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## Bounded domains with prescribed group of automorphisms

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### § 0. Introduction

By an automorphism of a complex manifold  $\Omega$  we mean a biholomorphic mapping  $f: \Omega \rightarrow \Omega$ . A classical result of H. Cartan (see [9]) states that for a bounded domain  $\Omega \subset \mathbb{C}^n$   $\text{Aut}(\Omega)$  has the structure of a Lie group. This is also the case if  $\Omega$  is a relatively compact domain in a Stein manifold.

Let  $\Omega \Subset \tilde{\Omega}$  be a relatively compact domain in a Stein manifold with a  $C^2$ , strongly pseudoconvex boundary. It is known [11] that if  $\text{Aut}(\Omega)$  is not compact then  $\Omega$  is biholomorphic to an open unit ball in  $\mathbb{C}^n$ . Thus the automorphism group of such a domain is either  $SU(n, 1)$  or a compact Lie group. It is natural to ask whether every compact group can appear as the automorphism group of such  $\Omega$ . For the case of the trivial group  $G = \{\text{id}\}$ , there are triply connected domains in  $\mathbb{C}$  with smooth boundary but with no nontrivial automorphisms. Finding contractible strongly pseudoconvex domains  $\Omega \subset \mathbb{C}^n$ ,  $n \geq 2$  with  $\text{Aut}(\Omega) = \{\text{id}\}$  is less easy, but it is possible to take  $\Omega$  to be a small, smooth perturbation of the ball  $B^n$  (see [4]). The next simplest case is  $G = T^1$ , the circle group. There is no smoothly bounded Riemann surface  $M$  with  $\text{Aut}(M) = T^1$ , but an appropriate domain may be constructed in  $\mathbb{C}^2$  (Proposition 1.3). In this paper (Theorems 1, 2) we show how to construct a smoothly bounded domain  $\Omega$  in  $\mathbb{C}^n$  whose group of biholomorphisms is any prescribed compact group  $G$ . If  $G$  is connected our construction (§ 3) is quite explicit:

**THEOREM 1.** *Let  $G$  be a connected compact Lie Group and  $G_{\mathbb{C}}$  its complexification. Then there exists a strongly pseudoconvex domain  $\Omega \subset G_{\mathbb{C}}$  (or  $\Omega \subset G_{\mathbb{C}} \times \mathbb{C}$  in case the center of  $G$  is one dimensional) with real analytic boundary so that  $\text{Aut}(\Omega) = G$ , acting by left translations.*

The object in constructing  $\Omega \subset G_{\mathbb{C}}$  is to keep it invariant under left translations by  $G$  while ruling out additional symmetries. If  $G$  acts on a complex

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manifold  $M \neq G_{\mathbb{C}}$  by biholomorphisms it may happen that no such  $\Omega \subset M$  exists (see example 3.0).

The following two theorems were first obtained by Saerens and Zame [12] independently of our Theorem 1, but the proofs we give in § 4 are shorter and more elementary in nature.

**THEOREM 2.** *Let  $G$  be any compact Lie group. Then there is a strongly pseudoconvex domain  $\Omega \subset \mathbb{C}^n$  with real analytic boundary such that  $\text{Aut}(\Omega) = G$ .*

**THEOREM 3.** *Let  $G$  be a compact Lie group. Then there exists a surface  $\Sigma \subset \mathbb{R}^n$  which is an arbitrarily small smooth perturbation of the unit sphere  $S^{n-1} \subset \mathbb{R}^n$  whose group of isometries is linear and isomorphic to  $G$ . Moreover, if for some affine map  $T$  of  $\mathbb{R}^n$   $T\Sigma = \Sigma$  then  $T \in G \subset O(n)$*

*Remark.* The dimension  $n$  in the above two theorems may be taken to be  $n = k^2$  if  $G$  has a faithful action on  $\mathbb{R}^k$ .

Note that while Theorem 2 applies to disconnected Lie groups it only gives existence of required domains  $\Omega$  (typically with  $\dim \Omega \gg \dim G$ ). Its proof cannot be used to actually construct  $\Omega$  without prohibitive calculations.

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## Notation

Let  $G$  be a compact group and  $\mathfrak{g}$  its Lie algebra. We choose a faithful imbedding of  $G$  into some unitary group  $U(n)$ . Thus  $\mathfrak{g} \subset \mathfrak{U}(n)$  is a subalgebra of skew Hermitian matrices. We set  $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \oplus i\mathfrak{g}$  and  $G_{\mathbb{C}} \subset GL(n, \mathbb{C})$  the connected Lie group corresponding to  $\mathfrak{g}_{\mathbb{C}}$ . If  $G$  itself is connected then  $G \subset G_{\mathbb{C}}$  as a totally real submanifold of the Stein manifold  $G_{\mathbb{C}}$ . If  $\omega \subset \mathfrak{g}$  is a small neighborhood of  $0 \in \mathfrak{g}$  then  $\Omega = G \cdot \exp i\omega \approx G \times \omega$  is a tubular neighborhood of  $G$  in  $G_{\mathbb{C}}$ . Here  $\exp$  is the matrix exponential function. The groups  $L(G)$ ,  $R(G) \subset \text{Aut}(G_{\mathbb{C}})$  are the groups of left and right translations by  $G$ . If  $g \in G$ ,  $X, Y \in \mathfrak{g}$  we, as usual, define,  $\text{Ad}(g)X = gXg^{-1}$  and  $\text{ad}(X)(Y) = XY - YX$  (matrix multiplication).

## § 1. Tori

In this section we give examples of domains whose automorphism groups are  $T^n$ . First we consider a Reinhardt domain  $\Omega \subset \mathbb{C}^n$ , i.e.  $\Omega$  is invariant under  $(z_1, \dots, z_n) \rightarrow (e^{i\theta_1}z_1, \dots, e^{i\theta_n}z_n)$  for  $\theta_1, \dots, \theta_n \in \mathbb{R}$ . It is obvious that  $T^n$  is contained in the automorphism group of a Reinhardt domain. The logarithmic image of  $\Omega$  is

$$\omega = \text{Log}(\Omega) = \{(\xi_1, \dots, \xi_n) : (e^{\xi_1}, \dots, e^{\xi_n}) \in \Omega\}. \quad (1)$$

The automorphism group for certain  $\Omega$  is given as follows (see [1]).

**THEOREM 1.1.** *If  $\Omega$  is a Reinhardt domain, and if  $\text{Log}(\Omega)$  is a bounded convex domain in  $\mathbb{R}^n$ , then  $\text{Aut}(\Omega)$  consists of transformations of the form*

$$(z_1, \dots, z_n) \rightarrow (c_1 z^{m_1}, \dots, c_n z^{m_n}) \quad (2)$$

where the matrix  $M$  with rows  $m_1, \dots, m_n$  belongs to  $GL(n, \mathbb{Z})$ .

It is evident that a mapping of the form (2) will map  $\Omega$  to  $\Omega$  if and only if  $T\xi = M\xi + \log |c|$  is an affine self-mapping of  $\omega$ .

**COROLLARY 1.2.** *Let  $\omega \subset \mathbb{R}^n$  be a bounded convex domain, and let  $\Omega \subset \mathbb{C}^n$  be the Reinhardt domain with  $\text{Log}(\Omega) = \omega$ . If  $\omega$  has no nontrivial affine self-mappings, then  $\text{Aut}(\Omega) = T^n$ .*

If  $n \geq 2$ , then it is clear that a “generic” domain  $\omega$  in  $\mathbb{R}^n$  has no affine self-mappings. This is not true for  $n = 1$ , since every interval in  $\mathbb{R}$  has an (affine) inversion.

Let  $D \subset \mathbb{C}$  be a smoothly bounded triply connected domain with  $\text{Aut}(D) = id$ . Let  $0 < r_1(z) < r_2(z)$  be continuous functions on  $\bar{D}$  and set

$$\Omega = \{(z, w) \in D \times \mathbb{C} : r_1(z) < |w| < r_2(z)\}.$$

**PROPOSITION 1.3.** *Let  $D \subset \mathbb{C}$  be a smoothly bounded triply connected domain with  $\text{Aut}(D) = id$ . If we choose  $r_1(z), r_2(z)$  such that  $r_1(z)r_2(z)$  is not the modulus of an analytic function on  $D$ , then with  $\Omega$  as above,  $\text{Aut}(\Omega) = T^1$ .*

It is clear that we may arrange for  $\Omega$  to have real analytic, strongly pseudoconvex boundary.

For the proof we will use invariant 2-forms, as in [2]. We may choose  $a_1$ ,

$a_2 \in \mathbb{C}$  such that

$$T_j = \frac{dz \wedge dw}{(z - a_j)w}, \quad j = 1, 2$$

are linearly independent cohomology classes. If  $[T_j]$  is the set of holomorphic 2-forms cohomologous to  $T_j$ , then there exists a unique  $\omega_{T_j}$  which minimizes the  $L^2$ -norm  $\|\omega\| = |\int_{\Omega} \omega \wedge \bar{\omega}|^{1/2}$  over  $[T_j]$ . We may write

$$\omega_{T_j} = \sum_{k=-\infty}^{\infty} f_j^k(z) w^k dz \wedge dw.$$

Since  $T_j$  is independent of the rotation  $(z, w) \rightarrow (z, e^{i\theta}w)$ , so is  $\omega_{T_j}$ . Thus

$$\omega_{T_j} = f_j(z) w^{-1} dz \wedge dw.$$

If  $f \in \text{Aut}(\Omega)$ , then

$$\frac{\omega_{f^*T_1}}{\omega_{f^*T_2}} = f^* \left( \frac{\omega_{T_1}}{\omega_{T_2}} \right). \quad (3)$$

By the arguments above,  $\omega_{T_1}/\omega_{T_2} = m(z)$  is a nonconstant meromorphic function, and the left hand side of (3) is another meromorphic function,  $\tilde{m}(z)$ . Thus writing  $f(z, w) = (f_1(z, w), f_2(z, w))$  we have

$$\tilde{m}(z) = m(f_1(z, w))$$

and so  $f_1(z, w)$  depends on the variable  $z$  alone. We conclude, then, that  $f$  induces a mapping of the vertical fibers  $\Omega_{z_0} = \{(z, w) \in \Omega : z = z_0\}$  of  $\Omega$ . Thus  $f_1$  is an automorphism of  $D$ , and therefore  $f_1(z, w) = z$ .

We conclude from this, that  $f_2(z, w)$  must be an automorphism of the fiber

$$\Omega_z = \{w \in \mathbb{C} : r_1(z) < |w| < r_2(z)\}.$$

Therefore either

$$f_2(z, w) = e^{i\theta(z)} w$$

or

$$f_2(z, w) = c(z)/w.$$

In the first case,  $\theta(z) = \theta_0$  is constant, since it must be real-valued and holomorphic. In the second case, however,  $c(z)$  is a holomorphic function and

$$r_1(z) = |c(z)|/r_2(z).$$

which is a contradiction.

## § 2. Simple and connected groups

Given a simple compact group  $G$  we construct (in Lemma 2.3) a domain  $\Omega \subset G_{\mathbb{C}}$  in the complexification of  $G$  and prove (Proposition 2.6) that the connected component of the identity  $\text{Aut}(\Omega)_0 = G$ . At first we shall assume that  $G$  is compact connected and semi-simple. Simplicity of  $G$  is necessary only in Proposition 2.6.

The Killing form  $\kappa(x, y) = \text{Tr}(\text{ad } x \text{ ad } y)$  is negative definite on  $\mathfrak{g}$ , the Lie algebra of  $G$ . In this section we shall use  $-\kappa$  as the inner product on  $\mathfrak{g}$ . This inner product on the left invariant vector fields gives a biinvariant metric on  $G$ .

Let  $\text{Aut}(\mathfrak{g})$  be the Lie group of Lie algebra automorphisms of  $\mathfrak{g}$ , and  $\text{Ad}(G) \subset \text{Aut}(\mathfrak{g})$  be the image of  $G$  under the adjoint representation i.e. the inner automorphisms of  $\mathfrak{g}$ . Recall [7] that  $\text{Aut}(\mathfrak{g})/\text{Ad}(G)$  is a finite group (of order  $\leq 6$  if  $\mathfrak{g}$  is simple) and that each  $\sigma \in \text{Aut}(\mathfrak{g})$  preserves the Killing form which we will write as  $\text{Aut}(\mathfrak{g}) \subset O(\mathfrak{g})$ , the orthogonal transformations on  $\mathfrak{g}$ . If  $I \in O(\mathfrak{g})$  is the identity map we readily observe:

LEMMA 2.1.  $-I \notin \text{Aut}(\mathfrak{g})$ .

We proceed to construct the domain  $\Omega \subset G_{\mathbb{C}}$ . Let  $\{e_1, \dots, e_d\}$  be an orthonormal basis for  $\mathfrak{g}$  and let  $x_1, \dots, x_d$  be coordinates with respect to this basis.

LEMMA 2.2. *On  $\mathfrak{g} \setminus \{0\}$  we may find a smooth function  $\psi(x)$  with the following properties:*

- i)  $\psi(\lambda x) = |\lambda| \psi(x)$  for all  $\lambda \in \mathbb{R}$
- ii)  $\psi \circ g = \psi$ ,  $g \in O(\mathfrak{g})$  implies  $g = \pm I$ .

*Proof.* Let  $\psi_0(x) = \sum_{i=1}^d x_i^4$ . Note that the maxima of  $\psi_0$  on the unit sphere  $S^{d-1} \subset \mathfrak{g}$  are at  $\pm e_i$ ,  $i = 1, 2, \dots, d$ . Set  $v = (1, 2, 3, \dots, d)$  and

$$\psi_{\epsilon}(x) = \frac{\psi_0(x) + \epsilon \langle x, v \rangle^4}{|x|^3}.$$

For small enough  $\epsilon > 0$  there will be local maxima of  $\psi_{\epsilon}$  at  $\pm \tilde{e}_i$  with  $\tilde{e}_i$  very close to  $e_i$ , and with

$$\langle \tilde{e}_i, \tilde{e}_j \rangle > 0 \quad \text{for all } i, j. \tag{4}$$

Further, it is clear that for small enough  $\epsilon$  we will have

$$\psi_{\epsilon}(\tilde{e}_i) \neq \psi_{\epsilon}(\tilde{e}_j), \quad \text{if } i \neq j. \tag{5}$$

Thus if we fix  $\epsilon > 0$  with above properties and assume that  $\psi_\epsilon \circ g = \psi_\epsilon$ ,  $g \in O(\mathcal{g})$  we obtain, using (5), that  $g(\tilde{e}_i) = \pm \tilde{e}_i$ . Finally, using (4), we must have  $g(\tilde{e}_i) = \tilde{e}_i$  or  $g(\tilde{e}_i) = -\tilde{e}_i$  for all  $i = 1, 2, \dots, d$ .

**LEMMA 2.3.** *There exists a domain  $\omega \subset \mathcal{g}$  such that*

- i)  $\omega = -\omega$
- ii)  $\Omega = G \cdot \exp(i\omega)G_{\mathbb{C}}$  is strongly pseudoconvex and smoothly bounded.
- iii) If  $\sigma \in \text{Aut}(\mathcal{g})$  and  $\sigma(\omega) = \omega$ , then  $\sigma = I$ .

*Proof.* Let  $\psi(x)$  be as in Lemma 2.2 and set

$$\omega_{\epsilon, \delta} = \left\{ x \in \mathcal{g} : \sum_{i=1}^d x_i^2 + \delta \psi(x) < \epsilon^2 \right\}.$$

From Lemmas 2.1 and 2.2 it follows that for  $\epsilon, \delta > 0$  properties i) and iii) hold. For  $\epsilon \gg \delta > 0$  sufficiently small property ii) holds as well.

For the next lemma we observe that any group automorphism  $h$  of  $G$  extends to (a holomorphic) automorphism of  $G_{\mathbb{C}}$ , here denoted also by  $h$ .

**LEMMA 2.4.** *Let  $\Omega$  be as in Lemma 2.3. Suppose  $h$ , an automorphism of  $G$ , and  $X \in \omega$  have the property that*

$$R_{\exp iX} \circ h(\Omega) = \Omega.$$

*Then  $X = 0$  and  $h$  is the identity automorphism.*

*Proof.* The differential  $dh \in \text{Aut}(\mathcal{g}) \subset \text{Aut}(\mathcal{g}_{\mathbb{C}})$ . We thus have

$$h(\exp iY) = \exp(i dh(Y)), \quad Y \in \mathcal{g}.$$

In  $G_{\mathbb{C}}$  consider the curve  $\gamma(t) = \exp itX$ . Since  $dh(\omega) = -dh(\omega)$  it follows that

$$\{t \in \mathbb{R} : \gamma(t) \in \Omega\} = (-a, a), \quad a > 0$$

is a symmetric interval. Next we observe that if  $X \neq 0$  the set

$$\begin{aligned} \{t \in \mathbb{R} : \gamma(t) \in R_{\exp iX} \circ h(\Omega)\} &= \{t \in \mathbb{R} : \gamma(t) \exp(-tX) \in h(\Omega)\} \\ &= \{t \in \mathbb{R} : (t-1)X \in dh(\omega)\} \end{aligned}$$

is of the form  $(-b+1, b+1)$  and thus not symmetric. This contradiction shows that  $X = 0$  and Lemma 2.3 then forces  $dh = I$ .

**COROLLARY 2.5.** *Suppose  $R_z(\Omega) = \Omega$ ,  $z \in G_{\mathbb{C}}$ . Then  $z \in Z(G_{\mathbb{C}})$ , the center of  $G_{\mathbb{C}}$ .*

*Proof.* Write  $z = g \exp iX$ ,  $g \in G$ ,  $X \in \mathfrak{g}$ . Since  $L_{g^{-1}}(\Omega) = \Omega$  we have that  $R_{\exp iX} \circ h(\Omega) = \Omega$ , where  $h(x) = g^{-1}xg$  is an inner automorphism of  $G$ . Lemma 1.4 then implies that  $X = 0$  and  $gx = xg$  for all  $x \in G$  and thus, by extending holomorphically, for all  $x \in G_{\mathbb{C}}$ .  $\square$

**PROPOSITION 2.6.** *Let  $G$  be a simple connected Lie group and  $\Omega \subset G_{\mathbb{C}}$  as constructed in Lemma 2.3. Then the connected component of the identity is  $\text{Aut}(\Omega) = L(G)$ .*

*Proof.* Recall that  $d = \dim G$ . Since  $\Omega \subset G_{\mathbb{C}}$  is a small tubular neighborhood of  $G$ , we have  $H_d(\Omega, \mathbb{Z}) = \mathbb{Z}$ . By Lemma 2.3 of [3] there exists an orbit of  $\text{Aut}(\Omega)$  in  $\Omega$  whose dimension is at most  $d$ . Since  $L(G) \subset \text{Aut}(\Omega)$  that orbit must be a finite union of  $G$  orbits, and any of these are stable under  $\text{Aut}(\Omega)_0$ . So suppose  $G \cdot x_0$  is  $\text{Aut}(\Omega)_0$  stable for some  $x_0 \in \Omega$ . Restricting the Bergmann metric  $ds^2$  to the manifold  $G \cdot x_0 \simeq G$  we see that  $\text{Aut}(\Omega)_0$  is naturally a subgroup of the connected component  $I_0(G, ds^2)$  of the isometry group. By Theorem 1 of [10] it now follows that any  $f \in \text{Aut}(\Omega)_0$  is of the form

$$f(g \cdot x_0) = agb \cdot x_0, \quad g \in G$$

for some  $a, b \in G$ . Extending holomorphically to  $g \in G_{\mathbb{C}}$  we see that  $f = L_a \circ R_{x_0^{-1}bx_0}$ , and hence  $R_{x_0^{-1}bx_0}(\Omega) = \Omega$ . By Corollary 2.5  $b \in Z(G_{\mathbb{C}})$ , and so  $f = L_{ab} \in L(G)$ .  $\square$

### § 3. Connected groups and proof of Theorem 1

**PROPOSITION 3.1.** *Let  $G$  be a compact, connected Lie group. Then there exists a piecewise strongly pseudoconvex domain  $\Omega \subset G_{\mathbb{C}}$ , (or  $\Omega \subset G_{\mathbb{C}} \times \mathbb{C}$  in case the center of  $G$  is one dimensional) such that  $G = \text{Aut}(\Omega)$ .*

One may contemplate constructing such  $\omega$  domain  $\Omega$  inside other complex manifolds that posses a natural  $G$  action. The following example shows that achieving  $G = \text{Aut}(\Omega)$  may be impossible.

**EXAMPLE 3.0.** Let  $G = SO(3)$  act on the complex sphere

$$\Sigma = \{z \in \mathbb{C}^3 \mid \sum z_i^2 = 1\}$$

Every  $G$  orbit on  $\Sigma$  intersects the curve

$$\alpha(t) = (\cosh t, i \sinh t, 0)$$

at  $\alpha(\pm s)$  for some  $s$ . Consequently the only  $G$  invariant pseudoconvex domains in  $\Sigma$  are  $\Omega_R = \{z \in \Sigma \mid |z| < R\}$ . These domains are also  $O(3)$  invariant.

*Proof of 3.1.* Any compact connected Lie group  $G$  is of the form

$$G = T^l \times G_1 \times \cdots \times G_k / H \quad \text{where} \quad G_1, \dots, G_k \text{ are simple,}$$

1-connected and connected,  $H \subset Z(T^l \times G_1 \times \cdots \times G_k)$  is finite, and  $H \cap T^l = \{e\}$ .

In the following we will denote  $G_1 \times \cdots \times G_k$  by  $G_s$ . Let  $\Omega^0$  be a domain with  $\text{Aut}(\Omega^0) = T^l$ , as constructed in Section 1. For each simple factor  $G_j$  let  $\Omega^j$  the domain constructed in Lemma 2.3. Moreover, we may arrange our choice of  $\omega_i$ 's so that  $\Omega_i$  is not biholomorphically equivalent to  $\Omega_j$  if  $i \neq j$ . To see this, we need only to note that if we shrink  $\omega_j$ , then we obtain a biholomorphically inequivalent  $\Omega_j$  (see, for instance, Theorem 3.3 of [2]). Now set

$$D = \Omega_0 \times \Omega^1 \times \cdots \times \Omega^k.$$

We note that  $D$  is biholomorphic to a domain in  $(T^l \times G_1 \times \cdots \times G_k)_{\mathbb{C}}$  of the form  $(T^l \times G_s) \cdot \exp \{i(\omega^0 \times \omega^1 \times \cdots \times \omega^k)\}$ . By our choice of  $D$  and theorem of H. Cartan [9]

$$\text{Aut}(D) = T^l \times \text{Aut}(\Omega^1) \times \text{Aut}(\Omega^2) \times \cdots \times \text{Aut}(\Omega^k). \quad (6)$$

Next set  $\Omega = D / L(H)$ . Again, biholomorphically

$$\Omega = G \cdot \exp \{i(\omega^0 \times \omega^1 \times \cdots \times \omega^k)\} \subset G_{\mathbb{C}}.$$

If  $f \in \text{Aut}(\Omega)_0$  then it is homotopic to the identity and may thus be lifted to  $\tilde{f} \in \text{Aut}(D)_0$ . By (6) and Proposition 2.6  $\tilde{f} = L_{\tilde{g}}$ ,  $\tilde{g} \in T^l \times G^1 \times \cdots \times G^k$  and thus  $f = L_g$  for some  $g \in G$ . Hence  $L(G) = \text{Aut}(\Omega)_0$  is a normal subgroup of  $\text{Aut}(\Omega)$ . Therefore if  $h \in \text{Aut}(\Omega)$

$$hL_gh^{-1} = L_{\chi(g)}, \quad \chi \in \text{Aut}(G),$$

that is for any  $x \in \Omega$

$$h(g \cdot x) = \chi(g) \cdot h(x).$$

Setting  $x = e \in G$ ,  $h = R_{h(e)} \circ \chi$  on  $G \subset \Omega$  which gives  $h = R_{h(e)} \circ \chi$  on  $\Omega$  after extending holomorphically to  $\chi \in \text{Aut}(G_{\mathbb{C}})$ . By composing  $h$  with a suitable left translation  $L_g$ ,  $g \in G$  we may assume that  $h(e) = \exp iX$ ,  $X \in \mathfrak{g}$ . We write the Lie algebra of  $\mathfrak{g}$  as  $\mathfrak{g}_0 + \mathfrak{g}_s$  where  $\mathfrak{g}_s$  is the Lie algebra of  $G_s$  and  $\mathfrak{g}_0$  is the center of  $\mathfrak{g}$ , i.e. the Lie algebra of  $T^l$ . The differential of  $\chi$  must preserve this decomposition, so  $d\chi = d\chi_0 \circ d\chi_s = d\chi_s \circ d\chi_0$ . Similarly we can write  $X = X_0 + X_s$  so  $\exp iX_0 \exp iX_s = \exp iX_s \exp iX_0$ . We conclude that translation by  $X_0$  followed by  $d\chi_0$  preserves  $\omega^0$  and thus by assumption on  $\omega^0 X_0 = 0$  and  $d\chi_0 = I$ . Finally as in

the proof of Lemma 2.4 we must have  $X_s = 0$  and  $d\chi_s(\omega^1 \times \cdots \times \omega^s) = \omega^1 \times \cdots \times \omega^2$ . Our choice of these domains forces first  $d\chi_s(\omega^j) = \omega^j$  and then  $d\chi_s = I$ .  $\square$

To complete the proof of Theorem 1 we now apply the semicontinuity theorem of Greene and Krantz [6] to smoothen the domain  $\Omega$ . Let  $r(z)$  be a  $G$  invariant strongly plurisubharmonic exhaustion function of  $\Omega$ . For large  $\lambda$

$$\Omega_\lambda = \{z \in \Omega : r(z) < \lambda\}$$

is strongly pseudoconvex with smooth real analytic boundary. Evidently  $G \subset \text{Aut}(\Omega_\lambda)$ . Lemma 3.2 below shows that  $\text{Aut}(\Omega_\lambda)$  is a normal family of groups in the sense of Greene and Krantz, and thus by their semicontinuity theorem  $\text{Aut}(\Omega_\lambda) \subset G$  for  $\lambda$  sufficiently large. The proof of the theorem is now complete.

**LEMMA 3.2.** *Let  $(\lambda_j)$  be a sequence converging to  $+\infty$  and let  $\varphi_j \in \text{Aut}(\Omega_{\lambda_j})$ . Then there exists a subsequence  $\{\varphi_{j_k}\}$  converging uniformly on compact sets to an element  $\varphi \in \text{Aut}(\Omega)$ .*

*Proof.* Since  $\Omega$  is bounded we may assume: (by extracting a subsequence) that  $\{\varphi_j\}$  converges uniformly on compact subsets to a holomorphic  $\psi \in \Omega \rightarrow \bar{\Omega}$ . By a theorem of H. Cartan [9] either  $\psi \in \text{Aut}(\Omega)$  or  $\psi(\Omega) \subset \partial\Omega$ . We now show, arguing as in [3], that the latter case is impossible. Recall that by construction  $\Omega$  is covered by a product of bounded domains. By lifting our maps we may assume that  $\Omega$  itself is a product

$$\Omega = \Omega^0 \times \Omega^1 \times \cdots \times \Omega^k.$$

Suppose  $\psi(\Omega) \cap \partial\Omega^0 \times \Omega^1 \times \cdots \times \Omega^k \neq 0$ . Then, since  $\partial\Omega^0$  is strongly pseudoconvex,  $\psi(\Omega) \subset \{p_0\} \times \Omega^1 \times \cdots \times \Omega^k$  for some  $p_0 \in \partial\Omega^0$ . Now let  $U$  be a contractible neighborhood of  $p_0$  in  $\Omega^0$ , and let  $T$  be a compactly supported cycle representing a nontrivial class in  $H_q(\Omega)$ , where  $q = \dim_{\mathbb{C}} \Omega$ . For large  $j$  we have  $\varphi_j(T) \subset U \times \Omega^1 \times \cdots \times \Omega^k$  which is homologically trivial in dimension  $q$ . On the other hand  $\varphi_j$  is a diffeomorphism and hence  $\psi_j(T)$  cannot be a boundary in  $\Omega$  for  $j$  large.

## § 4. Existence proofs

In this section we prove Theorems 2 and 3.

**PROPOSITION 4.1.** *Let  $G$  be a compact Lie group. Then there exists an*

*orthogonal action of  $G$  on  $\mathbb{R}^n$  with the following properties*

- (i) *If  $H \subset O(n)$  is a subgroup such that  $Hx = Gx$  for all  $x \in \mathbb{R}^n$ , then  $H = G$ .*
- (ii) *There exists a set  $F \subset \mathbb{R}^n$  consisting of finitely many  $G$ -orbits such that if  $g \in O(n)$  and  $gF = F$ , then  $g \in G$ .*

*Proof.* Let  $G$  be faithfully imbedded in  $O(k)$ , and let  $G$  act diagonally on

$$\mathbb{R}^n = \mathbb{R}^k \oplus \cdots \oplus \mathbb{R}^k \quad (k \text{ times}). \quad (*)$$

First we show that (i) is satisfied for this action. By assumption, the decomposition of  $\mathbb{R}^n$  in  $(*)$  is also  $H$ -invariant. For  $v_1, \dots, v_k \in \mathbb{R}^k$ , we write  $v = (v_1, \dots, v_k) \in \mathbb{R}^n$  by the decomposition  $(*)$ . For  $h \in H$ , and  $u = (v, \dots, v) \in \mathbb{R}^n$ , there exists by assumption  $g \in G$  such that

$$hu = gu = (w, \dots, w).$$

Thus we conclude that  $H$  acts diagonally on the decomposition  $(*)$ . Finally, if  $\{e_1, \dots, e_k\}$  is a basis of  $\mathbb{R}^k$ , we set  $v = (e_1, \dots, e_k)$  to see that  $hv = gv$  implies that  $h = g$ .

For part (ii), we construct a sequence of sets  $F_j$ ,  $j = 1, 2, 3, \dots$ , with the following properties:

1.  $F_j$  is the union of  $j$   $G$ -orbits.
2. If  $H_j = \{g \in O(n) : gF_j = F_j\}$ , and if  $H_j \neq G$ , then  $H_j \supsetneq H_{j+1}$ .

Since the  $H_j$  are compact and each contains  $G$ , we must have  $H_l = G$  for some  $l$ . Indeed, at each step either the dimension or the number of components must decrease.

Now fix  $\epsilon > 0$ , pick  $x_1$  of length  $1 + \epsilon$ , and set  $F_1 = Gx_1$ . We proceed inductively, under the assumption that  $H_j \neq G$ . By part (i) there exists a point  $x_{j+1}$  such that  $H_j x_{j+1} \neq Gx_{j+1}$ . We then set

$$F_{j+1} = F_j \cup Gx_{j+1}.$$

Since we may take  $\|x_{j+1}\| > \|x\|$  for all  $x \in F_j$ , we see that  $H_{j+1}F_j \subset F_j$ , and thus  $H_{j+1} \supsetneq H_j$ .

**PROPOSITION 4.2.** *Let  $G$  be a compact Lie group. There exists an orthogonal action of  $G$  on  $\mathbb{R}^n$  and a  $G$  invariant domain  $\omega \subset \mathbb{R}^n$  which is a small, smooth perturbation of the unit ball with the property that  $g\omega = \omega$  and  $g$  affine implies  $g \in G$ .*

*Proof.* Let  $Gx_1 \cup \cdots \cup Gx_l$  denote the set obtained in (ii) of Proposition 1. For any  $\delta > 0$ , we may assume that  $1 + \delta > |x_1| > |x_2| > \cdots > |x_l| > 1$ . From  $S^{n-1}$  we remove a small tubular neighborhood  $V_j$  of  $|x_j|^{-1} \cdot Gx_j$  such that the area of  $V_j$  is small and such that  $\bar{V}_i \cap \bar{V}_j = \emptyset$  for  $i \neq j$ .

Now we may make a small smooth perturbation of  $S^{n-1}$  of the form

$$\Sigma = \{r(x)x : x \in S^{n-1}\}$$

where  $r(x)$  is a smooth function on  $S^{n-1}$  with  $r \geq 1$ , and  $r(x) = 1$  for  $x \notin \bigcup_{j=1}^l V_j$ . Let us write

$$\omega = \{x \in \mathbb{R}^n : |x| < r(x/|x|)\}.$$

Before we specify  $r(x)$  more precisely, let us note that if  $h$  is an affine transformation of  $\mathbb{R}^n$  with  $h(\Sigma) = \Sigma$ , then  $h \in O(n)$ . To see this, write

$$\omega_1 = \{x \in \mathbb{R}^n : |x| < 1, x/|x| \notin V_1 \cup \dots \cup V_l\}.$$

Thus  $\omega_1$  is a conical subset of  $\omega$  generated by the complement of  $V_1 \cup \dots \cup V_l$ . Since  $h(\omega) = \omega$ ,  $h$  must preserve volume. And since the volume of  $\omega - \omega_1$  is small,  $h(\omega_1) \cap \omega_1$  contains an open set. It follows, then, that  $|h(x)| = 1$  for  $x$  in an open subset of  $S^{n-1}$ . We conclude, then, that  $h \in O(n)$ .

Let  $\chi \in C^\infty(\mathbb{R})$  be monotone decreasing with  $\chi(0) = 1$ ,  $\chi'(0) < 0$  and  $\chi = 0$  on  $[1, \infty)$ . We define

$$r(x) = 1 + (|x_j| - 1)\chi(M \operatorname{dist}^2(x, |x_j|^{-1}Gx_j))$$

for  $x \in V_j$  and  $r = 1$  elsewhere on  $S^{n-1}$ . For  $M$  sufficiently large,  $r$  is smooth. Choosing  $\delta > 0$  sufficiently small, we have  $r$  close to 1.

Now if  $h \in O(n)$  and  $h\Sigma = \Sigma$ , then  $h$  must map  $Gx_j$  to a portion of  $\Sigma$  with distance  $|x_j|$  to the origin. At the same time,  $h$  must map  $Gx_j$  to a portion of  $\Sigma$  where the distance to the origin takes a local maximum. Thus  $h(Gx_j) \subset Gx_j$ . We conclude from Proposition 1, then, that  $h \in G$ .

*Proof of Theorem 3.* We let  $\omega$  be the domain obtained in Proposition 2, and let  $\Sigma = \partial\omega$ . If  $\omega$  is sufficiently close to the unit ball, then  $\Sigma$  is positively curved. Thus  $\Sigma$  is rigid, and any isometry  $g$  of  $\Sigma$  extends to an isometry of  $\mathbb{R}^n$  (cf. [8]). It follows that  $g \in G$ , and thus  $G$  is the group of isometries of  $\Sigma$ .

*Proof of Theorem 2.* Let  $\omega \subset \mathbb{R}^n$  be the domain from Theorem 1, and let

$$\Omega = (\omega + i\mathbb{R}^n) - V,$$

where

$$V = \{z_1^2 + \dots + z_{n+1}^2 = \frac{1}{2}\}.$$

We claim that  $\operatorname{Aut}(\Omega) = G$ . Since  $G \subset O(n)$ , it follows that  $\operatorname{Aut}(\Omega) \supset G$ . On the other hand,  $\omega$  is contained in a proper cone, and thus is biholomorphic to a bounded domain. Thus any  $f \in \operatorname{Aut}(\Omega)$  extends to a holomorphic mapping  $f \in \operatorname{Aut}(\omega + i\mathbb{R}^n)$ . By the Corollary to Theorem 1 of [5] or by [13]  $f(z)$  is of the

form

$$f(z) = Az + b + ic$$

where  $A \in GL(n, \mathbb{R})$ , and  $b, c \in \mathbb{R}^n$ . Since  $Az + b$  maps  $\omega$  to itself, it follows from Proposition 2 that  $b = 0$  and  $A$  represents an orthogonal transformation in  $G$ . Thus  $A$  maps  $V$  to itself, but it is evident that  $V \neq V + ic$  if  $c \neq 0$ . We conclude, then, that  $f \in G$ .

To complete the proof of Theorem 2, we now smoothen the domain  $\Omega$ , as in the proof of Theorem 1. The only difference is that in the normal families argument we now use the fact that  $\Omega$  cannot be retracted to  $V$ , since  $H_n(V \cap \Omega, \mathbb{Z}) \neq 0$  but  $\Omega$  is contractible. We can then apply the Semicontinuity theorem of Greene and Krantz [6] to smoothen  $\Omega$ .

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