Isolated critical points of mappings from R4 to R2 and a natural splitting of the Milnor number of a classical fibered link. Part I: Basic theory; examples.

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Isolated critical points of mappings from $\mathbf{R}^4$ to $\mathbf{R}^2$ and a natural splitting of the Milnor number of a classical fibered link. Part I: Basic theory; examples

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§0. Introduction; statement of results.

From a fibered link $\mathcal{K} = (S^3, K)$ may be constructed a field $S_{\mathcal{K}}$ of (not everywhere tangent) 2-planes on S^3. When $\mathcal{K} = \mathcal{K}_F$ is the link of an isolated critical point of a map $F:\mathbf{R}^4 \to \mathbf{R}^2$, $S_{\mathcal{K}}$ is essentially the field of kernels of DF. Homotopically, $S_{\mathcal{K}}$ determines integers $\lambda(\mathcal{K})$ and $\rho(\mathcal{K})$.

THEOREM. $\lambda(\mathcal{K}) + \rho(\mathcal{K}) = \mu(\mathcal{K})$.

Here $\mu(\mathcal{K})$ is the Milnor number of $\mathcal{K}$, that is, the rank of the first homology of the fiber surface of $\mathcal{K}$. At least in some cases, this splitting of $\mu(\mathcal{K})$ corresponds to a geometrically natural direct sum decomposition of this homology group, into subgroups corresponding to "negative" (or "left-handed") and "positive" (or "right-handed") parts of the fiber surface, [12]. For instance, if $\mathcal{K}$ is a closed positive braid (such as the Lorenz links of dynamical systems [1], or the overlapping class of links of complex plane curve singularities – links $\mathcal{K}_F$ where $F:\mathbf{C}^2 \to \mathbf{C}$ is complex-analytic), then $\lambda(\mathcal{K}) = 0$ and $\rho(\mathcal{K}) = \mu(\mathcal{K})$, substantiating the intuition that such links are as positive as they can be. If $\mathcal{K}$ is the figure-8 knot, then $\lambda(\mathcal{K}) = 1 = \rho(\mathcal{K})$.

The Euler characteristic $1 - \mu(\mathcal{K})$ of a fiber surface of $\mathcal{K}$ can be computed by correctly counting the singularities of a vectorfield on the surface. There is a sense in which the extra information in the splitting $\mu(\mathcal{K}) = \lambda(\mathcal{K}) + \rho(\mathcal{K})$ comes from making this calculation "all around the circle" of fiber surfaces.

This is the first of several papers devoted to the study of λ, ρ, and related invariants. In this paper I develop the basic theory, and compute a number of examples.

More specifically, in §1, I construct the field $S_{\mathcal{K}}$, and two related (tangent) 2-plane fields $T^{\pm}_{\mathcal{K}}$, from an open-book structure on S^3 of type $\mathcal{K}$. A standard

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parallelization of S^3 permits one to consider $S_{\mathscr{K}}$ as a map $S^3 \to S^2 \times S^2$ and $T^{\pm}_{\mathscr{K}}$ as maps $S^3 \to S^2$; the Hopf invariants $(-\lambda(\mathscr{K}), \rho(\mathscr{K}))$ and $\tau^{\pm}(\mathscr{K})$ of these maps depend only on $\mathscr{K}$. In §2, the following results are obtained.

THEOREM 2.3. *For any fibered link $\mathscr{K}$, $\tau^+(\mathscr{K}) - \tau^-(\mathscr{K}) = 1 - \mu(\mathscr{K})$.*

THEOREM 2.5. *For any fibered link $\mathscr{K}$, $\lambda(\mathscr{K}) = -\tau^+(\mathscr{K})$ and $\rho(\mathscr{K}) = \tau^-(\mathscr{K}) + 1$.*

COROLLARY 2.6. *For any fibered link $\mathscr{K}$, $\lambda(\mathscr{K}) + \rho(\mathscr{K}) = \mu(\mathscr{K})$.*

In §3, I exploit the close relationship between fibered links and isolated critical points of mappings from $\mathbf{R}^4$ to $\mathbf{R}^2$ to give another way to calculate λ and ρ (in theory), which is used in examples in §4.

In [12], λ and ρ are computed for closed strict generalized homogeneous braids. In [13], the pair $\{\tau^{\pm}(\mathscr{K})\}$ of invariants of a fibered link $\mathscr{K}$ in S^3 is generalized to a set $\{\tau^s(\mathscr{K}) : s \in \{+, -\}^m\}$ of invariants of a fibered link $\mathscr{K}$ of m components in any closed 3-manifold. Work with Walter Neumann ([8], [9]) extends the definition of λ and ρ to higher dimensions, and studies their behavior (in dimension 3) under various geometric operations (cabling, connected sum, Murasugi sum, Stallings twists); it is shown, in particular, that *for many pairs of fibered links $\mathscr{K}$, $\mathscr{K}'$ in S^3, $\{\lambda(\mathscr{K}), \rho(\mathscr{K})\} \neq \{\lambda(\mathscr{K}'), \rho(\mathscr{K}')\}$ although $\mathscr{K}$ and $\mathscr{K}'$ have identical Seifert forms and algebraic monodromy.* A projected future paper generalizes λ and ρ to non-fibered links, using [2].

An interesting phenomenon (still unexplained as of December, 1986) is that in no known example is $\lambda(\mathscr{K})$ or $\rho(\mathscr{K})$ negative.

§1. Some plane fields associated to a classical fibered link; the invariants λ, ρ, and $\tau^{\pm}$.

The 3-sphere S^3 is the boundary of the unit 4-disk D^4 of $\mathbf{C}^2$. The 1-sphere S^1 is the boundary of the unit disk D^2 of $\mathbf{C}$. These spaces are oriented by the usual conventions; $\mathbf{R}^4$ (resp. $\mathbf{R}^2$) is the oriented real vectorspace underlying $\mathbf{C}^2$ (resp. $\mathbf{C}$); $\mathbf{H}$, the quaternions, is oriented by the usual identification of its underlying vectorspace with $\mathbf{R}^4$. The oriented span of a k-frame $(U_1, \ldots, U_k)$ is $\langle U_1, \ldots, U_k \rangle$.

A *link* $\mathscr{K}$ is a pair (S^3, K) where K is an oriented closed smooth 1-submanifold of S^3 (if K is connected, $\mathscr{K}$ is also called a *knot*); $\mathscr{K}$ is *fibered* if there exist a closed disk-bundle neighborhood $N(K)$ of K in S^3 with a smooth trivialization

$$\psi: N(K) \to D^2$$

and a smooth fibration over S^1 of the link exterior $E(K) = S^3 \setminus \operatorname{Int} N(K)$

$$\phi: E(K) \to S^1$$

such that $\psi \mid \partial N(K) = \phi \mid \partial N(K)$. Such maps ψ, ϕ glue together to give a piecewise smooth map

$$\pi: S^3 \to D^2$$

which is called an *open-book structure on* S^3, of *type* $\mathcal{K}$. (This summary treatment follows [8]; cf. also [4].) If $\mathcal{K}$ is a fibered link, then all open-book structures of type $\mathcal{K}$ are equivalent in quite a strong sense [4], and the ambient isotopy type of $\mathcal{K}$ determines the ambient isotopy type of any *fiber surface* of $\mathcal{K}$ – i.e., a fiber of (any) ϕ. The *Milnor number* $\mu(\mathcal{K})$ of $\mathcal{K}$ is the rank of the first homology of a fiber surface of $\mathcal{K}$.

Associated to a fibered link $\mathcal{K}$ are maps $S_{\mathcal{K}}$, $T^+_{\mathcal{K}}$, and $T^-_{\mathcal{K}}$ from S^3 to $G = G^+(2, 4)$, the Grassmann manifold of oriented 2-planes in $\mathbf{R}^4$, which will be defined in terms of certain vectorfields.

Let π be an open-book structure of type $\mathcal{K}$. Then $\pi/|\pi|: S^3 \setminus K \to S^1$ is a fibration (by fiber surfaces to which open collars are attached piecewise-smoothly) extending ϕ. It is always possible to take π such that $\pi/|\pi|$ is smooth, and we do so. Let U be the field of unit tangent vectors on $S^3 \setminus K$ perpendicular to the plane $\ker(D(\pi/|\pi|))$, so oriented as to point to the positive side of the fiber surfaces. Let V be the field of unit tangent vectors on $N(K)$ which span the line $\ker(D\pi)$, so oriented as to induce the given orientation on $K = \pi^{-1}(0)$. In $N(K) \setminus K$, where both are defined, the vectorfields U and V are orthogonal; let $U \times V$ be the field of unit vectors, tangent to S^3, such that the orthonormal 3-frame $(U, V, U \times V)$ gives the orientation of S^3. Finally, let W be the field of unit inward normal vectors on S^3 (so the orthonormal 4-frame $(U, V, U \times V, W)$ gives the orientation of $\mathbf{C}^2$).

DEFINITION 1.1. *For $Q \in E(K)$, $S_{\mathcal{K}}(Q) = T^+_{\mathcal{K}}(Q) = T^-_{\mathcal{K}}(Q)$ is the oriented tangent plane to the fiber surface $\phi^{-1}(\phi(Q))$ through Q. For $Q \in N(K) \setminus K$, $S_{\mathcal{K}}(Q) = \langle V(Q), |\pi(Q)| (U \times V)(Q) + (1 - |\pi(Q)|^2)^{1/2} W(Q) \rangle$ and $T^\pm_{\mathcal{K}}(Q) = \langle |\pi(Q)| V(Q) \mp (1 - |\pi(Q)|^2)^{1/2} U(Q), (U \times V)(Q) \rangle$. For $Q \in K$, $S_{\mathcal{K}}(Q) = \langle V(Q), W(Q) \rangle$ and $T^\pm_{\mathcal{K}}(Q)$ is the plane tangent to S^3 and orthogonal to V at Q, so oriented that $\pm V$ points to its positive side.*

It is easy to see that the maps $S_{\mathcal{K}}$ and $T_{\mathcal{K}}^{\pm}$ are continuous, and that their homotopy classes in $\pi_3(G)$ depend only on $\mathcal{K}$ and not on the choice of π (so the notation is not too abusive).

It is well known that G is diffeomorphic to $S^2 \times S^2$, so that $\pi_3(G)$ is isomorphic to $\mathbf{Z} \oplus \mathbf{Z}$. To obtain an explicit isomorphism (and, thus, integer invariants of $\mathcal{K}$), we use quaternions to give an explicit diffeomorphism, with pleasant properties which will facilitate later computations. Let 1, $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ be the standard orthonormal basis of $\mathbf{H}$; in terms of the identification of $\mathbf{H}$ with $\mathbf{C}^2$ already established, we have $1 = (1, 0)$, $\mathbf{i} = (i, 0)$, $\mathbf{j} = (0, 1)$, and $\mathbf{k} = (0, i)$ (note that we distinguish the quaternion $\mathbf{i}$ from the complex number i by boldface). The tangent space of S^3 at 1 is $\langle \mathbf{i}, \mathbf{j}, \mathbf{k} \rangle$, the *pure quaternions*; let $S^2 = S^3 \cap \langle \mathbf{i}, \mathbf{j}, \mathbf{k} \rangle$ be the oriented sphere of unit pure quaternions. Let $\mathcal{P}: \mathbf{H} \to \langle \mathbf{i}, \mathbf{j}, \mathbf{k} \rangle$ be the "pure part" mapping; write $\operatorname{conj}: \mathbf{H} \to \mathbf{H}$ for quaternionic conjugation; let $\mathcal{U}: \mathbf{H} \backslash \{0\} \to S^3 : Q \mapsto Q / \|Q\|$. For $Q \in \mathbf{H}$, let $L_Q: \mathbf{H} \to \mathbf{H}$ (resp., R_Q) be the real-linear map $A \mapsto QA$ (resp. $A \mapsto AQ$).

It is a fact that S^2 consists precisely of the square roots of $-1 \in H$. Hence each $\mathbf{p} \in S^2$ determines two complex structures on $\mathbf{R}^4$, with structure maps respectively $L_{\mathbf{p}}$ and $R_{\mathbf{p}}$; call $(\mathbf{R}^4, L_{\mathbf{p}})$ (resp. $(\mathbf{R}^4, R_{\mathbf{p}})$) the **p**-*left* (resp. **p**-*right*) *complex structure*. (The **i**-left complex structure is the original structure of $\mathbf{R}^4$ as $\mathbf{C}^2$; the **i**-right complex structure is as it were the direct sum $\mathbf{C} \oplus \bar{\mathbf{C}}$.) Each **p**-left (resp. **p**-right) complex structure determines a subset of G, namely, the oriented real 2-planes which are left (resp. right) **p**-stable – that is, which are complex lines in that structure. (Note that if a plane is left or right **p**-stable with one orientation, then the same plane is left or right $-\mathbf{p}$-stable with the opposite orientation.) There are, in fact, well-defined maps l and r from G to S^2 such that, for every $W \in G$, W is a complex line in the $l(W)$-left complex structure and in the $r(W)$-right complex structure, and in those structures only.

LEMMA 1.2. *If* (A, B) *is a 2-frame, then* $l(\langle A, B \rangle) = \mathcal{U}\mathcal{P}(B \operatorname{conj}(A))$ *and* $r(\langle A, B \rangle) = \mathcal{U}\mathcal{P}(\operatorname{conj}(A) B)$. *In complex coordinates, if* $A = (z_1, w_1)$, $B = (z_2, w_2)$, *then*

$$l(\langle A, B \rangle) = \mathcal{U}\mathcal{P}(\bar{z}_1 z_2 + \bar{w}_1 w_2, z_1 w_2 - w_1 z_2),$$
$$r(\langle A, B \rangle) = \mathcal{U}\mathcal{P}(\bar{z}_1 z_2 + w_1 \bar{w}_2, z_1 \bar{w}_2 - \bar{w}_1 z_2).$$

The pair $(l, r): G \to S^2 \times S^2$ *is a diffeomorphism.*

The compositions $(l, r) \circ S_{\mathcal{K}}$, $(l, r) \circ T_{\mathcal{K}}^{+}$, and $(l, r) \circ T_{\mathcal{K}}^{-}$ map S^3 to $S^2 \times S^2$ and so provide elements of $\pi_3(S^2 \times S^2) = \pi_3(S^2) \oplus \pi_3(S^2)$. Recall that the *Hopf* *invariant* $H(g)$ of a continuous map $g: S^3 \to S^2$ can be defined as follows: let $\mathbf{p}$ and $\mathbf{q}$ be distinct regular values of g_1, a map homotopic to g which is smooth near $g_1^{-1}(\{\mathbf{p}, \mathbf{q}\})$; then $H(g)$ is the linking number of the smooth links $g_1^{-1}(\mathbf{p})$ and $g_1^{-1}(\mathbf{q})$, where $g_1^{-1}(\mathbf{p})$ is oriented so that, if D is a small oriented normal disk intersecting it once positively, then $g_1 \mid D: D \to S^2$ preserves orientation (and similarly for $g_1^{-1}(\mathbf{q})$). The Hopf invariant of

$$\mathrm{Hopf}: S^3 \to S^2: (z, w) \mapsto ([|z|^2 - |w|^2]i, 2iz\bar{w}),$$

the *complex* (as opposed to conjugate-complex) *Hopf fibration,* is $+1$. The Hopf invariant of maps induces an isomorphism (denoted by the same name and letter) $H: \pi_3(S^2) \to \mathbf{Z}$. If $-: S^2 \to S^2$ is the antipodal map, then $H(-\circ g) = H(g)$ for any $g: S^3 \to S^2$, since fibers of $-\circ g$ are fibers of g with orientation reversed, and linking number is bilinear.

LEMMA 1.3. *If $T: S^3 \to G$ is such that $T(Q)$ is tangent to S^3 at Q for every $Q \in S^3$, then $H(r \circ T) = 1 + H(l \circ T)$.*

Proof. We first check a special case in which $H(l \circ T) = 0$. The tangent space to S^3 at Q is $\langle \mathbf{i}Q, \mathbf{j}Q, \mathbf{k}Q \rangle$. Let $T(Q) = \langle \mathbf{j}Q, \mathbf{k}Q \rangle$; then $(l \circ T)(Q) = \mathbf{i}$ is constant, so $H(l \circ T) = 0$. On the other hand, $(r \circ T)(Q) = \mathcal{U}\mathcal{P}(\mathrm{conj}\,(\mathbf{j}Q)\mathbf{k}Q) = Q^{-1}(-\mathbf{jk})Q = -\mathrm{Hopf}\,(Q)$, so $H(r \circ T) = 1$.

In general, let $T(Q) = \langle \mathbf{p}(Q)Q, \mathbf{q}(Q)Q \rangle$. Then

$$l(T(Q)) = \mathbf{q}(Q)Q\,\mathrm{conj}\,(\mathbf{p}(Q)Q) = -\mathbf{q}(Q)\mathbf{p}(Q) = \mathbf{p}(Q)\mathbf{q}(Q),$$
$$r(T(Q)) = \mathrm{conj}\,(\mathbf{p}(Q)Q)\mathbf{q}(Q)Q = -Q^{-1}\mathbf{p}(Q)\mathbf{q}(Q)Q,$$

so $r \circ T = Ad \circ (id, -\circ T)$, where $Ad: S^3 \times S^2 \to S^2: (Q, x) \mapsto Q^{-1}xQ$. Thus

$$\begin{aligned} H(r \circ T) &= H(Ad \circ (id, -\circ T)) = H(Ad_{\#}([id], [-\circ T])) \\ &= H(Ad_{\#}([id], [*])) + H([-\circ T])H(Ad_{\#}([*], [\mathrm{Hopf}])) \\ &= 1 + H(l \circ T) \end{aligned}$$

by the special case and the sentence preceding the lemma.

DEFINITION 1.4. *By $\lambda(\mathcal{K})$, $\rho(\mathcal{K})$, and $\tau^{\pm}(\mathcal{K})$ will be denoted the integers $-H(l \circ S_{\mathcal{K}})$ (note the sign!), $H(r \circ S_{\mathcal{K}})$, and $H(l \circ T^{\pm}_{\mathcal{K}})$, respectively.*

§2. Braided open-book structures; relations among $\tau^{\pm}$, λ, ρ, and μ.

Let $O=\{(z,0):|z|=1\}=\mathrm{Hopf}^{-1}(\mathbf{i})$, $O'=\{(0,w):|w|=1\}=\mathrm{Hopf}^{-1}(-\mathbf{i})$. Let D be a round disk on S^2 centered at $\mathbf{i}$, $N=N(O)=\mathrm{Hopf}^{-1}(D)$. The map $S^3\backslash O'\to O:(z,w)\mapsto(z/|z|,0)$ is a fibration by open great hemi-2-spheres; its restriction to N presents N as a disk-bundle neighborhood of O, with fibers *meridional disks of O*. Let R be the oriented unit tangent vectorfield to the oriented fibers of Hopf.

Let $\mathcal{K}'=(S^3,K')$ be any link. Then if $1>\varepsilon>0$, there is an ambient isotopic link $\mathcal{K}=(S^3,K)$ such that K is contained in $\operatorname{Int}N$ and, at each point of K, the component of R along the positively directed tangent line to K is at least $1-\varepsilon$. (Apply the classical lemma of Alexander to find an isotopy carrying K' onto a closed braid with axis O', on some number $n>0$ of strings; then use a "radial" isotopy in the open solid torus $S^3\backslash O'$ to make this closed braid lie arbitrarily C^1-close to O, the core of the solid torus, and note that $R\mid O$ is the field of unit tangent vectors to O.) In fact, K can be taken to have a disk-bundle neighborhood $N(K)$ that intersects each meridional disk of O in a union of n *meridional disks of K*, which can be taken to be round (in the spherical geometry of the meridional disk of O), with a trivialization $\psi:N(K)\to D^2$ such that the component of R along the oriented line $\ker(D\psi)$ is at least $1-\varepsilon$.

If, further, $\mathcal{K}$ is fibered, then there are such a trivialization of $N(K)$ as above, and a fibration ϕ of $E(K)$ over S^1, which glue together to give an open-book structure π of type $\mathcal{K}$ such that π is smooth and $\mathbf{i}$ and $-\mathbf{i}$ are regular values of $l\circ S$, $r\circ S$, $l\circ T^+$, and $r\circ T^-$ (where $S=S_{\mathcal{K}}$ and $T^+=T^{\pm}_{\mathcal{K}}$ are the plane fields constructed from $\mathcal{K}$ as in §1). We will call such a π a *braided open-book structure.*

LEMMA 2.1. *Let π be a braided open-book structure. Then*: (1) *each of* $(l\circ S)^{-1}(-\mathbf{i})$, $(r\circ S)^{-1}(\mathbf{i})$, $(l\circ T^+)^{-1}(\mp\mathbf{i})$, *and* $(r\circ T^+)^{-1}(\pm\mathbf{i})$ *has empty intersection with* $N(K)$ (*in particular, it is homologous to* 0 *in* $N(K)$); (2) *the naturally oriented* 1-*submanifold* $N(K)\cap(l\circ S)^{-1}(\mathbf{i})$ (*resp.* $N(K)\cap(r\circ S)^{-1}(-\mathbf{i})$; $N(K)\cap(l\circ \mathrm{T}^{\pm})^{-1}(\pm\mathbf{i})$; $N(K)\cap(r\circ T^{\pm})^{-1}(\mp\mathbf{i})$) *of* $N(K)$ *is homologous to* K (*resp.* $-K$; $\pm K$; $\mp K$) *in* $N(K)$.

Proof. By using R in place of V, construct plane fields $\tilde{S}$ and $\tilde{T}^{\pm}$ on $N(K)$. Note that $W=\mathbf{i}R$. Using 1.2, one calculates that $l\circ\tilde{S}$, $r\circ\tilde{S}$, $l\circ\tilde{T}^{\pm}$, and $r\circ\tilde{T}^{\pm}$ have non-negative $\mathbf{i}$ component in $N(K)$; it follows easily that the $\mathbf{i}$ components of $l\circ S$, $r\circ S$, $l\circ T^{\pm}$, and $r\circ T^{\pm}$ are bounded away from -1, establishing the statements in (1).

To verify (2), first note that (the fundamental class of) K generates $H_1(N(K);\mathbf{Z})$, so what has to be determined in each case is what integer multiple of K the 1-submanifold in question represents, and this is given by its linking number with the boundary of any one of the meridional disks K. An argument similar to that given for (1) can now be applied.

DEFINITION 2.2. *Let* $\operatorname{pos}(\mathscr{K})$ (*resp.* $\operatorname{neg}(\mathscr{K})$) *denote the oriented* 1-*submanifold* $E(K)\cap(l\circ S)^{-1}(\mathbf{i})=E(K)\cap(l\circ T^{\pm})^{-1}(\mathbf{i})$ (*resp.* $E(K)\cap(l\circ S)^{-1}(-\mathbf{i})=E(K)\cap(l\circ T^{\pm})^{-1}(-\mathbf{i})$) *of* $E(K)$.

THEOREM 2.3. *For any fibered link* $\mathscr{K}$, $\tau^+(\mathscr{K})-\tau^-(\mathscr{K})=1-\mu(\mathscr{K})$.

Proof. We may assume we have a braided open book structure π of type $\mathscr{K}$. Then $\tau^+(\mathscr{K})=H(l\circ T^+)=lk((l\circ T^+)^{-1}(\mathbf{i}),\ (l\circ T^+)^{-1}(-\mathbf{i}))$. By 2.1-2, this equals $lk(K+\operatorname{pos}(\mathscr{K}),\operatorname{neg}(\mathscr{K}))$. Similarly $\tau^-(\mathscr{K})=H(l\circ T^-)=lk((l\circ T^-)^{-1}(\mathbf{i}),\ (l\circ T^-)^{-1}(-\mathbf{i}))=lk(\operatorname{pos}(\mathscr{K}),-K+\operatorname{neg}(\mathscr{K}))$. Thus $\tau^+(\mathscr{K})-\tau^-(\mathscr{K})=lk(K,\operatorname{pos}(\mathscr{K})+\operatorname{neg}(\mathscr{K}))$. Now, the linking number of K with an oriented 1-submanifold L of $S^3\backslash K$ is equal to the inter-section number of L with any Seifert surface of K. It follows that $lk(K,\operatorname{pos}(\mathscr{K})+\operatorname{neg}(\mathscr{K}))$ is the intersection number of $\operatorname{pos}(\mathscr{K})+\operatorname{neg}(\mathscr{K})$ with a fiber surface F of $\mathscr{K}$, that is, the algebraic number of points of F where the tangent plane to F is left $\pm\mathbf{i}$-stable. These points are exactly the zeroes of a certain tangent vectorfield on F (namely, the orthogonal projection of R into the tangent plane to F), and the sum of the indices of the zeroes of that vectorfield equals $1-\mu(\mathscr{K})$, the Euler characteristic of F. The theorem follows upon observing that the multiplicity assigned to a point of $(\operatorname{pos}(\mathscr{K})+\operatorname{neg}(\mathscr{K}))\cap F$ by the orientation of $\operatorname{pos}(\mathscr{K})+\operatorname{neg}(\mathscr{K})$ is the index of that vectorfield at the point.

Remark 2.4. In some vague sense, the new information in the splitting of $1-\mu(\mathscr{K})$ as $\tau^+(\mathscr{K})-\tau^-(\mathscr{K})$ is coming from carrying out the vectorfield argument "all around the circle" of fiber surfaces.

THEOREM 2.5. *For any fibered link* $\mathscr{K}$, $\lambda(\mathscr{K})=-\tau^+(\mathscr{K})$ *and* $\rho(\mathscr{K})=\tau^-(\mathscr{K})+1$.

COROLLARY 2.6. *For any fibered link* $\mathscr{K}$, $\lambda(\mathscr{K})+\rho(\mathscr{K})=\mu(\mathscr{K})$.

Proof of 2.5. By 2.1-2, $\lambda(\mathscr{K})=-H(l\circ S)=-lk(K+\operatorname{pos}(\mathscr{K}),\ \operatorname{neg}(\mathscr{K}))=-\tau^+(\mathscr{K})$. Just for this proof, let $\operatorname{POS}(\mathscr{K})=E(K)\cap(r\circ S)^{-1}(\mathbf{i})$, $\operatorname{NEG}(\mathscr{K})=E(K)\cap(r\circ S)^{-1}(-\mathbf{i})$. Then $\rho(\mathscr{K})=H(r\circ S)=lk(-K+\operatorname{POS}(\mathscr{K}),\operatorname{NEG}(\mathscr{K}))=H(r\circ T^-)=H(l\circ T^-)+1=\tau^-(\mathscr{K})+1$ (using 1.3).

§3. Isolated critical points; λ and ρ as intersection numbers

Let U be an open neighborhood of a point $\mathbf{x}$ in $\mathbf{R}^4$; let $f: U \to \mathbf{R}^2$ be continuous at $\mathbf{x}$ and smooth in $U \backslash \{\mathbf{x}\}$. Denote by Df the (real) differential of f, which we take to be a smooth mapping from $U \backslash \{\mathbf{x}\}$ into the space of 2-by-4 matrices, the rows of $Df(\mathbf{y})$ being the gradients at $\mathbf{y}$ of the components of f. As usual, $\mathbf{y}$ is called a *regular point* of f if $Df(\mathbf{y})$ has rank 2, a *critical point* otherwise. Slightly extending the standard usage, we will call $\mathbf{x}$ an *isolated critical point* of f if, for $\varepsilon > 0$ sufficiently small, every $\mathbf{y}$ with $0 < \|\mathbf{x} - \mathbf{y}\| \le \varepsilon$ is a regular point of f. (So, if it happens that f is smooth at $\mathbf{x}$ and $\mathbf{x}$ is a regular point of f, by this usage $\mathbf{x}$ is also an isolated critical point of f.) By "counting constants" one finds that the expected dimension of the set of critical points of a smooth mapping is 1; thus, the (genuinely critical) isolated critical points are unusual, but of correspondingly great interest.

If $\mathbf{y}$ is a regular point of f, then the matrix $Df(\mathbf{y})$ considered as an ordered pair of rows is a 2–frame, and so $\langle Df \rangle$ is a smooth mapping from the set of regular points of f into the Grassmann manifold G.

DEFINITION 3.1. *Let* $\mathbf{x}$ *be an isolated critical point of* f. *For* $\varepsilon > 0$ *small enough that* $\mathbf{y}$ *is a regular point for* $0 < \|\mathbf{x} - \mathbf{y}\| \le \varepsilon$, *let* $E: S^3 \to S^3(\mathbf{x}, \varepsilon): \mathbf{u} \mapsto \mathbf{x} + \varepsilon \mathbf{u}$; *then define* $\lambda(f; \mathbf{x}) = -H(l \circ \langle Df \rangle \circ E)$, $\rho(f; \mathbf{x}) = H(r \circ \langle Df \rangle \circ E)$. (Clearly these do depend only on f and $\mathbf{x}$.)

Two basic facts about isolated critical points are relevant here: (A) when f is sufficiently well-behaved (e.g., real-polynomial) near its isolated critical point $\mathbf{x}$, there is an associated "local link" $\mathcal{K}(f; \mathbf{x})$, well-defined up to ambient isotopy, and $\mathcal{K}(f; \mathbf{x})$ is fibered; (B) conversely, given a fibered link $\mathcal{K}$ in S^3, there may be constructed a well-behaved $f_{\mathcal{K}}: \mathbf{R}^4 \to \mathbf{R}^2$ with an isolated critical point at $\mathbf{0}$, such that $\mathcal{K}(f_{\mathcal{K}}; \mathbf{0}) = \mathcal{K}$. More details will be recalled shortly.

Remark 3.2. Milnor [7, Sect. 10] proved (A) for polynomial mappings; Kauffman and Neumann [4] extracted from his proof the relevant property of real-polynomial maps, which they called "tameness", see 3.3. So far as I know, (B) was shown first by Looijenga [6], and rederived in [4] (see 3.7). All this work is in general dimensions. None of it describes λ or ρ.

If $\mathbf{x}$ is a regular point of f and U is a sufficiently small open neighborhood of $\mathbf{x}$, then $U \cap f^{-1}(f(\mathbf{x}))$ is a smooth 2-submanifold of U. If $\mathbf{x}$ is an isolated critical point of (even a smooth) f, then this generally fails (but not always: cf. examples in [7]); all we can say is that, for suitably small U, the level set $f^{-1}(f(\mathbf{x}))$

intersects U in a "2-submanifold with an isolated singularity at $\mathbf{x}$". To give a more precise description of this singularity, we have to impose extra hypotheses on f in a neighborhood of $\mathbf{x}$.

DEFINITION 3.3 ([4]). *Let* $\mathbf{x}$ *be an isolated critical point of* f. *Then* f *is tame at* $\mathbf{x}$ *if, for all sufficiently small* $\varepsilon > 0$,

(A) *the level set* $f^{-1}(f(\mathbf{x}))$ *intersects* $S^3(\mathbf{x}; \varepsilon)$ *transversely,*

(B) *for all sufficiently small* $\delta = \delta(\varepsilon) > 0$ *the intersection* $f^{-1}(D^2(f(\mathbf{x}); \delta)) \cap D^4(\mathbf{x}; \varepsilon)$ *is a 4-ball, smooth except for corners along* $f^{-1}(S^1(f(\mathbf{x}); \delta)) \cap S^3(\mathbf{x}; \varepsilon)$. *If* f *is tame at* $\mathbf{x}$, *then (for any sufficiently small* $\varepsilon > 0$) *let* $\mathcal{K}(f; \mathbf{x}) = (S^3, E^{-1}(S^3(\mathbf{x}; \varepsilon) \cap f^{-1}(f(\mathbf{x}))))$; *this is the* local link *of* f *at* $\mathbf{x}$.

Remarks 3.4. (1) As given in [4], the definition of "tame" includes the (inessential) further hypothesis that f is smooth at $\mathbf{x}$. (2) A mapping can have an isolated critical point at which it is smooth but not tame. (3) Hypothesis (A) of 3.3 already ensures that $\mathcal{K}(f; \mathbf{x})$ is well-defined (up to ambient isotopy); hypothesis (B) ensures that $\mathcal{K}(f; \mathbf{x})$ is fibered. (4) As remarked in [4], the proof of the "fibration theorem for real singularities" in [7] consists of showing that a real polynomial mapping is tame at an isolated critical point.

CONSTRUCTION 3.5. Let $\pi : S^3 \to D^2$ be a smooth open book structure of type $\mathcal{K}$. Define cone $\pi : \mathbf{R}^4 \to \mathbf{R}^2$ by $(\text{cone } \pi)(\mathbf{y}) = \|\mathbf{y}\| \, \pi(\mathbf{y}/\|\mathbf{y}\|)$ if $\mathbf{y} \neq \mathbf{0}$, $(\text{cone } \pi)(\mathbf{0}) = (0, 0)$. Then the only critical point of cone π is $\mathbf{0}$, cone π is tame at $\mathbf{0}$, and $\mathcal{K}(\text{cone } \pi; \mathbf{0}) = \mathcal{K}$.

Remarks 3.6. (1) This construction is a stripped-down version of the original one in [6]. (Looijenga showed that, by an appropriate choice of π, cone π can be taken to be a real polynomial in $\mathbf{x}$ and $\|\mathbf{x}\|$, and thus real-algebraic, though typically not smooth but merely continuous at $\mathbf{0}$; when, however, $\mathcal{K}$ is antipodally equivariant – in particular if it is a connected sum of some fibered knot with itself – then cone π can be taken to be a polynomial in $\mathbf{x}$ alone. It was to this case that Looijenga drew explicit attention.) (2) By replacing $\|\mathbf{x}\|$ with a smooth, monotone function of $\|\mathbf{x}\|$ infinitely flat at 0, cf. [4], cone π can be assumed smooth (but transcendental) at $\mathbf{0}$.

PROPOSITION 3.7. *If* $\mathcal{K}$ *is a fibered link with smooth open-book structure* π, *then* $\lambda(\mathcal{K}) = \lambda(\text{cone } \pi; \mathbf{0})$ *and* $\rho(\mathcal{K}) = \rho(\text{cone } \pi; \mathbf{0})$. *If* $\mathbf{x}$ *is an isolated critical point of* f *and* f *is tame at* $\mathbf{x}$ *then* $\lambda(\mathcal{K}(f; \mathbf{x})) = \lambda(f; \mathbf{x})$ and $\rho(\mathcal{K}(f; \mathbf{x})) = \rho(f; \mathbf{x})$.

Proof. We may assume π is braided (§2). By taking the ε in the definition of "braided open-book structure" sufficiently small, one may make the 2-plane fields $S_{\mathcal{K}}$ and $\langle D(\text{cone}\,\pi)\rangle \mid S^3$ arbitrarily close; so they are homotopic.

Let crit (f) denote the set of critical points of f. For each $\mathbf{p} \in S^2$, consider the sets

$$L^*(f, \mathbf{p}) = \text{crit}\,(f) \cup (l \circ \langle Df\rangle)^{-1}(\mathbf{p}),$$
$$R^*(f, \mathbf{p}) = \text{crit}\,(f) \cup (r \circ \langle Df\rangle)^{-1}(\mathbf{p}).$$

If $A(\mathbf{y})$ and $B(\mathbf{y})$ denote the rows of $Df(\mathbf{y})$, considered as quaternions, then (cf. 1.2)

$$L^*(f, \mathbf{p}) \cup L^*(f, -\mathbf{p}) = \{\mathbf{y} : \mathcal{P}(\mathbf{p}\mathcal{P}(B(\mathbf{y})(\text{conj}\,A(\mathbf{y})))) = \mathbf{0}\},$$
$$R^*(f, \mathbf{p}) \cup R^*(f, -\mathbf{p}) = \{\mathbf{y} : \mathcal{P}(\mathbf{p}\mathcal{P}((\text{conj}\,A(\mathbf{y}))B(\mathbf{y}))) = \mathbf{0}\},$$

while $L^*(f, \mathbf{p}) = \{\mathbf{y} \in L^*(f, \mathbf{p}) \cup L^*(f, -\mathbf{p}) : \mathbf{p}\mathcal{P}(B(\mathbf{y})(\text{conj}\ A(\mathbf{y}))) \leq 0\}$ (and so on), when we identify $\langle 1\rangle \subset H$ with $\mathbf{R}$. Note that, for $\mathbf{p} \neq \mathbf{q}$, $L^*(f, \mathbf{p}) \cap L^*(f, \mathbf{q}) = \text{crit}\,(f) = R^*(f, \mathbf{p}) \cap R^*(f, \mathbf{q})$. (Note also that, though $L^*(f, \mathbf{p}) \cup L^*(f, -\mathbf{p})$ and $R^*(f, \mathbf{P}) \cup R^*(f, -\mathbf{p})$ are level sets of mappings to a 3-dimensional vectorspace, their expected codimension is not 3 but 2 because of the Plücker conditions.)

Now suppose crit $(f) \cap D^4(\mathbf{x}; \varepsilon) \subset \{\mathbf{x}\}$.

HYPOTHESIS 3.8. $\mathbf{p}$, $\mathbf{q} \in S^2$, $\mathbf{p} \neq \mathbf{q}$, are such that (with respect to some convenient theory of geometric cycles representing ordinary homology over Z) the set $(l \circ \langle Df\rangle)^{-1}(\mathbf{p}) \cap S^3(\mathbf{x}; \varepsilon)$ (resp. $(l \circ \langle Df\rangle)^{-1}(\mathbf{q}) \cap S^3(\mathbf{x}; \varepsilon)$; $(r \circ \langle Df\rangle)^{-1}(\mathbf{p}) \cap S^3(\mathbf{x}; \varepsilon)$; $(r \circ \langle Df\rangle)^{-1}(\mathbf{q}) \cap S^3(\mathbf{x}; \varepsilon)$) is the support of an absolute 1-cycle in $S^3(\mathbf{x}; \varepsilon)$ which bounds a relative 2-cycle in $D^4(\mathbf{x}; \varepsilon)$ supported by $L^*(f, \mathbf{p}) \cap D^4(\mathbf{x}; \varepsilon)$ (resp. $L^*(f, \mathbf{q}) \cap D^4(\mathbf{x}; \varepsilon)$; $R^*(f, \mathbf{p}) \cap D^4(\mathbf{x}; \varepsilon)$; $R^*(f, \mathbf{q}) \cap D^4(\mathbf{x}; \varepsilon)$). (We will use the same symbols for the cycles and their supports.)

PROPOSITION 3.9. *Under Hypothesis* 3.8, $\lambda(f; \mathbf{x})$ (resp. $\rho(f; \mathbf{x})$) *is the homological intersection number at* $\mathbf{x}$ *of* $L^*(f, p) \cap D^4(\mathbf{x}; \varepsilon)$ *and* $L^*(f, \mathbf{q}) \cap D^4(\mathbf{x}; \varepsilon)$ (*resp.* $R^*(f, \mathbf{p}) \cap D^4(\mathbf{x}; \varepsilon)$ *and* $R^*(f, \mathbf{q}) \cap D^4(\mathbf{x}; \varepsilon)$).

Proof. This is a tautology, given the definitions of λ and ρ as Hopf invariants and the relationship between linking numbers and intersection numbers.

Remarks 3.10. (1) The point of 3.8-9 is that frequently 3.8 can be verified, as, for instance, in the examples in §4. (2) One might conjecture that, for any f with an isolated critical point at f, 3.8 holds for almost all pairs $(\mathbf{p}, \mathbf{q})$. Certainly it seems reasonable to expect, of a given f, that for almost all $\mathbf{p}$ the sets $L^*(f, \mathbf{p})$ and $R^*(f, \mathbf{p})$ are "2-manifolds with isolated singularities at $\mathbf{x}$". Perhaps some sort of higher-order tameness should be defined. (3) If f is a real-polynomial mapping, then, for any $\mathbf{p}$, $L^*(f, \mathbf{p}) \cup L^*(f, -\mathbf{p})$ is a real-algebraic set and $L^*(f, \mathbf{p})$ is semi-algebraic (of course the same goes for R^*). Suppose $\mathbf{x}$ is an isolated critical point of f and that $\mathbf{p}, \mathbf{q} \in S^2$, $\mathbf{q} \neq \pm\mathbf{p}$, are such that both $L^*(f, \mathbf{p}) \cup L^*(f, -\mathbf{p})$ and $L^*(f, \mathbf{q}) \cup L^*(f, -\mathbf{q})$ are purely 2-dimensional near $\mathbf{x}$. Then, near $\mathbf{x}$, each of $L^*(f, \mathbf{p})$, $L^*(f, -\mathbf{p})$, $L^*(f, \mathbf{q})$, and $L^*(f, -\mathbf{q})$ is the cone on some singular-link-with-integer-multiplicities, so 3.8 holds. One might be tempted, therefore, to reason from the distributive law that "$4\lambda(f, \mathbf{x})$ is the *real-algebro-geometric* intersection number at $\mathbf{x}$ of the real-algebraic surfaces $L^*(f, \mathbf{p}) \cup L^*(f, -\mathbf{p})$ and $L^*(f, \mathbf{q}) \cup L^*(f, -\mathbf{q})$". It seems hard to make that statement true inside real algebraic geometry! (Real-algebraic cycles are naturally oriented over $\mathbf{Z}/2\mathbf{Z}$ rather than over $\mathbf{Z}$. In the present case, even if each of the algebraic sets $L^*(f, \mathbf{p}) \cup L^*(f, -\mathbf{p})$ is purely 2-dimensional, giving them local $\mathbf{Z}$-orientations algebro-geometrically is complicated by the fact that the parameter space for this family of surfaces is the non-orientable real projective plane RP^2 rather than S^2; cf. the last sentence of 4.1.) Perhaps there exists (I have not been able to learn of it) an applicable theory of integer intersection numbers, *calculable inside real semi-algebraic geometry*, and giving the correct topological answers? (4) As mentioned in §0, I know (as of December, 1986) of no example of a fibered link $\mathcal{K}$ with $\lambda(\mathcal{K}) < 0$. Especially if no such *link* exists, it would be interesting to known whether there exists a *function* f with an isolated critical point $\mathbf{x}$ such that $\lambda(f; \mathbf{x}) < 0$.

§4. Examples

Most of the examples in this section involve complex analyticity somehow, so we begin by introducing some complex machinery.

MACHINERY 4.1. At a point where $F: \mathbf{C}^2 \to \mathbf{C}$ is smooth, the *complex differential* $D_{\mathbf{C}}F$ is the complex row vector $[F_z F_{\bar{z}} F_w F_{\bar{w}}]$, where $F_z = (F_x - iF_y)/2$, $F_{\bar{z}} = (F_x + iF_y)/2i$, etc., and subscripts indicate partial differentiation. In terms of $D_{\mathbf{C}}F$, the real differential matrix DF is

$$\begin{bmatrix} \operatorname{Re}(F_z + F_{\bar{z}}) & \operatorname{Re}(iF_z - iF_{\bar{z}}) & \operatorname{Re}(F_w + F_{\bar{w}}) & \operatorname{Re}(iF_w - iF_{\bar{w}}) \\ \operatorname{Im}(F_z + F_{\bar{z}}) & \operatorname{Im}(iF_z - iF_{\bar{z}}) & \operatorname{Im}(F_w + F_{\bar{w}}) & \operatorname{Im}(iF_w - iF_{\bar{w}}) \end{bmatrix}.$$

As in 1.2, we see that at a regular point of F, $l(\langle DF\rangle)$ is the unit vector of

$$(|F_z|^2-|F_{\bar z}|^2+|F_w|^2-|F_{\bar w}|^2)\mathbf{i}-2\,\mathrm{Im}\,(F_z\overline{F_{\bar w}}-\overline{F_{\bar z}}F_w)\mathbf{j}-2\,\mathrm{Re}\,(F_z\overline{F_{\bar w}}-\overline{F_{\bar z}}F_w)\mathbf{k};\quad (*)$$

similarly, at a regular point of F, $r(\langle DF\rangle)$ is the unit vector of

$$(|F_z|^2-|F_{\bar z}|^2+|F_{\bar w}|^2-|F_w|^2)\mathbf{i}-2\,\mathrm{Im}\,(F_z\overline{F_w}-\overline{F_{\bar z}}F_{\bar w})\mathbf{j}-2\,\mathrm{Re}\,(F_z\overline{F_w}-\overline{F_{\bar z}}F_{\bar w})\mathbf{k};\quad (**)$$

and (if F is smooth everywhere) crit (F) is defined by the vanishing of either $(*)$ or $(**)$.

In this complex context, we will have a particular interest in $L^*(F,\mathbf{i})\cup L^*(F,-\mathbf{i})$ and $R^*(F,\mathbf{i})\cup R^*(F,-\mathbf{i})$. As sets,

$$L^*(F,\pm\mathbf{i})=\{F_z\overline{F_{\bar w}}-\overline{F_{\bar z}}F_w=0,\ \pm(|F_z|^2-|F_{\bar z}|^2+|F_w|^2-|F_{\bar w}|^2)\geqslant 0\}$$

and

$$R^*(F,\pm\mathbf{i})=\{F_z\overline{F_w}-\overline{F_{\bar z}}F_{\bar w}=0,\ \pm(|F_z|^2-|F_{\bar z}|^2-|F_w|^2+|F_{\bar w}|^2)\geqslant 0\}.$$

Suppose $F_z\overline{F_{\bar w}}-\overline{F_{\bar z}}F_w$ and $F_z\overline{F_w}-\overline{F_{\bar z}}F_{\bar w}$ are products of complex analytic functions and conjugates of complex analytic functions. Then their level sets, where they are 2-dimensional, are equipped with natural integer multiplicities; in particular this is true of the sets of zeroes, and so at any isolated critical point of F near which $L^*(F,\pm\mathbf{i})$ and $R^*(F,\pm\mathbf{i})$ are 2-dimensional, Hypothesis 3.8 will be satisfied (with $\mathbf{p}=\mathbf{i}$, $\mathbf{q}=-\mathbf{i}$). Note, however, that the multiplicity assigned by the defining function must be twisted by the sign of $\mathbf{i}$ to give the multiplicity needed for 3.8 (consider the local coordinates on S^2 given by stereographic projection from the two poles $\mathbf{i}$ and $-\mathbf{i}$).

EXAMPLE 4.2. Let $f:\mathbf{C}^2\to\mathbf{C}$ be a complex polynomial. If f is squarefree, then any critical point (z,w) is necessarily isolated. Claim: in this case, $\lambda(\mathcal{K}(f;(z,w)))=0$. Proof: at any regular point of f, ker Df is a complex line, so $l\circ\langle Df\rangle$ is identically $\mathbf{i}$ and $\lambda(\mathcal{K}(f;(z,w))=\lambda(f;(z,w))$ is the Hopf invariant of a constant.

It follows from 2.6 that $\rho(\mathcal{K}(f;(z,w)))=\mu(\mathcal{K}(f;(z,w)))$. In fact, $R^*(f;-\mathbf{i})$ is the complex plane curve $\{f_z=0\}$ with the opposite orientation to that given by its complex structure, and $R^*(f;\mathbf{i})$ is the conjugate-complex plane curve $\{\overline{f_w}=0\}$ with the orientation given by its conjugate-complex structure; the intersection number at (z,w) of these cycles is then $(-1)\cdot(-1)=1$ times the intersection number at (z, w) of the complex plane curves $\{f_z = 0\}$ and $\{f_w = 0\}$, which is Milnor's definition of $\mu(\mathcal{K}(f; (z, w)))$ [7, p. 59].

Remarks 4.3. (1) The links $\mathcal{K}(f; (z, w))$ are well understood (cf. [3], [5], etc.); they are (quite restricted) iterated torus links, and also (very special) closed strictly positive braids. In [12] it is shown that $\lambda(\mathcal{K}) = 0$, $\rho(\mathcal{K}) = \mu(\mathcal{K})$ for any closed strictly positive braid $\mathcal{K}$. (More generally, if $\mathcal{K}$ is a closed strictly homogeneous braid [14] or even a closed generalized strictly homogeneous braid, then $\lambda(\mathcal{K})$ and $\rho(\mathcal{K})$ are the negative and positive parts of $\mu(\mathcal{K})$ in an obvious sense.) In [9], λ and ρ are calculated for all fibered iterated torus links. (2) 4.2 substantiates the intuition that the link of a complex plane curve singularity (or any closed strictly positive braid) is somehow "as positive as it can be". It should be contrasted with the fact that, though the symmetrized Seifert form of such a link has non-negative signature, [11], it is only rarely positive-definite – for complex plane curves, this happens exactly when the singularity is "simple" in the sense of Arnol'd. (Actually, the sign convention in [11] is unusual; with the more standard one, a closed positive braid has non-positive signature.)

EXAMPLE 4.4. Let $\mathrm{Rev}: S^3 \to S^3$ be an orientation-reversing diffeomorphism. The mirror image of a link $\mathcal{K} = (S^3, K)$ is the link $\mathrm{Rev}\,\mathcal{K} = (S^3, \mathrm{Rev}\,K)$. Claim: if $\mathcal{K}$ is fibered, then $\lambda(\mathrm{Rev}\,\mathcal{K}) = \rho(\mathcal{K})$ (so also $\rho(\mathrm{Rev}\,\mathcal{K}) = \lambda(\mathcal{K})$). Proof: this is a simple calculation from the formulas in 2.3 and 2.5-6. (More generally, if $f: \mathbf{H} \to \mathbf{H}$ has an isolated critical point at $\mathbf{0}$, then $\mathrm{conj} \circ f$ has also and $\lambda(\mathrm{conj} \circ f; \mathbf{0}) = \rho(f; \mathbf{0})$, by consideration of the effect of conj on $\pi_3(G)$; of course, $\mathcal{K}(\mathrm{conj} \circ f; \mathbf{0})$ is a mirror image of $\mathcal{K}(f; \mathbf{0})$.) In particular, if $\mathcal{K}$ is amphicheiral (i.e., isotopic to its mirror image) then $\lambda(\mathcal{K}) = \rho(\mathcal{K})$.

EXAMPLE 4.5. The figure-8 knot $\mathcal{K}$ is amphicheiral, and $\mu(\mathcal{K}) = 2$, so by 4.4, $\lambda(\mathcal{K}) = 1 = \rho(\mathcal{K})$. Now, $\mathcal{K}$ is a closed homogeneous braid (the closure of the homogeneous braid word $\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}$ in the 3-string braid group B_3), and the techniques of [14] could also be brought to bear. But it most entertaining to calculate $\lambda(\mathcal{K})$ and $\rho(\mathcal{K})$ by 4.1.

First let $F(z, w) = w^3 - 3|z|^2(1 + z - \bar{z})w - 2(z + \bar{z})$. Then $D_{\mathbf{C}}F(z, w) = [-3w(\bar{z} + 2|z|^2 - \bar{z}^2) - 2 \quad -3w(z + z^2 - 2|z|^2) - 2 \quad 3w^2 - 3|z|^2(1 + z - \bar{z})\ 0]$ so $L^*(F, \mathbf{i}) \cup L^*(F, -\mathbf{i}) = \{(-3\bar{w}(\bar{z} + \bar{z}^2 - 2|z|^2) - 2)(w^2 - |z|^2(1 + z - \bar{z})) = 0\}$ and $R^*(F, \mathbf{i}) \cup R^*(F, -\mathbf{i}) = \{(-3w(\bar{z} + 2|z|^2 - \bar{z}^2) - 2)(\bar{w}^2 - |z|^2(1 + z - \bar{z})) = 0\}$. For $|z|^2 + |w|^2$ small, $(-3\bar{w}(\bar{z} + \bar{z}^2 - 2|z|^2) - 2)(-3w(\bar{z} + 2|z|^2 - \bar{z}^2) - 2) \neq 0$, so at a point of $\mathrm{crit}(F)$ near $(0, 0)$, $w^2 = |z|^2(1 + z - \bar{z})$, $w = \pm|z| + o(|z|^3)$; then $|F_z|^2 - |F_{\bar{z}}|^2 + |F_w|^2 - |F_{\bar{w}}|^2 = \pm 4\ |z|\ \mathrm{Re}\ (\bar{z} - z + 4|z|^2 - \bar{z}^2 - z^2) + o(|z|^4) =$

$\pm 8\,|z|\,((\mathrm{Re}\,z)^2+3(\mathrm{Im}\,z)^2)+o(|z|^4)$. Since $(\mathrm{Re}\,z)^2+3(\mathrm{Im}\,z)^2$ is positive-definite, the critical point of F at $(0,0)$ is isolated. Also, $L^*(F, \pm\mathbf{i})$ (resp. $R^*(F, \pm\mathbf{i})$) is well approximated near $(0,0)$ by $\{w=\pm|z|\}$ (resp. $\{w=\mp|z|\}$). These cycles have intersection number 0 at $(0,0)$, so $\lambda(F;(0,0))=0=\rho(F;(0,0))$, so $\mu(\mathscr{K}(F;(0,0)))=0$; though $(0,0)$ is a genuine critical point, $\mathscr{K}(F;(0,0))$ is unknotted.

Let $G(z,w)=F(z^2,w)$; again $(0,0)$ is an isolated critical point;

$$L^*(G,\mathbf{i})\cup L^*(G,-\mathbf{i})=\{(2\bar{z}[-3\bar{w}(\bar{z}^2+\bar{z}^4-2\,|z|^4)-2])\\ \times(w^2-|z|^4\,(1+z^2-\bar{z}^2))=0\}$$

and

$$R^*(G,\mathbf{i})\cup R^*(G,-\mathbf{i})=\{(2z[-3w(\bar{z}^2+2\,|z|^4-\bar{z}^4)-2])(\bar{w}^2-|z|^4\,(1+z^2-\bar{z}^2))\}$$

so one quickly calculates $\lambda(G;(0,0))=1=\rho(G;(0,0))$, $\mu(\mathscr{K}(G;(0,0)))=2$. Now, $K(G;(0,0))$ double-covers $K(F;(0,0))$, which is connected, so it has 1 or 2 components – but $\mu(\mathscr{K}(G;(0,0))$ is even, so $K(G;(0,0))$ has an odd number of components. Thus $K(G;(0,0))$ is connected and $\mathscr{K}(G;(0,0))$ is a knot. It must be the figure-8 knot. (Only three fibered *knots* have Milnor number 2 – the two trefoils and the figure-8 knot. One trefoil is $\mathscr{K}(z^2+w^3;(0,0))$; $\lambda(\mathscr{K}(z^2+w^3;(0,0))=0$ by 4.2. The other trefoil is $\mathrm{Rev}\,\mathscr{K}(z^2+w^3;(0,0))$; $\rho(\mathrm{Rev}\,\mathscr{K}(z^2+w^3;(0,0)))=0$ by 4.4.)

Of course it is easy enough to see directly that $\mathscr{K}(G;(0,0))$ is a figure-8 knot, by considering the closed braid cut out by $G=0$ in a sufficiently small bidisk boundary $\{(z,w):|z|\le\varepsilon,\ |w|\le\varepsilon'\}$, which is readily seen to be the closure of $\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}$.

Remark 4.6. Perron was the first to give a real-polynomial mapping $\mathbf{R}^4\to\mathbf{R}^2$ having an isolated critical point with local link the figure-8 knot, [10]. His polynomial is somewhat more complicated than that in 4.5, and in particular has resisted my occasional attempts to use it to calculate λ and ρ; the "half-complex" nature of F and G (the vanishing of their $\bar{w}$-derivatives) is a great simplification.

EXAMPLE 4.7. Let $f(z,w)=z^2+w^3$, $g(z,w)=z^3+w^2$, $f=f\bar{g}$. Then

$$D_{\mathbf{C}}F(z,w)=[2z\bar{g}\ 3\bar{z}^2f\ 3w^2\bar{g}\ 2\bar{w}f],$$
$$L^*(F,\mathbf{i})\cup L^*(F,-\mathbf{i})=\{(4zw-9z^2w^2)\bar{f}\bar{g}=0\},$$
$$R^*(F,\mathbf{i})\cup R^*(F,-\mathbf{i})=\{6z\bar{w}^2\,|g|^2-6z^2\bar{w}\,|f|^2=0\}.$$

The origin is an isolated critical point of F. 4.1 applies directly to calculate $\lambda: L^*(\mathbf{i}) = \{z\bar{f} = 0\}$ and $-L^*(F, -\mathbf{i}) = \{w\bar{g}(4-9zw) = 0\}$ as cycles, so $\lambda(F; (0,0)) = 1$. (Of course the complex curve $4 - 9zw = 0$ doesn't pass through $(0,0)$ so it isn't involved in the calculation.) 4.1 doesn't quite apply to calculate ρ, since (as a set) $R^*(F, \mathbf{i}) = \{\bar{w}\,|g|^2 - z\,|f|^2 = 0\}$, and $\bar{w}\,|g|^2 - z\,|f|^2$ isn't just the product of some complex analytic and some conjugate-analytic factors. But one may verify that there is a neighborhood of $(0,0)$ which has the same intersection with $R^*(F, \mathbf{i})$ as it has with $\{w - \bar{z} = 0\}$, and then calculate $\rho(F; (0,0)) = 2$ (since $-R^*(F, -\mathbf{i}) = \{z\bar{w} = 0\}$ as a cycle).

This example can be generalized. Let a, b, c, d be positive integers, $f(z, w) = z^a + w^b$, $g(z, w) = z^c + w^d$, $F(z, w) = f(z, w)g(\bar{z}, \bar{w})$, $G(z, w) = f(z, w)g(z, \bar{w})$. Then $\mathcal{K}(F; (0,0))$ and $\mathcal{K}(G; (0,0))$ are certain iterated torus links (of GCD (a, b) + GCD $(c, d) > 1$ components). For most choices of a, b, c, d, the critical point of F (resp. G) at $(0,0)$ is isolated so $\mathcal{K}(F; (0,0))$ (resp. $\mathcal{K}(G; (0,0))$) is a fibered link; the invariants $\lambda(F; (0,0)), \ldots, \rho(G; (0,0))$ can be calculated. Typically such a link is neither (isotopic to) the link of a complex plane curve singularity nor (isotopic to) the mirror image of such a link; this is detected by λ and ρ without recourse to the classification of links of curve singularities. Note that for certain bad choices of exponents, the critical point of F or of G at $(0,0)$ will not be isolated; e.g., $a = c$ is bad for F, and $b = d$ is bad for both F and G. Note also that it can be determined just which of the links $\mathcal{K}(F; (0,0))$ and $\mathcal{K}(G; (0,0))$ are, and are not, fibered – for instance, by using the calculus of splice diagrams [3]. Interestingly, it appears that whenever $\mathcal{K}(F; (0,0))$ is fibered, in fact F has an isolated critical point at $(0,0)$, and likewise for G (cf. [13]).

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REFERENCES

[1] JOAN BIRMAN and R. F. WILLIAMS, "*Knotted periodic orbits in dynamical systems.* I: Lorenz equations", *Topology 22* (1983).

[2] DAVID GABAI, Lectures at MSRI, Spring 1985.

[3] DAVID EISENBUD and WALTER D. NEUMANN, *Three-dimensional link theory and invariants of plane curve singularities, Ann. Math. Studies 110* (1985), Princeton Univ. Press.

[4] LOUIS H. KAUFFMAN and WALTER D. NEUMANN, "Products of knots, branched fibrations and sums of singularities", *Topology 16* (1977), 369–393.

[5] LÊ DŨNG TRÁNG, "Sur les noeuds algébriques", *Comp. Math. 25* (1972), 282–322.

[6] E. LOOIJENGA, "A note on polynomial isolated singularities", *Indag. Math. 33* (1971), 418–421.

[7] J. MILNOR, *Singular Points of Complex Hypersurfaces, Ann. Math. Studies 61* (Princeton University Press, 1971).

[8] WALTER NEUMANN and LEE RUDOLPH, "Unfoldings in knot theory", *Math. Annalen* (1987).

[9] ——, "Computing the enhanced Milnor number", in preparation (1987).

[10] B. PERRON, "Le Noeud 'Huit' est Algébrique Réel", *Invent. Math. 65* (1982), 441–451.

[11] LEE RUDOLPH, "Nontrivial positive braids have positive signature", *Topology 21* (1982), 325–327.

[12] ——, "Isolated critical points of mappings from $\mathbf{R}^4$ to $\mathbf{R}^2$ and a natural splitting of the Milnor number of a classical fibred link. Part II: Totally braided open-book structures and closed generalized homogeneous braids, with a remark on the generalized Jones polynomial of some homogeneous braids", *Geometry & Topology: Manifolds, Varieties, and Knots: Proceedings of the* 1985 *Georgia Topology Conference* (C. McCrory and T. Shifrin, eds.), Marcel Dekker, New York (1986), 251–263.

[13] ——, "Mutually braided open books and new invariants of fibered links", to appear in *Contemp. Math.* (Proceedings of 1986 Santa Cruz conference on Artin's Braid Group, A. Libgober, ed.).

[14] J. R. STALLINGS, "Constructions of fibred knots and links", *Proc. Symp. Pure Math.* XXXII, part 2 (1978), 55–60.

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