot \mathcal{F}$, where $\mathbb{H}$ is the augmentation ideal in $\mathbb{Q}H$. Then $\mathcal{F}^\wedge$ is $\mathcal{F}/\mathbb{H} \cdot \mathcal{F} = \mathbb{Q} \otimes_{QH} \mathcal{F}$; it is also shown there that $\mathcal{F}^\wedge$ is a direct summand (over $\mathbb{Q}$) of $\tau\mathbb{Z}G_S$).

Thus we may define a new trace function $t: \mathbb{Z}G_S \to \mathcal{F}^\wedge$ via T followed by the natural projection $\mathcal{F} \to \mathcal{F}^\wedge$. Let $[f]$ denote the image of $f \in \mathcal{F}$ in $\mathcal{F}^\wedge$. We will show that this trace function t "extends" the function T_1 given above, in certain cases.

Let $\langle A \rangle$ denote the conjugation classes in G determined by the elements of A; for each $a \in A$, $\langle a \rangle$ is the conjugation class in G defined by a. Let $tA: \mathbb{Z}G \to \mathbb{Z}\langle A \rangle$ be the trace map determined by restricting to those conjugation classes in $\langle A \rangle$.

Let $\alpha: \mathbb{Z}\langle A \rangle \to \mathcal{F}^\wedge$ be the map defined by sending $\langle a \rangle \mapsto [a]$. If we let $\gamma: \mathbb{Z}\langle A \rangle \to \tau\mathbb{Z}G$ be the natural split injection into $\tau\mathbb{Z}G$, l be the localization $\mathbb{Z}G \to \mathbb{Z}G_S$ and $r: \tau(\mathbb{Z}G_S) \to \mathcal{F}^\wedge$ be the projection induced by the projection T above, then one sees easily that $\alpha = r \circ \tau(l) \circ \gamma$.

LEMMA 2.0. *If P is any finitely generated projective $\mathbb{Z}G$-module, then* $\alpha(\rho_{tA}(P)) = \rho_t(P_S)$.

Proof. This follows from the definition because, if e is the defining idempotent for P, then e_S is the defining idempotent for P_S. ■

DEFINITION. We say that the Hattori–Stallings rank $r_G P$ is **carried by conjugacy classes of finite order** if, for each finitely generated projective $\mathbb{Z}G$-module P, the coordinate $r_G P(\langle x \rangle)$ of $r_G P$ on the conjugacy class $\langle x \rangle$ is trivial except for elements $x \in G$ of finite order.

LEMMA 2.1. *If the Hattori–Stallings rank is carried by conjugacy classes of finite order, then the rank ρ_t is really given by the Kaplansky rank κ, i.e., if $\beta: \mathbb{Z} \to \mathcal{F}^\wedge$ is given by $1 \mapsto [1]$, then $\beta(\kappa P) = \rho_t(P_S)$.*

Proof. By lemma 2.0, $\rho_t(P_S) = \alpha(\rho_{tA} P)$. But each conjugacy class $\langle a \rangle \neq \langle 1 \rangle$ in $\mathbb{Z}\langle A \rangle$ consists of elements of infinite order, so $\rho_{tA} P = \kappa P \cdot \langle 1 \rangle$. ■

A result of B. Eckmann [E] shows that the Hattori–Stallings rank (over $\mathbb{Q}G$, and hence over $\mathbb{Z}G$) is carried by elements of finite order if G is one of the following types of groups:

(a) solvable groups G

(b) linear groups $G \subseteq GL_r(F)$ where F is a field of characteristic 0.

(c) groups of cohomology dimension $cd_{\mathbb{Q}}G \leq 2$.
provided G has finite homology dimension over $\mathbb{Q}$.

Furthermore, if G is a residually finite group, then P. Linnell has shown that the Hattori–Stallings rank (over $\mathbb{Z}G$) is concentrated on $\langle 1 \rangle$ [L].

Some properties of the rank ρ_t are given in the following

PROPOSITION 2.2. *Let P and Q be finitely generated projective $\mathbb{Z}G_S$-modules. Then*

K_S(a): *$\rho_t P$ is a member of $\mathcal{F}^\wedge$.*

K_S(b): $\rho_t(P \otimes Q) = \rho_t P + \rho_t Q$.

If $[1] \neq 0$ in $\mathcal{F}^\wedge$, and P is a stably-free $\mathbb{Z}G_S$-module, then

K_S(c): *$\rho_t P = k \cdot [1]$ with $k \in \mathbb{Z}$ and $0 \leq k \leq n(P)$, where $n(P)$ is the minimal number of generators of P as a $\mathbb{Z}G_S$-module and $k \in \mathbb{Z}$ is the stable-free rank.*

K_S(d): $\rho_t P = 0 \Leftrightarrow P = 0$.

K_S(e): $\rho_t P = n(P) \cdot [1] \Leftrightarrow P \approx \mathbb{Z}G_S^{n[P]}$.

Proof. Statements (a) and (b) are clear. Statement (c) follows from (b) and the fact that $\rho_t \mathbb{Z}G = [1]$. We will show statement (d). Statement (e) then follows from (d). Because P is stably-free we see that the following sequence is exact for some positive integer n:

$$P \rightarrowtail \mathbb{Z}G_S^n \twoheadrightarrow Q$$

with Q stably-free. Then $\rho_t P = 0$ yields that $\rho_t Q = n \cdot [1]$ (here is where we use that fact that $[1] \neq 0$, because then $[1]$ has infinite order in $\tau \mathbb{Z}G_S$); we may assume that, in fact, Q is free of rank n (perhaps by replacing n by $n + k$). Then the Kaplansky property L4 implies that $P = 0$. ■

Question. Do K_S(c), (d) and (e) hold without the assumption that P is stably-free?

DEFINITION 2.3. We say that the finitely generated $\mathbb{Z}G$-module M is **pre-projective** (respectively, **pre-stably free**) if the localization M_S is a projective (respectively, stably-free) $\mathbb{Z}G_S$-module. For example, if X is a $[G, m]$-complex, then the mth homotopy group $\pi_m X$ is a pre-projective $\mathbb{Z}G$-module. We define the **Kaplansky rank** $\kappa_A M$ of a pre-projective module M to be $\kappa_A M = p_t M_S$. Of course, if M_S is stably-free, then $\kappa_A M$ is an integer multiple of $[1]$. It is not known to me whether or not the Kaplansky rank is independent of the choice of A.

COROLLARY 2.4. *Let M be a pre-stably-free $\mathbb{Z}G$-module. Then $\kappa M = 0$ iff the localization $M_S = 0$. If M is a submodule of a free $\mathbb{Z}G$-module, then $M = 0$.*

Proof. The first statement is just a special case of (d) above. The second follows from lemma 1.3. □

3. Application to aspherical complexes

We say that a $[G, 2]$-complex X has the **Whitehead condition** (**WC**) if either X is aspherical or, if X is not aspherical, then whenever X is the subcomplex of an $[H, 2]$-complex Y, Y is not aspherical (see [BD] and [BDS] for reference). A **group G** is **WC** if every $[G, 2]$-complex satisfies WC. For any group G, let $\boldsymbol{P_1 G}$ denote the **maximal perfect subgroup** of G. The following theorem is an improvement over several theorems in [BD] and [BDS].

THEOREM 3.1. *Let G be a finitely presented R-group which has a normal abelian torsion-free subgroup not contained in P_1G. Then G has* WC.

Proof. The deficiency of G is ≤ 1. If X is a $[G, 2]$-complex, then the Euler characteristic $\chi X \geqq 0$, with X aspherical iff $\chi X = 0$. Suppose $\chi X > 0$ and X is a subcomplex of an $[H, 2]$-complex Y. We will show that Y is not aspherical. Suppose that Y were aspherical. Then it follows from [BD] that there is a non-trivial perfect subgroup P in G such that the cohomological dimension $cd(G/P) \leqq 2$. Furthermore, G/P has type FL with $\chi(G/P) = \chi X > 0$. Now the hypothesis implies that G/P is an R-group, which is impossible (because G/P is an R-group and FL implies that $\chi(G/P) = 0$; see the proof of theorem 1.2). ■

We can now improve corollary 3.7 of [BDS] to read: *if G is the finitely presented fundamental group of a non-aspherical subcomplex $X < Y$ of an aspherical* 2*-dimensional complex, then G has a non-trivial, superperfect, normal C-subgroup* (*see* [BD] *for a definition of C-subgroup*) *P with respect to $C_*\tilde{X} \to \mathbb{Z}$. Moreover, $cdG/P \leqq 2$ and the center of G is contained in P.* See also Corollary 4.7 of [BD].

We also note the following peculiar corollary: *For any finitely presented group G the cartesian product $\mathbb{Z} \times G$ has* WC.

Jonathan Hillman (private communication) has pointed out that 3.1 improves another corollary of [BDS], namely Corollary 5.2.

THEOREM 3.2. *If G is a* 2*-ended group, then G has* WC.

Proof. If G is a 2-ended group which doesn't have WC, then by Corollary 5.2 of [BDS] we have the exact sequence $P \rightarrowtail G \twoheadrightarrow \mathbb{Z}$, where P is a finite perfect group and the deficiency of G is 1. But it is easy to see that, because P is finite, G has an infinite cyclic central element. Thus, by 1.4, G has WC. ∎

4. Application to deficiency

Throughout this section we assume that the NATF-subgroup A in G has been chosen once and for all and that $0 \neq [1] \in \mathscr{F}^\wedge$ (this happens iff $[1] \in \mathscr{F}^\wedge$ has infinite order, see [DF] for details and examples). For any $[G, 2]$-complex X, let X_L denote the covering of X corresponding to the subgroup L. We say that G is **L-Cockcroft** if there is a $[G, 2]$-complex X such that the Hurewicz map $\pi_2 X \to H_2 X_L$ is trivial. Such a space X is also called L-Cockcroft. If X is a *finite* $[G, 2]$-complex, then we say that it is a **$[G, 2]_f$-complex,** If L is a *normal* subgroup of G, the **weight of L in G** (denoted by $wt_G L$) is the minimal number of elements which normally generate L (in G). In this section we show the following.

THEOREM 4.1. *Let $1 \to L \to G \to H \to 1$ be an exact sequence of groups with*
(a) *G finitely presented,*
(b) *H an R-group,*
(c) *the weight $wt_G L$ finite, and*
(d) *the $\mathbb{Z}H$-module $H_1 L$ localizing to zero.*
Then the deficiency $\operatorname{def} G$ *of G is ≤ 1, and is equal to 1 iff G is L-Cockcroft and $H_2 L = 0$.*

Notice that the above hypothesis 4.1(d) is satisfied if $H_1 L$ is finitely generated as an abelian group or is a torsion group. Also, 4.1(c) and (d) are satisfied if L is finitely generated. In particular, let $L = 1$. The theorem then says that any R-group G has deficiency 1 iff G is 1-Cockcroft; i.e., G has geometric dimension $\leqq 2$. This is the Rosset–Hillman theorem. The proof of Theorem 4.1 will be given at the end of the section.

For example, if $G = gp\{a, b \colon (a^r b^{-s})^q\}$ $(r, s, q > 0)$, $H = gp\{a, b \colon a^r b^{-s}\}$, and $G \twoheadrightarrow H$ is the map induced by the identity on the generators, then the kernel L is normally generated by the element $l = a^r b^{-s}$ and $H_1 L$ is a torsion group generated by all the conjugates of l. Notice that $a^r = b^s$ generates an infinite cyclic central subgroup in H. Hence, $\operatorname{def} G = 1$ implies that G is L-Cockcroft and $H_2 L = 0$ (the latter also follows from a result of Fischer, Karrass, and Solitar [FKS] that says that the normal subgroup L is a free product of cyclic groups.

Moreover, E. Dyer and A. Vasquez [DV_2] have an explicit construction of a $K(G, 1)$-space from which it is easy to see that G is L-Cockcroft).

We first prove the following

THEOREM 4.2. *Let $L \rightarrowtail G \twoheadrightarrow H$ be an exact sequence of groups and homomorphisms, with G finitely presented, H an R-group with* NATF*-subgroup A, and $\mathrm{wt}_G L < \infty$. Further assume that, for $S = \mathbb{Z}A - 0$, the $\mathbb{Z}H_S$-modules $(H_iL)_S$ are all projective for $i = 1, 2, 3$, and that $(H_3L)_S$ is finitely generated as a $\mathbb{Z}H_S$-module. Let $\kappa_i = \kappa_A H_i L$. Then, for any $[G, 2]_f$-complex X, we have*

$$\kappa_A(\mathbb{Z} \otimes_L \pi_2 X) = \kappa_3 - \kappa_2 + \kappa_1 + \chi X \cdot [1] \in \mathscr{F}^\wedge.$$

Proof. Let X_L denote the covering of X corresponding to the subgroup L. Let $\{l_\alpha \mid \alpha \in \mathscr{A}\}$ be a set of elements of L whose normal closure (in G) is equal to L. Assume that $|\mathscr{A}| < \infty$. Use the elements l_α to add 2-cells e_α to X to obtain the space Y containing X as a subcomplex. If let $\mathbb{Z}H^{\mathscr{A}}$ denote the direct sum of $|\mathscr{A}|$ copies of $\mathbb{Z}H$, then the following sequences of $\mathbb{Z}H$-modules are exact:

$$0 \to H_3L \to \mathbb{Z} \otimes_L \pi_2 X_L \to H_2 X_L \to H_2 L \to 0, \tag{4.3}$$

$$0 \to H_2 X_L \to \pi_2 Y \to \mathbb{Z}H^{\mathscr{A}} \to H_1 L \to 0. \tag{4.4}$$

Sequence (4.3) is a restatement of two classical theorems of Hopf relating the second and third homology groups of L to the homology of a 2-complex X_L and its universal cover $\tilde{X} = \tilde{X}_L$. Notice that the complex X_L can be identified as a subcomplex of the universal cover $\tilde{Y}$ of Y. The second sequence is a restatement of the homology sequence of the pair $(\tilde{Y}, X_L)$.

Because $|\mathscr{A}| < \infty$. we see that H_1L is a finitely generated $\mathbb{Z}H$-module. We also see that $K = (H_2X_L)_S$ is a finitely generated projective, because $(\pi_2 Y)_S$ is a stably-free $\mathbb{Z}G_S$-module. By localizing (4.4) at S, it is evident that $\kappa_A H_2 X_L - \kappa_A H_1 L = \chi X \cdot [1]$. It then follows from (4.3) that $W_S = Z \otimes_L (\pi_2 X)_S$ is projective and that

$$\kappa_A H_3 L - \kappa_A H_2 L = \kappa_A W - \kappa_A H_2 X_L.$$

The equality follows. ■

COROLLARY 4.5. *In addition, suppose that $(H_iL)_S = 0$ for $i = 1, 2, 3$. Then $\kappa_A(\mathbb{Z} \otimes_L \pi_2 X) = \chi X \cdot [1] \in \mathscr{F}^\wedge$ is a non-negative integer multiple of* [1].

Proof. The equality follows from theorem 4.4. In this case, $(\pi_2 Y)_S$ is stably-free $\Rightarrow (H_2 X_L)_S$ stably free $\Rightarrow (\mathbb{Z} \otimes_L \pi_2 X)_S = M_S$ stably free. Hence, $\kappa_A M$ is a non-negative integer multiple of $[1]$. ■

Note that in this case there is an exact sequence $M_S \rightarrowtail (\pi_2 Y)_S \twoheadrightarrow \mathbb{Z}H_S^{\mathscr{A}}$ of stably-free modules. Note also that if $\chi X = 0$, than $\kappa_A M = 0$, and hence $M_S = 0$. If $H_3 L = 0$, then M is a submodule of a free $\mathbb{Z}H$-module (namely, $C_2 \tilde{Y}$) and hence by lemma 1.3, $M = 0$. This says that $\pi_2 X$ is a **perfect $\mathbb{Z}L$-module** ($\mathbb{Z} \otimes_{\mathbb{Z}L} \pi_2 X = 0$) and therefore also a perfect $\mathbb{Z}G$-module. I do not know of a non-trivial example of a $[G, 2]$-complex whose second homotopy group is a perfect $\mathbb{Z}G$-module.

We say that a $[G, 2]_f$-complex is **minimal** if it has the minimum Euler characteristic among all such complexes.

Example 4.6(?). Let $L \rightarrowtail G \twoheadrightarrow H$ be an exact sequence of groups with $\operatorname{def} G = 1$, $H_3 L = 0$, $cdG > 2$, H an R-group, and $wt_G L$ finite. If $(H_1 L)_S = 0$ and X is a minimal $[G, 2]_f$-complex, then $\mathbb{Z} \otimes_L \pi_2 X = 0$, while $\pi_2 X \neq 0$. Does such an example exist?

A $[G, 2]$-complex with a *single zero cell* will be called a **$[G, 2]^*$-complex.** Any $[G, 2]$-complex has the homotopy type of a $[G, 2]^*$-complex by simply factoring out a maximal tree in the 1-skeleton.

If X is any $[G, 2]_f$-complex and $C_* \tilde{X} \to \mathbb{Z}$ is the augmented cellular chain complex of the universal cover $\tilde{X}$ of X, considered as a complex of $\mathbb{Z}G$-modules, then $R_X = \ker \{C_1 \tilde{X} \to C_0 \tilde{X}\}$ is called a **relation module** corresponding to X.

THEOREM 4.7. *Suppose that $L \rightarrowtail G \twoheadrightarrow H$ is an exact sequence of groups with G finitely presented, H an R-group having an* NATF-*subgroup A, and $wt_G L < \infty$. Let X be a minimal $[G, 2]_f^*$-complex. In addition, let $(H_1 L)_S$ and $(H_2 L)_S$ be projective $\mathbb{Z}H_S$-modules, $\kappa_i = \kappa_A H_i L$, m (respectively n) be the rank over $\mathbb{Z}G$ of $C_1 \tilde{X}$ (respectively $C_2 \tilde{X}$), and $N = \mathbb{Z} \otimes_L R_X$.*

(a) *Then N is a finitely generated projective $\mathbb{Z}H_S$-module and*

$$\kappa_A N = (m - 1) \cdot [1] + \kappa_2 - \kappa_1.$$

(b) *If N is a stably-free $\mathbb{Z}H_S$-module, then $\kappa_A N = k \cdot [1]$ and*

$$\operatorname{def} G (= m - n) \leq k.$$

(c) *Furthermore, if X is L-Cockcroft, then N is free of rank n and in this case,*

$$\operatorname{def} G \cdot [1] = [1] + \kappa_1 - \kappa_2 \in \mathscr{F}^\wedge.$$

Proof. One sees easily that, if IG denotes that augmentation ideal inside $\mathbb{Z}G$, then $(\mathbb{Z} \otimes_L IG)_S \approx (H_1L)_S \oplus \mathbb{Z}H_S$. Thus, $(H_1L)_S$ is projective $\Leftrightarrow (\mathbb{Z} \otimes_L IG)_S$ is and both are finitely generated if G is. By tensoring the map $\delta_2 : C_2\tilde{X} \approx \mathbb{Z}G^n \to C_1\tilde{X} \approx \mathbb{Z}G^m$ with $\mathbb{Z} \otimes_L$-, we obtain the map d_2. One then shows that $(\operatorname{im} d_2)_S \oplus (\mathbb{Z} \otimes_L IG)_S \approx \mathbb{Z}H^m$ and that $N_S \approx (\operatorname{im} d_2)_S \oplus (H_2L)_S$. The same argument as in 4.2 shows that $(H_2L)_S$ is finitely generated. The calculation of $\kappa_A N$ follows. If X is L-Cockcroft then it follows that the boundary map $i \otimes \delta_2 : \mathbb{Z} \otimes_L C_2\tilde{X} \to N$ is an isomorphism; hence, $\kappa_A N = n \cdot [1]$. The computation for the deficiency of G is a result of the formula $\operatorname{def} G = 1 - \chi X$. ■

Note that the formula $\operatorname{def} G \cdot [1] = [1] + \kappa_1 - \kappa_2$ is analogous to the formula $\operatorname{def} G \leqq \operatorname{rank}_{\mathbb{Z}} H_1G -$(minimum number of generators of H_2G). One calls the group **efficient** if the latter inequality is an equality. Equality in the former case might be called ***L*-efficient**.

Proof of 4.1. If we assume that $(H_1L)_S = 0$ (this is so if H_1L is finitely generated as an abelian group or is a torsion group), then we may prove a theorem with no assumed conditions on H_2L. The proof of theorem 4.7 above shows the statement: G is L-Cockcroft and $H_2L = 0 \Rightarrow \operatorname{def} G = 1$ (i.e., $\kappa_1 = \kappa_2 = 0$ and use the stably-free rank). We will show the converse. Let $U = H_2X_L$. Because $(H_1L)_S = 0$, the following sequence is split exact:

$$0 \to U_S \to (\pi_2 Y)_S \to \mathbb{Z}H_S^k \to 0.$$

Then $\operatorname{def} G = 1$ implies that $\chi X = 0$, so $\chi Y =$ stably-free rank of $(\pi_2 Y)_S = k = wt_G L$. Hence, by the Kaplansky property L4, we have that $U_S = 0$. But U is a submodule of the free $\mathbb{Z}H$-module $C_2\tilde{Y}$, hence $U = 0$. Thus G is L-Cockcroft and $H_2L = 0$. This proves theorem 4.1. ■

Example 4.8. Let G' denote the commutator subgroup of the finitely presented group G. Then if $H_1G = G/G'$ is infinite and $(H_1G')_S = 0$, then (by 4.1) we have that $\operatorname{def} G \leq 1$. The deficiency is equal to 1 iff G is G'-Cockcroft and $H_2G' = 0$.

Example 4.9. Let G be any finitely presented group with commutator subgroup G' finitely generated. Consider G'', the second derived group of G. The group $G/G'' = H$ has H_1G' as a normal abelian subgroup and $wt_G G''$ is finite (G' is finitely generated implies that $wt_{G'} G'' < \infty$. Thus $wt_G G'' < \infty$). We assume that H_1G' is torsion free, so that H is an R-group. Now if $(H_1G'')_S = 0$, then the conclusions of theorem 4.1 hold for the sequence $G'' \rightarrowtail G \twoheadrightarrow G/G''$.

Notice that it follows from sequence 4.4 that if H is an R-group with $wt_G L < \infty$ and $H_2X_L = 0$, then the projective dimension of $(H_1L)_S \leq 1$. This follows because, in this case, the sequence $0 \to (\pi_2 X)_S \to (\mathbb{Z}H^k)_S \to (H_1L)_S \to 0$ is exact, with $(\pi_2 Y)_S$ finitely generated and stably-free.

We say that a group G is an ***E*-group** (with respect to the resolution $P_* \to \mathbb{Z}$) if H_1G is torsion free and for some projective $\mathbb{Z}G$-resolution $P_* \to \mathbb{Z}$ of the trivial module $\mathbb{Z}$, the homomorphism $\mathbb{Z} \otimes_{\mathbb{Z}G} d_2$: $\mathbb{Z} \otimes_{\mathbb{Z}G} P_2 \to \mathbb{Z} \otimes_{\mathbb{Z}G} P_1$ is a monomorphism. Such groups are studied in [St].

COROLLARIES 4.10. There are some interesting special cases of theorem 4.1.

First, assume that $\operatorname{def} G = 1$ and that H_1L is torsion-free. In this case L becomes an E-group [St] because $H_2X_L = 0$. Let P_1G denote the maximal perfect subgroup of G. Then one may apply the theory of Strebel's E-groups as in [BD], Section 4, to observe that

(*) $H_2P_1L = 0$, G/P_1L has cohomological dimension ≤ 2 and type FL, and that the Euler characteristic $\chi(G/P_1L) = 0$.

Furthermore, if $P_1L = 1$, then G has geometric dimension 2.

Secondly, a group U is said to be **locally indicable** if every nontrivial finitely generated subgroup of U has infinite abelianization. We consider the nonempty family $\mathcal{S}_L$ consisting of all normal subgroups V of a group L such that G/V is locally indicable ($G \in \mathcal{S}_L$). If we order $\mathcal{S}_L$ by inclusion, then it is easy to see that this family has a minimal element, call it $\mathbb{P}_A L$ (this is called the **Adams' subgroup** of L). Note that L is locally indicable iff $\mathbb{P}_A L = 1$. Then a similar argument to that given by Adams in [A] shows that (given the hypotheses of theorem 4.1 plus H_1L torsion free) (*) is true with P_1L replaced by $\mathbb{P}_A L$, provided $\mathbb{P}_A L$ is *perfect*. See also proposition 3.1 of [HS].

Assume that $L \rightarrowtail G \twoheadrightarrow H$ is an exact sequence of groups satisfying the hypotheses of theorem 4.1. If, in addition, L is locally indicable, then G has deficiency $1 \Leftrightarrow G$ has geometric dimension 2 and $H_2L = 0$. This follows because $\operatorname{def} G = 1$ (and X is the $[G, 2]$-complex having $\chi X = 0) \Leftrightarrow H_2X_L = 0$. This latter happens iff $1 \otimes_L \partial_2 : \mathbb{Z} \otimes_L C_2 \to \mathbb{Z} \otimes_L C_1$ is monic. Then apply the fact that local indicibility of L yields that ∂_2 is monic as well.

For example, L could be a classical knot or link group, or a finitely generated torsion-free 1-relator group. These groups are known to be locally indicable [H].

Example 4.11. We give an application of theorem 4.1 to the Whitehead problem. A normal subgroup $L \trianglelefteq G$ is small if $wt_G L < \infty$ and $H_1 L$ is finitely generated as an abelian group.

THEOREM 4.12. *Let $K \rightarrowtail G \twoheadrightarrow Q$ be an exact sequence of groups with Q an R-group, G finitely presented, and K small. Furthermore, suppose that K contains the maximal perfect subgroup $P_1 G$ of G. Then either of the following two hypotheses implies that G has WC.*

(a) $H_2 K = 0$ *and* $\operatorname{def} G < 1$, *or*
(b) $H_2 K \neq 0$ *and* $\operatorname{def} G = 1$.

Proof. If G does not have WC then there is a non-trivial perfect normal subgroup $P \trianglelefteq G$ so that G is P-Cockcroft. Because K contains $P_1 G$, than K contains P. Thus G is K-Cockcroft. If in addition, $H_2 K = 0$, then $\operatorname{def} G = 1$; if $H_2 K \neq 0$, then $\operatorname{def} G < 1$, by theorem 4.1. These contradict hypotheses (a) or (b). ∎

For example, let $G(\alpha)$ be the αth term of the derived series of G, where α is any ordinal number. Suppose for some ordinal α, the abelian group $G(\alpha)/G(\alpha+1)$ is non-trivial and torsion-free and that $G(\alpha)$ is small. Then if $H_2 G(\alpha) = 0$ and $\operatorname{def} G < 1$, it follows that G has WC.

The **parity of a normal subgroup** K in G is the truth value of the statement

$$\mathbb{P}_K : H_2 K = 0 \text{ and } G \text{ is } K\text{-Cockcroft}.$$

Suppose G is a finitely presented group which admits a surjection $\varphi : G \twoheadrightarrow Q$ with Q an R-group and $K = \ker \varphi$ small. Then any other surjection of G onto an R-group with small kernel K' has the parity of K and K' the same, depending only on the deficiency of G.

5. Application to cohomological dimension

In this section we give an algebraic analog to theorem 4.1. The crucial step is to define the sequence 4.4 without the use of complexes.

Let $\mathbb{P} : K \rightarrowtail P_2 \to P_1 \to P_0 \twoheadrightarrow \mathbb{Z}$ be an exact sequence of $\mathbb{Z}G$-modules, where each P_i is a finitely generated projective. We assume that there is an exact sequence of groups $L \rightarrowtail G \twoheadrightarrow H$ where H is an R-group with NATF-subgroup A. The existence of the sequence $\mathbb{P}$ says that G is nearly finitely presentable. We further assume that H_1L is finitely generated as a $\mathbb{Z}H$-module and that it localizes to zero. Let the integer k denote the minimal number of generators of H_1L as a $\mathbb{Z}H$-module and choose a surjection $p:\mathbb{Z}H^k \twoheadrightarrow H_1L$.

By tensoring $\mathbb{P}$ with $\mathbb{Z}\otimes_L$- and letting $C_i = \ker 1\otimes d_i$ we obtain the exact sequence $C_2 \rightarrowtail \mathbb{Z}\otimes_L P_2 \to C_1 \to H_1L$. Hence there is a map $g:\mathbb{Z}H^k \to \mathbb{Z}\otimes_L P_1$ whose image is into C_1 and is onto H_1L. It is clear, then, that $\operatorname{im} g + \operatorname{im} 1\otimes d_2 = C_1$. Thus the sequence

$$0 \to B \to \mathbb{Z}\otimes_L P_2 \oplus \mathbb{Z}H^k \dashrightarrow \mathbb{Z}\otimes_L P_1 \to \mathbb{Z}\otimes_L P_0 \to \mathbb{Z} \to 0$$

is exact, where the map $\dashrightarrow$ is given by $1\otimes d_2 + g$. Here, $B = \ker\{1\otimes d_2 + g\}$. The following lemma is easily proved.

LEMMA 5.1. *Let $r:U\to V$ and $u:W\to V$ be module homomorphisms and $h = r+u: U\oplus W \to V$. Then, if $K = \ker h$, the following sequence is exact*:

$$0 \to \ker r \to K \to W \to \operatorname{im} u/(\operatorname{im} u \cap \operatorname{im} r) \to 0.$$

where $K\to W$ is the projection $U\oplus W \to W$ restricted to K and the map with domain W is induced by u. ■

If we let $r = 1\otimes d_2: \mathbb{Z}\otimes_L P_2 \to C_1$ and $u:\mathbb{Z}H^k \to C_1$. Then $H_1L \approx \ker 1\otimes d_1/\operatorname{im} r \approx (\operatorname{im} r + \operatorname{im} u)/\operatorname{im} r \approx \operatorname{im} u/(\operatorname{im} u \cap \operatorname{im} r)$ and we obtain the exact sequence (generalizing 4.4):

$$C_2 \rightarrowtail B \to \mathbb{Z}H^k \twoheadrightarrow H_1L.$$

One may also show that the analog of 4.3 is exact:

$$H_3L \rightarrowtail \mathbb{Z}\otimes_L K \to C_2 \twoheadrightarrow H_2L.$$

Now if $(H_1L)_S = 0$, then the argument of theorem 4.1 yields an element $\kappa_A B = k\cdot[1] + \kappa_2 - \kappa_1 + \kappa_0 \in \mathcal{F}^\wedge$, where $\kappa_i = \kappa_A(\mathbb{Z}\otimes_L P_i)$, and $\kappa_A C_2 = \kappa_A B - k\cdot[1]$. It doesn't seem that (in general) $\kappa_A C_2$ has anything to do with the Euler character $\kappa P_2 - \kappa P_1 + \kappa P_0$ ($\in \mathbb{Z}$) of $\mathbb{P}$. To record the dependence of $\kappa_A C_2$ on L and $\mathbb{P}$ let us denote $\kappa_2 - \kappa_1 + \kappa_0$ by $\chi_G(\mathbb{P}, L)$. We can now state the following.

THEOREM 5.2: (a) *Let L be a normal subgroup of a group G such that*

$G/L=H$ is an R-group. To each NATF-*subgroup A and each partial finitely generated resolution $\mathbb{P}$ we can associate the element $\chi_G(\mathbb{P}, L)=\kappa_2-\kappa_1+\kappa_0 \in \mathcal{F}^\wedge$.*

(b) *Now let H_1L be finitely generated as a $\mathbb{Z}H$-module, and $(H_1L)_S=0$. If $[1]\neq 0 \in \mathcal{F}^\wedge$, then $(C_2)_S$ is a finitely generated projective $\mathbb{Z}G_S$-module and $\kappa_A C_2=\chi_G(\mathbb{P}, L)$.*

(c) *If the $\mathbb{Z}H_S$-module $(C_2)_S$ is stably-free then $\chi_G(\mathbb{P}, L) \geqq n \cdot [1]$ and $n \geq 0$.*

(d) *Finally, if $\mathbb{P}$ is stably-free and L is locally indicable (or L has no perfect subgroups and H_1L is torsion-free), then $K=0$ iff $\chi_G(\mathbb{P}, L)=0$;* ■

Note 5.3. One could remove the hypothesis "stably-free" in 5.2(d) if one could show, for a finitely generated projective $\mathbb{Z}G_S$-module P, that $\kappa_A P=0 \Rightarrow P=0$ (see proposition 2.2).

Note 5.4. Notice that the hypothesis in 4.1 that $wt_G L<\infty$ has been replaced in 5.2 by the weaker hypothesis that H_1L is finitely generated as a $\mathbb{Z}H$-module. However the conclusion of 5.2 is weaker, as well.

Note 5.5. If the partial resolution $\mathbb{P}: P_2 \to P_1 \to P_0 \to \mathbb{Z}$ is *free* and finitely generated, let $\mu_2\mathbb{P}=r_2-r_1+r_0$, where $r_i=\operatorname{rank}_{\mathbb{Z}G} P_i$. We let $\mu_2 G$ be the minimum of the set of numbers $\mu_2\mathbb{P}$, where $\mathbb{P}$ ranges over all such free finitely generated partial resolutions of length 2 [Sw]. Then we may recast 4.1 in the following form:

THEOREM 5.6. *If $L \rightarrowtail G \twoheadrightarrow H$ is an exact sequence of groups with H an R-group, $(H_1L)_S=0$, μ_2G defined, and $[1]\neq 0$. Then $\mu_2 G \geq 0$. Also, $\mu_2G=0 \Leftrightarrow$ there exists a partial free finitely generated $\mathbb{Z}G$-resolution $\mathbb{P}$ such that $C_2=0$. If L is locally indicable (or if L has no perfect subgroups and H_1L is torsion free), then $\mu_2G=0 \Leftrightarrow cdG \leq 2$, G has type FL, and $H_2L=0$ (compare with [D_3]).* ■

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University of Oregon,
Eugene, Oregon 97403 U.S.A.

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