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The non-vanishing of the deviations of a local ring

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Let R be a local noetherian ring with maximal ideal m and residue field \mathbf{k} . Then (cf. [3], [6]) $\text{Tor}_*^R(\mathbf{k}, \mathbf{k})$ has the structure of a free divided powers algebra on a graded \mathbf{k} -vector space $V = \bigoplus_i V_i$. In particular the Poincaré series for R has the form

$$\sum_{i=0}^{\infty} [\dim \text{Tor}_i^R(\mathbf{k}, \mathbf{k})] t^i = \frac{\prod_{j=1}^{\infty} (1 + t^{2j+1})^{\dim V_{2j+1}}}{\prod_{j=1}^{\infty} (1 - t^{2j})^{\dim V_{2j}}} \tag{1}$$

The integers $e_j = \dim V_j$ are called the *deviations* of R . The equation above shows they are completely determined by the betti numbers $\dim \text{Tor}_i^R(\mathbf{k}, \mathbf{k})$, and conversely. Moreover ([1], [11], [12]) the Yoneda Ext-algebra, $\text{Ext}_R^*(\mathbf{k}, \mathbf{k})$, is naturally the universal enveloping algebra of a graded Lie algebra L_R dual to V , and hence

$$e_i = \dim L_R^i, \quad \text{all } i.$$

Let \hat{R} denote the completion of R with respect to the powers of m . By the Cohen structure theorem, \hat{R} has the form \tilde{R}/I where \tilde{R} is a regular local noetherian ring (with maximal ideal \tilde{m}) and $I \subset \tilde{m}^2$. We call R a *weak complete intersection* if I is generated by a regular sequence.

Now in [3] Assmus proves the following

THEOREM A (Assmus). *The following conditions are equivalent:*

- (i) R is a weak complete intersection.
- (ii) $e_j = 0, j \geq 3$.
- (iii) $e_3 = 0$.

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This raised the question of whether or not any deviation could vanish if R was not a weak complete intersection.

A first step was taken by Gulliksen [7], [8] who showed that infinitely many e_{2k} must be nonzero for non weak complete intersections. Subsequently Avramov and Halperin [5] showed that only finitely many e_i could be zero. Moreover in special cases (eg. Jacobsson [9], Löfwall [10]) it was known that no deviation was zero.

In this paper we completely settle the question with

THEOREM B. *Suppose R is not a weak complete intersection. Then no deviation can vanish:*

$$e_i(R) \geq 1, \quad i \geq 1.$$

The rest of this paper is devoted to the proof of Theorem B, which depends on a variation of an idea (special variables) of André [2], and an adaptation of the minimal models of Avramov [4].

Our first observation is that betti numbers (and hence deviations are unchanged if we complete R , and hence we may assume without loss of generality that $R = \tilde{R}/I$, \tilde{R} regular, $I \subset \tilde{m}^2$, as above. We make this assumption henceforth.

The next step is to build a suitable DGA model for R . This involves the process, introduced by Tate [13], of “adjoining freely commuting variables” which in our case may be either exterior, symmetric or divided power variables. To simplify we shall use “ X ” to mean an exterior or symmetric variable and “ Y ” to mean an exterior or divided power variable. More precisely we establish the *Notation convention*. Let X_1, \dots, X_i (resp. Y_1, \dots, Y_j) denote symbols of degrees p (resp., q). Denote by $\Lambda(X_1, \dots, X_i)$ the symmetric (resp. exterior) algebra on the free \mathbb{Z} -module with basis X_γ if p is even (resp. odd). Denote by $\Gamma(Y_1, \dots, Y_j)$ the free divided powers (resp. exterior) algebra on the free \mathbb{Z} -module with basis Y_γ if q is even (resp. odd).

Then, if A is any graded algebra (commutative in the graded sense) we adopt the notation

$$A[X_1, \dots, X_i] = A \otimes_{\mathbb{Z}} \Lambda(X_1, \dots, X_i)$$

and

$$A[Y_1, \dots, Y_j] = A \otimes_{\mathbb{Z}} \Gamma(Y_1, \dots, Y_j)$$

and we say we have *adjoined variables* X_γ (or Y_γ) to A .

We now fix an integer q arbitrarily and construct a sequence $A(0) \subset A(1) \subset \dots$ of DGA's augmented to R . Indeed we set $A(0) = \tilde{R}$ and let $\pi: A(0) \rightarrow R$ be the quotient map. Then, choosing representatives $x_{11}, \dots, x_{1n_1} \in I$ of a \mathbf{k} -basis of $I/\tilde{m} \cdot I$ we set

$$A(1) = \tilde{R}[X_{11}, \dots, X_{1n_1}]; \quad dX_{1i} = x_{1i}$$

Extend π (uniquely) to $A(1)$ by setting $\pi(X_{1i}) = 0$. Then $H_0(\pi): H_0(A(1)) \xrightarrow{\cong} R$.

Suppose by induction that $A(k)$, $1 \leq k \leq i - 1$, are constructed and satisfy $H_0(A(1)) = H_0(A(k))$ and $H_l(A(k)) = 0$, $1 \leq l < k$. Choose cycles z_{i1}, \dots, z_{in_i} representing a \mathbf{k} -basis of $H_{i-1}(A(i-1))/\tilde{m} \cdot H_{i-1}(A(i-1))$ and define $A(i)$ by

$$A(i) = \begin{cases} A(i-1)[X_{i1}, \dots, X_{in_i}]; & dX_{ij} = z_{ij} \text{ if } i < q. \\ A(i-1)[Y_{i1}, \dots, Y_{in_i}]; & dY_{ij} = z_{ij} \text{ if } i \geq q. \end{cases}$$

The differential in $A(i)$ is then determined by the requirement that $A(i)$ be a DGA and (in the second case when i is even) that

$$d(\gamma^s Y_{ij}) = z_{ij} \otimes \gamma^{s-1} Y_{ij}, \quad s \geq 1,$$

γ^s denoting the divided power operations.

Finally, set $A = \varinjlim_i A(i)$ and note that π extends uniquely to $\pi: A \rightarrow R$ with

$H(\pi): H(A) \xrightarrow{\cong} R$. We say A is a model for R with switching degree q .

PROPOSITION 1. *Let A be as above and set $\bar{q} = q$ if q is even and $\bar{q} = q + 1$ if q is odd. Let $m = \dim m/m^2 = \dim \tilde{m}/\tilde{m}^2$. Then*

- (i) *For any $\alpha_0, \dots, \alpha_m \in H_+(\mathbf{k} \otimes_{A(q-1)} A)$, $\alpha_0 \cdot \dots \cdot \alpha_m = 0$.*
- (ii) *The integers n_i satisfy*

$$n_i = e_{i+1}, \quad 1 \leq i < 2\bar{q}, \quad \text{and} \quad n_{2\bar{q}} \leq e_{2\bar{q}+1}.$$

Proof. We construct inductively the commutative DGA diagram below in which the vertical arrows induce homology isomorphisms. Indeed let $y_{01}, \dots, y_{0m} \in \tilde{m}$ represent a basis of \tilde{m}/\tilde{m}^2 and set: $B(1) = A[y_{01}, \dots, y_{0m}]; dY_{0j} = y_{0j}$. Because \tilde{R} is regular the augmentation $\tilde{R}[Y_{01}, \dots, Y_{0j}] \rightarrow \mathbf{k}$ induces an isomorph-

$$\begin{array}{ccccccc}
 B(0) & \longrightarrow & B(1) & \longrightarrow & \cdots & B(i) & \cdots \longrightarrow & B(q) \\
 \parallel & & \downarrow \simeq & & & \downarrow \simeq & & \downarrow \simeq \\
 A & \longrightarrow & \mathbf{k} \otimes_{\tilde{R}} A & \longrightarrow & \cdots & \mathbf{k} \otimes_{A^{(i-1)}} A & \cdots \longrightarrow & \mathbf{k} \otimes_{A^{(q-1)}} A
 \end{array} \tag{2}$$

ism of homology. Because $B(1)$ is a free $\tilde{R}[Y_{01}, \dots, Y_{0m}]$ module the induced morphism

$$B(1) \rightarrow \mathbf{k} \otimes_{\tilde{R}[Y_{01}, \dots, Y_{0m}]} B(1) = \mathbf{k} \otimes_{\tilde{R}} A$$

also induces an isomorphism of homology.

Suppose next that the diagram has been constructed up through $B(i)$. Note that

$$\mathbf{k} \otimes_{A^{(i-1)}} A = \mathbf{k}[X_{i1}, \dots, X_{in_i}][\cdots \cdots \cdots]$$

Thus the X_{ij} ($1 \leq j \leq n_i$) are cycles representing a basis of $H_i(\mathbf{k} \otimes_{A^{(i-1)}} A)$. Choose cycles $u_{ij} \in B(i)$ mapping to the X_{ij} ($1 \leq j \leq n_i$) and set $B(i+1) = B(i)[Y_{i1}, \dots, Y_{in_i}]$; $dY_{ij} = u_{ij}$. When i is odd set $d(\gamma^s Y_{ij}) = u_{ij} \cdot \gamma^{s-1} Y_{ij}$.

We have then the sequence of DGA morphisms

$$\begin{aligned}
 B(i+1) &= B(i)[Y_{i1}, \dots, Y_{in_i}] \\
 &\rightarrow (\mathbf{k} \otimes_{A^{(i-1)}} A)[Y_{i1}, \dots, Y_{in_i}] \quad (dY_{ij} = X_{ij}) \\
 &= \mathbf{k}[X_{i1}, \dots, X_{in_i}, Y_{i1}, \dots, Y_{in_i}][\cdots \cdots \cdots] \\
 &\rightarrow \mathbf{k} \otimes_{\mathbf{k}[X_{i1}, \dots, Y_{in_i}]} \mathbf{k}[X_{i1}, \dots, Y_{in_i}][\cdots \cdots \cdots] \\
 &= \mathbf{k} \otimes_{A^{(i)}} A.
 \end{aligned}$$

The first arrow induces an isomorphism of homology because $B(i) \rightarrow \mathbf{k} \otimes_{A^{(i-1)}} A$ does, and the second induces an isomorphism of homology because $\mathbf{k}[X_{i1}, \dots, Y_{in_i}] \rightarrow \mathbf{k}$ does. This completes the construction of (2).

Again, because each $B(i)$ is a free A -module and because $A \twoheadrightarrow R$ induces an isomorphism $H(A) \xrightarrow{\cong} R$ it follows that for $0 \leq i \leq q-1$

$$B(i+1) \rightarrow R \otimes_A B(i+1) = R[Y_{01}, \dots, Y_{in_i}]$$

induces an isomorphism of homology. In particular $R \otimes_A B(1)$ is just the Koszul complex, K^R , and $R \otimes_A B(i+1)$ is obtained from $R \otimes_A B(i)$ by adjoining a minimal number of exterior or divided power variables so as to kill $H_i(R \otimes_A B(i))$. Thus by adjoining such variables in degrees $> q$ to $R \otimes_A B(q)$, we get Tate's acyclic closure C , of R .

Now Gulliksen's theorem [6] asserts that $d(C) \supseteq_m C$. Since $H_+(C) = 0$ it

follows that $(\ker d)_+ \subset {}_m C$. Since C is a free $R \otimes_A B(q)$ module it follows that

$$(\ker d)_+ \cap (R \otimes_A B(q)) \subset {}_m \otimes_A B(q).$$

The argument of Gulliksen [7; Lemma 1] now shows that the product of $m + 1$ homology classes in $H_+(R \otimes_A B(q))$ is zero – in view of the homology isomorphisms above this proves part (i) of the proposition.

Moreover, according to Gulliksen [6; Corollary 1], e_i is the number of variables in C of degree i . But for $1 \leq i < q$, the number of variables Y_{ij} of degree $i + 1$ is n_i , and so

$$n_i = e_{i+1} \quad 1 \leq i < q. \tag{3}$$

We complete the proof of part (ii) of the proposition by considering a model A' for R as above, but with switching degree $2\bar{q} + 1$. By (3), applied to A' , the number of variables of degree j in A' is e_{j+1} , $1 \leq j \leq 2\bar{q}$. Thus (ii) will be established once we show that A' has the same number of (resp., at least as many) variables as A in degrees $< 2\bar{q}$ (resp., $2\bar{q}$).

We may, clearly, take $A'(q - 1) = A(q - 1)$. Suppose by induction that for some $r \geq q$, A' and A have the same number of variables in degrees $\leq r - 1$, and that there is a DGA morphism $\phi : A'(r - 1) \rightarrow A(r - 1)$ which is an isomorphism in degrees $< 2\bar{q}$.

If $r < 2\bar{q}$ then $H_{r-1}(\phi)$ is necessarily an isomorphism, and so we may choose $A'(r) = A'(r - 1)[X_{r1}, \dots, X_{rm}]$ with $H_{r-1}(\phi) : \text{class}(dX_{ri}) \mapsto \text{class}(dY_{ri})$. Thus A' and A have the same number of variables in degree r . Moreover, because the X_{ri} are either polynomial or exterior variables, we may extend ϕ to a DGA morphism $\phi : A'(r) \rightarrow A(r)$ by setting $\phi(X_{ri}) = Y_{ri}$. Since

$$A(r)_{<2\bar{q}} \subset A(r - 1) \oplus \bigoplus_{j=1}^{n_r} A(r - 1) \cdot Y_{rj}$$

(and similarly for $A'(r)$) it follows that ϕ is an isomorphism in degrees $< 2\bar{q}$.

Finally, suppose $r = 2\bar{q}$. Since ϕ is an isomorphism in degrees $< 2\bar{q}$ it follows that $H_{r-1}(\phi)$ is surjective. In this case the number of variables of degree r to be adjoined to $A'(r - 1)$ is at least as large as the number adjoined to $A(r - 1)$. This completes the proof. ■

We now establish Theorem B by supposing $e_3 \neq 0$ and some $e_i = 0$ ($i > 3$) and deducing a contradiction. Let $s \geq 3$ be the least integer for which $e_{s+1} = 0$. There are two cases.

Case I. $s = 2k + 1$. Let A be a model for R as above with switching degree $2k$.

By Proposition 1 (ii),

$$n_{2k} = e_s > 0 \quad \text{and} \quad n_{2k+1} = e_{s+1} = 0.$$

We construct a $\text{deg } -2k$ derivation, θ , of the DGA, (A, d) such that

$$\theta(Y_{2k,1}) = 1 \quad \text{and} \quad \theta(\gamma^q Y_{2k,1}) = \gamma^{q-1} Y_{2k,1}. \tag{4}$$

Indeed, (4), together with the conditions

$$\theta(A_{<2k}) = 0 \quad \text{and} \quad \theta(\gamma^q Y_{2k,i}) = 0, \quad i > 1,$$

defines a derivation of the DGA, $A(2k)$. Since n_{2k+1} is zero $A(2k) = A(2k + 1)$. Suppose θ is extended to some $A(j - 1)$, $j - 1 \geq 2k + 1$. Then $\theta(dY_{ji})$ is a cycle in A_{j-2k-1} . Because $j > (2k + 1)$ and because $H_+(A) = 0$ (since $H(A) = R$) it follows that $\theta(dY_{ji}) = d\Phi_i$, some $\Phi_i \in A_{j-2k}$.

Thus we may extend θ to $A(j)$ by setting

$$\theta(Y_{ji}) = \Phi_i \quad (\text{if } j \text{ is odd})$$

and

$$\theta(\gamma^q Y_{ji}) = \Phi_i \cdot \gamma^{q-1} Y_{ji}, \quad q \geq 1, \quad (\text{if } j \text{ is even}).$$

Finally, observe that θ factors to give a derivation $\bar{\theta}$ of the DGA $\mathbf{k} \otimes_{A(2k-1)} A$. In this quotient DGA, the elements $\gamma^q Y_{2k,1}$ are cycles. It follows from (4) that

$$\bar{\theta}^q(\gamma^q Y_{2k,1}) = 1$$

and hence $\gamma^q Y_{2k+1}$ represents a non-zero homology class for each q . But if $\text{char } \mathbf{k} = p$ or 0 then in $\mathbf{k}[Y_{2k,1}]$, $\gamma^{(1+p+p^2+\dots+p^m)}(Y_{2k,1})$ is a scalar multiple of $Y_{2k,1} \cdot \gamma^p(Y_{2k,1}) \cdot \dots \cdot \gamma^{p^m}(Y_{2k,1})$. Thus this latter element also represents a non-zero homology class, which contradicts part (i) of Proposition 1.

Case II. $s = 2k + 2$ ($k \geq 1$). Again let A be a model for R as above with switching degree $2k$. By Proposition 1 (ii),

$$n_{2k} = e_{s-1} > 0 \quad \text{and} \quad n_{2k+2} \leq e_{s+1} = 0. \tag{5}$$

Let $y_1, \dots, y_m \in \mathfrak{m}$ represent a \mathbf{k} -basis for $\mathfrak{m}/\mathfrak{m}^2$ and consider the differential

A -module (free as an A -module)

$$M = A \oplus A \cdot Y_1 \oplus \cdots \oplus A \cdot Y_m; \quad dY_i = y_i.$$

Now the quotient module M/A is isomorphic (as differential modules) with a direct sum of copies of A shifted in degree by 1. Since $H(A) = R$ is concentrated in degree zero, $H(M/A)$ is concentrated in degree 1. From the short exact sequence $A \rightarrow M \rightarrow M/A$ and that fact that $y_1, \dots, y_m \in \text{Im} d$ we deduce that $H(M) = H_0(M) \oplus H_1(M)$, and that $H_0(M) = \mathbf{k}$.

Using these facts we construct a derivation θ of degree $-2k$ from the DGA, A , to the differential A -module M . (M is a right A -module via $m \cdot a = (-1)^{\deg m \deg a} a \cdot m$.) Indeed we set $\theta(A_{2k-1}) = 0$ and

$$\theta(\gamma^q Y_{2k,i}) = \begin{cases} \gamma^{q-1} Y_{2k,i} & i = 1 \\ 0 & i > 1. \end{cases} \quad (6)$$

This defines θ in $A(2k)$.

Next, for each $Y_{2k+1,i}$ we have $\theta(dY_{2k+1,i}) \in \bar{m}$ and hence $\theta(dY_{2k+1,i}) = d\Phi_i$ for some $\Phi_i \in \bigoplus_j \bar{R} \cdot Y_j$. Extend θ to $A(2k+1)$ by setting $\theta(Y_{2k+1,i}) = \Phi_i$.

Now because of (5), $A(2k+1) = A(2k+2)$. Assume we have extended θ to $A(j-1)$, some $j-1 \geq 2k+2$. Then $\theta(dY_{ji}) \in M_{j-2k-1}$ and $j-2k-1 \geq 2$. Our calculations above ($H(M) = H_0(M) \oplus H_1(M)$) thus imply that $\theta(dY_{ji}) = d\Psi_i$ and we can extend θ to $A(j)$ by setting

$$\theta(Y_{ji}) = \Psi_i \quad \text{if } j \text{ is odd,}$$

and

$$\theta(\gamma^q Y_{ji}) = \gamma^{q-1}(Y_{ji}) \cdot \Psi_i, \quad q \geq 1, \quad \text{if } j \text{ is even.}$$

Finally, we extend the projection $A \rightarrow \mathbf{k} \otimes_{A(2k-1)} A$ to a map $\rho: M \rightarrow \mathbf{k} \otimes_{A(2k-1)} A$ of differential A -modules by setting $\rho(A \cdot Y_i) = 0$. The derivation $\rho \circ \theta$ factors to yield a derivation $\bar{\theta}$ of the DGA $\mathbf{k} \otimes_{A(2k-1)} A$ and we obtain a contradiction exactly as in case I.

This completes the proof. ■

REFERENCES

- [1] M. ANDRÉ, *Hopf algebras with divided powers*, J. Algebra 18 (1971), 19–50.
- [2] M. ANDRÉ, *Le caractere additif des deviations des anneaux locaux*, Comment. Math. Helv. 57 (1982) 648–675.

- [3] E. F. ASSMUS, *On the homology of local rings*, III. J. Math. 3 (1959) 187–199.
- [4] L. L. AVRAMOV, *Local algebras and rational homotopy*, in *Homotopie Algèbrique et Algèbre Locale*, Asterisque 113–114, (1984) 15–43.
- [5] L. AVRAMOV and S. HALPERIN, *On the non-vanishing of cotangent cohomology*, Comment. Math. Helv. 62/2, (1987) 169–184.
- [6] T. H. GULLIKSEN, *A proof of the existence of minimal R-algebra resolutions*, Acta Math. 120 (1968) 53–58.
- [7] T. H. GULLIKSEN, *A homological characterization of local complete intersections*, Compositio Math. 23 (1971) 251–255.
- [8] T. H. GULLIKSEN, *On the deviations of a local ring*, Math. Scand. 47 (1980) 5–20.
- [9] C. JACOBSSON, *On the positivity of the deviations of a local ring*, Math. Reports 15 (1982) University of Stockholm.
- [10] C. LÖFWALL, *On the centre of graded Lie algebras*, in *Homotopie Algèbrique et Algèbre Locale*, Asterisque 113–114 (1984) 263–267.
- [11] J. W. MILNOR and J. C. MOORE, *On the structure of Hopf algebras*, Annals of Math. 81 (1965) 211–264.
- [12] G. SJÖDIN, *Hopf algebras and derivations*, J. Algebra 64 (1980) 218–229.
- [13] J. TATE, *Homology of noetherian rings and local rings*, III. J. Math. 1 (1957), 14–25.

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