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Localization of group rings and applications to 2-complexes

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In this paper we use recent results of S. Rosset [R] on the localization of group rings to give applications to the theory of 2-dimensional CW-complexes and related fields. If G is a group, we let $\mathbb{Z}G$ denote the integral group ring of G . If A is a non-trivial normal abelian torsion-free subgroup of G (In this case we say that G is a **Rosset group** or just an **R-group**, for short), we let S denote the multiplicatively closed subset $\mathbb{Z}A \setminus 0$ and localize $\mathbb{Z}G \rightarrow \mathbb{Z}G_S$ so as to invert the elements of S .

The first application is concerned with extending the Kaplansky rank κP (see [DV]) for finitely generated projective $\mathbb{Z}G$ -modules P to projective $\mathbb{Z}G_S$ -modules. This extension has a number of interesting applications because many $\mathbb{Z}G$ -modules (such as the second homotopy module of a 2-complex) become projective upon localization.

The second application generalizes a theorem of Hillman [H]. If X is a connected 2-complex with fundamental group isomorphic to G (in this case X is called a **[G, 2]-complex**) and L is a subgroup of G , let X_L denote the covering of X corresponding to L . We say that X is **L-Cockcroft** if the Hurewicz map $\pi_2 X \rightarrow H_2 X_L$ is trivial. Let $H_i L$ denote the i th-homology group of L with coefficients in the trivial $\mathbb{Z}G$ -module \mathbb{Z} . If L is a normal subgroup of a group G , the **weight of L in G** (denoted by $wt_G L$) is the minimal number of elements whose normal closure in G is L .

THEOREM 1. *Suppose $L \twoheadrightarrow G \twoheadrightarrow H$ is an exact sequence of groups with H a Rosset group, G finitely presented, $H_1 L$ finitely generated as an abelian group and $wt_G L$ finite. Let X be any $[G, 2]$ -complex. Then the Euler characteristic $\chi X \geq 0$, with $\chi X = 0$ iff X is L -Cockcroft and $H_2 L = 0$.*

COROLLARY 2. *In addition to the above hypotheses, if either*
 (a) *$H_1 L$ is torsion-free and L has no perfect subgroups*

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or

(b) L is locally indicable,
then the $[G, 2]$ -complex X is aspherical iff $\chi X = 0$ and $H_2L = 0$.

Let G be a finitely presented group with $H = H_1G$ infinite and $L = G'$, the derived group of G . Furthermore, assume that H_1G' is finitely generated as an abelian group. By theorem 1, G always has deficiency ≤ 1 . If, in addition, G has no perfect subgroups (e.g., if G is residually nilpotent) and H_1G' is torsion-free, then it follows from corollary 2 that G has deficiency 1 iff it has geometric dimension 2 and $H_2L = 0$.

The outline of the paper is as follows. In section 1 we describe the localization results of S. Rosset and in section 2 we give the extension of Kaplansky's invariant to projectives over localized rings. In section 3 we apply the earlier results to shed new light on the aspherical question of J. H. C. Whitehead: is every connected subcomplex of an aspherical 2-complex aspherical? Section 4 contains the proof of theorem 1 while in section 5 we derive an algebraic analog of theorem 1.

1. Localization of certain group rings

In this section we describe recent results of S. Rosset [R]. Let G be a group and let A be a non-trivial torsion-free abelian normal subgroup of G . If a group G has such a normal subgroup, we will say that G is an **R -group**. Then the set $S = \mathbb{Z}A - 0$ is a multiplicatively closed subset of the integral group ring $\mathbb{Z}G$ and satisfies the Ore conditions [P, page 146]. Thus there exists a left ring of fractions

$$\mathbb{Z}G_S = \{\beta^{-1}\alpha \mid \alpha \in \mathbb{Z}G, \beta \in S\}$$

and a canonical injection $i: \mathbb{Z}G \rightarrow \mathbb{Z}G_S$ given by carrying $\alpha \rightarrow 1^{-1}\alpha$.

This localization has the following properties:

(L1) The right $\mathbb{Z}G$ -module $\mathbb{Z}G_S$ is flat.

(L2) If M is any left $\mathbb{Z}G$ -module, then the localization M_S of M is given by $M_S = \mathbb{Z}G_S \otimes_{\mathbb{Z}G} M$. If the underlying abelian group M^0 of M is finitely generated or consists only of elements of finite order, then $M_S = 0$.

(L3) The ring $\mathbb{Z}G_S$ has rank invariance for finitely generated free modules; i.e., if $\mathbb{Z}G_S^m \approx \mathbb{Z}G_S^n$, then $m = n$.

The property (L3) is proved via the stronger Kaplansky property:

(L4) Let $\varphi: \mathbb{Z}G_S^m \rightarrow \mathbb{Z}G_S^m$ be any *surjection* from a free $\mathbb{Z}G_S$ -module of rank m to itself. Then φ is an isomorphism, as well (see [R], theorem F).

Using properties L1–L3 S. Rosset [R] gives the following remarkable generalization of a theorem of D. Gottlieb ([G], [S]):

THEOREM 1.0. *If X is a finite aspherical complex whose fundamental group $\pi_1 X$ is an R -group, then the Euler characteristic $\chi(X) = 0$.*

In another paper [D₁] we show the following generalization of Rosset's theorem. This has also been discovered independently by L. Fornera in her Ph.D. thesis at ETH.

THEOREM 1.1. *Let X be a finite aspherical complex with $\pi_1 X = G$. Let $L \twoheadrightarrow G \twoheadrightarrow H$ be an exact sequence of groups with $H_* L$ finitely generated as an abelian group and H an R -group. Then $\chi(X) = 0$.*

DEFINITION 1.1. Let m be an integer ≥ 2 . A $[G, m]$ -complex is a connected CW-complex whose dimension is $\leq m$, whose fundamental group $\pi_1 X$ is isomorphic to G , and whose universal cover \tilde{X} is $(m - 1)$ -connected. For example, any connected 2-complex is a $[\pi_1 X, 2]$ -complex.

Combining the results of [R] and [H], we have the following.

THEOREM 1.2 (Hillman–Rosset). *Let X be a finite $[G, m]$ -complex whose fundamental group is an R -group. Then the Euler characteristic $\chi(X) \geq 0$. The Euler characteristic of X is zero iff X is aspherical.*

Before giving the proof, we give the following:

LEMMA 1.3. *Let M be a submodule of a free $\mathbb{Z}G$ -module F . Then $M_S = 0$ iff $M = 0$.*

Proof. The exact sequence $M \twoheadrightarrow F \twoheadrightarrow Q = F/M$ localizes to the exact sequence $M_S \twoheadrightarrow F_S \twoheadrightarrow Q_S$. The inclusion $F \rightarrow F_S$ induces an inclusion $M \rightarrow M_S$. The result follows. ■

Proof of the theorem. Let $C_* \tilde{X} \rightarrow \mathbb{Z}$ denote the augmented cellular chain complex of the universal cover \tilde{X} , considered as a sequence of finitely generated free $\mathbb{Z}G$ -modules. Let $K = \ker [d_m : C_m \rightarrow C_{m-1}]$ be the m th-homotopy group of X . Localize the exact sequence $K \rightarrow C_* \tilde{X} \rightarrow \mathbb{Z}$ to obtain the exact sequence of stably-free projectives $K_S \rightarrow C_* \tilde{X}_S \rightarrow 0$. Thus the rank of K_S as a stably-free $\mathbb{Z}G_S$ -module is $\chi(X)$, which must necessarily be ≥ 0 . If $\chi(X) = 0$, then $\text{rank } K_S = 0$. It follows from L4 that $K_S = 0$ and from the lemma that $K = 0$. ■

This theorem has two very lovely corollaries, the first of which was noted in [H]. We say that a finitely presented group G has **(finite) geometric dimension ≤ 2** if G admits a (finite) aspherical $[G, 2]$ -complex.

COROLLARY 1.4. *If G is a finitely presented R -group, then the deficiency of G is ≤ 1 . The deficiency of G is equal to 1 iff G has finite geometric dimension 2. ■*

COROLLARY 1.5. *If H is any finitely presented group, then the deficiency of the cartesian product $\mathbb{Z} \times H$ is ≤ 1 . The deficiency of $\mathbb{Z} \times H = 1$ iff H is free.*

Proof. By the previous corollary, we need only show that the geometric dimension $\mathbb{Z} \times H \leq 2$ iff H is free. First, if H is finitely generated and free, then the obvious presentation of $\mathbb{Z} \times H$ of deficiency 1 may be realized as an aspherical $[\mathbb{Z} \times H, 2]$ -complex. In order to see the converse, we apply the Lyndon–Hochschild–Serre spectral sequence to the split exact sequence $\mathbb{Z} \twoheadrightarrow \mathbb{Z} \times H \twoheadrightarrow H$. If M is any $\mathbb{Z}H$ -module, then we obtain the split exact sequence

$$H^3(H; M) \twoheadrightarrow H^3(\mathbb{Z} \times H; M) \twoheadrightarrow H^2(H; M).$$

Thus $H^3(\mathbb{Z} \times H, M) = 0$ implies that $H^2(H; M) = 0$. This says that H has cohomological dimension ≤ 1 . That H has cohomological dimension 1 follows because H is torsion free. Now H is free by the famous result of J. Stallings [S₂, p. 58]. ■

2. Extending the Kaplansky invariant

In this section we show how to use the results of [D₂] to extend the invariant of I. Kaplansky (see [DV]) to localized group rings. We assume that the group G has a non-trivial normal abelian torsion-free subgroup A (we call such an A an **NATF-subgroup**). Let $S = \mathbb{Z}A - 0$ and localize $\mathbb{Z}G \rightarrow \mathbb{Z}G_S$. References for this section include [S], [D₂], [DV], and [P].

For any ring R , a **trace function on R** is a linear map $T: R \rightarrow B$, where B is an abelian group such that, for each $r, s \in R$, $T(rs) = T(sr)$. If we define the set $[R, R]$ to be the subgroup generated by the Lie brackets $[r, s] = rs - sr$, then the **universal trace function** is given by $T_u: R \rightarrow \tau R = R/[R, R]$. Any trace function T on R may be extended in the usual way to any $n \times n$ -matrix $M = [m_{ij}]$ over R via the formula $T(M) = \sum T(m_{ii})$. Any trace function T has the properties (a) $T(M + N) = T(M) + T(N)$ and (b) $T(PQ) = T(QP)$, where M, N are

$n \times n$ -matrices, P is an $m \times n$ -matrix, and Q is an $n \times m$ -matrix over R . Also, $T(1_n) = n \cdot T(1)$, provided R has a multiplicative identity 1, and 1_n is the identity $n \times n$ -matrix over R .

If G is a group and $\mathbb{Z}G$ is the integral group ring, then the universal trace group $\tau\mathbb{Z}G$ is easy to describe. Let CG denote the set of conjugacy classes of G . Then the group $\tau\mathbb{Z}G$ is equal to the free abelian group $\mathbb{Z}CG$ generated by the set CG . For an element $x \in G$, let $\langle x \rangle \in CG$ denote the conjugacy class of the element x .

The trace function $T_1: \mathbb{Z}G \rightarrow \mathbb{Z}$ is given in either of two (equivalent) ways. First, for any $v \in \mathbb{Z}G$, let $T_1(v)$ be the coefficient of 1 in v . Secondly it can be described as the coefficient of $\langle 1 \rangle$ in $T_u(v)$.

Following [S] we extend the trace T to any endomorphism $f: R^n \rightarrow R^n$ by choosing a basis for R^n and defining $T(f)$ to be the trace of the matrix M of f with respect to this basis. This is independent of the choice of basis. Further, if P is any finitely generated projective R -module, choose an integer $n \geq 0$ and an idempotent endomorphism $e: R^n \rightarrow R^n$ whose image is isomorphic to P . Define the **rank of P** with respect to T to be $T(e)$. See [S] for the proof that this is well defined. We denote this rank by $\rho_T P$. If $R = \mathbb{Z}G$ and $T = T_1$, we denote this rank as κP . This is the **Kaplansky rank** (it is called iP in [DV]). The rank (**Hattori–Stallings**) for the universal trace function $T_u: \mathbb{Z}G \rightarrow \tau\mathbb{Z}G$ is usually denoted by $r_G P$.

The Kaplansky rank is known to have the following properties (see [DV]).

K(a) κP is an integer ≥ 0 .

K(b) If P and Q are finitely generated projective $\mathbb{Z}G$ -modules, then $\kappa(P \oplus Q) = \kappa P + \kappa Q$.

K(c) If $n(P)$ is the minimum number of generators of P as a $\mathbb{Z}G$ -module, then $\kappa P \leq n(P)$.

K(d) $\kappa P = 0$ iff $P = 0$.

K(e) $\kappa P = n(P)$ iff $P \approx \mathbb{Z}G^{n(P)}$.

Now let $S = \mathbb{Z}A - 0$ and localize $\mathbb{Z}G$ to $\mathbb{Z}G_S$ via the inclusion map i . Let H be the quotient G/A and $\pi: G \rightarrow H$ be the natural surjection. For any element $h \in H$ and $a \in A$, let $h * a$ denote the action induced by conjugation by any preimage of h under π (that is, if $\pi g = h$, then $h * a = g \cdot a \cdot g^{-1}$). This makes A into a $\mathbb{Z}H$ -module. In this case, we will give a complete description of a direct summand \mathcal{F}^\wedge of $\tau\mathbb{Z}G_S$. The proofs for this description are given in [DF].

First, let \mathcal{F} denote the quotient field of $\mathbb{Z}A$. It is easy to see that, by choosing a set E of right coset generators for H in G (let $1 \in E$), the ring $\mathbb{Z}G_S$ is an \mathcal{F} -module and that it is \mathcal{F} -isomorphic to the vector space $\mathcal{F}(E)$ with natural basis E . Consider the projection $T: \mathbb{Z}G_S \rightarrow \mathcal{F} \cdot 1 = \mathcal{F}$ of the ring onto the coordinate corresponding to $1 \in E$. Note that $\mathbb{Z}G_S$ and $L = [\mathbb{Z}G_S, \mathbb{Z}G_S]$ are \mathbb{Q} -vector spaces,

where \mathbb{Q} is the rational numbers. Factoring out by the image of L under T defines the vector space \mathcal{F}^\wedge . (It is shown in [DF] that $T(L)$ is precisely the \mathbb{Q} -subspace $\mathbb{H} \cdot \mathcal{F}$, where \mathbb{H} is the augmentation ideal in $\mathbb{Q}H$. Then \mathcal{F}^\wedge is $\mathcal{F}/\mathbb{H} \cdot \mathcal{F} = \mathbb{Q} \otimes_{\mathbb{Q}H} \mathcal{F}$; it is also shown there that \mathcal{F}^\wedge is a direct summand (over \mathbb{Q}) of $\tau\mathbb{Z}G_S$).

Thus we may define a new trace function $t: \mathbb{Z}G_S \rightarrow \mathcal{F}^\wedge$ via T followed by the natural projection $\mathcal{F} \rightarrow \mathcal{F}^\wedge$. Let $[f]$ denote the image of $f \in \mathcal{F}$ in \mathcal{F}^\wedge . We will show that this trace function t “extends” the function T_1 given above, in certain cases.

Let $\langle A \rangle$ denote the conjugation classes in G determined by the elements of A ; for each $a \in A$, $\langle a \rangle$ is the conjugation class in G defined by a . Let $t_A: \mathbb{Z}G \rightarrow \mathbb{Z}\langle A \rangle$ be the trace map determined by restricting to those conjugation classes in $\langle A \rangle$.

Let $\alpha: \mathbb{Z}\langle A \rangle \rightarrow \mathcal{F}^\wedge$ be the map defined by sending $\langle a \rangle \mapsto [a]$. If we let $\gamma: \mathbb{Z}\langle A \rangle \rightarrow \tau\mathbb{Z}G$ be the natural split injection into $\tau\mathbb{Z}G$, l be the localization $\mathbb{Z}G \rightarrow \mathbb{Z}G_S$ and $r: \tau(\mathbb{Z}G_S) \rightarrow \mathcal{F}^\wedge$ be the projection induced by the projection T above, then one sees easily that $\alpha = r \circ \tau(l) \circ \gamma$.

LEMMA 2.0. *If P is any finitely generated projective $\mathbb{Z}G$ -module, then $\alpha(\rho_{t_A}(P)) = \rho_t(P_S)$.*

Proof. This follows from the definition because, if e is the defining idempotent for P , then e_S is the defining idempotent for P_S . ■

DEFINITION. We say that the Hattori–Stallings rank $r_G P$ is **carried by conjugacy classes of finite order** if, for each finitely generated projective $\mathbb{Z}G$ -module P , the coordinate $r_G P(\langle x \rangle)$ of $r_G P$ on the conjugacy class $\langle x \rangle$ is trivial except for elements $x \in G$ of finite order.

LEMMA 2.1. *If the Hattori–Stallings rank is carried by conjugacy classes of finite order, then the rank ρ_t is really given by the Kaplansky rank κ , i.e., if $\beta: \mathbb{Z} \rightarrow \mathcal{F}^\wedge$ is given by $1 \mapsto [1]$, then $\beta(\kappa P) = \rho_t(P_S)$.*

Proof. By lemma 2.0, $\rho_t(P_S) = \alpha(\rho_{t_A} P)$. But each conjugacy class $\langle a \rangle \neq \langle 1 \rangle$ in $\mathbb{Z}\langle A \rangle$ consists of elements of infinite order, so $\rho_{t_A} P = \kappa P \cdot \langle 1 \rangle$. ■

A result of B. Eckmann [E] shows that the Hattori–Stallings rank (over $\mathbb{Q}G$, and hence over $\mathbb{Z}G$) is carried by elements of finite order if G is one of the following types of groups:

- (a) solvable groups G
- (b) linear groups $G \subseteq GL_r(F)$ where F is a field of characteristic 0.

(c) groups of cohomology dimension $cd_{\mathbb{Q}}G \leq 2$.

provided G has finite homology dimension over \mathbb{Q} .

Furthermore, if G is a residually finite group, then P. Linnell has shown that the Hattori–Stallings rank (over $\mathbb{Z}G$) is concentrated on $\langle 1 \rangle$ [L].

Some properties of the rank ρ_t are given in the following

PROPOSITION 2.2. *Let P and Q be finitely generated projective $\mathbb{Z}G_S$ -modules. Then*

K_S (a): $\rho_t P$ is a member of \mathcal{F}^\wedge .

K_S (b): $\rho_t(P \otimes Q) = \rho_t P + \rho_t Q$.

If $[1] \neq 0$ in \mathcal{F}^\wedge , and P is a stably-free $\mathbb{Z}G_S$ -module, then

K_S (c): $\rho_t P = k \cdot [1]$ with $k \in \mathbb{Z}$ and $0 \leq k \leq n(P)$, where $n(P)$ is the minimal number of generators of P as a $\mathbb{Z}G_S$ -module and $k \in \mathbb{Z}$ is the stable-free rank.

K_S (d): $\rho_t P = 0 \Leftrightarrow P = 0$.

K_S (e): $\rho_t P = n(P) \cdot [1] \Leftrightarrow P \approx \mathbb{Z}G_S^{n(P)}$.

Proof. Statements (a) and (b) are clear. Statement (c) follows from (b) and the fact that $\rho_t \mathbb{Z}G = [1]$. We will show statement (d). Statement (e) then follows from (d). Because P is stably-free we see that the following sequence is exact for some positive integer n :

$$P \rightarrow \mathbb{Z}G_S^n \rightarrow Q$$

with Q stably-free. Then $\rho_t P = 0$ yields that $\rho_t Q = n \cdot [1]$ (here is where we use that fact that $[1] \neq 0$, because then $[1]$ has infinite order in $\tau \mathbb{Z}G_S$); we may assume that, in fact, Q is free of rank n (perhaps by replacing n by $n + k$). Then the Kaplansky property L4 implies that $P = 0$. ■

Question. Do K_S (c), (d) and (e) hold without the assumption that P is stably-free?

DEFINITION 2.3. We say that the finitely generated $\mathbb{Z}G$ -module M is **pre-projective** (respectively, **pre-stably free**) if the localization M_S is a projective (respectively, stably-free) $\mathbb{Z}G_S$ -module. For example, if X is a $[G, m]$ -complex, then the m th homotopy group $\pi_m X$ is a pre-projective $\mathbb{Z}G$ -module. We define the **Kaplansky rank** $\kappa_A M$ of a pre-projective module M to be $\kappa_A M = p_t M_S$. Of course, if M_S is stably-free, then $\kappa_A M$ is an integer multiple of $[1]$. It is not known to me whether or not the Kaplansky rank is independent of the choice of A .

COROLLARY 2.4. *Let M be a pre-stably-free $\mathbb{Z}G$ -module. Then $\kappa M = 0$ iff the localization $M_S = 0$. If M is a submodule of a free $\mathbb{Z}G$ -module, then $M = 0$.*

Proof. The first statement is just a special case of (d) above. The second follows from lemma 1.3. \square

3. Application to aspherical complexes

We say that a $[G, 2]$ -complex X has the **Whitehead condition (WC)** if either X is aspherical or, if X is not aspherical, then whenever X is the subcomplex of an $[H, 2]$ -complex Y , Y is not aspherical (see [BD] and [BDS] for reference). A **group G is WC** if every $[G, 2]$ -complex satisfies WC. For any group G , let P_1G denote the **maximal perfect subgroup** of G . The following theorem is an improvement over several theorems in [BD] and [BDS].

THEOREM 3.1. *Let G be a finitely presented R -group which has a normal abelian torsion-free subgroup not contained in P_1G . Then G has WC.*

Proof. The deficiency of G is ≤ 1 . If X is a $[G, 2]$ -complex, then the Euler characteristic $\chi X \geq 0$, with X aspherical iff $\chi X = 0$. Suppose $\chi X > 0$ and X is a subcomplex of an $[H, 2]$ -complex Y . We will show that Y is not aspherical. Suppose that Y were aspherical. Then it follows from [BD] that there is a non-trivial perfect subgroup P in G such that the cohomological dimension $cd(G/P) \leq 2$. Furthermore, G/P has type FL with $\chi(G/P) = \chi X > 0$. Now the hypothesis implies that G/P is an R -group, which is impossible (because G/P is an R -group and FL implies that $\chi(G/P) = 0$; see the proof of theorem 1.2). \blacksquare

We can now improve corollary 3.7 of [BDS] to read: *if G is the finitely presented fundamental group of a non-aspherical subcomplex $X < Y$ of an aspherical 2-dimensional complex, then G has a non-trivial, superperfect, normal C -subgroup (see [BD] for a definition of C -subgroup) P with respect to $C_*\tilde{X} \rightarrow \mathbb{Z}$. Moreover, $cdG/P \leq 2$ and the center of G is contained in P . See also Corollary 4.7 of [BD].*

We also note the following peculiar corollary: *For any finitely presented group G the cartesian product $\mathbb{Z} \times G$ has WC.*

Jonathan Hillman (private communication) has pointed out that 3.1 improves another corollary of [BDS], namely Corollary 5.2.

THEOREM 3.2. *If G is a 2-ended group, then G has WC.*

Proof. If G is a 2-ended group which doesn't have WC, then by Corollary 5.2 of [BDS] we have the exact sequence $P \twoheadrightarrow G \twoheadrightarrow \mathbb{Z}$, where P is a finite perfect group and the deficiency of G is 1. But it is easy to see that, because P is finite, G has an infinite cyclic central element. Thus, by 1.4, G has WC. ■

4. Application to deficiency

Throughout this section we assume that the NATF-subgroup A in G has been chosen once and for all and that $0 \neq [1] \in \mathcal{F}^\wedge$ (this happens iff $[1] \in \mathcal{F}^\wedge$ has infinite order, see [DF] for details and examples). For any $[G, 2]$ -complex X , let X_L denote the covering of X corresponding to the subgroup L . We say that G is **L -Cockcroft** if there is a $[G, 2]$ -complex X such that the Hurewicz map $\pi_2 X \rightarrow H_2 X_L$ is trivial. Such a space X is also called L -Cockcroft. If X is a *finite* $[G, 2]$ -complex, then we say that it is a **$[G, 2]_f$ -complex**. If L is a *normal* subgroup of G , the **weight of L in G** (denoted by $wt_G L$) is the minimal number of elements which normally generate L (in G). In this section we show the following.

THEOREM 4.1. *Let $1 \rightarrow L \rightarrow G \rightarrow H \rightarrow 1$ be an exact sequence of groups with*

- (a) G finitely presented,
- (b) H an R -group,
- (c) the weight $wt_G L$ finite, and
- (d) the $\mathbb{Z}H$ -module $H_1 L$ localizing to zero.

Then the deficiency $\text{def } G$ of G is ≤ 1 , and is equal to 1 iff G is L -Cockcroft and $H_2 L = 0$.

Notice that the above hypothesis 4.1(d) is satisfied if $H_1 L$ is finitely generated as an abelian group or is a torsion group. Also, 4.1(c) and (d) are satisfied if L is finitely generated. In particular, let $L = 1$. The theorem then says that any R -group G has deficiency 1 iff G is 1-Cockcroft; i.e., G has geometric dimension ≤ 2 . This is the Rosset–Hillman theorem. The proof of Theorem 4.1 will be given at the end of the section.

For example, if $G = gp\{a, b: (a^r b^{-s})^q\}$ ($r, s, q > 0$), $H = gp\{a, b: a^r b^{-s}\}$, and $G \twoheadrightarrow H$ is the map induced by the identity on the generators, then the kernel L is normally generated by the element $l = a^r b^{-s}$ and $H_1 L$ is a torsion group generated by all the conjugates of l . Notice that $a^r = b^s$ generates an infinite cyclic central subgroup in H . Hence, $\text{def } G = 1$ implies that G is L -Cockcroft and $H_2 L = 0$ (the latter also follows from a result of Fischer, Karrass, and Solitar [FKS] that says that the normal subgroup L is a free product of cyclic groups.

Moreover, E. Dyer and A. Vasquez [DV₂] have an explicit construction of a $K(G, 1)$ -space from which it is easy to see that G is L -Cockcroft).

We first prove the following

THEOREM 4.2. *Let $L \twoheadrightarrow G \twoheadrightarrow H$ be an exact sequence of groups and homomorphisms, with G finitely presented, H an R -group with NATF-subgroup A , and $\text{wt}_G L < \infty$. Further assume that, for $S = \mathbb{Z}A - 0$, the $\mathbb{Z}H_S$ -modules $(H_i L)_S$ are all projective for $i = 1, 2, 3$, and that $(H_3 L)_S$ is finitely generated as a $\mathbb{Z}H_S$ -module. Let $\kappa_i = \kappa_A H_i L$. Then, for any $[G, 2]_f$ -complex X , we have*

$$\kappa_A(\mathbb{Z} \otimes_L \pi_2 X) = \kappa_3 - \kappa_2 + \kappa_1 + \chi X \cdot [1] \in \mathcal{F}^\wedge.$$

Proof. Let X_L denote the covering of X corresponding to the subgroup L . Let $\{l_\alpha \mid \alpha \in \mathcal{A}\}$ be a set of elements of L whose normal closure (in G) is equal to L . Assume that $|\mathcal{A}| < \infty$. Use the elements l_α to add 2-cells e_α to X to obtain the space Y containing X as a subcomplex. If let $\mathbb{Z}H^{\mathcal{A}}$ denote the direct sum of $|\mathcal{A}|$ copies of $\mathbb{Z}H$, then the following sequences of $\mathbb{Z}H$ -modules are exact:

$$0 \rightarrow H_3 L \rightarrow \mathbb{Z} \otimes_L \pi_2 X_L \rightarrow H_2 X_L \rightarrow H_2 L \rightarrow 0, \quad (4.3)$$

$$0 \rightarrow H_2 X_L \rightarrow \pi_2 Y \rightarrow \mathbb{Z}H^{\mathcal{A}} \rightarrow H_1 L \rightarrow 0. \quad (4.4)$$

Sequence (4.3) is a restatement of two classical theorems of Hopf relating the second and third homology groups of L to the homology of a 2-complex X_L and its universal cover $\tilde{X} = \tilde{X}_L$. Notice that the complex X_L can be identified as a subcomplex of the universal cover \tilde{Y} of Y . The second sequence is a restatement of the homology sequence of the pair (\tilde{Y}, X_L) .

Because $|\mathcal{A}| < \infty$, we see that $H_1 L$ is a finitely generated $\mathbb{Z}H$ -module. We also see that $K = (H_2 X_L)_S$ is a finitely generated projective, because $(\pi_2 Y)_S$ is a stably-free $\mathbb{Z}G_S$ -module. By localizing (4.4) at S , it is evident that $\kappa_A H_2 X_L - \kappa_A H_1 L = \chi X \cdot [1]$. It then follows from (4.3) that $W_S = \mathbb{Z} \otimes_L (\pi_2 X)_S$ is projective and that

$$\kappa_A H_3 L - \kappa_A H_2 L = \kappa_A W - \kappa_A H_2 X_L.$$

The equality follows. ■

COROLLARY 4.5. *In addition, suppose that $(H_i L)_S = 0$ for $i = 1, 2, 3$. Then $\kappa_A(\mathbb{Z} \otimes_L \pi_2 X) = \chi X \cdot [1] \in \mathcal{F}^\wedge$ is a non-negative integer multiple of $[1]$.*

Proof. The equality follows from theorem 4.4. In this case, $(\pi_2 Y)_S$ is stably-free $\Rightarrow (H_2 X_L)_S$ stably free $\Rightarrow (\mathbb{Z} \otimes_L \pi_2 X)_S = M_S$ stably free. Hence, $\kappa_A M$ is a non-negative integer multiple of [1]. ■

Note that in this case there is an exact sequence $M_S \twoheadrightarrow (\pi_2 Y)_S \twoheadrightarrow \mathbb{Z}H_S^{\text{st}}$ of stably-free modules. Note also that if $\chi X = 0$, then $\kappa_A M = 0$, and hence $M_S = 0$. If $H_3 L = 0$, then M is a submodule of a free $\mathbb{Z}H$ -module (namely, $C_2 \tilde{Y}$) and hence by lemma 1.3, $M = 0$. This says that $\pi_2 X$ is a **perfect $\mathbb{Z}L$ -module** ($\mathbb{Z} \otimes_{\mathbb{Z}L} \pi_2 X = 0$) and therefore also a perfect $\mathbb{Z}G$ -module. I do not know of a non-trivial example of a $[G, 2]$ -complex whose second homotopy group is a perfect $\mathbb{Z}G$ -module.

We say that a $[G, 2]_f$ -complex is **minimal** if it has the minimum Euler characteristic among all such complexes.

Example 4.6(?). Let $L \twoheadrightarrow G \twoheadrightarrow H$ be an exact sequence of groups with $\text{def } G = 1$, $H_3 L = 0$, $cd G > 2$, H an R -group, and $wt_G L$ finite. If $(H_1 L)_S = 0$ and X is a minimal $[G, 2]_f$ -complex, then $\mathbb{Z} \otimes_L \pi_2 X = 0$, while $\pi_2 X \neq 0$. Does such an example exist?

A $[G, 2]$ -complex with a *single zero cell* will be called a **$[G, 2]^*$ -complex**. Any $[G, 2]$ -complex has the homotopy type of a $[G, 2]^*$ -complex by simply factoring out a maximal tree in the 1-skeleton.

If X is any $[G, 2]_f$ -complex and $C_* \tilde{X} \rightarrow \mathbb{Z}$ is the augmented cellular chain complex of the universal cover \tilde{X} of X , considered as a complex of $\mathbb{Z}G$ -modules, then $R_X = \ker \{C_1 \tilde{X} \rightarrow C_0 \tilde{X}\}$ is called a **relation module** corresponding to X .

THEOREM 4.7. *Suppose that $L \twoheadrightarrow G \twoheadrightarrow H$ is an exact sequence of groups with G finitely presented, H an R -group having an NATF-subgroup A , and $wt_G L < \infty$. Let X be a minimal $[G, 2]_f^*$ -complex. In addition, let $(H_1 L)_S$ and $(H_2 L)_S$ be projective $\mathbb{Z}H_S$ -modules, $\kappa_i = \kappa_A H_i L$, m (respectively n) be the rank over $\mathbb{Z}G$ of $C_1 \tilde{X}$ (respectively $C_2 \tilde{X}$), and $N = \mathbb{Z} \otimes_L R_X$.*

(a) *Then N is a finitely generated projective $\mathbb{Z}H_S$ -module and*

$$\kappa_A N = (m - 1) \cdot [1] + \kappa_2 - \kappa_1.$$

(b) *If N is a stably-free $\mathbb{Z}H_S$ -module, then $\kappa_A N = k \cdot [1]$ and*

$$\text{def } G (= m - n) \leq k.$$

(c) Furthermore, if X is L -Cockcroft, then N is free of rank n and in this case,

$$\text{def } G \cdot [1] = [1] + \kappa_1 - \kappa_2 \in \mathcal{F}^\wedge.$$

Proof. One sees easily that, if IG denotes that augmentation ideal inside $\mathbb{Z}G$, then $(\mathbb{Z} \otimes_L IG)_S \approx (H_1L)_S \oplus \mathbb{Z}H_S$. Thus, $(H_1L)_S$ is projective $\Leftrightarrow (\mathbb{Z} \otimes_L IG)_S$ is and both are finitely generated if G is. By tensoring the map $\delta_2: C_2\tilde{X} \approx \mathbb{Z}G^n \rightarrow C_1\tilde{X} \approx \mathbb{Z}G^m$ with $\mathbb{Z} \otimes_{L^-}$, we obtain the map d_2 . One then shows that $(\text{im } d_2)_S \oplus (\mathbb{Z} \otimes_L IG)_S \approx \mathbb{Z}H^m$ and that $N_S \approx (\text{im } d_2)_S \oplus (H_2L)_S$. The same argument as in 4.2 shows that $(H_2L)_S$ is finitely generated. The calculation of $\kappa_A N$ follows. If X is L -Cockcroft then it follows that the boundary map $i \otimes \delta_2: \mathbb{Z} \otimes_L C_2\tilde{X} \rightarrow N$ is an isomorphism; hence, $\kappa_A N = n \cdot [1]$. The computation for the deficiency of G is a result of the formula $\text{def } G = 1 - \chi X$. ■

Note that the formula $\text{def } G \cdot [1] = [1] + \kappa_1 - \kappa_2$ is analogous to the formula $\text{def } G \leq \text{rank}_{\mathbb{Z}} H_1G - (\text{minimum number of generators of } H_2G)$. One calls the group **efficient** if the latter inequality is an equality. Equality in the former case might be called **L -efficient**.

Proof of 4.1. If we assume that $(H_1L)_S = 0$ (this is so if H_1L is finitely generated as an abelian group or is a torsion group), then we may prove a theorem with no assumed conditions on H_2L . The proof of theorem 4.7 above shows the statement: G is L -Cockcroft and $H_2L = 0 \Rightarrow \text{def } G = 1$ (i.e., $\kappa_1 = \kappa_2 = 0$ and use the stably-free rank). We will show the converse. Let $U = H_2X_L$. Because $(H_1L)_S = 0$, the following sequence is split exact:

$$0 \rightarrow U_S \rightarrow (\pi_2 Y)_S \rightarrow \mathbb{Z}H_S^k \rightarrow 0.$$

Then $\text{def } G = 1$ implies that $\chi X = 0$, so $\chi Y = \text{stably-free rank of } (\pi_2 Y)_S = k = \text{wt}_G L$. Hence, by the Kaplansky property L4, we have that $U_S = 0$. But U is a submodule of the free $\mathbb{Z}H$ -module $C_2\tilde{Y}$, hence $U = 0$. Thus G is L -Cockcroft and $H_2L = 0$. This proves theorem 4.1. ■

Example 4.8. Let G' denote the commutator subgroup of the finitely presented group G . Then if $H_1G = G/G'$ is infinite and $(H_1G')_S = 0$, then (by 4.1) we have that $\text{def } G \leq 1$. The deficiency is equal to 1 iff G is G' -Cockcroft and $H_2G' = 0$.

Example 4.9. Let G be any finitely presented group with commutator subgroup G' finitely generated. Consider G'' , the second derived group of G . The

group $G/G'' = H$ has H_1G' as a normal abelian subgroup and $wt_G G''$ is finite (G' is finitely generated implies that $wt_G G'' < \infty$. Thus $wt_G G'' < \infty$). We assume that H_1G' is torsion free, so that H is an R -group. Now if $(H_1G'')_S = 0$, then the conclusions of theorem 4.1 hold for the sequence $G'' \twoheadrightarrow G \twoheadrightarrow G/G''$.

Notice that it follows from sequence 4.4 that if H is an R -group with $wt_G L < \infty$ and $H_2X_L = 0$, then the projective dimension of $(H_1L)_S \leq 1$. This follows because, in this case, the sequence $0 \rightarrow (\pi_2 X)_S \rightarrow (\mathbb{Z}H^k)_S \rightarrow (H_1L)_S \rightarrow 0$ is exact, with $(\pi_2 Y)_S$ finitely generated and stably-free.

We say that a group G is an **E -group** (with respect to the resolution $P_* \rightarrow \mathbb{Z}$) if H_1G is torsion free and for some projective $\mathbb{Z}G$ -resolution $P_* \rightarrow \mathbb{Z}$ of the trivial module \mathbb{Z} , the homomorphism $\mathbb{Z} \otimes_{\mathbb{Z}G} d_2: \mathbb{Z} \otimes_{\mathbb{Z}G} P_2 \rightarrow \mathbb{Z} \otimes_{\mathbb{Z}G} P_1$ is a monomorphism. Such groups are studied in [St].

COROLLARIES 4.10. There are some interesting special cases of theorem 4.1.

First, assume that $\text{def } G = 1$ and that H_1L is torsion-free. In this case L becomes an E -group [St] because $H_2X_L = 0$. Let P_1G denote the maximal perfect subgroup of G . Then one may apply the theory of Strebel's E -groups as in [BD], Section 4, to observe that

(*) $H_2P_1L = 0$, G/P_1L has cohomological dimension ≤ 2 and type FL , and that the Euler characteristic $\chi(G/P_1L) = 0$.

Furthermore, if $P_1L = 1$, then G has geometric dimension 2.

Secondly, a group U is said to be **locally indicable** if every nontrivial finitely generated subgroup of U has infinite abelianization. We consider the nonempty family \mathcal{S}_L consisting of all normal subgroups V of a group L such that G/V is locally indicable ($G \in \mathcal{S}_L$). If we order \mathcal{S}_L by inclusion, then it is easy to see that this family has a minimal element, call it $\mathbb{P}_A L$ (this is called the **Adams' subgroup** of L). Note that L is locally indicable iff $\mathbb{P}_A L = 1$. Then a similar argument to that given by Adams in [A] shows that (given the hypotheses of theorem 4.1 plus H_1L torsion free) (*) is true with P_1L replaced by $\mathbb{P}_A L$, provided $\mathbb{P}_A L$ is *perfect*. See also proposition 3.1 of [HS].

Assume that $L \twoheadrightarrow G \twoheadrightarrow H$ is an exact sequence of groups satisfying the hypotheses of theorem 4.1. If, in addition, L is locally indicable, then G has deficiency $1 \Leftrightarrow G$ has geometric dimension 2 and $H_2L = 0$. This follows because $\text{def } G = 1$ (and X is the $[G, 2]$ -complex having $\chi X = 0$) $\Leftrightarrow H_2X_L = 0$. This latter happens iff $1 \otimes_L \partial_2: \mathbb{Z} \otimes_L C_2 \rightarrow \mathbb{Z} \otimes_L C_1$ is monic. Then apply the fact that local indicability of L yields that ∂_2 is monic as well.

For example, L could be a classical knot or link group, or a finitely generated torsion-free 1-relator group. These groups are known to be locally indicable [H].

Example 4.11. We give an application of theorem 4.1 to the Whitehead problem. A normal subgroup $L \trianglelefteq G$ is small if $wt_G L < \infty$ and $H_1 L$ is finitely generated as an abelian group.

THEOREM 4.12. *Let $K \twoheadrightarrow G \twoheadrightarrow Q$ be an exact sequence of groups with Q an R -group, G finitely presented, and K small. Furthermore, suppose that K contains the maximal perfect subgroup $P_1 G$ of G . Then either of the following two hypotheses implies that G has WC.*

- (a) $H_2 K = 0$ and $\text{def } G < 1$, or
- (b) $H_2 K \neq 0$ and $\text{def } G = 1$.

Proof. If G does not have WC then there is a non-trivial perfect normal subgroup $P \trianglelefteq G$ so that G is P -Cockcroft. Because K contains $P_1 G$, then K contains P . Thus G is K -Cockcroft. If in addition, $H_2 K = 0$, then $\text{def } G = 1$; if $H_2 K \neq 0$, then $\text{def } G < 1$, by theorem 4.1. These contradict hypotheses (a) or (b). ■

For example, let $G(\alpha)$ be the α th term of the derived series of G , where α is any ordinal number. Suppose for some ordinal α , the abelian group $G(\alpha)/G(\alpha+1)$ is non-trivial and torsion-free and that $G(\alpha)$ is small. Then if $H_2 G(\alpha) = 0$ and $\text{def } G < 1$, it follows that G has WC.

The **parity of a normal subgroup K** in G is the truth value of the statement

$$\mathbb{P}_K : H_2 K = 0 \text{ and } G \text{ is } K\text{-Cockcroft.}$$

Suppose G is a finitely presented group which admits a surjection $\varphi : G \twoheadrightarrow Q$ with Q an R -group and $K = \ker \varphi$ small. Then any other surjection of G onto an R -group with small kernel K' has the parity of K and K' the same, depending only on the deficiency of G .

5. Application to cohomological dimension

In this section we give an algebraic analog to theorem 4.1. The crucial step is to define the sequence 4.4 without the use of complexes.

Let $\mathbb{P} : K \twoheadrightarrow P_2 \rightarrow P_1 \rightarrow P_0 \twoheadrightarrow \mathbb{Z}$ be an exact sequence of $\mathbb{Z}G$ -modules, where each P_i is a finitely generated projective. We assume that there is an exact

sequence of groups $L \twoheadrightarrow G \twoheadrightarrow H$ where H is an R -group with NATF-subgroup A . The existence of the sequence \mathbb{P} says that G is nearly finitely presentable. We further assume that H_1L is finitely generated as a $\mathbb{Z}H$ -module and that it localizes to zero. Let the integer k denote the minimal number of generators of H_1L as a $\mathbb{Z}H$ -module and choose a surjection $p: \mathbb{Z}H^k \twoheadrightarrow H_1L$.

By tensoring \mathbb{P} with $\mathbb{Z} \otimes_L -$ and letting $C_i = \ker 1 \otimes d_i$ we obtain the exact sequence $C_2 \twoheadrightarrow \mathbb{Z} \otimes_L P_2 \rightarrow C_1 \rightarrow H_1L$. Hence there is a map $g: \mathbb{Z}H^k \rightarrow \mathbb{Z} \otimes_L P_1$ whose image is into C_1 and is onto H_1L . It is clear, then, that $\text{im } g + \text{im } 1 \otimes d_2 = C_1$. Thus the sequence

$$0 \rightarrow B \rightarrow \mathbb{Z} \otimes_L P_2 \oplus \mathbb{Z}H^k \rightarrow \mathbb{Z} \otimes_L P_1 \rightarrow \mathbb{Z} \otimes_L P_0 \rightarrow \mathbb{Z} \rightarrow 0$$

is exact, where the map \rightarrow is given by $1 \otimes d_2 + g$. Here, $B = \ker \{1 \otimes d_2 + g\}$. The following lemma is easily proved.

LEMMA 5.1. *Let $r: U \rightarrow V$ and $u: W \rightarrow V$ be module homomorphisms and $h = r + u: U \oplus W \rightarrow V$. Then, if $K = \ker h$, the following sequence is exact:*

$$0 \rightarrow \ker r \rightarrow K \rightarrow W \rightarrow \text{im } u / (\text{im } u \cap \text{im } r) \rightarrow 0.$$

where $K \rightarrow W$ is the projection $U \oplus W \rightarrow W$ restricted to K and the map with domain W is induced by u . ■

If we let $r = 1 \otimes d_2: \mathbb{Z} \otimes_L P_2 \rightarrow C_1$ and $u: \mathbb{Z}H^k \rightarrow C_1$. Then $H_1L \approx \ker 1 \otimes d_1 / \text{im } r \approx (\text{im } r + \text{im } u) / \text{im } r \approx \text{im } u / (\text{im } u \cap \text{im } r)$ and we obtain the exact sequence (generalizing 4.4):

$$C_2 \twoheadrightarrow B \rightarrow \mathbb{Z}H^k \twoheadrightarrow H_1L.$$

One may also show that the analog of 4.3 is exact:

$$H_3L \twoheadrightarrow \mathbb{Z} \otimes_L K \rightarrow C_2 \twoheadrightarrow H_2L.$$

Now if $(H_1L)_S = 0$, then the argument of theorem 4.1 yields an element $\kappa_A B = k \cdot [1] + \kappa_2 - \kappa_1 + \kappa_0 \in \mathcal{F}^\wedge$, where $\kappa_i = \kappa_A(\mathbb{Z} \otimes_L P_i)$, and $\kappa_A C_2 = \kappa_A B - k \cdot [1]$. It doesn't seem that (in general) $\kappa_A C_2$ has anything to do with the Euler character $\kappa P_2 - \kappa P_1 + \kappa P_0$ ($\in \mathbb{Z}$) of \mathbb{P} . To record the dependence of $\kappa_A C_2$ on L and \mathbb{P} let us denote $\kappa_2 - \kappa_1 + \kappa_0$ by $\chi_G(\mathbb{P}, L)$. We can now state the following.

THEOREM 5.2: (a) *Let L be a normal subgroup of a group G such that*

$G/L = H$ is an R -group. To each NATF-subgroup A and each partial finitely generated resolution \mathbb{P} we can associate the element $\chi_G(\mathbb{P}, L) = \kappa_2 - \kappa_1 + \kappa_0 \in \mathcal{F}^\wedge$.

(b) Now let H_1L be finitely generated as a $\mathbb{Z}H$ -module, and $(H_1L)_S = 0$. If $[1] \neq 0 \in \mathcal{F}^\wedge$, then $(C_2)_S$ is a finitely generated projective $\mathbb{Z}G_S$ -module and $\kappa_A C_2 = \chi_G(\mathbb{P}, L)$.

(c) If the $\mathbb{Z}H_S$ -module $(C_2)_S$ is stably-free then $\chi_G(\mathbb{P}, L) \cong n \cdot [1]$ and $n \geq 0$.

(d) Finally, if \mathbb{P} is stably-free and L is locally indicable (or L has no perfect subgroups and H_1L is torsion-free), then $K = 0$ iff $\chi_G(\mathbb{P}, L) = 0$; ■

Note 5.3. One could remove the hypothesis “stably-free” in 5.2(d) if one could show, for a finitely generated projective $\mathbb{Z}G_S$ -module P , that $\kappa_A P = 0 \Rightarrow P = 0$ (see proposition 2.2).

Note 5.4. Notice that the hypothesis in 4.1 that $wt_G L < \infty$ has been replaced in 5.2 by the weaker hypothesis that H_1L is finitely generated as a $\mathbb{Z}H$ -module. However the conclusion of 5.2 is weaker, as well.

Note 5.5. If the partial resolution $\mathbb{P}: P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow \mathbb{Z}$ is free and finitely generated, let $\mu_2 \mathbb{P} = r_2 - r_1 + r_0$, where $r_i = \text{rank}_{\mathbb{Z}G} P_i$. We let $\mu_2 G$ be the minimum of the set of numbers $\mu_2 \mathbb{P}$, where \mathbb{P} ranges over all such free finitely generated partial resolutions of length 2 [Sw]. Then we may recast 4.1 in the following form:

THEOREM 5.6. *If $L \twoheadrightarrow G \twoheadrightarrow H$ is an exact sequence of groups with H an R -group, $(H_1L)_S = 0$, $\mu_2 G$ defined, and $[1] \neq 0$. Then $\mu_2 G \geq 0$. Also, $\mu_2 G = 0 \Leftrightarrow$ there exists a partial free finitely generated $\mathbb{Z}G$ -resolution \mathbb{P} such that $C_2 = 0$. If L is locally indicable (or if L has no perfect subgroups and H_1L is torsion free), then $\mu_2 G = 0 \Leftrightarrow cdG \leq 2$, G has type FL, and $H_2L = 0$ (compare with [D₃]). ■*

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