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## The non-vanishing of the deviations of a local ring

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Let  $R$  be a local noetherian ring with maximal ideal  $m$  and residue field  $\mathbf{k}$ . Then (cf. [3], [6])  $\text{Tor}_*^R(\mathbf{k}, \mathbf{k})$  has the structure of a free divided powers algebra on a graded  $\mathbf{k}$ -vector space  $V = \bigoplus_i V_i$ . In particular the Poincaré series for  $R$  has the form

$$\sum_{i=0}^{\infty} [\dim \text{Tor}_i^R(\mathbf{k}, \mathbf{k})] t^i = \frac{\prod_{j=1}^{\infty} (1 + t^{2j+1})^{\dim V_{2j+1}}}{\prod_{j=1}^{\infty} (1 - t^{2j})^{\dim V_{2j}}} \tag{1}$$

The integers  $e_j = \dim V_j$  are called the *deviations* of  $R$ . The equation above shows they are completely determined by the betti numbers  $\dim \text{Tor}_i^R(\mathbf{k}, \mathbf{k})$ , and conversely. Moreover ([1], [11], [12]) the Yoneda Ext-algebra,  $\text{Ext}_R^*(\mathbf{k}, \mathbf{k})$ , is naturally the universal enveloping algebra of a graded Lie algebra  $L_R$  dual to  $V$ , and hence

$$e_i = \dim L_R^i, \quad \text{all } i.$$

Let  $\hat{R}$  denote the completion of  $R$  with respect to the powers of  $m$ . By the Cohen structure theorem,  $\hat{R}$  has the form  $\tilde{R}/I$  where  $\tilde{R}$  is a regular local noetherian ring (with maximal ideal  $\tilde{m}$ ) and  $I \subset \tilde{m}^2$ . We call  $R$  a *weak complete intersection* if  $I$  is generated by a regular sequence.

Now in [3] Assmus proves the following

**THEOREM A (Assmus).** *The following conditions are equivalent:*

- (i)  $R$  is a weak complete intersection.
- (ii)  $e_j = 0, j \geq 3$ .
- (iii)  $e_3 = 0$ .

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This raised the question of whether or not any deviation could vanish if  $R$  was not a weak complete intersection.

A first step was taken by Gulliksen [7], [8] who showed that infinitely many  $e_{2k}$  must be nonzero for non weak complete intersections. Subsequently Avramov and Halperin [5] showed that only finitely many  $e_i$  could be zero. Moreover in special cases (eg. Jacobsson [9], Löfwall [10]) it was known that no deviation was zero.

In this paper we completely settle the question with

**THEOREM B.** *Suppose  $R$  is not a weak complete intersection. Then no deviation can vanish:*

$$e_i(R) \geq 1, \quad i \geq 1.$$

The rest of this paper is devoted to the proof of Theorem B, which depends on a variation of an idea (special variables) of André [2], and an adaptation of the minimal models of Avramov [4].

Our first observation is that betti numbers (and hence deviations are unchanged if we complete  $R$ , and hence we may assume without loss of generality that  $R = \tilde{R}/I$ ,  $\tilde{R}$  regular,  $I \subset \tilde{m}^2$ , as above. We make this assumption henceforth.

The next step is to build a suitable DGA model for  $R$ . This involves the process, introduced by Tate [13], of “adjoining freely commuting variables” which in our case may be either exterior, symmetric or divided power variables. To simplify we shall use “ $X$ ” to mean an exterior or symmetric variable and “ $Y$ ” to mean an exterior or divided power variable. More precisely we establish the *Notation convention*. Let  $X_1, \dots, X_i$  (resp.  $Y_1, \dots, Y_j$ ) denote symbols of degrees  $p$  (resp.,  $q$ ). Denote by  $\Lambda(X_1, \dots, X_i)$  the symmetric (resp. exterior) algebra on the free  $\mathbb{Z}$ -module with basis  $X_\gamma$  if  $p$  is even (resp. odd). Denote by  $\Gamma(Y_1, \dots, Y_j)$  the free divided powers (resp. exterior) algebra on the free  $\mathbb{Z}$ -module with basis  $Y_\gamma$  if  $q$  is even (resp. odd).

Then, if  $A$  is any graded algebra (commutative in the graded sense) we adopt the notation

$$A[X_1, \dots, X_i] = A \otimes_{\mathbb{Z}} \Lambda(X_1, \dots, X_i)$$

and

$$A[Y_1, \dots, Y_j] = A \otimes_{\mathbb{Z}} \Gamma(Y_1, \dots, Y_j)$$

and we say we have *adjoined variables*  $X_\gamma$  (or  $Y_\gamma$ ) to  $A$ .

We now fix an integer  $q$  arbitrarily and construct a sequence  $A(0) \subset A(1) \subset \dots$  of DGA's augmented to  $R$ . Indeed we set  $A(0) = \tilde{R}$  and let  $\pi: A(0) \rightarrow R$  be the quotient map. Then, choosing representatives  $x_{11}, \dots, x_{1n_1} \in I$  of a  $\mathbf{k}$ -basis of  $I/\tilde{m} \cdot I$  we set

$$A(1) = \tilde{R}[X_{11}, \dots, X_{1n_1}]; \quad dX_{1i} = x_{1i}$$

Extend  $\pi$  (uniquely) to  $A(1)$  by setting  $\pi(X_{1i}) = 0$ . Then  $H_0(\pi): H_0(A(1)) \xrightarrow{\cong} R$ .

Suppose by induction that  $A(k)$ ,  $1 \leq k \leq i - 1$ , are constructed and satisfy  $H_0(A(1)) = H_0(A(k))$  and  $H_l(A(k)) = 0$ ,  $1 \leq l < k$ . Choose cycles  $z_{i1}, \dots, z_{in_i}$  representing a  $\mathbf{k}$ -basis of  $H_{i-1}(A(i-1))/\tilde{m} \cdot H_{i-1}(A(i-1))$  and define  $A(i)$  by

$$A(i) = \begin{cases} A(i-1)[X_{i1}, \dots, X_{in_i}]; & dX_{ij} = z_{ij} \text{ if } i < q. \\ A(i-1)[Y_{i1}, \dots, Y_{in_i}]; & dY_{ij} = z_{ij} \text{ if } i \geq q. \end{cases}$$

The differential in  $A(i)$  is then determined by the requirement that  $A(i)$  be a DGA and (in the second case when  $i$  is even) that

$$d(\gamma^s Y_{ij}) = z_{ij} \otimes \gamma^{s-1} Y_{ij}, \quad s \geq 1,$$

$\gamma^s$  denoting the divided power operations.

Finally, set  $A = \varinjlim_i A(i)$  and note that  $\pi$  extends uniquely to  $\pi: A \rightarrow R$  with

$H(\pi): H(A) \xrightarrow{\cong} R$ . We say  $A$  is a model for  $R$  with switching degree  $q$ .

**PROPOSITION 1.** *Let  $A$  be as above and set  $\bar{q} = q$  if  $q$  is even and  $\bar{q} = q + 1$  if  $q$  is odd. Let  $m = \dim m/m^2 = \dim \tilde{m}/\tilde{m}^2$ . Then*

- (i) *For any  $\alpha_0, \dots, \alpha_m \in H_+(\mathbf{k} \otimes_{A(q-1)} A)$ ,  $\alpha_0 \cdot \dots \cdot \alpha_m = 0$ .*
- (ii) *The integers  $n_i$  satisfy*

$$n_i = e_{i+1}, \quad 1 \leq i < 2\bar{q}, \quad \text{and} \quad n_{2\bar{q}} \leq e_{2\bar{q}+1}.$$

*Proof.* We construct inductively the commutative DGA diagram below in which the vertical arrows induce homology isomorphisms. Indeed let  $y_{01}, \dots, y_{0m} \in \tilde{m}$  represent a basis of  $\tilde{m}/\tilde{m}^2$  and set:  $B(1) = A[y_{01}, \dots, y_{0m}]; dY_{0j} = y_{0j}$ . Because  $\tilde{R}$  is regular the augmentation  $\tilde{R}[Y_{01}, \dots, Y_{0j}] \rightarrow \mathbf{k}$  induces an isomorph-

$$\begin{array}{ccccccc}
 B(0) & \longrightarrow & B(1) & \longrightarrow & \cdots & B(i) & \cdots \longrightarrow & B(q) \\
 \parallel & & \downarrow \simeq & & & \downarrow \simeq & & \downarrow \simeq \\
 A & \longrightarrow & \mathbf{k} \otimes_{\tilde{R}} A & \longrightarrow & \cdots & \mathbf{k} \otimes_{A^{(i-1)}} A & \cdots \longrightarrow & \mathbf{k} \otimes_{A^{(q-1)}} A
 \end{array} \tag{2}$$

ism of homology. Because  $B(1)$  is a free  $\tilde{R}[Y_{01}, \dots, Y_{0m}]$  module the induced morphism

$$B(1) \rightarrow \mathbf{k} \otimes_{\tilde{R}[Y_{01}, \dots, Y_{0m}]} B(1) = \mathbf{k} \otimes_{\tilde{R}} A$$

also induces an isomorphism of homology.

Suppose next that the diagram has been constructed up through  $B(i)$ . Note that

$$\mathbf{k} \otimes_{A^{(i-1)}} A = \mathbf{k}[X_{i1}, \dots, X_{in_i}][\cdots \cdots \cdots]$$

Thus the  $X_{ij}$  ( $1 \leq j \leq n_i$ ) are cycles representing a basis of  $H_i(\mathbf{k} \otimes_{A^{(i-1)}} A)$ . Choose cycles  $u_{ij} \in B(i)$  mapping to the  $X_{ij}$  ( $1 \leq j \leq n_i$ ) and set  $B(i+1) = B(i)[Y_{i1}, \dots, Y_{in_i}]$ ;  $dY_{ij} = u_{ij}$ . When  $i$  is odd set  $d(\gamma^s Y_{ij}) = u_{ij} \cdot \gamma^{s-1} Y_{ij}$ .

We have then the sequence of DGA morphisms

$$\begin{aligned}
 B(i+1) &= B(i)[Y_{i1}, \dots, Y_{in_i}] \\
 &\rightarrow (\mathbf{k} \otimes_{A^{(i-1)}} A)[Y_{i1}, \dots, Y_{in_i}] \quad (dY_{ij} = X_{ij}) \\
 &= \mathbf{k}[X_{i1}, \dots, X_{in_i}, Y_{i1}, \dots, Y_{in_i}][\cdots \cdots \cdots] \\
 &\rightarrow \mathbf{k} \otimes_{\mathbf{k}[X_{i1}, \dots, Y_{in_i}]} \mathbf{k}[X_{i1}, \dots, Y_{in_i}][\cdots \cdots \cdots] \\
 &= \mathbf{k} \otimes_{A^{(i)}} A.
 \end{aligned}$$

The first arrow induces an isomorphism of homology because  $B(i) \rightarrow \mathbf{k} \otimes_{A^{(i-1)}} A$  does, and the second induces an isomorphism of homology because  $\mathbf{k}[X_{i1}, \dots, Y_{in_i}] \rightarrow \mathbf{k}$  does. This completes the construction of (2).

Again, because each  $B(i)$  is a free  $A$ -module and because  $A \twoheadrightarrow R$  induces an isomorphism  $H(A) \xrightarrow{\cong} R$  it follows that for  $0 \leq i \leq q-1$

$$B(i+1) \rightarrow R \otimes_A B(i+1) = R[Y_{01}, \dots, Y_{in_i}]$$

induces an isomorphism of homology. In particular  $R \otimes_A B(1)$  is just the Koszul complex,  $K^R$ , and  $R \otimes_A B(i+1)$  is obtained from  $R \otimes_A B(i)$  by adjoining a minimal number of exterior or divided power variables so as to kill  $H_i(R \otimes_A B(i))$ . Thus by adjoining such variables in degrees  $> q$  to  $R \otimes_A B(q)$ , we get Tate's acyclic closure  $C$ , of  $R$ .

Now Gulliksen's theorem [6] asserts that  $d(C) \supseteq_m C$ . Since  $H_+(C) = 0$  it

follows that  $(\ker d)_+ \subset {}_m C$ . Since  $C$  is a free  $R \otimes_A B(q)$  module it follows that

$$(\ker d)_+ \cap (R \otimes_A B(q)) \subset {}_m \otimes_A B(q).$$

The argument of Gulliksen [7; Lemma 1] now shows that the product of  $m + 1$  homology classes in  $H_+(R \otimes_A B(q))$  is zero – in view of the homology isomorphisms above this proves part (i) of the proposition.

Moreover, according to Gulliksen [6; Corollary 1],  $e_i$  is the number of variables in  $C$  of degree  $i$ . But for  $1 \leq i < q$ , the number of variables  $Y_{ij}$  of degree  $i + 1$  is  $n_i$ , and so

$$n_i = e_{i+1} \quad 1 \leq i < q. \tag{3}$$

We complete the proof of part (ii) of the proposition by considering a model  $A'$  for  $R$  as above, but with switching degree  $2\bar{q} + 1$ . By (3), applied to  $A'$ , the number of variables of degree  $j$  in  $A'$  is  $e_{j+1}$ ,  $1 \leq j \leq 2\bar{q}$ . Thus (ii) will be established once we show that  $A'$  has the same number of (resp., at least as many) variables as  $A$  in degrees  $< 2\bar{q}$  (resp.,  $2\bar{q}$ ).

We may, clearly, take  $A'(q - 1) = A(q - 1)$ . Suppose by induction that for some  $r \geq q$ ,  $A'$  and  $A$  have the same number of variables in degrees  $\leq r - 1$ , and that there is a DGA morphism  $\phi : A'(r - 1) \rightarrow A(r - 1)$  which is an isomorphism in degrees  $< 2\bar{q}$ .

If  $r < 2\bar{q}$  then  $H_{r-1}(\phi)$  is necessarily an isomorphism, and so we may choose  $A'(r) = A'(r - 1)[X_{r1}, \dots, X_{rm}]$  with  $H_{r-1}(\phi) : \text{class}(dX_{ri}) \mapsto \text{class}(dY_{ri})$ . Thus  $A'$  and  $A$  have the same number of variables in degree  $r$ . Moreover, because the  $X_{ri}$  are either polynomial or exterior variables, we may extend  $\phi$  to a DGA morphism  $\phi : A'(r) \rightarrow A(r)$  by setting  $\phi(X_{ri}) = Y_{ri}$ . Since

$$A(r)_{<2\bar{q}} \subset A(r - 1) \oplus \bigoplus_{j=1}^{n_r} A(r - 1) \cdot Y_{rj}$$

(and similarly for  $A'(r)$ ) it follows that  $\phi$  is an isomorphism in degrees  $< 2\bar{q}$ .

Finally, suppose  $r = 2\bar{q}$ . Since  $\phi$  is an isomorphism in degrees  $< 2\bar{q}$  it follows that  $H_{r-1}(\phi)$  is surjective. In this case the number of variables of degree  $r$  to be adjoined to  $A'(r - 1)$  is at least as large as the number adjoined to  $A(r - 1)$ . This completes the proof. ■

We now establish Theorem B by supposing  $e_3 \neq 0$  and some  $e_i = 0$  ( $i > 3$ ) and deducing a contradiction. Let  $s \geq 3$  be the least integer for which  $e_{s+1} = 0$ . There are two cases.

*Case I.*  $s = 2k + 1$ . Let  $A$  be a model for  $R$  as above with switching degree  $2k$ .

By Proposition 1 (ii),

$$n_{2k} = e_s > 0 \quad \text{and} \quad n_{2k+1} = e_{s+1} = 0.$$

We construct a  $\text{deg } -2k$  derivation,  $\theta$ , of the DGA,  $(A, d)$  such that

$$\theta(Y_{2k,1}) = 1 \quad \text{and} \quad \theta(\gamma^q Y_{2k,1}) = \gamma^{q-1} Y_{2k,1}. \tag{4}$$

Indeed, (4), together with the conditions

$$\theta(A_{<2k}) = 0 \quad \text{and} \quad \theta(\gamma^q Y_{2k,i}) = 0, \quad i > 1,$$

defines a derivation of the DGA,  $A(2k)$ . Since  $n_{2k+1}$  is zero  $A(2k) = A(2k + 1)$ . Suppose  $\theta$  is extended to some  $A(j - 1)$ ,  $j - 1 \geq 2k + 1$ . Then  $\theta(dY_{ji})$  is a cycle in  $A_{j-2k-1}$ . Because  $j > (2k + 1)$  and because  $H_+(A) = 0$  (since  $H(A) = R$ ) it follows that  $\theta(dY_{ji}) = d\Phi_i$ , some  $\Phi_i \in A_{j-2k}$ .

Thus we may extend  $\theta$  to  $A(j)$  by setting

$$\theta(Y_{ji}) = \Phi_i \quad (\text{if } j \text{ is odd})$$

and

$$\theta(\gamma^q Y_{ji}) = \Phi_i \cdot \gamma^{q-1} Y_{ji}, \quad q \geq 1, \quad (\text{if } j \text{ is even}).$$

Finally, observe that  $\theta$  factors to give a derivation  $\bar{\theta}$  of the DGA  $\mathbf{k} \otimes_{A(2k-1)} A$ . In this quotient DGA, the elements  $\gamma^q Y_{2k,1}$  are cycles. It follows from (4) that

$$\bar{\theta}^q(\gamma^q Y_{2k,1}) = 1$$

and hence  $\gamma^q Y_{2k,1}$  represents a non-zero homology class for each  $q$ . But if  $\text{char } \mathbf{k} = p$  or  $0$  then in  $\mathbf{k}[Y_{2k,1}]$ ,  $\gamma^{(1+p+p^2+\dots+p^m)}(Y_{2k,1})$  is a scalar multiple of  $Y_{2k,1} \cdot \gamma^p(Y_{2k,1}) \cdot \dots \cdot \gamma^{p^m}(Y_{2k,1})$ . Thus this latter element also represents a non-zero homology class, which contradicts part (i) of Proposition 1.

*Case II.*  $s = 2k + 2$  ( $k \geq 1$ ). Again let  $A$  be a model for  $R$  as above with switching degree  $2k$ . By Proposition 1 (ii),

$$n_{2k} = e_{s-1} > 0 \quad \text{and} \quad n_{2k+2} \leq e_{s+1} = 0. \tag{5}$$

Let  $y_1, \dots, y_m \in \mathfrak{m}$  represent a  $\mathbf{k}$ -basis for  $\bar{\mathfrak{m}}/\bar{\mathfrak{m}}^2$  and consider the differential

$A$ -module (free as an  $A$ -module)

$$M = A \oplus A \cdot Y_1 \oplus \cdots \oplus A \cdot Y_m; \quad dY_i = y_i.$$

Now the quotient module  $M/A$  is isomorphic (as differential modules) with a direct sum of copies of  $A$  shifted in degree by 1. Since  $H(A) = R$  is concentrated in degree zero,  $H(M/A)$  is concentrated in degree 1. From the short exact sequence  $A \rightarrow M \rightarrow M/A$  and that fact that  $y_1, \dots, y_m \in \text{Im} d$  we deduce that  $H(M) = H_0(M) \oplus H_1(M)$ , and that  $H_0(M) = \mathbf{k}$ .

Using these facts we construct a derivation  $\theta$  of degree  $-2k$  from the DGA,  $A$ , to the differential  $A$ -module  $M$ . ( $M$  is a right  $A$ -module via  $m \cdot a = (-1)^{\deg m \deg a} a \cdot m$ .) Indeed we set  $\theta(A_{2k-1}) = 0$  and

$$\theta(\gamma^q Y_{2k,i}) = \begin{cases} \gamma^{q-1} Y_{2k,i} & i = 1 \\ 0 & i > 1. \end{cases} \quad (6)$$

This defines  $\theta$  in  $A(2k)$ .

Next, for each  $Y_{2k+1,i}$  we have  $\theta(dY_{2k+1,i}) \in \tilde{m}$  and hence  $\theta(dY_{2k+1,i}) = d\Phi_i$  for some  $\Phi_i \in \bigoplus_j \tilde{R} \cdot Y_j$ . Extend  $\theta$  to  $A(2k+1)$  by setting  $\theta(Y_{2k+1,i}) = \Phi_i$ .

Now because of (5),  $A(2k+1) = A(2k+2)$ . Assume we have extended  $\theta$  to  $A(j-1)$ , some  $j-1 \geq 2k+2$ . Then  $\theta(dY_{ji}) \in M_{j-2k-1}$  and  $j-2k-1 \geq 2$ . Our calculations above ( $H(M) = H_0(M) \oplus H_1(M)$ ) thus imply that  $\theta(dY_{ji}) = d\Psi_i$  and we can extend  $\theta$  to  $A(j)$  by setting

$$\theta(Y_{ji}) = \Psi_i \quad \text{if } j \text{ is odd,}$$

and

$$\theta(\gamma^q Y_{ji}) = \gamma^{q-1}(Y_{ji}) \cdot \Psi_i, \quad q \geq 1, \quad \text{if } j \text{ is even.}$$

Finally, we extend the projection  $A \rightarrow \mathbf{k} \otimes_{A(2k-1)} A$  to a map  $\rho: M \rightarrow \mathbf{k} \otimes_{A(2k-1)} A$  of differential  $A$ -modules by setting  $\rho(A \cdot Y_i) = 0$ . The derivation  $\rho \circ \theta$  factors to yield a derivation  $\bar{\theta}$  of the DGA  $\mathbf{k} \otimes_{A(2k-1)} A$  and we obtain a contradiction exactly as in case I.

This completes the proof. ■

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