Zeitschrift: Commentarii Mathematici Helvetici

Herausgeber: Schweizerische Mathematische Gesellschaft

Band: 62 (1987)

Artikel: Reduction of isolated singularities.

Autor: Dieterich, Ernst

DOI: https://doi.org/10.5169/seals-47368

Comment. Math. Helvetici 62 (1987) 654-676 0010-2571/87/040654-23$01.50 + 0.20/0

Reduction of isolated singularities

ERNST DIETERICH

Let $f(X)=f(X_0, \ldots, X_d)$ be a polynomial in $\mathbb{C}[X_0, \ldots, X_d]$, such that the hypersurface $H \subset \mathbb{A}^{d+1}(\mathbb{C})$ given by $f(X)$ has an isolated singularity at 0, and let $\Lambda=\hat{\mathcal{O}}_{H,0}=\mathbb{C}[[X_0, \ldots, X_d]]/(f(X))$ be the complete local ring of H at 0. It is known that $(H, 0)$ is a simple singularity (i.e. there exist only finitely many isomorphism classes of singularities in the semiuniversal deformation of $(H, 0)$) if and only if Λ is of finite type (i.e. there exist only finitely many isomorphism classes of indecomposable Cohen–Macaulay Λ-modules) [Kn 85], [Bu/Gr/Schr 86]. Moreover, if Λ is of finite type then all indecomposable Cohen–Macaulay Λ-modules are classified [Gr/Kn 85], [He 78], [Kn 85], the Auslander–Reiten quiver of Λ is determined and is known to be closely related to the Dynkin diagram which corresponds to the simple singularity $(H, 0)$ [Di/Wi 86], [Au 86], [Kn 85].

Motivated by this recent development I have begun to study the category of Cohen–Macaulay Λ-modules in case Λ is of infinite type. In this respect the present article contains the following two main results.

THEOREM I. *Let $\Lambda=\mathbb{C}[[X_0, \ldots, X_d]]/(f(X))$ be the complete local ring of a nonsimple isolated hypersurface singularity. Let $\mathcal{A}(\Lambda)$ be the Auslander–Reiten quiver of Λ, and denote by $\mathcal{C}$ the connected component of $\mathcal{A}(\Lambda)$ which contains $[\Lambda]$. Then*

$$A(\Lambda)\backslash\mathcal{C} \cong \dot{\bigcup_{i\in I}} \mathbb{Z}\mathbb{A}_\infty/\langle \tau^{n(i)}\rangle,$$

where I is an index set, and $n(i) \in \{1, 2\}$ for all $i \in I$. Moreover, if d is even then $n(i)=1$ for all $i \in I$.

THEOREM II. *Let $\Lambda=\mathbb{C}[[X_0, \ldots, X_d]]/(f(X))$ be the complete local ring of a nonsimple isolated hypersurface singularity. Then there exists an arithmetic sequence of natural numbers $r, 2r, 3r, \ldots$ such that for each mr $(m \in \mathbb{N})$ there exists an infinite sequence $(M_{m,n})_{n\in\mathbb{N}}$ of indecomposable pairwise nonisomorphic Cohen–Macaulay Λ-modules all of which have rank mr.*

The general idea underlying the proof of Theorems I and II is to make use of techniques which have been developed in representation theory of artin algebras. This strategy, while it cannot be carried out in a straightforward way, turns out to work in case there exists an ideal $\mathcal{I}$ in Λ such that $\Lambda/\mathcal{I}$ is artinian and the functor $\Lambda/\mathcal{I} \otimes_{\Lambda}$, with the category of Cohen–Macaulay Λ-modules as domain, reflects isomorphisms, preserves indecomposability and separates isomorphism classes. We call such an ideal $\mathcal{I}$ a reduction ideal of Λ. This leads to the question of existence of a reduction ideal. Generalizing an approach (known as Maranda's Theorem) which gives a positive answer to the analogous question for lattices over orders, we obtain a criterion for the existence of a reduction ideal in the following situation. Let R be a commutative noetherian complete local Cohen–Macaulay ring, with unique maximal ideal $\mathfrak{m}$, and let Λ be an R-algebra (not assumed to be commutative) which is finitely generated as R-module. Denote by $\operatorname{mod} \Lambda$ the category of all finitely generated left Λ-modules, and by $\operatorname{mod}_R \Lambda$ the category of all finitely generated left Λ-modules which are projective as R-modules. Call the annihilator ideal of the functor $\operatorname{Ext}^1_\Lambda(\ ,\ ) : \operatorname{mod}_R \Lambda \times \operatorname{mod} \Lambda \to \operatorname{mod} R$ *the Ext-annihilating ideal of* Λ in R. Then the following criterion holds.

THEOREM III. *If the Ext-annihilating ideal of Λ in R is $\mathfrak{m}$-primary, then there exists a reduction ideal of Λ.*

This raises the question of estimating the Ext-annihilating ideal of an R-algebra Λ, a problem which also seems to be of independent interest. There are two classes of algebras for which we can prove that the Ext-annihilating ideal is $\mathfrak{m}$-primary, namely a) for isolated singularities of finite type (not assumed to be commutative), and b) for the complete local rings of isolated Cohen–Macaulay singularities on an affine algebraic variety over an algebraically closed field. This leads to the following results which are related to Theorems I and II.

THEOREM IV. *Let R be a commutative noetherian complete regular local ring, and let Λ be an R-algebra which is finitely generated free as R-module. Assume that Λ is of finite type. Then each connected component $\mathscr{C}$ of the stable Auslander–Reiten quiver of Λ is of the form $\mathscr{C} \cong \mathbb{Z}\Delta/G$, where Δ is a Dynkin diagram and G is a group of automorphisms of $\mathbb{Z}\Delta$.*

If R is an algebraically closed field, then we recover Riedtmann's well-known Theorem [Rie 80]. The stable Auslander–Reiten quivers of the simple isolated hypersurface singularities, mentioned at the beginning, also fall under the situation described in Theorem IV. But much more generally, Theorem IV states that Dynkin diagrams always appear in connection with isolated singularities of finite type (commutative or noncommutative, in arbitrary dimensions), as soon as the Auslander–Reiten quiver contains a stable point.

THEOREM V. *Let* $I \subset \mathbb{C}[[X_1, \ldots, X_n]]$ *be an ideal, such that* $\Lambda = \mathbb{C}[[X_1, \ldots, X_n]]/I$ *is an isolated Cohen–Macaulay singularity. If* Λ *is of infinite type, then there exists an infinite sequence* $(M_i)_{i \in \mathbb{N}}$ *of indecomposable Cohen–Macaulay* Λ*-modules such that* $\operatorname{rank} M_i < \operatorname{rank} M_{i+1}$, *for all* $i \in \mathbb{N}$.

Using different methods, Herzog and Sanders recently have obtained results which are related to Theorem V [He/Sa 85].

THEOREM VI. *Let* $\Lambda = \mathbb{C}[[X_1, \ldots, X_n]]/I$ *be as in Theorem V. Let* $\mathscr{C}$ *be a connected component of the stable Auslander–Reiten quiver of* Λ *such that* $\mathscr{C}$ *contains a periodic point. Then* $\mathscr{C} \cong \mathbb{Z}\Delta/G$, *where* Δ *is either a Dynkin diagram or* $\mathbb{A}_\infty$, *and* G *is a group of automorphisms of* $\mathbb{Z}\Delta$.

Specializing Theorem VI to isolated hypersurface singularities we obtain Theorem I, and combining Theorem I with the main result of [Bu/Gr/Schr 86] we obtain Theorem II.

For definition and basic combinatorial structure of the Auslander–Reiten quiver of an isolated singularity, as well as for terminology and notation related to this concept, the reader is referred to [Di 86]. Auslander's characterization of isolated singularities via existence of Auslander–Reiten sequences [Au 84], together with the combinatorial results of Happel, Preiser and Ringel [Hap/Pr/Rin 79], [Hap/Pr/Rin 80] yields as an easy consequence the fundamental structure theorem for connected components of the stable Auslander–Reiten quiver which contain a periodic point [Di 86, Theorem 3].

Much of the present article is based on this structure theorem. In section 1 we study consequences which may be drawn from it, in case the isolated singularity contains a reduction ideal. In section 2 we turn to the problem of existence of a reduction ideal and we prove for a rather general class of algebras (which includes the class of isolated singularities) the sufficient existence criterion in terms of the Ext-annihilating ideal mentioned above. Section 3 is devoted to estimating the Ext-annililating ideal for two classes of isolated singularities. The results exhibited above then follow as easy consequences. Theorems I, . . . , VI appear in the text as Theorems 19, 20, 7, 9, 16, 17.

I am grateful to Claus Michael Ringel for clarifying remarks on stable valued translation quivers, and I am grateful to Gert-Martin Greuel for discussing with me the Ext-annihilators of isolated hypersurface singularities. I am particularly indebted to Maurice Auslander for numerous discussions on Ext-annihilators of affine-algebraic isolated singularities. In fact, almost all of section 3.2 leading up to Proposition 14, has been outlined to me by him. I also would like to thank the Deutsche Forschungsgemeinschaft for financial support.

0. Preliminaries

Throughout this article, modules are understood to be left modules, and maps are written on the left of the argument. For any ring A, we write $\operatorname{Mod} A$ for the category of all left A-modules, $\operatorname{mod} A$ for the category of all finitely generated left A-modules, and $\operatorname{gldim} A$ for the left global dimension of A. For any $M \in \operatorname{Mod} A$, $\operatorname{pd}(M)$ denotes the projective dimension of M, and $\Omega^n(M)$ is the n-th *syzygy* module of M. If V is a vector-space over a skewfield $\mathfrak{k}$, then we write $[V:\mathfrak{k}]$ for the dimension of V over $\mathfrak{k}$. For any object C in any category $\mathscr{C}$ we denote by $[C]$ the isomorphism class of C, and by $[\mathscr{C}]$ the set of all isomorphism classes of $\mathscr{C}$. The symbol $\subset$ means inclusion or equality. We agree that $\mathbb{N} = \{1, 2, 3, \ldots\}$, whereas $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$.

We write $k[[X_1, \ldots, X_n]]$ for the ring of formal power series in n variables $X_1, \ldots, X_n$ over a field k. Cohen–Macaulay modules over a commutative noetherian local ring are always understood to be maximal Cohen–Macaulay modules, i.e. the depth of the module equals the Krull dimension of the ring. For any commutative ring S, we write $\operatorname{Spec}(S)$ for the spectrum of S, $\operatorname{Max}(S)$ for the maximal spectrum of S, $\operatorname{Reg}(S)$ for the regular locus of S, and $\operatorname{Sing}(S)$ for the singular locus of S. The dimension of S is understood to be the Krull dimension of S, and is denoted by $\dim S$.

Throughout, R denotes a commutative noetherian complete local ring, and Λ denotes an R-algebra which is finitely generated as R-module. Let $\mathscr{L}$ be the class of all R-algebras which arise in this way. Usually we shall consider algebras from subclasses of $\mathscr{L}$ by assuming in addition, for example, that R is Cohen–Macaulay or even regular, or else that Λ is finitely generated free as R-module or even a commutative local Cohen–Macaulay ring. But, unless otherwise stated, Λ is not assumed to be commutative. We write $\mathfrak{m}$ for the unique maximal ideal of R, d for the dimension of R, and, in case R is a domain, K for the field of fractions of R. General elements in $\operatorname{Spec}(R)$ are denoted by small german letters such as $\mathfrak{p}$, $\mathfrak{q}$, etc. In case Λ is commutative, elements in $\operatorname{Spec}(\Lambda)$ are denoted by capital german letters such as $\mathfrak{P}$, $\mathfrak{Q}$, etc. In case Λ is commutative local, we write $\mathfrak{M}$ for the unique maximal ideal of Λ. For any $\Lambda \in \mathscr{L}$ and $M, N \in \operatorname{mod} \Lambda$ we set $\underline{\operatorname{Hom}}_\Lambda(M, N) = \operatorname{Hom}_\Lambda(M, N)/\operatorname{Hom}'_\Lambda(M, N)$, where $\operatorname{Hom}'_\Lambda(M, N)$ consists of all homomorphisms in $\operatorname{Hom}_\Lambda(M, N)$ which factor through a projective Λ-module.

Given an R-algebra Λ in $\mathscr{L}$, we denote by $\operatorname{mod}_R \Lambda$ (emphasizing the subscript

R) the full subcategory of $\operatorname{mod} \Lambda$ consisting of all objects which are projective as R-modules. The study of the category $\operatorname{mod}_R \Lambda$ will be our main objective. We write $\operatorname{ind} \Lambda$ for the full subcategory of $\operatorname{mod} \Lambda$ consisting of all indecomposable objects of $\operatorname{mod} \Lambda$, and similarly we write $\operatorname{ind}_R \Lambda$ for the full subcategory of $\operatorname{mod}_R \Lambda$ consisting of all indecomposable objects of $\operatorname{mod}_R \Lambda$. (For any twosided ideal $\mathcal{J}$ contained in the radical of Λ, Λ is complete with respect to the $\mathcal{J}$-adic topology and idempotents can be lifted from $\Lambda/\mathcal{J}$ to Λ. Consequently every object in $\operatorname{ind} \Lambda$ has local endomorphism ring, and therefore Krull–Schmidt's Theorem holds in $\operatorname{mod} \Lambda$ as well as in $\operatorname{mod}_R \Lambda$.)

For any $M \in \operatorname{mod}_R \Lambda$ we set $\rho(M) = \rho_\Lambda(M) = [M/\mathfrak{m}M : R/\mathfrak{m}]$, and we call $\rho(M)$ the R-rank of M. Usually ρ will be considered as a function on $[\operatorname{ind}_R \Lambda]$. An R-algebra Λ in $\mathcal{L}$ is said to be of *finite type* in case $[\operatorname{ind}_R \Lambda]$ is a finite set, respectively of *infinite type* in case $[\operatorname{ind}_R \Lambda]$ is an infinite set. With any R-algebra Λ in $\mathcal{L}$ we associate two sets of natural numbers $P_\infty(\Lambda) \subset P(\Lambda) \subset \mathbb{N}$ as follows. Consider the rank function $\rho_\Lambda : [\operatorname{ind}_R \Lambda] \to \mathbb{N}$, and set $P(\Lambda) = \operatorname{im}(\rho_\Lambda)$, respectively $P_\infty(\Lambda) = \{r \in \mathbb{N} \mid \rho_\Lambda^{-1}(r) \text{ is infinite}\}$. Now let $\mathcal{A}$ be a subclass of $\mathcal{L}$. We say that *the first Brauer–Thrall conjecture is true* for $\mathcal{A}$ in case for all $\Lambda \in \mathcal{A}$, if Λ is of infinite type then $P(\Lambda)$ is infinite. We say that *the second Brauer–Thrall conjecture is true for* $\mathcal{A}$ in case for all $\Lambda \in \mathcal{A}$, if Λ is of infinite type then $P_\infty(\Lambda)$ is infinite.

Let Λ be an algebra in $\mathcal{L}$, with regular ground ring R, and such that Λ is finitely generated free as R-module. Following Auslander [Au 84] we say that Λ is nonsingular if $\operatorname{gldim} \Lambda = \dim R$, respectively isolated singular if $\operatorname{gldim} \Lambda \neq \dim R$ and $\operatorname{gldim} \Lambda_{\mathfrak{p}} = \dim R_{\mathfrak{p}}$ for all $\mathfrak{p} \in \operatorname{Spec}(R) \backslash \{\mathfrak{m}\}$, respectively nonisolated singular in all other cases.

A special but important subclass of $\mathcal{L}$ frequently arises in the following way. Suppose we are given a commutative noetherian complete local ring S. Then there exists a commutative noetherian complete regular local subring $R \subset S$ such that S is finitely generated as R-module. R is called a *Noether normalization* of S. In this situation, the category $\operatorname{mod}_R S$ coincides with the category of Cohen–Macaulay S-modules, and S is finitely generated free as R-module if and only if S is a Cohen-Macaulay ring. For commutative noetherian complete local Cohen–Macaulay rings S with Noether normalization $R \subset S$, the notion of nonsingularity, isolated singularity and nonisolated singularity, as defined above, coincide with the corresponding notions from commutative algebra.

1. Isolated singularities with reduction ideal

We assume that R is a commutative noetherian complete local ring, and that Λ is an R-algebra which is finitely generated as R-module.

DEFINITION. A twosided ideal $\mathcal{I}$ of Λ is called a reduction ideal of Λ if it has the following properties.

(a) $\mathcal{I} \subset m\Lambda$.

(b) $\Lambda/\mathcal{I}$ is artinian.

(c) The functor $\mathcal{F}_{\mathcal{I}} = \Lambda/\mathcal{I} \otimes_\Lambda : \mathrm{mod}_R \Lambda \to \mathrm{mod}(\Lambda/\mathcal{I})$ preserves indecomposability and separates isomorphism classes.

If there exists a reduction ideal $\mathcal{I}$, then we call $\mathcal{F}_{\mathcal{I}}$ its *reduction functor.* It reduces the dimension of the ground ring from d to 0, it maps nonisomorphisms to nonisomorphisms, and it induces an inclusion mapping between the sets of isomorphism classes of indecomposable objects, $\bar{\mathcal{F}}_{\mathcal{I}} : [\mathrm{ind}_R \Lambda] \hookrightarrow [\mathrm{ind}(\Lambda/\mathcal{I})]$. Since $\mathrm{mod}(\Lambda/\mathcal{I})$ is an abelian category in which all objects have finite length, we have much better knowledge about $\mathrm{mod}(\Lambda/\mathcal{I})$ than about $\mathrm{mod}_R \Lambda$. As we shall see, some of this information can be transferred from $\mathrm{mod}(\Lambda/\mathcal{I})$ to $\mathrm{mod}_R \Lambda$, by way of the reduction functor $\mathcal{F}_{\mathcal{I}}$.

We give some examples of reduction ideals, in case R is commutative noetherian complete regular local and Λ is finitely generated free as R-module.

(1) If $d = 0$, then (0) is a reduction ideal of Λ.

(2) If $d = 1$ and $K \otimes_R \Lambda$ is a separable K-algebra, then $mc\Lambda$ is a reduction ideal of Λ, where c is the conductor in R of a maximal order $\Lambda' \supset \Lambda$ into Λ; alternatively, $mh\Lambda$ is a reduction ideal of Λ, where h is the Higman ideal of Λ. (See [Cu/Re 81] or [Ro/Hu 70] as general references for the case $d = 1$.)

(3) If Λ is nonsingular, then $m\Lambda$ is a reduction ideal of Λ because all objects in $\mathrm{mod}_R \Lambda$ are projective.

The following statements (4) and (5) will be proved in sections 2 and 3 (see Theorem 7, Proposition 8 and Corollary 15).

(4) If Λ is an isolated singularity of finite type, then Λ has a reduction ideal.

(5) If k is an algebraically closed field, $I \subset \mathbb{C}[[X_1, \ldots, X_n]]$ an ideal such that $\Lambda = \mathbb{C}[[X_1, \ldots, X_n]]/I$ is an isolated Cohen–Macaulay singularity, then Λ has a reduction ideal.

More generally it would be interesting to characterize those isolated singularities which have a reduction ideal. On the other hand, for nonisolated singularities we have only counterexamples so far: If Λ is a nonisolated hypersurface singularity of type $\mathbb{A}_\infty$ or $\mathbb{D}_\infty$ (in the language of [Bu/Gr/Schr 86], i.e.

$$\Lambda = \mathbb{C}[[X_0, \ldots, X_d]]/(X_1^2 + \cdots + X_d^2)$$

or

$$\Lambda = \mathbb{C}[[X_0, \ldots, X_d]]/(X_0 X_1^2 + X_2^2 + \cdots + X_d^2),$$

then the classification of $\mathrm{mod}_R\,\Lambda$ shows that Λ has no reduction ideal. Also observe that any power of a reduction ideal is again a reduction ideal. Therefore the set of all reduction ideals never contains a minimal element (unless Λ is artinian). On the contrary it would be interesting to find the maximal elements of this set, in case it is nonempty.

Whereas in sections 2 and 3 we will be concerned with the question of existence of a reduction ideal, in the remaining part of this section we will investigate properties of isolated singularities which have a reduction ideal. In this respect the following generalization of a Lemma of Harada and Sai [Har/Sa 70], [Rin 79] is fundamental.

LEMMA 1. *Let Λ be an R-algebra as above, given by a ring homomorphism $\phi: R \to \Lambda$. Assume that Λ has a reduction ideal $\mathcal{J}$. Set $\imath = \phi^{-1}(\mathcal{J})$ and $\ell =$* $\mathrm{length}_R\,(R/\imath)$. *If $M_1 \xrightarrow{\psi_1} M_2 \xrightarrow{\psi_2} \cdots \xrightarrow{\psi_{2^\ell - 1}} M_{2^\ell}$ is a chain of $2^\ell - 1$ nonisomorphisms in $\mathrm{mod}_R\,\Lambda$, such that all modules $M_1, \ldots, M_{2^\ell}$ are indecomposable and of R-*rank $\leqq \ell$, *then the image of the composed morphism $\psi_{2^\ell-1} \cdots \psi_1$ is contained in $\mathcal{J}M_{2^\ell}$.*

Proof. Applying the reduction functor, we obtain a chain

$$\mathcal{F}_{\mathcal{J}}(M_1) \xrightarrow{\mathcal{F}_{\mathcal{J}}(\psi_1)} \mathcal{F}_{\mathcal{J}}(M_2) \xrightarrow{\mathcal{F}_{\mathcal{J}}(\psi_2)} \cdots \xrightarrow{\mathcal{F}_{\mathcal{J}}(\psi_{2^\ell-1})} \mathcal{F}_{\mathcal{J}}(M_{2^\ell})$$

of $2^\ell - 1$ nonisomorphisms in $\mathrm{mod}\,(\Lambda/\mathcal{J})$, such that all modules $\mathcal{F}_{\mathcal{J}}(M_1), \ldots, \mathcal{F}_{\mathcal{J}}(M_{2^\ell})$ are indecomposable. Because $\Lambda/\mathcal{J}$ is artinian and $\mathrm{length}_{\Lambda/\mathcal{J}\Lambda}\,(\mathcal{F}_{\mathcal{J}}(M_i)) \leqq \ell$ for all $i = 1, \ldots, 2^\ell$, we know from the Lemma of Harada and Sai that the composed morphism $\mathcal{F}_{\mathcal{J}}(\psi_{2^\ell-1}) \cdots \mathcal{F}_{\mathcal{J}}(\psi_1)$ is zero. But this is equivalent to the fact that $\mathrm{im}\,(\psi_{2^\ell-1} \cdots \psi_1) \subset \mathcal{J}M_{2^\ell}$. q.e.d.

Recall that the R-algebra Λ is said to be *connected* if $\Lambda = \Lambda_1 \oplus \Lambda_2$ implies that either $\Lambda_1 = 0$ or $\Lambda_2 = 0$, for every decomposition of Λ into a direct sum of twosided ideals Λ_1 and Λ_2. Denote the functor $\Lambda/m\Lambda \otimes_\Lambda : \mathrm{mod}_R\,\Lambda \to \mathrm{mod}_{R/m}\,(\Lambda/m\Lambda)$ by $\mathcal{F}_m$. Then Λ is connected if and only if for any two indecomposable projective Λ-modules P and P' there exists a sequence $P = P_0, P_1, \ldots, P_n = P'$ of indecomposable projective Λ-modules such that, for all $i = 0, \ldots, n-1$, either

$$\mathcal{F}_m(\mathrm{Hom}_\Lambda\,(P_i, P_{i+1})) \neq 0 \quad \text{or} \quad \mathcal{F}_m(\mathrm{Hom}_\Lambda\,(P_{i+1}, P_i)) \neq 0.$$

PROPOSITION 2. *Let Λ be an isolated singularity or nonsingular. Assume* *that Λ is connected and has a reduction ideal $\mathcal{I}$. Let $\mathscr{C} = (\mathscr{C}_0, \mathscr{C}_1)$ be a connected component of the Auslander–Reiten quiver $\mathscr{A}(\Lambda) = (\mathscr{A}_0, \mathscr{A}_1)$, on which the rank function $\rho : \mathscr{C}_0 \to \mathbb{N}$ is bounded. Then $\mathscr{A}(\Lambda) = \mathscr{C}$, and $\mathscr{A}(\Lambda)$ is finite.*

In case $d = 0$, this result is due to Auslander [Au 74]. We closely follow his proof, adapting it to our more general situation by way of the reduction functor $\mathscr{F}_{\mathcal{I}}$.

Proof. Let Λ, $\mathcal{I}$ and $\mathscr{C}$ be given as in the Proposition. Let $\mathscr{b}$ be an upper bound for ρ on $\mathscr{C}_0$ and set $\ell = \operatorname{length}_R (R/\imath)$, $\imath = \mathcal{I} \cap R$. We claim that the following statement is true.

(*) If M and N are indecomposable objects in $\operatorname{mod}_R \Lambda$ such that $\mathscr{F}_{\mathcal{I}}(\operatorname{Hom}_\Lambda (M, N)) \neq 0$, then $[M] \in \mathscr{C}_0$ if and only if $[N] \in \mathscr{C}_0$.

Proof of (*): Suppose M and N are indecomposable objects in $\operatorname{mod}_R \Lambda$ such that $\mathscr{F}_{\mathcal{I}}(\operatorname{Hom}_\Lambda (M, N)) \neq 0$, and suppose that $[M] \in \mathscr{C}_0$, but $[N] \notin \mathscr{C}_0$. Choose $\psi \in \operatorname{Hom}_\Lambda (M, N)$ such that $\operatorname{im} \psi \not\subset \mathcal{I}N$. Since every indecomposable object in $\operatorname{mod}_R \Lambda$ has a source morphism, and in view of the connection between source morphisms and irreducible morphisms (see statements (1) and (2) preceding Proposition 2 in [Di 86]) we obtain that, for all $c \in \mathbb{N}$, ψ can be factored as

$$\psi = \sum_{i=1}^{n} \xi_i \zeta_i,$$

where $X_i = \operatorname{domain}(\xi_i)$ is an indecomposable object in $\mathscr{C}_0$, and ζ_i is a composition of c irreducible morphisms in $\mathscr{C}_1$, for all $i = 1, \ldots, n$. On setting $c = 2^{\ell\mathscr{b}}$ we deduce from Lemma 1 that $\operatorname{im} \psi \subset \mathcal{I}N$, which contradicts our choice of ψ. Arguing with sink maps instead of source maps one proves dually that $[N] \in \mathscr{C}_0$ implies $[M] \in \mathscr{C}_0$.

Now choose $[M] \in \mathscr{C}_0$ arbitrary and let $P_M \twoheadrightarrow M$ be a projective cover of M. Then (*) implies that there exists a projective point in $\mathscr{C}_0$. Since Λ is connected, (*) implies that all projective points of $\mathscr{A}_0$ are in $\mathscr{C}_0$. Let $[N] \in \mathscr{A}_0$ arbitrary and let $P_N \twoheadrightarrow N$ be a projective cover of N. Then (*) implies that $[N] \in \mathscr{C}_0$. This proves $\mathscr{A}(\Lambda) = \mathscr{C}$.

On the other hand, the factorization property of Auslander–Reiten sequences together with Lemma 1 shows that for every point $[N] \in \mathscr{A}_0$ there exists a chain $[P] = [N_1] \to [N_2] \to \cdots \to [N_c] = [N]$ of arrows in $\mathscr{A}_1$, such that P is indecomposable projective and $c < 2^{\ell\mathscr{b}}$. Since $\mathscr{A}(\Lambda)$ is locally finite it follows that $\mathscr{A}(\Lambda)$ is finite. q.e.d.

As an immediate consequence of Proposition 2 we obtain the following statements on isolated singularities with reduction ideal.

COROLLARY 3. *The first Brauer–Thrall conjecture is true for isolated singularities with reduction ideal.*

Proof. Let Λ be an isolated singularity with reduction ideal, and assume that Λ is of infinite type. Then there exists a connected algebra-component Λ' of Λ which is of infinite type. By Proposition 2, the rank function is unbounded on each connected component of $\mathscr{A}(\Lambda')$. q.e.d.

COROLLARY 4. *Let Λ be an isolated singularity with reduction ideal. Let $\mathscr{C} = (\mathscr{C}_0, \mathscr{C}_1)$ be a connected component of the stable Auslander–Reiten quiver $\mathscr{A}_s(\Lambda)$ and assume that $\mathscr{C}_0$ contains a periodic point. Then the Cartan class of $\mathscr{C}$ is either a Dynkin diagram or $\mathbb{A}_\infty$.*

Proof. Let $\bar{\mathscr{C}}$ be the connected component of $\mathscr{A}(\Lambda)$ which contains $\mathscr{C}$. If $\bar{\mathscr{C}} \neq \mathscr{C}$, then the rank function $\rho: \mathscr{C}_0 \to \mathbb{N}$ is not additive on $\mathscr{C}$, so the Cartan class of $\mathscr{C}$ is either a Dynkin diagram or $\mathbb{A}_\infty$, by [Di 86, Theorem 3]. If $\bar{\mathscr{C}} = \mathscr{C}$, then $\mathscr{C}$ is a connected component of $\mathscr{A}(\Lambda)$ which contains no projective point. Therefore ρ is unbounded on $\mathscr{C}_0$, by Proposition 2, so the Cartan class of $\mathscr{C}$ is $\mathbb{A}_\infty$, by [Di 86, Theorem 3]. q.e.d.

2. Construction of reduction ideals via annihilators of Ext

Throughout this section, let R be a commutative noetherian complete local ring and let Λ be an R-algebra which is finitely generated as R-module. Within this general setup we turn to the question of existence of a reduction ideal. Generalizing an approach which goes back to Maranda [Ma 53] we shall prove the following existence criterion: If R is Cohen–Macaulay and the annihilator of the functor $\mathrm{Ext}^1_\Lambda(\ ,\ ): \mathrm{mod}_R \Lambda \times \mathrm{mod}\, \Lambda \to \mathrm{mod}\, R$ is m-primary, then there exists a reduction ideal of Λ.

With any element $r \in m$ we associate the category $\mathrm{mod}_{R/rR}(\Lambda/r\Lambda)$ given by the factoralgebra $\Lambda/r\Lambda$, and the factorcategory $(\mathrm{mod}_R \Lambda)/\mathscr{H}_r$ given by the system of relations $\mathscr{H}_r = \{r \,\mathrm{Hom}_\Lambda(M, N) \mid M, N \in \mathrm{mod}_R \Lambda\}$. By definition, the objects of $(\mathrm{mod}_R \Lambda)/\mathscr{H}_r$ are the objects of $\mathrm{mod}_R \Lambda$, and the morphism set in $(\mathrm{mod}_R \Lambda)/\mathscr{H}_r$ from M to N is given by $\mathrm{Hom}_\Lambda(M, N)/r \,\mathrm{Hom}_\Lambda(M, N)$. Note the difference: whereas morphisms in $\mathrm{mod}_{R/rR}(\Lambda/r\Lambda)$ are Λ-linear maps between residue class modules, morphisms in $(\mathrm{mod}_R \Lambda)/\mathscr{H}_r$ are residue classes of Λ-linear maps between modules. Given any full subcategory $\mathscr{S}$ of $\mathrm{mod}_R \Lambda$, we consider the functor $\mathscr{F}_r: \mathscr{S} \to \mathrm{mod}_{R/rR}(\Lambda/r\Lambda)$ given by $\mathscr{F}_r = \Lambda/r\Lambda \otimes_\Lambda$, and the canonical functor $\mathcal{R}_r : \mathcal{S} \to (\mathrm{mod}_R\, \Lambda)/\mathcal{H}_r$. For $S = \mathrm{mod}_R\, \Lambda$, the following facts are easily verified.

(i) The functor $\mathcal{R}_r$ is a representation equivalence (i.e. $\mathcal{R}_r$ is full, dense and isomorphism-reflecting).

(ii) There exists a uniquely determined functor $\bar{\mathcal{F}}_r : (\mathrm{mod}_R\, \Lambda)/\mathcal{H}_r \to \mathrm{mod}_{R/rR}\, (\Lambda/r\Lambda)$ such that $\mathcal{F}_r = \bar{\mathcal{F}}_r \mathcal{R}_r$.

(iii) If r is a nonzerodivisor of R, then $\bar{\mathcal{F}}_r$ is faithful.

LEMMA 5. *Let $\mathcal{S}$ be a full subcategory of* $\mathrm{mod}_R\, \Lambda$. *Let a be a nonunit and nonzerodivisor of R such that $a\, \mathrm{Ext}^1_\Lambda(M, N) = 0$, for all $M, N \in \mathcal{S}$. Denote by $\bar{a}$ the residue class of a in R/a^2R. Then, for all $M, N \in \mathcal{S}$ and for each morphism $g \in \mathrm{Hom}_{\Lambda/a^2\Lambda}(M/a^2M, N/a^2N)$ there exists a morphism $f \in \mathrm{Hom}_\Lambda(M, N)$ such that $\mathcal{F}_a(f) = \mathcal{F}_{\bar{a}}(g)$.*

Proof. Let $M, N \in \mathcal{S}$ and $g \in \mathrm{Hom}_{\Lambda/a^2\Lambda}(M/a^2M, N/a^2N)$ be given. Because M is R-projective, there exists an R-linear map $\tilde{f} : M \to N$ which lifts g. Then $\mathrm{im}\,(\lambda\tilde{f} - \tilde{f}\lambda) \subset a^2N$, for all $\lambda \in \Lambda$. Since a^2 is a nonzerodivisor of N, for all $\lambda \in \Lambda$ there exists a unique R-linear map $F_\lambda : M \to N$ such that $a^2F_\lambda = \lambda\tilde{f} - \tilde{f}\lambda$. It turns our that $F = \{F_\lambda\}_{\lambda \in \Lambda} : \Lambda \to \mathrm{Hom}_R(M, N)$ is a derivation. In view of the isomorphisms

$$\begin{aligned} \mathrm{Ext}^1_\Lambda(M, N) &\cong H^1(\Lambda, \mathrm{Hom}_R(M, N)) \\ &\cong \mathrm{Der}\,(\Lambda, \mathrm{Hom}_R(M, N))/\mathrm{In}\,\mathrm{Der}\,(\Lambda, \mathrm{Hom}_R(M, N)), \end{aligned}$$

our assumption $a\, \mathrm{Ext}^1_\Lambda(M, N) = 0$ implies that aF is an inner derivation. Hence there exists an R-linear map $h : M \to N$ such that $aF_\lambda = \lambda h - h\lambda$, for all $\lambda \in \Lambda$. Then $\lambda\tilde{f} - \tilde{f}\lambda = a^2F_\lambda = a(\lambda h - h\lambda)$ implies that $f = \tilde{f} - ah : M \to N$ is a Λ-linear map, and $\mathcal{F}_a(f) = \mathcal{F}_{\bar{a}}(g)$. q.e.d.

PROPOSITION 6. *Let $\mathcal{S}$ be a full subcategory of* $\mathrm{mod}_R\, \Lambda$. *Let a be a nonunit and nonzerodivisor of R such that $a\, \mathrm{Ext}^1_\Lambda(M, N) = 0$, for all $M, N \in \mathcal{S}$. Then the functor $\mathcal{F}_{a^2} : \mathcal{S} \to \mathrm{mod}_{R/a^2R}\, (\Lambda/a^2\Lambda)$ preserves indecomposability and separates isomorphism classes.*

Proof. Let $M \in \mathcal{S}$ be given and assume that $\mathcal{F}_{a^2}(M)$ decomposes properly. Choose an idempotent $g \in \mathrm{End}_{\Lambda/a^2\Lambda}(M/a^2M)$ which is different from 0 and different from 1. By Lemma 5 there exists an endomorphism $f \in \mathrm{End}_\Lambda M$ such that $\mathcal{F}_a(f) = \mathcal{F}_{\bar{a}}(g)$. Since g is idempotent, $\mathcal{F}_a(f)$ is also idempotent. Since $\mathcal{F}_a(f) = \bar{\mathcal{F}}_a\mathcal{R}_a(f)$ and $\bar{\mathcal{F}}_a$ is faithful, $\mathcal{R}_a(f)$ is an idempotent element in $(\mathrm{End}_\Lambda M)/a(\mathrm{End}_\Lambda M)$ which can be lifted to an idempotent element $\bar{f}$ in

$\mathrm{End}_\Lambda M$. It is easily seen that $\bar{f}$ is different from 0 and different from 1, because otherwise g would be equal to 0 or 1, contradicting our choice of g. Hence M decomposes properly.

Let $M, N \in \mathcal{S}$ be given and assume that $\mathcal{F}_{a^2}(M) \cong \mathcal{F}_{a^2}(N)$. Choose an isomorphism $g \in \mathrm{Hom}_{\Lambda/a^2\Lambda}(M/a^2M, N/a^2N)$. By Lemma 5 there exists a morphism $f \in \mathrm{Hom}_\Lambda(M, N)$ such that $\mathcal{F}_a(f) = \mathcal{F}_{\bar{a}}(g)$. Since g is an isomorphism, $\mathcal{F}_a(f)$ is also an isomorphism. Since a is an element in $\mathfrak{m}$, it follows that f is an isomorphism. Hence $M \cong N$. q.e.d.

Proposition 6 is known as "Maranda's Theorem" in case $d = 1$ and $\mathcal{S} = \mathrm{mod}_R \Lambda$. Originally it has been proved by Maranda for the group ring of a finite group over the ring of p-adic integers [Ma 53], and later it has been generalized by D. G. Higman to arbitrary orders over complete discrete valuation rings [Hi 60]. In generalizing Maranda's Theorem to algebras over higher dimensional ground rings, as formulated in Proposition 6, I drew much benefit from the beautiful presentation of this topic given in [Cu/Re 81] for the case $d = 1$. The reason for considering arbitrary full subcategories $\mathcal{S}$ of $\mathrm{mod}_R \Lambda$ will become clear in the sequel, when we shall apply Proposition 6 inductively.

DEFINITION. Let Λ be an R-algebra as above. We define the Ext-annihilating ideal of Λ to be the annihilator ideal (in R) of the bifunctor $\mathrm{Ext}^1_\Lambda(\ ,\ ): \mathrm{mod}_R \Lambda \times \mathrm{mod}\,\Lambda \to \mathrm{mod}\, R$. We denote the Ext-annihilating ideal of Λ by $\mathfrak{a}$.

Note the asymmetry in the product category $\mathrm{mod}_R \Lambda \times \mathrm{mod}\,\Lambda$ which, for definition of the Ext-annihilating ideal, we choose as domain of the bifunctor $\mathrm{Ext}^1_\Lambda(\ ,\ )$. Observe that, if R is Cohen–Macaulay and $\mathfrak{a}$ is $\mathfrak{m}$-primary, then there exist plenty of maximal R-regular sequences which are contained in $\mathfrak{a}$. (Choose any system of parameters $x_1, \ldots, x_d$ of R. Then $x_1^n, \ldots, x_d^n$ is a maximal R-regular sequence for all $n \in \mathbb{N}$, and there exists $n_0 \in \mathbb{N}$ such that $x_1^n, \ldots, x_d^n$ is contained in $\mathfrak{a}$ for all $n \geqq n_0$.)

Given any finite set of elements $\{r_1, \ldots, r_n\}$ in R, we denote by $(r_1, \ldots, r_n)$ the R-ideal generated by $\{r_1, \ldots, r_n\}$, and we denote by $(r_1, \ldots, r_n)\Lambda$ the twosided Λ-ideal generated by $\{r_1, \ldots, r_n\}$.

THEOREM 7. *Let R be a commutative noetherian complete local Cohen–Macaulay ring and let Λ be an R-algebra which is finitely generated as R-module. Assume that the* Ext-*annihilating ideal $\mathfrak{a}$ of Λ is $\mathfrak{m}$-primary. Then for every maximal R-regular sequence $a_1, \ldots, a_d$ contained in $\mathfrak{a}$, $(a_1^2, \ldots, a_d^2)\Lambda$ is a reduction ideal of Λ.*

Proof. Let $a_1, \ldots, a_d$ be a maximal R-regular sequence which is contained in $\mathfrak{a}$, and set $\mathcal{I} = (a_1^2, \ldots, a_d^2)\Lambda$. From our assumptions on $a_1, \ldots, a_d$ it follows immediately that $\mathcal{I} \subset \mathfrak{m}\Lambda$ and that $\Lambda/\mathcal{I}$ is artinian. For investigation of the functor $\mathcal{F}_{\mathcal{I}} = \Lambda/\mathcal{I} \otimes_\Lambda : \operatorname{mod}_R \Lambda \to \operatorname{mod}(\Lambda/\mathcal{I})$ we introduce a sequence of functors, associated with the given R-regular sequence as follows. Set $\mathfrak{a}_0 = (0)$ and $\mathfrak{a}_i = (a_1^2, \ldots, a_i^2)$, for all $i = 1, \ldots, d$. Let $\mathcal{S}_i$ be the full subcategory of $\operatorname{mod}_{R/\mathfrak{a}_i}(\Lambda/\mathfrak{a}_i\Lambda)$ which is given by the class of objects $\{M/\mathfrak{a}_iM \mid M \in \operatorname{mod}_R \Lambda\}$, for all $i = 0, \ldots, d$. We consider the sequence of functors $\mathcal{F}_{a_i^2}: \mathcal{S}_{i-1} \to \mathcal{S}_i$ given by $\mathcal{F}_{a_i^2} = \Lambda/a_i^2\Lambda \otimes_\Lambda$, where $i = 1, \ldots, d$. Then the following statements hold for each $i = 0, \ldots, d-1$.

(i) $R/\mathfrak{a}_i$ is a complete local ring, $\Lambda/\mathfrak{a}_i\Lambda$ is an $R/\mathfrak{a}_i$-algebra which is finitely generated as $R/\mathfrak{a}_i$-module, and $\mathcal{S}_i$ is a full subcategory of $\operatorname{mod}_{R/\mathfrak{a}_i}(\Lambda/\mathfrak{a}_i\Lambda)$.

(ii) The residue class $a_{i+1} + \mathfrak{a}_i$ is a nonunit and nonzero divisor of $R/\mathfrak{a}_i$.

(iii) $(a_{i+1} + \mathfrak{a}_i) \operatorname{Ext}^1_{\Lambda/\mathfrak{a}_i\Lambda}(M/\mathfrak{a}_iM, N/\mathfrak{a}_iN) = 0$ for all $M, N \in \operatorname{mod}_R \Lambda$. (Assertion (i) follows trivially from our assumptions and definitions. Since $a_1^2, \ldots, a_d^2$ is an R-regular sequence in $\mathfrak{m}$, we obtain assertion (ii) and the isomorphism

$$\operatorname{Ext}^1_{\Lambda/\mathfrak{a}_i\Lambda}(M/\mathfrak{a}_iM, N/\mathfrak{a}_iN) \cong \operatorname{Ext}^1_\Lambda(M, N/\mathfrak{a}_iN), \quad \text{for all} \quad M, N \in \operatorname{mod}_R \Lambda.$$

Now assertion (iii) follows in view of $a_{i+1}^2 \in \mathfrak{a}$ and the definition of $\mathfrak{a}$.) Due to (i)–(iii) the hypotheses of Proposition 6 are satisfied for each of the functors $\mathcal{F}_{a_i^2}: \mathcal{S}_{i-1} \to \mathcal{S}_i$. Therefore $\mathcal{F}_{a_i^2}$ preserves indecomposability and separates isomorphism classes, for all $i = 1, \ldots, d$. On the other hand $\mathcal{F}_{\mathcal{I}} \cong \mathcal{F}_{a_d^2} \cdot \ldots \cdot \mathcal{F}_{a_1^2}$, and therefore $\mathcal{F}_{\mathcal{I}}$ preserves indecomposability and separates isomorphism classes. q.e.d.

3. Isolated singularities with $\mathfrak{m}$-primary Ext-annihilating ideal

This section is mainly devoted to showing for two classes of isolated singularities that their Ext-annihilating ideals are $\mathfrak{m}$-primary, namely

a) for isolated singularities of finite type, and

b) for isolated Cohen–Macaulay singularities which are of the form $\Lambda = k[[X_1, \ldots, X_n]]/I$, where k is an algebraically closed field and $I \subset k[[X_1, \ldots, X_n]]$ an ideal.

Once this is established, all results of section 1 apply to any isolated singularity which belongs to a) or b), by Theorem 7. As a consequence we obtain the results announced in the introduction.

3.1. Isolated singularities of finite type

Let R be a commutative noetherian complete regular local ring and let Λ be an R-algebra which is finitely generated free as R-module.

PROPOSITION 8. *If Λ is an isolated singularity of finite type, then the* Ext-*annihilating ideal of Λ is $\mathfrak{m}$-primary.*

Proof. Let $M_1, \ldots, M_n$ be a set of representatives for the isomorphism classes of indecomposable objects in $\operatorname{mod}_R \Lambda$, and set $M = \oplus_{i=1}^n M_i$. To each pair $(X, Y) \in \operatorname{mod}_R \Lambda \times \operatorname{mod}_R \Lambda$ we assign the R – ideal $\mathfrak{a}_{X,Y} = \operatorname{ann}_R (\underline{\operatorname{Hom}}_\Lambda (X, Y))$. Then by [Au 84, main theorem], for every pair (X, Y) there exists a number $\ell = \ell_{X,Y} \in \mathbb{N}$ such that $\mathfrak{m}^\ell \subset \mathfrak{a}_{X,Y}$. Moreover, the following inclusions are easily verified.

(1) $\mathfrak{a}_{M,M} \subset \bigcap_{i,j=1}^n \mathfrak{a}_{M_i,M_j}$

(2) $(\mathfrak{a}_{X,Y} \cap \mathfrak{a}_{X,Z}) \subset \mathfrak{a}_{X,Y\oplus Z}$, for all $X, Y, Z \in \operatorname{mod}_R \Lambda$.

(3) $(\mathfrak{a}_{X,Z} \cap \mathfrak{a}_{Y,Z}) \subset \mathfrak{a}_{X\oplus Y,Z}$, for all $X, Y, Z \in \operatorname{mod}_R \Lambda$.

Properties (1)–(3) imply that $\mathfrak{a}_{M,M} \subset \bigcap_{X \in \operatorname{mod}_R \Lambda} \mathfrak{a}_{X,X}$. Hence we obtain $\mathfrak{m}^\ell \subset \mathfrak{a}_{M,M} \subset \mathfrak{a}$, where $\ell = \ell_{M,M}$ and where $\mathfrak{a}$ is the Ext-annihilating ideal of Λ, as defined in section 2. In addition, $\mathfrak{a} \subset \mathfrak{m}$, because otherwise Λ would be nonsingular. Therefore $\mathfrak{a}$ is $\mathfrak{m}$-primary. q.e.d.

THEOREM 9. *Let Λ be an isolated singularity of finite type. Then the Cartan class of any connected component $\mathscr{C} = (\mathscr{C}_0, \mathscr{C}_1)$ of the stable Auslander–Reiten quiver $\mathscr{A}_s(\Lambda)$ is a Dynkin diagram.*

Proof. By Proposition 10 and Theorem 7, Λ has a reduction ideal. Since Λ is assumed to be of finite type, each point of $\mathscr{C}_0$ is periodic, and the Cartan class of $\mathscr{C}$ cannot be $\mathbb{A}_\infty$. Therefore, by Corollary 4, the Cartan class of $\mathscr{C}$ is a Dynkin diagram. q.e.d.

Remark. If Λ is of finite type, then it has to be an isolated singularity or nonsingular [Au 84]. Therefore in Theorem 9 we may as well omit the hypothesis that Λ is an isolated singularity.

3.2. Commutative local isolated Cohen–Macaulay singularities

Assume that k is an algebraically closed field, that $V \subset \mathbb{A}^n(k)$ is an affine algebraic variety of dimension d, and that $0 \in V$ is an isolated Cohen–Macaulay singularity of V. Let $\Lambda = \hat{\mathcal{O}}_{V,0}$ be the complete local ring of V at 0, and let $R \subset \Lambda$ be a chosen Noether normalization. We call such an R-algebra Λ an *affine-algebraic isolated singularity.* An affine-algebraic isolated singularity Λ is an isolated singularity in the sense of section 0, and $\operatorname{mod}_R \Lambda$ is the category of Cohen–Macaulay Λ-modules. Our aim is to study the Ext-annihilating ideal $\mathfrak{a}$ of

A. However, much of what follows will be formulated in a more general setting. Ultimately, all results proved for affine-algebraic isolated singularities generalize to arbitrary isolated singularities which arise as factorring of the formal power series ring over k, by Artin's Theorem [Ar 69].

We first consider the noncomplete situation. Let $A = k[X_1, \ldots, X_n]/I$ be the affine coordinate ring of $V \subset \mathbb{A}^n(k)$. Let $A^e = A \otimes_k A$ be the enveloping algebra of A, and let $A^e_{\mathcal{P}} = A_{\mathcal{P}} \otimes_k A_{\mathcal{P}}$ be the enveloping algebra of $A_{\mathcal{P}}$, for any $\mathcal{P} \in \operatorname{Spec}(A)$.

PROPOSITION 10. *For all* $\mathcal{P} \in \operatorname{Reg}(A)$, *we have the inequality* $\operatorname{pd}_{A^e_{\mathcal{P}}} A_{\mathcal{P}} \leqq d$.

Proof. We first prove the statement for maximal ideals. For any $\mathcal{M} \in \operatorname{Max}(A) \cap \operatorname{Reg}(A)$, set $d' = \dim A_{\mathcal{M}}$ and let $\varepsilon : A^e_{\mathcal{M}} \twoheadrightarrow A_{\mathcal{M}}$ be the augmentation map. Then $\mathcal{Q} := \varepsilon^{-1}(\mathcal{M}A_{\mathcal{M}})$ is a maximal ideal in $A^e_{\mathcal{M}}$, and we have the following facts.

(1) $A^e_{\mathcal{M}}$ is noetherian.

(2) $(A^e_{\mathcal{M}})_{\mathcal{Q}}$ is a commutative noetherian regular local ring of dimension $2d'$.

(3) $A_{\mathcal{M}}/\mathcal{M}A_{\mathcal{M}} \cong (A^e_{\mathcal{M}})_{\mathcal{Q}}/\mathcal{Q}(A^e_{\mathcal{M}})_{\mathcal{Q}}$, as fields and as $(A^e_{\mathcal{M}})_{\mathcal{Q}}$-modules.

(4) $(A_{\mathcal{M}})_{\mathcal{Q}'} = \begin{cases} A_{\mathcal{M}} & \text{if } \mathcal{Q}' = \mathcal{Q} \\ 0 & \text{if } \mathcal{Q}' \in \operatorname{Max}(A^e_{\mathcal{M}}) \backslash \{\mathcal{Q}\}. \end{cases}$

We indicate the proof of (1)–(4). On setting $T = \{s \otimes t \mid s, t \in A \backslash \mathcal{M}\} \subset A^e$, we have that $A^e_{\mathcal{M}} \cong T^{-1}(A^e)$. On the other hand, A^e is isomorphic to the coordinate ring of $V \times V \subset \mathbb{A}^n(k) \times \mathbb{A}^n(k)$, and therefore is noetherian. This proves (1). Let $p \in V$ be the regular point corresponding to $\mathcal{M}$. Then (p, p) is a regular point on $V \times V$. The local ring of $V \times V$ at (p, p) is isomorphic to $(A^e_{\mathcal{M}})_{\mathcal{Q}}$, and its residue class field is isomorphic to $A_{\mathcal{M}}/\mathcal{M}A_{\mathcal{M}}$. This proves (2) and (3). For any $\mathcal{Q}' \in \operatorname{Max}(A^e_{\mathcal{M}})$, if $\mathcal{Q}' \supset \ker \varepsilon$ then $\mathcal{Q}' = \mathcal{Q}$. Therefore, if $\mathcal{Q}' \in \operatorname{Max}(A^e_{\mathcal{M}}) \backslash \{\mathcal{Q}\}$ then there exists an element $s \in \ker \varepsilon \backslash \mathcal{Q}'$ such that $sA_{\mathcal{M}} = 0$. Hence $(A_{\mathcal{M}})_{\mathcal{Q}'} = 0$, which proves (4).

Using (1)–(4), together with standard arguments on the projective dimension of a finitely generated module over a regular local ring, we obtain the following equalities.

$$\begin{aligned} \operatorname{pd}_{A^e_{\mathcal{M}}} A_{\mathcal{M}} &= \sup_{\mathcal{Q}' \in \operatorname{Max}(A^e_{\mathcal{M}})} \{\operatorname{pd}_{(A^e_{\mathcal{M}})_{\mathcal{Q}'}} (A_{\mathcal{M}})_{\mathcal{Q}'}\} \\ &= \operatorname{pd}_{(A^e_{\mathcal{M}})_{\mathcal{Q}}} A_{\mathcal{M}} \\ &= \operatorname{pd}_{(A^e_{\mathcal{M}})_{\mathcal{Q}}} A_{\mathcal{M}}/\mathcal{M}A_{\mathcal{M}} - d' \\ &= \operatorname{pd}_{(A^e_{\mathcal{M}})_{\mathcal{Q}}} (A^e_{\mathcal{M}})_{\mathcal{Q}}/\mathcal{Q}(A^e_{\mathcal{M}})_{\mathcal{Q}} - d' \\ &= 2d' - d' = d' \leqq d. \end{aligned}$$

Now let $\mathcal{P} \in \operatorname{Reg}(A)$ be arbitrary. Then there exists a maximal ideal $\mathcal{M} \in \operatorname{Reg}(A)$ containing $\mathcal{P}$. Thus $A_{\mathcal{P}} \cong T^{-1}A_{\mathcal{M}}$, where $T = \{s \otimes t \mid s, t \in A_{\mathcal{M}} \setminus \mathcal{P}A_{\mathcal{M}}\} \subset A^e_{\mathcal{M}}$. Therefore $\operatorname{pd}_{A^e_{\mathcal{P}}} A_{\mathcal{P}} \leqq \operatorname{pd}_{A^e_{\mathcal{M}}} A_{\mathcal{M}} \leqq d$. q.e.d.

COROLLARY 11. *Let $\mathcal{M}$ be the maximal ideal of A which corresponds to the isolated singular point $0 \in V$. Then for all $\mathcal{P}' \in \operatorname{Spec}(A_{\mathcal{M}}) \setminus \{\mathcal{M}A_{\mathcal{M}}\}$, we have the inequality*

$$\operatorname{pd}_{(A_{\mathcal{M}})^e_{\mathcal{P}'}} (A_{\mathcal{M}})_{\mathcal{P}'} \leqq d.$$

Proof. Let $\mathcal{P}' \in \operatorname{Spec}(A_{\mathcal{M}}) \setminus \{\mathcal{M}A_{\mathcal{M}}\} = \operatorname{Reg}(A_{\mathcal{M}})$. Then $\mathcal{P}' = \mathcal{P}A_{\mathcal{M}}$ for some $\mathcal{P} \in \operatorname{Reg}(A)$. Applying Proposition 10 we obtain that

$$\operatorname{pd}_{(A_{\mathcal{M}})^e_{\mathcal{P}'}} (A_{\mathcal{M}})_{\mathcal{P}'} = \operatorname{pd}_{A^e_{\mathcal{P}}} A_{\mathcal{P}} \leqq d. \quad \text{q.e.d.}$$

For the next step we adopt a more general setting. We assume now that S is any commutative noetherian local Cohen–Macaulay ring, with $\mathcal{M} = \operatorname{rad} S$ and $d = \dim S$, subject to the following conditions.

(a) S contains an algebraically closed field k such that $S^e = S \otimes_k S$ is noetherian.

(b) $\operatorname{Sing}(S) = \{M\}$.

(c) $\operatorname{pd}_{S^e_{\mathcal{P}}} S_{\mathcal{P}} \leqq d$, for all $\mathcal{P} \in \operatorname{Reg}(S)$.

We denote the class of all such rings S by $\mathcal{S}$. If $V \subset \mathbb{A}^n(k)$ is an affine algebraic variety and $0 \in V$ is an isolated Cohen–Macaulay singularity, as above, then the local ring $\mathcal{O}_{V,0}$ of V at 0 belongs to $\mathcal{S}$, due to Corollary 11. For any local ring S in $\mathcal{S}$, let $\hat{S}$ be the completion of S and set $\hat{\mathcal{M}} = \operatorname{rad} \hat{S}$. Further, let $0 \to J \to S^e \xrightarrow{\varepsilon} S \to 0$ be the augmentation sequence of S^e, and set $M = \Omega^d(S)$ (the d-th *syzygy* of S in $\operatorname{mod} S^e$). Finally, let $\mathcal{A}$ be the ideal in S which is given by $\mathcal{A} = \operatorname{ann}_S (\underline{\operatorname{End}}_{S^e} M / J\, \underline{\operatorname{End}}_{S^e} M)$.

PROPOSITION 12. *For any local ring S in $\mathcal{S}$, the ideal $\mathcal{A} \subset S$ has the following properties.*

(i) *$\mathcal{M}^n \subset \mathcal{A}$, for some $n \in \mathbb{N}$.*

(ii) *$\mathcal{A} \operatorname{Ext}^{d+1}_S (X, Y) = 0$, for all $X, Y \in \operatorname{Mod} S$.*

Proof. For brevity we write $E_M = \operatorname{End}_{S^e} M$, $E'_M = \{\phi \in E_M \mid \phi$ factors through a projective S^e-module$\}$, and $\underline{E}_M = E_M / E'_M = \underline{\operatorname{End}}_{S^e} M$. Since $\underline{E}_M / J\underline{E}_M \cong E_M / (JE_M + E'_M)$, we have $\mathcal{A}E_M \subset JE_M + E'_M$. For any $\mathcal{P} \in \operatorname{Reg}(S)$ set $T = T(\mathcal{P}) = \{s \otimes t \mid s, t \in S \setminus \mathcal{P}\} \subset S^e$. Then $T^{-1}M \cong \Omega^d(S_{\mathcal{P}})$. From property (c) of S we obtain that $T^{-1}M$ is a projective $T^{-1}S^e$-module, and hence that $T^{-1}(\underline{E}_M) = 0$, for all $\mathcal{P} \in \operatorname{Reg}(S)$.

We now turn to the proof of (i) and (ii). For all $\mathscr{P} \in \mathrm{Reg}(S)$ we have that $(E_M/JE_M)_{\mathscr{P}} \cong T^{-1}(E_M/JE_M) \cong T^{-1}(E_M)/T^{-1}(JE_M) = 0$. Hence $\mathrm{supp}_S(E_M/JE_M) \subset \{\mathscr{M}\}$, and therefore $\mathscr{M}^n \subset \mathrm{ann}_S(E_M/JE_M) = \mathscr{A}$, for some $n \in \mathbb{N}$.

For all $X, Y \in \mathrm{Mod}\, S$ we have isomorphisms $\mathrm{Ext}_S^{d+1}(X, Y) \cong \mathrm{Ext}_{S^e}^{d+1}(S, \mathrm{Hom}_k(X, Y)) \cong \mathrm{Ext}_{S^e}^1(M, \mathrm{Hom}_k(X, Y))$ [Ca/Ei 56, IX, Corollary 4.4]. Using the composed isomorphism and the inclusion $\mathscr{A}E_M \subset JE_M + E'_M$ we obtain that $\mathscr{A}\,\mathrm{Ext}_S^{d+1}(X, Y) \cong \mathscr{A}\,\mathrm{Ext}_{S^e}^1(M, \mathrm{Hom}_k(X, Y)) \subset J\,\mathrm{Ext}_{S^e}^1(M, \mathrm{Hom}_k(X, Y)) + E'_M\,\mathrm{Ext}_{S^e}^1(M, \mathrm{Hom}_k(X, Y))$. In this sum, the first summand is isomorphic to $J\,\mathrm{Ext}_S^{d+1}(X, Y)$, hence zero, whereas the second summand vanishes by definition of E'_M. q.e.d.

We continue to work over rings S in $\mathscr{S}$. However, we point out that the following Lemma 13 is valid for any commutative noetherian local ring S.

LEMMA 13. (i) $\mathrm{Hom}_{\hat{S}}(X, Y) = \mathrm{Hom}_S(X, Y)$, *for all* $X, Y \in \mathrm{Mod}\,\hat{S}$.

(ii) $\mathrm{Ext}_S^i(P, Y) = 0$, *for all* $i \in \mathbb{N}$ *and* $P, Y \in \mathrm{mod}\,\hat{S}$, *with* P *a projective* $\hat{S}$*-module.*

(iii) $\mathrm{Ext}_{\hat{S}}^i(X, Y) = \mathrm{Ext}_S^i(X, Y)$, *for all* $i \in \mathbb{N}_0$ *and* $X, Y \in \mathrm{mod}\,\hat{S}$.

Proof. (i) Let $X, Y \in \mathrm{Mod}\,\hat{S}$. Clearly we have an inclusion $\mathrm{Hom}_{\hat{S}}(X, Y) \hookrightarrow \mathrm{Hom}_S(X, Y)$, given by restriction of scalars, and we have to show surjectivity of this inclusion. So let $\phi \in \mathrm{Hom}_S(X, Y)$, $x \in X$ and $\hat{s} \in \hat{S}$. Since S is dense in $\hat{S}$, there exists a convergent sequence $(s_i)_{i\in\mathbb{N}} \subset S$ such that $\lim_{i\to\infty} s_i = \hat{s}$. Then

$$\phi(\hat{s}x) = \phi\left(\left(\lim_{i\to\infty} s_i\right)x\right) = \phi\left(\lim_{i\to\infty}(s_i x)\right) = \lim_{i\to\infty}\phi(s_i x)$$

$$= \lim_{i\to\infty}(s_i\phi(x)) = \left(\lim_{i\to\infty} s_i\right)\phi(x) = \hat{s}\phi(x).$$

Hence $\phi \in \mathrm{Hom}_{\hat{S}}(X, Y)$.

(ii) Let $i \in \mathbb{N}$ and $P, Y \in \mathrm{mod}\,\hat{S}$, with P a projective $\hat{S}$-module. Then for all $B, I \in \mathrm{Mod}\,\hat{S}$, with I an injective $\hat{S}$-module, we have $\mathrm{Ext}_S^i(P, \mathrm{Hom}_{\hat{S}}(B, I)) \cong \mathrm{Hom}_{\hat{S}}(\mathrm{Tor}_i^S(P, B), I)$ [Ca/Ei 56, VI, Proposition 5.1]. Since P is a flat S-module, $\mathrm{Tor}_i^S(P, B) = 0$ and therefore $\mathrm{Ext}_S^i(P, \mathrm{Hom}_{\hat{S}}(B, I)) = 0$. Now if we choose I to be the injective hull of $\hat{S}/\hat{\mathscr{M}}$ and $B = \mathrm{Hom}_{\hat{S}}(Y, I)$, then $\mathrm{Hom}_{\hat{S}}(B, I) = \mathrm{Hom}_{\hat{S}}(\mathrm{Hom}_{\hat{S}}(Y, I) \cong Y$. Therefore $\mathrm{Ext}_S^i(P, Y) = 0$.

(iii) Given $X, Y \in \mathrm{mod}\,\hat{S}$, let $\mathbb{P}: \cdots \to P_1 \to P_0 \to X \to 0$ be an $\hat{S}$-projective resolution of X, and let $\mathbb{P}_1: \cdots \to P_2 \to P_1 \to \Omega(X) \to 0$ be the $\hat{S}$-projective resolution of $\Omega(X)$, obtained from $\mathbb{P}$ by shifting. Denote by $\mathbb{P}'$, respectively $\mathbb{P}'_1$, the acyclic S-complex which arises from $\mathbb{P}$, respectively $\mathbb{P}_1$, by restriction of scalars. Then for all $i \in \mathbb{N}$, $\operatorname{Ext}^i_{\hat{S}}(X, Y) = H^i(\operatorname{Hom}_{\hat{S}}(\mathbb{P}, Y)) = H^i(\operatorname{Hom}_S(\mathbb{P}', Y))$, by (i). It remains to show that $\operatorname{Ext}^i_S(X, Y) \cong H^i(\operatorname{Hom}_S(\mathbb{P}', Y))$, for all $i \in \mathbb{N}$.

Applying $\operatorname{Hom}_S(\ , Y)$ to the short exact sequence $0 \to \Omega(X) \xrightarrow{\iota} P_0 \to X \to 0$, we obtain the long exact sequence $0 \to \operatorname{Hom}_S(X, Y) \to \operatorname{Hom}_S(P_0, Y) \to \operatorname{Hom}_S(\Omega(X), Y) \to \operatorname{Ext}^1_S(X, Y) \to \operatorname{Ext}^1_S(P_0, Y) \to \cdots$. By (ii), $\operatorname{Ext}^i_S(P_0, Y) = 0$ for all $i \in \mathbb{N}$. Therefore $\operatorname{Ext}^1_S(X, Y) \cong \operatorname{Hom}_S(\Omega(X), Y)/\operatorname{Hom}_S(P_0, Y)\iota \cong H^1(\operatorname{Hom}_S(\mathbb{P}', Y))$. Moreover, if $\operatorname{Ext}^i_S(X, Y) \cong H^i(\operatorname{Hom}_S(\mathbb{P}', Y)$ for some $i \in \mathbb{N}$, then $\operatorname{Ext}^{i+1}_S(X, Y) = \operatorname{Ext}^i_S(\Omega(X), Y) \cong H^i(\operatorname{Hom}_S(\mathbb{P}'_1, Y)) = H^{i+1}(\operatorname{Hom}_S(\mathbb{P}', Y))$. Hence $\operatorname{Ext}^i_S(X, Y) \cong H^i(\operatorname{Hom}_S(\mathbb{P}', Y))$ for all $i \in \mathbb{N}$, by induction on i. q.e.d.

PROPOSITION 14. *For any local ring S in $\mathcal{S}$, the ideal $\hat{\mathcal{A}} = \mathcal{A}\hat{S} \subset \hat{S}$ has the following properties.*

(i) *$\hat{\mathcal{A}}$ is $\hat{\mathcal{M}}$-primary.*

(ii) *$\hat{\mathcal{A}} \operatorname{Ext}^1_{\hat{S}}(X, Y) = 0$ for all $X, Y \in \operatorname{mod} \hat{S}$, with X Cohen–Macaulay.*

Proof. Choose a Noether normalization $R \subset \hat{S}$. Then, since $\hat{S}$ is Cohen–Macaulay and $\operatorname{Sing}(\hat{S}) = \{\hat{\mathcal{M}}\}$, the R-algebra $\hat{S}$ is an isolated singularity in the sense of section 0. In particular, $\hat{S}$ is an R-order in the sense of [Au 78], and the category of Cohen–Macaulay $\hat{S}$-modules coincides with the category of $\hat{S}$-lattices in the sense of [Au 78]. Therefore, for each Cohen–Macaulay $\hat{S}$-module X we have that $\operatorname{Ext}^i_{\hat{S}}(\operatorname{Tr} X, \hat{S}) = 0$ for all $i = 1, \ldots, d$, where $\operatorname{Tr} X$ denotes the transpose of X [Au 78, Proposition 7.5]. Thus X is d-torsionfree in the language of [Au/Br 69], and hence there exists $X' \in \operatorname{mod} \hat{S}$ such that $X \cong \Omega^d(X')$, by [Au/Br 69, Theorem 2.17].

Now let $X, Y \in \operatorname{mod} \hat{S}$, with X Cohen–Macaulay, and let $X' \in \operatorname{mod} \hat{S}$ such that $X \cong \Omega^d(X')$. Then, applying Proposition 12 and Lemma 13, (iii), we obtain that

$$\hat{\mathcal{A}} \operatorname{Ext}^1_{\hat{S}}(X, Y) \cong \hat{\mathcal{A}} \operatorname{Ext}^1_{\hat{S}}(\Omega^d(X'), Y) \cong \hat{\mathcal{A}} \operatorname{Ext}^{d+1}_{\hat{S}}(X', Y) \cong \mathcal{A} \operatorname{Ext}^{d+1}_S(X', Y) = 0.$$

This proves (ii).

It follows that $\hat{\mathcal{A}} \subset \hat{\mathcal{M}}$, because otherwise $\hat{S}$ would be nonsingular. Moreover, $\hat{\mathcal{M}}^n \subset \hat{\mathcal{A}}$ for some $n \in \mathbb{N}$, by Proposition 12 and by definition of $\hat{\mathcal{A}}$. Therefore $\hat{\mathcal{A}}$ is $\hat{\mathcal{M}}$-primary, which proves (i). q.e.d.

For the remainder of this subsection we turn to isolated Cohen–Macaulay singularities of the form $\Lambda = k[[X_1, \ldots, X_n]]/I$, I an ideal in $k[[X_1, \ldots, X_n]]$, and draw the main conclusions from Proposition 14.

COROLLARY 15. *Let $I \subset k[[X_1, \ldots, X_n]]$ be an ideal, such that $\Lambda = k[[X_1, \ldots, X_n]]/I$ is an isolated Cohen–Macaulay singularity. Then the* Ext-*annihilating ideal $\mathfrak{a}$ of Λ in R is $\mathfrak{m}$-primary.*

Proof. By Artin's Theorem [Ar 69, Theorem 3.8] there exists an affine-algebraic isolated singularity $(V, 0)$ such that $\Lambda \cong \hat{\mathcal{O}}_{V,0}$. From Corollary 11 we know that $\mathcal{O}_{V,0} \in \mathcal{S}$. By Proposition 14, there exists $n \in \mathbb{N}$ such that $\hat{\mathcal{M}}^n \subset \hat{\mathcal{A}}$ and $\mathfrak{m}^n = (\hat{\mathcal{M}} \cap R)^n \subset (\hat{\mathcal{M}}^n \cap R) \subset (\hat{\mathcal{A}} \cap R) \subset \mathfrak{a}$. Moreover, $\mathfrak{a} \subset \mathfrak{m}$, because otherwise Λ would be nonsingular. Hence $\mathfrak{a}$ is $\mathfrak{m}$-primary. q.e.d.

The following two Theorems are immediate consequences of Corollary 15, Theorem 7, and Corollary 3 respectively Corollary 4.

THEOREM 16. *Let $I \subset k[[X_1, \dots, X_n]]$ be an ideal, such that $\Lambda = k[[X_1, \dots, X_n]]/I$ is an isolated Cohen–Macaulay singularity. Then the first Brauer–Thrall conjecture is true for Λ.*

THEOREM 17. *Let $I \subset k[[X_1, \dots, X_n]]$ be an ideal, such that $\Lambda = k[[X_1, \dots, X_n]]/I$ is an isolated Cohen–Macaulay singularity. Let $\mathcal{C} = (\mathcal{C}_0, \mathcal{C}_1)$ be a connected component of the stable Auslander–Reiten quiver $\mathcal{A}_s(\Lambda)$, and assume that $\mathcal{C}_0$ contains a periodic point. Then the Cartan class of $\mathcal{C}$ is either a Dynkin diagram or $\mathbb{A}_\infty$.*

3.3. *Isolated hypersurface singularities*

Throughout this subsection we assume that k is an algebraically closed field, and $f(X) = f(X_0, \dots, X_d)$ is a polynomial in $k[X_0, \dots, X_d]$ such that the hypersurface $H \subset \mathbb{A}^{d+1}(k)$ defined by $f(X)$ has an isolated singularity at 0. Let $\Lambda = \hat{\mathcal{O}}_{H,0} \cong k[[X_0, \dots, X_d]]/(f(X))$ be the complete local ring of H at 0, and let $R \subset \Lambda$ be a chosen Noether normalization. We call such an R-algebra Λ an *isolated hypersurface singularity*.

Because an isolated hypersurface singularity is an affine-algebraic isolated singularity in the sense of section 3.2, Theorems 16 and 17 are true for isolated hypersurface singularities. However, it is interesting to see that for isolated hypersurface singularities there is a much straighter way of deducing Theorems 16 and 17. Namely we have the following result which has been pointed out to me by G.-M. Greuel and F.-O. Schreyer.

PROPOSITION 18. *Let $\Lambda = k[[X_0, \dots, X_d]]/(f(X))$ be an isolated hypersurface singularity, with unique maximal ideal $\mathcal{M}$. Let $J_f = (\partial f/\partial X_0, \dots, \partial f/\partial X_d)$ be the Jacobi ideal of $f(X)$ in $k[[X_0, \dots, X_d]]$, and let $\mathcal{J}_f = (J_f + (f(X)))/(f(X))$ be its image in Λ. Then the ideal $\mathcal{J}_f$ has the following properties.*

(i) *$\mathcal{J}_f$ is $\mathcal{M}$-primary.*

(ii) *$\mathcal{J}_f \operatorname{Ext}^1_\Lambda(C, Y) = 0$, for all $C \in \operatorname{mod}_R \Lambda$ and $Y \in \operatorname{mod} \Lambda$.*

(Assertion (i) follows from Jacobi's Criterion because Λ is an isolated singularity. Assertion (ii) can easily be proved by calculating $\operatorname{Ext}^1_\Lambda(C, Y)$ via the Λ-projective resolution of C which is given by the matrix factorization of $f(X)$ corresponding to C [Ei 80, § 6]. We leave the details to the interested reader). Now Corollary 15 and Theorems 16 and 17 for isolated hypersurface singularities follow from Proposition 18 in the same way as for affine-algebraic isolated singularities they follow from Proposition 14.

We proceed to show that for isolated hypersurface singularities Theorems 16 and 17 can be strengthened considerably.

THEOREM 19. *Let $\Lambda = k[[X_0, \ldots, X_d]]/(f(X))$ be an isolated hypersurface singularity of dimension d which is of infinite type. Then the following statements hold.*

(i) *The Auslander–Reiten quiver of Λ is of the form $\mathscr{A}(\Lambda) \cong \mathscr{C} \,\dot\cup\, (\dot\bigcup_{i\in I} \mathbb{Z}\mathbb{A}_\infty/\langle \tau^{n(i)}\rangle)$, where $\mathscr{C}$ is the connected component of $\mathscr{A}(\Lambda)$ which contains $[\Lambda]$, I is an index set, and $n(i) \in \{1, 2\}$ for all $i \in I$. Moreover, if d is even then $n(i) = 1$ for all $i \in I$.*

(ii) *The full subquiver of $\mathscr{C}$ which consists of all points different from $[\Lambda]$ is of the form $\mathscr{C}_s \cong \dot\bigcup_{j\in J} \mathbb{Z}\Delta_j/G_j$, where J is a finite index set and for all $j \in J$, Δ_j is either a Dynkin diagram or $\mathbb{A}_\infty$ and G_j is a group of automorphisms of $\mathbb{Z}\Delta_j$.*

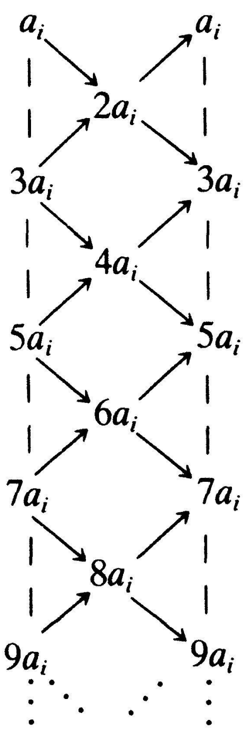
(iii) *If there is only one direct predecessor of $[\Lambda]$ in $\mathscr{A}(\Lambda)$, then the stable Auslander–Reiten quiver of Λ is of the form $\mathscr{A}_s(\Lambda) \cong \dot\bigcup_{i\in \hat I} \mathbb{Z}\mathbb{A}_\infty/\langle \tau^{n(i)}\rangle$, where $\hat I = I \,\dot\cup\, \{\iota\}$, and $n(i) \in \{1, 2\}$ for all $i \in \hat I$. Moreover, if d is even then $n(i) = 1$ for all $i \in \hat I$.*

Proof. Let M be any indecomposable nonprojective object in $\operatorname{mod}_R \Lambda$. Since Λ is a Gorenstein R-order in the sense of [Au 78], we have that $\tau([M]) = [\Omega^{2-d}(M)]$, by [Au 78, III, Proposition 1.8]. On the other hand, because Λ is a hypersurface, we also have that $[\Omega^2(M)] = [M]$, by [Ei 80, Theorem 6.1]. Therefore $\tau^2([M]) = [M]$, and if d is even then $\tau([M]) = [M]$. Then, since Λ is assumed to be of infinite type, Theorem 19 follows from Theorem 17 and Proposition 2. q.e.d.

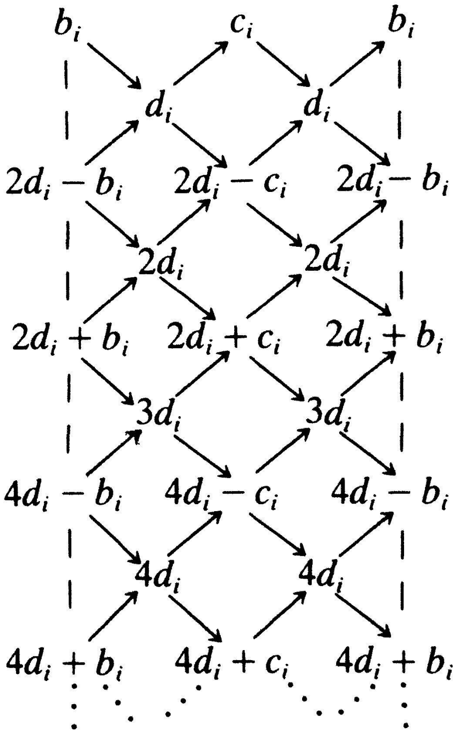
Remark. In the situation of Theorem 19 it is natural to ask for the index set I, the function $n: I \to \{1, 2\}$ and the component $\mathscr{C}$ associated with Λ. Knowledge of these data solves the classification problem of $\operatorname{mod}_R \Lambda$. It seems that a solution of this problem requires methods which are fundamentally different from those used in the present article. So far there is just one case in which a complete answer to this question is known: Let $\Lambda = \mathbb{C}[[X, Y]]/(f(X, Y))$ be the simple elliptic curve singularity of type $\tilde{\mathbb{E}}_8$, given by $f(X, Y) = Y(Y - X^2)(Y - aX^2)$, with $a \in \mathbb{C}\backslash\{0, 1\}$. Then there is only one direct predecessor of $[\Lambda]$ in $\mathscr{A}(\Lambda)$ and $\mathscr{A}_s(\Lambda) \cong \dot{\bigcup}_{i\in\hat{I}} \mathbb{Z}\mathbb{A}_\infty/\langle\tau^{n(i)}\rangle$, where $\hat{I} = \mathbb{P}^1(\mathbb{Q}) \times \mathbb{P}^1(\mathbb{C})$. Moreover, there are four "exceptional parameters" $\varepsilon_1, \ldots, \varepsilon_4$ in $\mathbb{P}^1(\mathbb{C})$ such that $n(i) = 1$ for all $i \in \mathbb{P}^1(\mathbb{Q}) \times (\mathbb{P}^1(\mathbb{C})\backslash\{\varepsilon_1, \ldots, \varepsilon_4\})$ and $n(i) = 2$ for all $i \in \mathbb{P}^1(\mathbb{Q}) \times \{\varepsilon_1, \ldots, \varepsilon_4\}$ [Di 85].

THEOREM 20. *The second Brauer–Thrall conjecture is true for isolated hypersurface singularities* $\Lambda = k[[X_0, \ldots, X_d]]/(f(X))$, *with* $\operatorname{char} k \neq 2$.

Proof. Let Λ be an isolated hypersurface singularity of infinite type, and let $\rho: [\operatorname{ind}_R \Lambda] \to \mathbb{N}$ be the rank function. Denote by $\mathscr{C}$ the unique component of $\mathscr{A}(\Lambda)$ which contains $[\Lambda]$. Then we know from Theorem 19 that $\mathscr{C}_s = \dot{\bigcup}_{j\in J} \mathscr{C}(j)$, where J is a finite index set and for all $j \in J$, $\mathscr{C}(j) \cong \mathbb{Z}\Delta_j/G_j$ with Δ_j either a Dynkin diagram or $\mathbb{A}_\infty$ and G_j a group of automorphisms of $\mathbb{Z}\Delta_j$. Note that if $\Delta_j = \mathbb{A}_\infty$, then ρ is unbounded on $\mathscr{C}(j) \cong \mathbb{Z}\mathbb{A}_\infty/\langle\tau^{n(j)}\rangle$, $n(j) \in \{1, 2\}$. (This follows from the structure of $\mathscr{C}(j)$ together with Lemma 1, by the same reasoning as in the proof of Proposition 2.) Moreover $\mathscr{A}(\Lambda) = \mathscr{C} \mathbin{\dot{\cup}} (\dot{\bigcup}_{i\in I} \mathscr{C}(i))$, where I is an index set and $\mathscr{C}(i) \cong \mathbb{Z}\mathbb{A}_\infty/\langle\tau^{n(i)}\rangle$ with $n(i) \in \{1, 2\}$, for all $i \in I$. We write $I_1 = \{i \in I \mid n(i) = 1\}$, $I_2 = \{i \in I \mid n(i) = 2\}$, $\mathscr{D}' = \dot{\bigcup}_{i\in I_1} \mathscr{C}(i)$, $\mathscr{D}'' = \dot{\bigcup}_{i\in I_2} \mathscr{C}(i)$, and $\mathscr{D} = \mathscr{D}' \mathbin{\dot{\cup}} \mathscr{D}'' = \dot{\bigcup}_{i\in I} \mathscr{C}(i)$. Since ρ is additive on $\mathscr{D}$, its values on each of the components $\mathscr{C}(i)$, $i \in I$, are given as follows. (We set $d_i = b_i + c_i$. Identify along the interrupted lines.)



$\mathscr{C}(i), n(i) = 1$



$\mathscr{C}(i), n(i) = 2$

We distinguish the following subsets of $\mathscr{D}_0 : A = (\mathscr{D}')_0$,

$$D = \{x \in (\mathscr{D}'')_0 \mid \rho(x) \in \{md_i \mid m \in \mathbb{N}, i \in I_2\}\},$$

$$M = \{x \in (\mathscr{D}'')_0 \mid \rho(x) \in \{b_i, c_i \mid i \in I_2\}\},$$

$$T = \{x \in (\mathscr{D}'')_0 \mid \rho(x) \in \{2md_i - b_i, 2md_i - c_i, 2md_i + b_i, 2md_i + c_i \mid m \in \mathbb{N}, i \in I_2\}\}.$$

Observe that $\mathscr{D}_0 = A \dot\cup D \dot\cup M \dot\cup T$. In the sequel, by an "infinite set $\{x_\nu\}$ of constant rank" we mean a subset $\{x_\nu \mid \nu \in \mathbb{N}\}$ of $[\mathrm{ind}_R \Lambda]$, whose elements x_ν are pairwise different, such that $\rho(x_\nu) = r$ for a fixed $r \in \mathbb{N}$ and for all $\nu \in \mathbb{N}$. We shall need the following auxiliary result.

(*) If there exists an infinite set $\{x_\nu\}$ of constant rank in $[\mathrm{ind}_R \Lambda]$, then there exists an infinite set $\{y_\nu\}$ of constant rank in $A \cup D$.

Proof of (*): Let $\{x_\nu\}$ be an infinite set of constant rank in $[\mathrm{ind}_R \Lambda]$. For all $j \in J$ with Δ_j a Dynkin diagram, $\mathscr{C}(j)$ is finite. For all $j \in J$ with $\Delta_j = \mathbb{A}_\infty$, ρ is unbounded on $\mathscr{C}(j)$ and is additive on a cofinite full subquiver of $\mathscr{C}(j) \cong \mathbb{Z}\mathbb{A}_\infty/\langle \tau^{n(j)} \rangle$, $n(j) \in \{1, 2\}$. Moreover, J is finite. Therefore $\{x_\nu\} \cap \mathscr{C}_0$ is finite. Hence, choosing a suitable subset of $\{x_\nu\}$, we obtain an infinite set $\{y_\nu\} \subset \mathscr{D}_0$ of constant rank. If $\{y_\nu\} \cap (A \cup D)$ is infinite, then (*) is proved. So assume that $\{y_\nu\} \cap (A \cup D)$ is finite. Then either $\{t_\nu\} = \{y_\nu\} \cap T$ is an infinite set of constant rank r, or $\{m_\nu\} = \{y_\nu\} \cap M$ is an infinite set of constant r. In the first case, let t'_ν be the unique direct predecessor of t_ν such that $\rho(t'_\nu) < r$. Then $\{t'_\nu\} \subset D$ is an infinite set of bounded rank, and therefore there exists an infinite subset $\{t''_\nu\} \subset \{t'_\nu\}$ such that $\{t''_\nu\}$ is of constant rank. In the second case, let m'_ν be the unique direct predecessor of m_ν. Then $\{m'_\nu\} \subset D$ and, because τ is given by Ω (see proof of Theorem 19), we have that $\rho(m'_\nu) \leqq r \cdot \rho(\Lambda)$. Hence $\{m'_\nu\}$ is an infinite set of bounded rank, and therefore there exists an infinite subset $\{m''_\nu\} \subset \{m_\nu\}$ such that $\{m''_\nu\}$ is of constant rank. This proves (*).

Now let Λ be an isolated hypersurface singularity of infinite type, with char $k \neq 2$. It is proven in [Bu/Gr/Schr 86] that there exists an infinite set $\{x_\nu\}$ of constant rank in $[\mathrm{ind}_R \Lambda]$. By (*) we conclude that there exists an infinite set $\{y_\nu\}$ of constant rank r in $A \cup D$. Now the structure of the components $\mathscr{C}(i)$, as pictured above, shows that for each $y_\nu = y_\nu^{(1)} \in (A \cup D) \cap \mathscr{C}(i)$ there exists a sequence $(y_\nu^{(\mu)})_{\mu \in \mathbb{N}} \subset (A \cup D) \cap \mathscr{C}(i)$ such that $\rho(y_\nu^{(\mu)}) = \mu r$, for all $\mu \in \mathbb{N}$. Therefore we obtain an infinite sequence $\{y_\nu^{(1)}\}, \{y_\nu^{(2)}\}, \{y_\nu^{(3)}\}, \ldots$ of infinite sets such that $\{y_\nu^{(\mu)}\}$ is of constant rank μr, for all $\mu \in \mathbb{N}$. q.e.d.

Remark. Theorem 20 remains valid in characteristic 2, if in addition either $d = 1$ or $\mathrm{mult}(f) \geqq 3$. Namely in this situation, if Λ is of infinite type, then there exists an infinite set $\{x_\nu\}$ of constant rank in $[\mathrm{ind}_R \Lambda]$ (see [Bu/Gr/Schr 86, proofs 3.1 and 3.5]), and we can continue to argue as above.

REFERENCES

[Ar 69] M. ARTIN, *Algebraic approximation of structures over complete local rings.* Publ. IHES *36*, 23–58 (1969).

[Au 74] M. AUSLANDER, *Representation theory of artin algebras II.* Comm. Alg. *1* (1974), 269–310.

[Au 78] M. AUSLANDER, *Functors and morphisms determined by objects.* Proceedings of a conference on representation theory, Philadelphia (1976). Marcel Dekker (1978), 1–245.

[Au 84] M. AUSLANDER, *Isolated singularities and existence of almost split sequences.* Notes by L. Unger. Proceedings of the fourth international conference on representations of algebras, Carleton University, Ottawa 84. Springer Lecture Notes 1178, 194–242.

[Au 86] M. AUSLANDER, *Rational singularities and almost split sequences.* Transactions of the American Math. Soc., *293*, 2 (1986), 511–531.

[Au/Br 69] M. AUSLANDER and M. BRIDGER, *Stable module theory.* Memoirs of the American Math. Soc., number 94, 1969.

[Bu/Gr/Schr 86] R. BUCHWEITZ and G.-M. GREUEL and F.-O. SCHREYER, *Cohen–Macaulay modules over hypersurface singularities II.* To appear.

[Ca/Ei 56] H. CARTAN and S. EILENBERG, *Homological algebra.* Princeton University Press, 1956.

[Cu/Re 81] C. W. CURTIS and I. REINER, *Methods of Representation Theory I,* Pure and applied mathematics (Wiley, London 1981).

[Di 85] E. DIETERICH, *Classification of the indecomposable representations of the cyclic group of order three in a complete discrete valuation ring of ramification degree four.* Preprint, Univ. Bielefeld, 1985.

[Di 86] E. DIETERICH, *The Auslander–Reiten quiver of an isolated singularity.* Proceedings of a conference on singularities, representations of algebras and vector bundles. Lambrecht, 1985. To appear in Springer Lecture Notes in Mathematics.

[Di/Wi 86] E. DIETERICH and A. WIEDEMANN: *The Auslander–Reiten quiver of a simple curve singularity.* Transactions of the American Math. Soc., *294*, 2 (1986), 455–475.

[Ei 80] D. EISENBUD, *Homological algebra on a complete intersection, with an application to group representations.* Transactions of the American Math. Soc., *260*, 1 (1980), 35–64.

[Gr/Kn 85] G.-M. GREUEL and H. KNÖRRER, *Einfache Kurvensingularitäten und torsionsfreie Moduln.* Math. Annalen *270* (1985), 417–425.

[Hap/Pr/Rin 79] D. HAPPEL and U. PREISER and C. M. RINGEL, *Vinberg's characterization of Dynkin diagrams using subadditive functions with application to DTr-periodic modules.* Proceedings of the third international conference on representations of algebras, Carleton University, Ottawa 1979. Springer Lecture Notes 832, 280–294.

[Hap/Pr/Rin 80] D. HAPPEL and U. PREISER and C. M. RINGEL, *Binary polyhedral groups and Euclidian diagrams.* Manuscripta Mathematica *31* (1980), 317–329.

[Har/Sa 70] M. HARADA and Y. SAI, *On categories of indecomposable modules I.* Osaka Journal of Math. *7* (1970), 323–344.

[He 78] J. HERZOG, *Ringe mit nur endlich vielen Isomorphieklassen von maximalen, unzerlegbaren Cohen–Macaulay Moduln.* Math. Annalen *233* (1978), 21–34.

[He/Sa 85] J. HERZOG and H. SANDERS, *Indecomposable syzygy modules of high rank on hypersurface rings.* To appear in Journal of Pure and Applied Algebra.

[Hi 60] D. G. HIGMAN, *On representations of orders over Dedekind domains.* Canadian Journal of Math. *12* (1960), 107–125.

[Kn 85] H. KNÖRRER, *Cohen–Macaulay modules on hypersurface singularities I.* Preprint, Universität Bonn, 1985.

[Ma 53] J. M. MARANDA, *On p-adic integral representations of finite groups.* Canadian Journal of Math. *5* (1953), 344–355.

[Rie 80] CHR. RIEDTMANN, *Algebren, Darstellungsköcher, Überlagerungen, und zurück.* Commentarii Math. Helv. *55* (1980), 199–224.

[Rin 79] C. M. RINGEL, *Report on the Brauer–Thrall conjectures.* Proceedings of the third international conference on representations of algebras, Carleton University, Ottawa 1979. Springer Lecture Notes 831, 104–136.

[Ro/Hu 70] K. W. ROGGENKAMP and V. HUBER-DYSON, *Lattices over orders I,* Springer Lecture Notes 115; II, Springer Lecture Notes 142.

Note added in proof

After finishing this article it has turned out that Y. Yoshino also has studied reductions of isolated singularities, simultaneously and independently. He considers the case where Λ is a commutative noetherian complete local Cohen–Macaulay ring which is an algebra over a perfect valuation field and an isolated singularity. Denote the class of all such algebras by $\mathscr{A}$. He shows that $\sum_R \mathscr{N}_R^\Lambda$ is an Ext-annihilating ideal for Λ, where R ranges over all Noether normalizations of Λ and $\mathscr{N}_R^\Lambda$ denotes the Noether different. From this he proves that the first Brauer–Thrall conjecture is true for $\mathscr{A}$. [Yuji Yoshino: Brauer–Thrall type theorem for maximal Cohen–Macaulay modules. Preprint, Nagoya University, 1986].

This generalizes Theorem 16 to the wider class of isolated singularities $\mathscr{A}$. Combining Yoshino's approach with Corollary 4, it is clear that Theorem 17 also holds more generally for all algebras $\Lambda \in \mathscr{A}$.

Moreover, it has been brought to my attention that K. W. Roggenkamp and A. Wiedemann also investigated generalizations of Maranda's Theorem, and obtained results which are related to Theorem 7 (unpublished).

Brandeis University
Department of Mathematics
Waltham, Mass. 02254
USA

Current address:
Max-Planck-Institut für Mathematik
Gottfried-Claren-Straße 26
D-5300 Bonn 3
FRG

Received July 30, 1986/December 30, 1986