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**Autor:** Oliver, Robert  
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## **$SK_1$ of finite group rings: V**

ROBERT OLIVER

We continue here the study of

$$SK_1(\mathbb{Z}G) = \text{Ker} [K_1(\mathbb{Z}G) \rightarrow K_1(\mathbb{Q}G)]$$

for finite  $G$ : the group shown by Wall [26] to be precisely the torsion subgroup of  $\text{Wh}(G)$ . In earlier papers in this series,  $SK_1(\mathbb{Z}G)$  has been studied via the extension

$$0 \rightarrow Cl_1(\mathbb{Z}G) \rightarrow SK_1(\mathbb{Z}G) \rightarrow \sum_{p \mid |G|} SK_1(\hat{\mathbb{Z}}_p G) \rightarrow 0; \tag{0.1}$$

where  $Cl_1(\mathbb{Z}G) \subseteq SK_1(\mathbb{Z}G)$  is the subgroup of elements described via  $K_2$  in localization sequences.

This paper contains the last step in deriving a combinatorial algorithm for describing the odd torsion in  $SK_1(\mathbb{Z}G)$ . By [17, Theorem 4.8],  $SK_1(\mathbb{Z}G)[\frac{1}{2}]$  splits naturally as a sum

$$SK_1(\mathbb{Z}G)[\frac{1}{2}] \cong Cl_1(\mathbb{Z}G)[\frac{1}{2}] \oplus \sum_{p>2} SK_1(\hat{\mathbb{Z}}_p G).$$

The groups  $SK_1(\hat{\mathbb{Z}}_p G)$  (also for  $p = 2$ ) are described by [15, Theorem 3] and [16, Theorem 2], in terms of  $H_2(Z_i)$  for certain subgroups  $Z_i \subseteq G$ . On the other hand, in [17], the problem of describing  $Cl_1(\mathbb{Z}G)_{(p)}$  for any odd prime  $p$  and any finite  $G$  is reduced to the case where  $G$  is a  $p$ -group (see [17, Theorem 4.8], and the discussion at the end of Section 3 below).

The following theorem is the central result of this paper, and gives a relatively simple way of describing  $Cl_1(\mathbb{Z}G)$  when  $G$  is a  $p$ -group (and  $p$  odd). Note that if  $G$  is any group, and  $G$  acts on  $\mathbb{Z}G$  by conjugation, then for any set  $S \subseteq G$  of conjugacy class representatives,

$$H_1(G; \mathbb{Z}G) \cong \sum_{h \in S} H_1(\mathbb{Z}_G(h)) \otimes \mathbb{Z}(h).$$

(If  $X \subseteq G$  is any conjugacy class, and  $h \in X$ , then  $Z(X) \cong \text{Ind}_{Z_G(h)}^G(\mathbb{Z})$  as  $\mathbb{Z}G$ -modules.) Thus,  $H_1(G; \mathbb{Z}G)$  is generated by elements  $g \otimes h$  for commuting  $g, h \in G$ .

**THEOREM 3.6.** *Fix an odd prime  $p$  and a  $p$ -group  $G$ . Write  $\mathbb{Q}G = \prod_{i=1}^k B_i$ , where each  $B_i$  is simple with center  $F_i$  and irreducible representation  $V_i$ . For each  $i$ , let  $(\mu_{F_i})_p$  be the group of  $p$ -th power roots of unity. Define*

$$\psi_G : H_1(G; \mathbb{Z}G) \rightarrow \prod_{i=1}^k (\mu_{F_i})_p,$$

where  $G$  acts on  $\mathbb{Z}G$  by conjugation, by setting

$$\psi_G(g \otimes h) = [\det_{F_i}(g, V_i^h)]_i \quad (g, h \in G, gh = hg, V_i^h = \{x \in V_i : hx = x\}).$$

Then  $Cl_1(\mathbb{Z}G) \cong \text{Coker}(\psi_G)$ .

Examples of computations of  $Cl_1(\mathbb{Z}G)$  using Theorem 3.6 for non-abelian  $G$  are given in Section 4. For abelian  $G$ , the isomorphism  $SK_1(\mathbb{Z}G) \cong \text{Coker}(\psi_G)$  is proven in [1, Theorem 1.8], and some examples of calculations of  $SK_1(\mathbb{Z}G)$  using that are given in Section 5 of the same paper.

Theorem 3.6 (and the other theorems referred to above) are stated, for simplicity, as describing the components of  $SK_1(\mathbb{Z}G)$  as abstract groups only. But the proofs also contain enough information so that one can take a specific element in  $SK_1(\mathbb{Z}G)$  (e.g., a specific element in  $\text{Coker}(\psi_G)$  as described above), and represent it by a matrix. The opposite problem, taking a specific matrix over  $\mathbb{Z}G$  and deciding how it sits in  $SK_1(\mathbb{Z}G)$  (if it does) is harder in general; the study in [20] of the Whitehead transfer homomorphism for oriented  $S^1$ -fiber bundles gives one example where this can be done.

In general, for any finite group  $G$ ,  $Cl_1(\mathbb{Z}G)$  is described by localization exact sequences

$$K_2^{\text{top}}(\hat{\mathbb{Z}}_p G) \xrightarrow{\varphi} C_p(\mathbb{Q}G) \xrightarrow{\varrho} Cl_1(\mathbb{Z}G)_{(p)} \rightarrow 0$$

for each prime  $p$ ; where for any maximal order  $\mathfrak{M} \subseteq \mathbb{Q}G$ :

$$\begin{aligned} C_p(\mathbb{Q}G) &\cong \varprojlim_n \text{Coker} [K_2(\mathfrak{M}) \rightarrow K_2(\mathfrak{M}/p^n \mathfrak{M})] \cong \varprojlim_n Cl_1(\mathfrak{M}; p^n \mathfrak{M}) \\ &\cong \text{Coker} \left[ K_2 \left( \mathfrak{M} \begin{bmatrix} 1 & \\ & p \end{bmatrix} \right) \rightarrow K_2^{\text{top}}(\hat{\mathbb{Q}}_p G) \right]_{(p)}. \end{aligned}$$

The  $C_p(\mathbb{Q}G)$  are described by the work of Bak and Rehmann on the congruence subgroup problem [3]. The remaining problem is then to find a set of generators for  $K_2^{\text{top}}(\hat{\mathbb{Z}}_p G)$ , or at least for its image in  $C_p(\mathbb{Q}G)$ . In the case of an odd prime  $p$  and a  $p$ -group  $G$ , the formula

$$Cl_1(\mathbb{Z}G) \cong \text{Coker} \left[ \psi_G : H_1(G; \mathbb{Z}G) \rightarrow \prod_{i=1}^k (\mu_{F_i})_p \right]$$

can be explained by noting that norm residue symbols define an isomorphism of  $C_p(\mathbb{Q}G)$  with  $\prod (\mu_{F_i})_p$ , and that  $H_1(G; \mathbb{Z}G) (\cong H_1(G; \hat{\mathbb{Z}}_p G))$  and  $K_2(\hat{\mathbb{Z}}_p G)$  both are closely related to the cyclic homology group  $HC_1(\hat{\mathbb{Z}}_p G)$  (see [21]).

The key new result here about generators for  $K_2(\hat{\mathbb{Z}}_p G)$  is:

**THEOREM 1.4.** *Let  $p$  be any prime, and fix a  $p$ -group  $G$  and an element  $z \in Z(G)$ . Then*

$$\begin{aligned} \text{Ker} [K_2^{\text{top}}(\hat{\mathbb{Z}}_p G) \rightarrow K_2^{\text{top}}(\hat{\mathbb{Z}}_p [G/z])] \\ = \langle \{g, 1 + \lambda(1 - z)^i h\} : \lambda \in \hat{\mathbb{Z}}_p, i \geq 1, g, h \in G, gh = hg \rangle. \end{aligned}$$

Since  $\text{Coker} [K_2^{\text{top}}(\hat{\mathbb{Z}}_p G) \rightarrow K_2^{\text{top}}(\hat{\mathbb{Z}}_p [G/z])]$  is also known in the above situation (see Proposition 2.1 below), it should in principle now be possible to inductively construct a set of generators for  $K_2^{\text{top}}(\hat{\mathbb{Z}}_p G)$ . Unfortunately, it's not always easy to explicitly lift elements from  $K_2^{\text{top}}(\hat{\mathbb{Z}}_p [G/z])$  to  $K_2^{\text{top}}(\hat{\mathbb{Z}}_p G)$ , even where they are known to lift. But such an inductive procedure does work to give generators for  $K_2^{\text{top}}(\hat{\mathbb{Z}}_p G)^+$  when  $p$  is odd, and this suffices when computing  $Cl_1(\mathbb{Z}G)$ .

Another consequence of Theorem 1.4 involves a comparison of  $Cl_1(RG)$  – when  $G$  is any finite group and  $R$  the ring of integers is some number field  $K \subseteq \mathbb{C}$  – with the “complex Artin cokernel”

$$A_{\mathbb{C}}(G) = \text{Coker} \left[ \sum \{R_{\mathbb{C}}(H) : H \subseteq G \text{ cyclic}\} \xrightarrow{\Sigma \text{Ind}} R_{\mathbb{C}}(G) \right].$$

Natural epimorphisms  $I_{RG} : A_{\mathbb{C}}(G) \twoheadrightarrow Cl_1(RG)$  are constructed, for such  $R$  and  $G$ , via localization sequences. Theorem 1.4 can then be applied to show that for any  $G$ ,  $I_{RG}$  is an isomorphism for  $R$  large enough. Thus,  $A_{\mathbb{C}}(G)$  represents the “largest possible”  $Cl_1(RG)$  when  $G$  is fixed and  $R$  varies. This is the second unexpected appearance of Artin cokernels when studying  $K_n(RG)$ : it was shown in [18] that  $D(\mathbb{Z}G)^+ \cong A_{\mathbb{Q}}(G)$  when  $G$  is a  $p$ -group and  $p$  any odd regular prime.

The obvious remaining question is: what about 2-power torsion in  $SK_1(\mathbb{Z}G)$ ? Unlike the case of odd torsion, this cannot be completely reduced to studying  $Cl_1(\mathbb{Z}G)$  for 2-groups  $G$ , but the results in [17] show that the main problem is with 2-groups. If  $G$  is a  $p$ -group (for any  $p$ ) and  $[G, G]$  is central and cyclic, then we can show that  $K_2^{\text{top}}(\hat{\mathbb{Z}}_p G)$  is generated by  $\{-1, -1\}$  and symbols  $\{g, u\}$  for  $g \in G$  and  $u \in (\hat{\mathbb{Z}}_p[Z_G(g)])^*$ ; and when  $p = 2$  this suffices to get a description of  $Cl_1(\mathbb{Z}G)$ . But there *are* 2-groups  $G$  for which  $K_2^{\text{top}}(\mathbb{Z}_2 G)$  is not generated by such symbols, and there may not be any simple algorithm for describing  $Cl_1(\mathbb{Z}G)$  in general. The best conjecture we have been able to make so far gives upper and lower bounds for  $Cl_1(\mathbb{Z}G)$ , bounds which differ by exponent two. The question of whether the inclusion  $Cl_1(\mathbb{Z}G)_{(2)} \subseteq SK_1(\mathbb{Z}G)_{(2)}$  ever fails to split is also still open.

The paper is organized as follows. Section 1 and 2 deal with the problems of finding generators for  $\text{Ker}(K_2(\hat{\mathbb{Z}}_p \alpha))$ , and of detecting  $\text{Coker}(K_2(\hat{\mathbb{Z}}_p \alpha))$ , respectively, when  $\alpha$  is a surjection of  $p$ -groups whose kernel is central and cyclic. This is applied in Section 3 to prove that  $Cl_1(\mathbb{Z}G) \cong \text{Coker}(\psi_G)$  when  $G$  is an odd  $p$ -group; and ways of using that to compute the odd torsion in  $Cl_1(\mathbb{Z}G)$  for arbitrary finite  $G$  are discussed. Examples are given in Section 4 to illustrate how Theorem 3.6 works in practice for computing  $Cl_1(\mathbb{Z}G)$ . Finally, in Section 5, the relationship between  $Cl_1(RG)$  and the complex Artin cokernel is studied, and the isomorphism  $Cl_1(RG) \cong A_c(G)$  proven for large  $R$ .

As for notation,  $C_n$  always denotes a (multiplicative) cyclic group of order  $n$ , and  $\zeta_n$  a primitive  $n$ -th root of unity. If  $F$  is any field, then  $\mu_F$  denotes the group of roots of unity in  $F$ , and  $(\mu_F)_p$  the group of  $p$ -th power roots of unity.

If  $R$  is a  $\hat{\mathbb{Q}}_p$ -algebra or a  $\hat{\mathbb{Z}}_p$ -order (e.g.,  $R = \hat{\mathbb{Q}}_p G$  or  $\hat{\mathbb{Z}}_p G$ ), then  $K_2(R)$  *always denotes the topological*  $K_2$ . The precise definition of these groups, and their occurrence in localization sequences, is described in [20]: in Theorem 2.1 and the preceding discussion (see also [3]). Here we just note that if  $R$  is a  $\hat{\mathbb{Z}}_p$ -order, then

$$K_2^{\text{top}}(R) \cong \varprojlim_n K_2(R/p^n R).$$

## Section 1

If  $R$  is a ring, and  $I \subseteq R$  is a 2-sided ideal, we define here

$$K_2(R, I) = \text{Ker}[K_2(R) \rightarrow K_2(R/I)].$$

A braid diagram analogous to that in [12, Remark 6.6] shows that for any ideals

$\bar{I} \subseteq I \subseteq R$ , there is an exact sequence

$$0 \rightarrow K_2(R, \bar{I}) \rightarrow K_2(R, I) \rightarrow K_2(R/\bar{I}, I/\bar{I}) \xrightarrow{\partial} K_1(R, \bar{I}) \rightarrow K_1(R, I) \rightarrow \dots$$

The main result of this section is to describe a set of generators for  $K_2(\hat{\mathbb{Z}}_p G, (1 - z))$ ; when  $p$  is any prime,  $G$  is any  $p$ -group, and  $z \in Z(G)$ . Three lemmas will first be needed.

**LEMMA 1.1.** *Fix a prime  $p$ , and a finite ring  $R$  of  $p$ -power order. Let  $J \subseteq R$  be the Jacobson radical, and let  $\{\alpha_1, \dots, \alpha_k\} \subseteq J$  be any set of elements such that  $\{p, \alpha_1, \dots, \alpha_k\}$  generates  $J$  (as an ideal). Then for any ideal  $I \subseteq J$  of  $R$  such that  $IJ = JI = 0$ , and such that  $I \subseteq \langle \alpha_1, \dots, \alpha_k \rangle_R$  if  $p = 2$ ,  $K_2(R, I)$  is generated by symbols of the form*

$$\{1 - \alpha_i, 1 - x\} : 1 \leq i \leq k, \quad x \in I. \tag{1}$$

*Proof.* We use the notation and relations for pointed bracket symbols in [25, Proposition 96–97]. By [17, Proposition 2.3],  $K_2(R, I)$  is generated by symbols of the form

$$\{1 - \alpha, 1 + x\} = \langle \alpha, 1 + x \rangle = \langle \alpha, x \rangle$$

for  $\alpha \in J$  and  $x \in I$  ( $\alpha x = x\alpha = 0$ ). Write  $\alpha = pr_0 + \alpha_1 r_1 + \dots + \alpha_k r_k$ ; so that

$$\begin{aligned} \langle \alpha, x \rangle &= \langle pr_0, x \rangle + \sum_{i=1}^k \langle \alpha_i r_i, x \rangle = \langle p, r_0 x \rangle + \sum_{i=1}^k \langle \alpha_i, r_i x \rangle \\ &= \sum_{i=1}^k \{1 - \alpha_i, 1 + r_i x\} + \langle p, r_0 x \rangle + p \langle -1, r_0 x \rangle \\ &= \sum_{i=1}^k \{1 - \alpha_i, 1 + r_i x\} + \langle p, r_0 x \rangle + \left\langle -p + \binom{p}{2} r_0 x, r_0 x \right\rangle \quad (x^2 = 0) \\ &= \sum_{i=1}^k \{1 - \alpha_i, 1 + r_i x\} + \left\langle \binom{p}{2} r_0 x, r_0 x \right\rangle. \quad (px \in JI = 0). \end{aligned}$$

If  $p$  is odd, then  $\binom{p}{2} x = 0$ , and we are done. If  $p = 2$  and  $I \subseteq \langle \alpha_1, \dots, \alpha_k \rangle_R$ , then the same procedure shows that for any  $x \in I$ ,  $\langle x, x \rangle$  is a sum of symbols of the form in (1).  $\square$

The following technical relation between symbols will be needed in the calculations.

LEMMA 1.2. *Let  $R$  be any ring. Fix  $a, u \in R^*$  and  $n \geq 2$  such that*

$$[a^n, u] = 1 = [a^i u a^{-i}, a^j u a^{-j}]$$

for any  $i, j$ . Then

$$\begin{aligned} \{a, u(a u a^{-1})(a^2 u a^{-2}) \cdots (a^{n-1} u a^{1-n})\} \\ = \{a^n, u\} + (n-1)\{u, u\} + \sum_{i=1}^{n-1} \{a^i u a^{-i}, u\}. \end{aligned}$$

*Proof.* In  $St(R)$ , set  $x = h_{12}(u)$ ,  $y = h_{13}(a)$ , and

$$T = (y x y^{-1})(y^2 x y^{-2}) \cdots (y^{n-1} x y^{1-n}).$$

Then

$$\begin{aligned} \{a, u(a u a^{-1}) \cdots (a^{n-1} u a^{1-n})\} &= [y, x T] \\ &= (y x y^{-1})(y T y^{-1}) T^{-1} x^{-1} = T (y^n x y^{-n}) T^{-1} x^{-1} = [T, y^n x y^{-n}] [y^n, x] \\ &= (\text{diag}(a u a^{-1} \cdot a^2 u a^{-2} \cdots a^{n-1} u a^{1-n}, u^{1-n}) * \text{diag}(u, u^{-1})) + \{a^n, u\} \\ &= \sum_{i=1}^{n-1} \{a^i u a^{-i}, u\} + \{u^{1-n}, u^{-1}\} + \{a^n, u\}. \end{aligned}$$

Here, for commuting matrices  $M, N \in E(R)$ ,  $M^* N \in K_2(R)$  denotes the commutator  $[\tilde{M}, \tilde{N}]$  of liftings to  $\tilde{M}, \tilde{N} \in St(R)$ .  $\square$

The third lemma will be needed when constructing filtrations of group rings by ideals. By a  $p$ -ring is meant the ring of integers in any finite extension of  $\mathbb{Q}_p$ .

LEMMA 1.3. *Fix a prime  $p$ , a  $p$ -group  $G$ , and some  $z \in Z(G)$ . Let  $p^n = |z|$ . Then, for any  $p$ -ring  $A$ , there are isomorphisms*

$$f_k: A/p^n[G/z] \xrightarrow{\cong} \frac{(1-z)^k A G}{(1-z)^{k+1} A G} \quad (k \geq 1)$$

and

$$f'_k: A/p[G/z] \xrightarrow{\cong} \frac{(1-z)^k A/p[G]}{(1-z)^{k+1} A/p[G]}; \quad (1 \leq k \leq p^n - 1)$$

both induced by sending  $\xi$  to  $(1-z)^k \xi$  for  $\xi \in AG$ .

*Proof.* Note first that for any  $\xi \in AG$ , and any  $k \geq 1$ ,

$$(1-z)^k p^n \xi \equiv (1-z)^k (1+z+z^2+\dots+z^{p^n-1}) \xi = 0 \pmod{(1-z)^{k+1} AG}. \tag{1}$$

Thus,  $(1-z)^k AG / (1-z)^{k+1} AG$  has exponent at most  $p^n$  for  $k \geq 1$ ; and is in particular finite. So the map

$$(1-z)^k : (1-z)AG \xrightarrow{\cong} (1-z)^{k+1} AG$$

is an isomorphism: it is clearly onto, and the groups are free  $A$ -modules of the same rank.

Thus, for  $\xi \in AG$  and  $k \geq 1$ , if  $(1-z)^k \xi = (1-z)^{k+1} \eta$  for some  $\eta \in AG$ , then  $(1-z)(\xi - (1-z)\eta) = 0$ , and so

$$\xi \in (1-z)\eta + (1+z+\dots+z^{p^n-1})AG \subseteq (1-z)AG + p^n AG.$$

Together with (1), this shows that  $(1-z)^k \xi \in (1-z)^{k+1} AG$  if and only if  $\xi \in p^n AG + (1-z)AG$ . So  $f_k$  is well defined and an isomorphism.

If  $1 \leq k \leq p^n - 1$ , and  $\xi' \in A/p[G]$  is such that  $(1-z)^k \xi' \in (1-z)^{k+1} A/p[G]$  then

$$(1+z+\dots+z^{p^n-1})\xi' = (1-z)^{p^n-1} \xi' \in (1-z)^{p^n} A/p[G] = 0;$$

and so  $\xi' \in (1-z)A/p[G]$ . The converse is clear, and so  $f'_k$  is a well defined isomorphism.  $\square$

The main result of this section can now be shown:

**THEOREM 1.4.** *Fix a prime  $p$ , an unramified  $p$ -ring  $A$ , a  $p$ -group  $G$ , and an element  $z \in Z(G)$ . Then*

$$K_2(AG, (1-z)AG) = \text{Ker} [K_2(AG) \rightarrow K_2(A[G/z])]$$

is a finite group, and is generated by symbols of the form

$$\{g, 1 - \lambda(1 - z)^i h\} : g, h \in G, [g, h] = 1, \lambda \in A, i \geq 1.$$

*Proof.* Let  $H_0 = \langle z \rangle$ , and fix a series of subgroups

$$H_0 \subseteq H_1 \subseteq \cdots \subseteq H_n = G$$

such that for each  $i = 1, \dots, n$ ,

$$H_i \triangleleft G \quad \text{and} \quad [H_i : H_{i-1}] = p.$$

For each  $i$ , fix  $z_i \in H_i \setminus H_{i-1}$ . Note that in  $G/H_{i-1}$ ,  $z_i$  is central of order  $p$ .

Let  $p^m = |z|$ . Define

$$S = \{(k; r, i_0, \dots, i_k) : 0 \leq k \leq n, i_0 \geq 1, 0 \leq r \leq m - 1, 0 \leq i_1, \dots, i_k \leq p - 1\}.$$

For each  $\sigma = (k; r, i_0, \dots, i_k) \in S$ , set  $k(\sigma) = k$ , and

$$X(\sigma) = p^r(1 - z)^{i_0}(1 - z_1)^{i_1} \cdots (1 - z_k)^{i_k} \in AG.$$

Define ideals  $I'(\sigma) \subseteq I(\sigma) \subseteq AG$  by setting

$$I'(\sigma) = \langle (1 - z)^{i_0+1}, p^{r+1}(1 - z)^{i_0}; p^r(1 - z)^{i_0}(1 - z_1)^{i_1} \cdots (1 - z_j)^{i_j+1}; 1 \leq j \leq k \rangle$$

and

$$I(\sigma) = I'(\sigma) + \langle X(\sigma) \rangle.$$

The idea now is to use  $S$  as a bookkeeping system for filtering the ideal  $(1 - z)AG$  into “pieces” small enough so that the theorem can be proven starting with Lemma 1.1. The following diagram gives a visual overview of this

filtration in the case where  $p = 3$ ,  $m = 2$ , and  $n = 2$  (i.e.,  $|z| = 9$  and  $|G| = 81$ ):

	$k(\sigma) = 0$	$k(\sigma) = 1$	$k(\sigma) = 2$	
$(1 - z)AG$	$(0; 0, 1)$	$(1; 0, 1, 0)$	$(2; 0, 1, 0, 0)$	
			$(2; 0, 1, 0, 1)$	
			$(2; 0, 1, 0, 2)$	
		$(1; 0, 1, 1)$	$(2; 0, 1, 1, 0)$	
			$(2; 0, 1, 1, 1)$	
			$(2; 0, 1, 1, 2)$	
		$(1; 0, 1, 2)$	$(2; 0, 1, 2, 0)$	
			$(2; 0, 1, 2, 1)$	
			$(2; 0, 1, 2, 2)$	
	$(0; 1, 1)$	$(1; 1, 1, 0)$	$(2; 1, 1, 0, 0)$	
			$(2; 1, 1, 0, 1)$	
			$(2; 1, 1, 0, 2)$	
$(1; 1, 1, 1)$		$(2; 1, 1, 1, 0)$		
		$(2; 1, 1, 1, 1)$		
		$(2; 1, 1, 1, 2)$		
$(1; 1, 1, 2)$		$(2; 1, 1, 2, 0)$		
		$(2; 1, 1, 2, 1)$		
		$(2; 1, 1, 2, 2)$		
$(1 - z)^2AG$	$(0; 0, 2)$	$(1; 0, 2, 0)$	$(2; 0, 2, 0, 0)$	
			$(2; 0, 2, 0, 1)$	
			$(2; 0, 2, 0, 2)$	
	$(1; 0, 2, 1)$	$(2; 0, 2, 1, 0)$		
		$(2; 0, 2, 1, 1)$		
		$(2; 0, 2, 1, 2)$		

(1)

The horizontal lines represent ideals in  $AG$ , ordered sequentially with the largest at the top. Each box represents some element  $\sigma \in S$ ; the horizontal line at the top of the box represents  $I(\sigma)$ , while the line at the bottom represents  $I'(\sigma)$ . That the  $I(\sigma)$  and  $I'(\sigma)$  actually do correspond with this picture will be shown in Step 2A below.

*Step 1.* We now show that for any  $\sigma \in S$ , there is an isomorphism

$$f_\sigma : A/p[G/H_{k(\sigma)}] \xrightarrow{\cong} I(\sigma)/I'(\sigma) \quad (2)$$

defined by setting  $f_\sigma([\xi]) = [X(\sigma) \cdot \xi]$  for  $\xi \in AG$ . This will be proven by induction on  $k = k(G)$ . If  $k = 0$ , so  $\sigma = (0; r, i)$  for some  $i \geq 1$  and  $0 \leq r \leq m - 1$ , then

$$(1 - z)^i AG / (1 - z)^{i+1} AG \cong A/p^m[G/Z] = A/p^m[G/H_0]$$

by Lemma 1.3; and so

$$I(\sigma)/I'(\sigma) = \frac{p^r(1 - z)^i AG + (1 - z)^{i+1} AG}{p^{r+1}(1 - z)^i AG + (1 - z)^{i+1} AG} \cong \frac{p^r A/p^m[G/H_0]}{p^{r+1} A/p^m[G/H_0]} \cong A/p[G/H_0].$$

Now assume  $k \geq 1$ , and write  $\sigma = (k; r, i_0, \dots, i_k)$ . Set

$$\hat{\sigma} = (k - 1; r, i_0, \dots, i_{k-1}) \in S.$$

By induction, we can assume that  $I(\hat{\sigma})/I'(\hat{\sigma}) \cong A/p[G/H_{k-1}]$ . By definition

$$I'(\sigma) = I'(\hat{\sigma}) + X(\hat{\sigma})(1 - z_k)^{i_k+1} AG,$$

$$I(\sigma) = I(\hat{\sigma}) + X(\hat{\sigma})(1 - z_k)^{i_k} AG,$$

$$I(\hat{\sigma}) = I'(\hat{\sigma}) + X(\hat{\sigma})AG.$$

Thus,  $I(\hat{\sigma}) \supseteq I(\sigma) \supseteq I'(\sigma) \supseteq I'(\hat{\sigma})$ ; and by Lemma 1.3:

$$I(\sigma)/I'(\sigma) \cong \frac{(1 - z_k)^{i_k} A/p[G/H_{k-1}]}{(1 - z_k)^{i_k+1} A/p[G/H_{k-1}]} \cong A/p[G/H_k].$$

(Recall that  $H_k = \langle H_{k-1}, z_k \rangle$ , and that  $0 \leq i_k \leq p - 1$ .)

*Step 2.* We next show that for any  $\sigma \in S$ ,

$$K_2(AG/I'(\sigma), I(\sigma)/I'(\sigma)) = \langle \{g, 1 - X(\sigma)\lambda h\} : [g, h] \in H_{k(\sigma)}, \lambda \in A \rangle. \quad (3)$$

This will be proven by downwards induction on  $k = k(\sigma)$ .

Note first that  $AG$  is a local ring with Jacobson radical

$$J(AG) = \langle p, 1 - g : g \in G \rangle. \tag{4}$$

If  $\sigma \in S$  and  $k(\sigma) = n$ , then  $H_n = G$ , and so  $I(\sigma)/I'(\sigma) \cong A/p$  by Step 1. In particular,

$$(I(\sigma)/I'(\sigma)) \cdot J(AG/I'(\sigma)) = 0 = J(AG/I'(\sigma)) \cdot (I(\sigma)/I'(\sigma)).$$

So (3) follows in this case from Lemma 1.1 (applied using  $\{1 - g : g \in G\}$  for the  $\alpha_i$ 's).

Now fix some  $\sigma = (k; r, i_0, \dots, i_k) \in S$ , where  $k < n$ . For each  $0 \leq i \leq p - 1$ , set

$$\sigma_i = (k + 1; r, i_0, \dots, i_k, i) \in S.$$

Assume inductively that (3) holds for the  $\sigma_i$ .

*Step 2A.* We now show that the  $I(\sigma_i) \supseteq I'(\sigma_i)$  and  $I(\sigma) \supseteq I'(\sigma)$  have the relations implied by diagram (1) above. By definition,  $I(\sigma_0) = I(\sigma)$  ( $X(\sigma_0) = X(\sigma)$ ). For any  $0 \leq i \leq p - 2$ ,

$$I'(\sigma_i) = I'(\sigma) + X(\sigma)(1 - z_{k+1})^{i+1}AG = I(\sigma_{i+1}). \tag{5}$$

Furthermore,

$$I'(\sigma_{p-1}) = I'(\sigma) + X(\sigma)(1 - z_{k+1})^pAG = I'(\sigma)$$

by (2): since  $X(\sigma)(1 - z_{k+1})^p = f((1 - z_{k+1})^p)$  and

$$(1 - z_{k+1})^p = (1 - z_{k+1}^p) = 0 \in A/p[G/H_k]. \quad (z_{k+1}^p \in H_k).$$

We thus have a filtration

$$I(\sigma) = I(\sigma_0) \supseteq I(\sigma_1) \supseteq \dots \supseteq I(\sigma_{p-1}) \supseteq I'(\sigma_{p-1}) = I'(\sigma); \tag{6}$$

and  $I(\sigma_i) = I'(\sigma_{i-1})$  for  $1 \leq i \leq p - 1$ .

*Step 2B.* For shortness in notation, we now write  $K_1(I)$ ,  $K_2(I)$  for  $K_1(R, I)$ ,  $K_2(R, I)$ :  $R$  is always a quotient ring of  $AG$ . We are assuming that (3) holds for

the  $\sigma_i$ ; i.e., that

$$K_2(I(\sigma_i)/I'(\sigma_i)) = \langle \{g, 1 - X(\sigma_i)\lambda h\} : [g, h] \in H_{k+1}, \lambda \in A \rangle \tag{7}$$

for each  $0 \leq i \leq p - 1$ . Let  $\{\lambda_1, \dots, \lambda_s\}$  be a  $\hat{\mathbb{Z}}_p$ -basis for  $A$ . Let  $h_1, \dots, h_t \in G$  be conjugacy class representatives (mod  $H_{k+1}$ ) for those elements such that  $[g_l, h_l] \in z_{k+1}H_k$  for some  $g_l \in G$ ; fix also such  $g_l$ . Then (7) takes the form

$$K_2(I(\sigma_i)/I'(\sigma_i)) = M_i + \langle \{g_l, 1 - X(\sigma_i)\lambda_j h_l\} : 1 \leq j \leq s, 1 \leq l \leq t \rangle; \tag{8}$$

where

$$M_i = \langle \{g, 1 - \lambda X(\sigma_i)h\} : [g, h] \in H_k, \lambda \in A \rangle. \tag{9}$$

*Step 2C.* Now assume that  $i < p - 1$ ; and consider the relative exact sequence

$$K_2(I(\sigma_i)/I'(\sigma_{i+1})) \rightarrow K_2(I(\sigma_i)/I'(\sigma_i)) \xrightarrow{\partial} K_1(I(\sigma_{i+1})/I'(\sigma_{i+1}))$$

(recall that  $I'(\sigma_i) = I(\sigma_{i+1})$ ). By (2) (and [24, Corollary 2.6]):

$$K_1(I(\sigma_{i+1})/I'(\sigma_{i+1})) \cong H_0(G; A/p[G/H_{k+1}]), \tag{10}$$

where  $G$  acts by conjugation. Furthermore, for  $1 \leq j \leq s, 1 \leq l \leq t$ ,

$$\begin{aligned} \partial(\{g_l, 1 - X(\sigma_i)\lambda_j h_l\}) &= [g_l, 1 - X(\sigma_i)\lambda_j h_l] = 1 - X(\sigma_i)\lambda_j(g_l h_l g_l^{-1} - h_l) \\ &= 1 + X(\sigma_i)(1 - z_{k+1})\lambda_j h_l = 1 + X(\sigma_{i+1})\lambda_j h_l \pmod{I'(\sigma_{i+1})} \end{aligned}$$

(recall that  $[g_l, h_l] \in z_{k+1}H_k$ ). By (10), these elements are all independent in  $K_1(I(\sigma_{i+1})/I'(\sigma_{i+1}))$ . So by (8) and (9),

$$\begin{aligned} \text{Im} [K_2(I(\sigma_i)/I'(\sigma)) \rightarrow K_2(I(\sigma_i)/I'(\sigma_i))] \\ &= \text{Im} [K_2(I(\sigma_i)/I'(\sigma_{i+1})) \rightarrow K_2(I(\sigma_i)/I'(\sigma_i))] \\ &= M_i = \langle \{g, 1 - \lambda X(\sigma_i)h\} : [g, h] \in H_k, \lambda \in A \rangle; \tag{11} \end{aligned}$$

all elements in  $M_i$  lift (using (2)) to  $K_2(I(\sigma_i)/I'(\sigma)) \subseteq K_2(I(\sigma)/I'(\sigma))$ .

Step 2D. By (8) and (11) (and (6)),

$$K_2(I(\sigma)/I'(\sigma)) = M + \langle \{g, 1 - \lambda X(\sigma)(1 - z_{k+1})^{p-1}h\} : [g, h] \in z_{k+1}H_k, \lambda \in A \rangle \tag{12}$$

where

$$\begin{aligned} M &= \langle \{g, 1 - \lambda X(\sigma)(1 - z_{k+1})^i h\} : 0 \leq i \leq p - 1, [g, h] \in H_k, \lambda \in A \rangle \\ &= \langle \{g, 1 - \lambda X(\sigma)h\} : [g, h] \in H_k, \lambda \in A \rangle. \end{aligned}$$

(Note that  $X(\sigma)^2 = 0$  in  $I(\sigma)/I'(\sigma)$ .) We want to show that  $K_2(I(\sigma)/I'(\sigma)) = M$ . Fix  $\lambda \in A$  and  $g, h \in G$  such that  $[g, h] \in z_{k+1}H_k$ , and set  $u = 1 - X(\sigma)\lambda h$ . Then

$$1 - X(\sigma)\lambda(1 - z_{k+1})^{p-1}h = \prod_{i=0}^{p-1} (1 - X(\sigma)\lambda z_{k+1}^i h) = \prod_{i=0}^{p-1} g^i u g^{-i} \in AG/I'(\sigma)$$

by (2) ( $I(\sigma)/I'(\sigma) \cong A/p[G/H_k]$ ). So by Lemma 1.2,

$$\begin{aligned} \{g, 1 - x(\sigma)\lambda(1 - z_{k+1})^{p-1}h\} &= \{g, u \cdot g u g^{-1} \cdots g^{p-1} u g^{1-p}\} \\ &= \{g^p, u\} + (p - 1)\{u, u\} + \sum_{j=1}^{p-1} \{g^j u g^{-j}, u\}. \end{aligned}$$

By definition,  $\{g^p, u\} \in M$ . For any  $0 \leq j \leq p - 1$ :

$$\begin{aligned} \{g^j u g^{-j}, u\} &= \{1 - X(\sigma)\lambda z_{k+1}^j h, 1 - X(\sigma)\lambda h\} \\ &= \left\{ 1 - (1 - z), 1 - X(\sigma) \frac{X(\sigma)}{1 - z} \lambda^2 z_{k+1}^j h^2 \right\} \in M \end{aligned}$$

(see [17, Lemma 2.2] for the last step). So from (12) we now get that  $K_2(I(\sigma)/I'(\sigma)) = M$ ; and this finishes the proof of (3).

Step 3. Now fix some  $i \geq 1$ . For any  $0 \leq r \leq m - 1$ , (3) applied to  $\sigma = (0; r, i)$  says that

$$\begin{aligned} K_2(p^r(1 - z)^i AG / \langle p^{r+1}(1 - z)^i, (1 - z)^{i+1} \rangle) \\ = \langle \{g, 1 - \lambda p^r(1 - z)^i h\} : [g, h] \in \langle z \rangle, \lambda \in A \rangle. \end{aligned}$$

For any such  $g, h,$  and  $\lambda,$  note that (in  $AG$ )

$$[g, 1 - \lambda p^r(1 - z)^i h] \equiv 0; 1 - \lambda p^r(1 - z)^i h \equiv (1 - \lambda(1 - z)^i h)^{p^r} \pmod{(1 - z)^{i+1}AG}.$$

It follows that

$$K_2((1 - z)^i AG / (1 - z)^{i+1} AG) = \langle \{g, 1 - \lambda(1 - z)^i h\} : [g, h] \in \langle z \rangle, \lambda \in A \rangle. \quad (13)$$

*Step 4.* The rest of the proof is analogous to Step 2B and 2C. Let  $\lambda_1, \dots, \lambda_s$  be a  $\hat{\mathbb{Z}}_p$ -basis for  $A,$  and let  $h_1, \dots, h_t \in G$  be conjugacy class representatives for  $G/z.$  For  $1 \leq l \leq t,$  choose  $g_l \in G$  so that  $[g_l, h_l] = z^{q_l},$  and  $1 \leq q_l \leq p^m = |z|$  is minimal. Then by (13),

$$K_2((1 - z)^i AG / (1 - z)^{i+1} AG) = N_i + \langle \{g_l, 1 - \lambda_j(1 - z)^i h_l\} : 1 \leq l \leq t, 1 \leq j \leq s \rangle, \quad (14)$$

where

$$N_i = \langle \{g, 1 - \lambda(1 - z)^i h\} : [g, h] = 1, \lambda \in A \rangle.$$

Consider the exact sequence

$$K_2\left(\frac{(1 - z)^i AG}{(1 - z)^{i+2} AG}\right) \rightarrow K_2\left(\frac{(1 - z)^i AG}{(1 - z)^{i+1} AG}\right) \xrightarrow{\partial} K_1\left(\frac{(1 - z)^{i+1} AG}{(1 - z)^{i+2} AG}\right). \quad (15)$$

For any  $j, l:$

$$\partial(\{g_l, 1 - \lambda_j(1 - z)^i h_l\}) = [g_l, 1 - \lambda_j(1 - z)^i h_l] = 1 + q_l \lambda_j(1 - z)^{i+1} h_l.$$

By Lemma 1.3, these elements are independent in

$$K_1((1 - z)^{i+1} AG / (1 - z)^{i+2} AG) \cong H_0(G; A/p^m[G])$$

and have order  $p^m/q_l$  ( $q_l$  is a power of  $p$ ). Furthermore, for each  $j$  and  $l,$   $[g_l^{p^m/q_l}, h_l] = 1,$  and so

$$p^m/q_l \cdot \{g_l, 1 - \lambda_j(1 - z)^i h_l\} \in N_i.$$

So by (14), and the exactness of (15),

$$\text{Im} \left[ K_2 \left( \frac{(1-z)^i AG}{(1-z)^{i+2} AG} \right) \rightarrow K_2 \left( \frac{(1-z)^i AG}{(1-z)^{i+1} AG} \right) \right] = \text{Ker} (\partial) = N_i.$$

Every element of  $N_i$  lifts to  $K_2((1-z)^i AG) \subseteq K_2((1-z)AG)$ . Thus, for any  $i \geq 1$ ,

$$K_2((1-z)^i AG) = K_2((1-z)^{i+1} AG) + \langle \{g, 1 - \lambda(1-z)^i h\} : gh = hg, \lambda \in A \rangle. \tag{16}$$

By induction, for any  $N > 1$ ,

$$K_2((1-z)AG) = K_2((1-z)^N AG) + \langle \{g, 1 - \lambda(1-z)^i h\} : gh = hg, \lambda \in A, 1 \leq i < N \rangle. \tag{17}$$

Let  $p^k = \exp(G)$ , and recall that  $|z| = p^m$ . Then  $p(1-z) \mid (1-z)^{p^m}$ , and so

$$1 + (1-z)^{(k+1)p^m} AG \subseteq 1 + p^{k+1}(1-z)AG \subseteq \{(1 + (1-z)\xi)^{p^k} : \xi \in AG\}.$$

Thus, for any commuting  $h, g \in G$ , any  $\lambda \in A$ , and any  $i \geq (k+1)p^m$ :

$$\{g, 1 - \lambda(1-z)^i h\} = \{g, (1 - (1-z)\xi)^{p^k}\} = \{g^{p^k}, 1 - (1-z)\xi\} = 0. \tag{some } \xi \in AG.$$

By (16), for any  $N > (k+1)p^m$ ,  $K_2((1-z)^{(k+1)p^m} AG) = K_2((1-z)^N AG)$ ; and so

$$K_2((1-z)^{(k+1)p^m}) = \lim_{\leftarrow N} K_2((1-z)^{(k+1)p^m} AG / (1-z)^N AG) = 0 \tag{18}$$

Equation (17) now takes the form

$$K_2((1-z)AG) = \langle \{g, 1 - \lambda(1-z)^i h\} : gh = hg, \lambda \in A, 1 \leq i < (k+1)p^m \rangle.$$

Furthermore, it suffices to take  $\lambda$  belonging to some  $\hat{\mathbb{Z}}_p$ -basis for  $A$ . This shows that  $K_2((1-z)AG)$  is generated by a finite set of elements of finite order, and is hence finite.  $\square$

With some more work, one can in fact show that  $K_2(AG, (1-z)AG)$  is

generated by symbols  $\{g, 1 - \lambda(1 - z)h\}$ , where  $gh = hg$  in  $G$  and  $\lambda$  lies in any fixed  $\hat{\mathbb{Z}}_p$ -basis for  $A$ .

One easy consequence of Theorem 1.4 is:

**THEOREM 1.5.** *For any prime  $p$ , any unramified  $p$ -ring  $A$ , and any  $p$ -group  $G$ ,  $K_2(AG)$  is finite.*

*Proof.* Fix some  $1 \neq z \in Z(G)$ . Then  $K_2(AG, (1 - z)AG)$  is finite by Theorem 1.4. We may assume inductively that  $K_2(A[G/z])$  is finite; and so  $K_2(AG)$  is also finite.  $\square$

In fact, using the results in [17], this can be extended to arbitrary finite  $G$ . Whether it is true for arbitrary  $\mathbb{Z}_p$ -orders, we do not know.

## Section 2

Theorem 1.4 gives a set of generators for  $\text{Ker}(K_2(A\alpha))$ , when  $\alpha: \tilde{G} \rightarrow G$  is a central extension of  $p$ -groups with cyclic kernel. In this section, we study  $\text{Coker}(K_2(A\alpha))$  when  $\text{Ker}(\alpha) \subseteq Z(\tilde{G})$ . This problem was studied in [19]:  $\text{Coker}(K_2(A\alpha))$  is described there for an arbitrary surjection  $\alpha$ , but only up to a mysterious contribution by  $H_3(G)$ . What we show here is that the  $H_3(G)$  contribution vanishes when  $\alpha$  is a central extension.

**PROPOSITION 2.1.** *Let  $p$  be any prime, let  $A$  be an unramified  $p$ -ring, and let  $\alpha: \tilde{G} \rightarrow G$  be any central extension of  $p$ -groups (i.e.,  $\text{Ker}(\alpha) \subseteq Z(\tilde{G})$ ). Then there is an exact sequence*

$$0 \rightarrow \text{Coker}(H_2(\alpha)) \xrightarrow{T_\alpha} \text{Coker}[K_2(A\alpha): K_2(A\tilde{G}) \rightarrow K_2(AG)] \xrightarrow{\Gamma_2^*(\alpha)} H_1(G; AG) / \langle g \otimes \lambda h : [\alpha^{-1}g, \alpha^{-1}h] = 1 \rangle.$$

Here,  $T_\alpha$  is included by the usual inclusion  $H_2(G) \rightarrow K_2(AG) / \{-1, G\}$ , and  $\Gamma_2^*(\alpha)$  is induced by the homomorphism

$$\Gamma_2^*(G): K_2(AG) \rightarrow H_1(G; AG) / \langle g \otimes \lambda g^n : g \in G, \lambda \in A, n \in \mathbb{Z} \rangle$$

of [19, Theorem 3.6]. In particular, for any  $g \in G$ ,  $H = Z_G(g)$ , and any

$u \in (AH)^*$ ,

$$\Gamma_2^*(\alpha)(\{g, u\}) = g \otimes \Gamma_H(u) \in H_1(G; AG) / \langle g \otimes \lambda h : [\alpha^{-1}g, \alpha^{-1}h] = 1 \rangle.$$

*Proof.* Define the group  $\hat{G}$  and the order  $\mathfrak{A}$  to be the pullbacks:

$$\begin{array}{ccc} \hat{G} & \xrightarrow{r_1} & \tilde{G} \\ \downarrow r_2 & & \downarrow \alpha \\ \tilde{G} & \xrightarrow{\alpha} & G \end{array} \qquad \begin{array}{ccc} \mathfrak{A} & \xrightarrow{\hat{r}_1} & A\tilde{G} \\ \downarrow \hat{r}_2 & & \downarrow A\alpha \\ A\tilde{G} & \xrightarrow{A\alpha} & AG. \end{array}$$

Set

$$I_1 = \text{Ker} [A\hat{G} \xrightarrow{Ar_1} A\tilde{G}], \quad I_2 = \text{Ker} [A\hat{G} \xrightarrow{Ar_2} A\tilde{G}], \quad I = \text{Ker} [A\tilde{G} \xrightarrow{A\alpha} AG].$$

Then  $\mathfrak{A} \cong A\hat{G}/(I_1 \cap I_2)$ ; and so by Lemma 2.4 in [16],

$$\mathfrak{A} \cong A\hat{G}/I_1I_2.$$

Step 1. By [26, Theorem 4.1],

$$\text{tors}(K_1(A\hat{G})) \cong \mu_A \times \hat{G}^{ab} \times SK_1(A\hat{G}); \tag{1}$$

where  $\mu_A$  denotes the group of roots of unity in  $A$ . We first claim that

$$\hat{G}^{ab} \hookrightarrow K_1(A\hat{G}/I_1I_2) \cong K_1(\mathfrak{A}) \tag{2}$$

is injective. To see this, let  $I(A\hat{G})$  denote the augmentation ideal of  $A\hat{G}$ . Then  $I(A\hat{G})^2 \supseteq I_1I_2$ , and by [19, Proposition 2.2]:

$$A\hat{G}/I(A\hat{G})^2 \cong A \times (A \otimes \hat{G}^{ab}).$$

The isomorphism identifies  $g \in \hat{G}^{ab}$  with  $(1, 1 \otimes g)$ , and so  $\hat{G}^{ab} \subseteq K_1(A\hat{G}/I(A\hat{G})^2)$ .

Now set  $K = \text{Ker}(\alpha) \cong \text{Ker}(r_1)$ , and consider the following diagram:

$$\begin{array}{ccccccccc} H_2(\hat{G}) & \longrightarrow & H_2(\tilde{G}) & \xrightarrow{\delta_{r_1}} & K & \longrightarrow & \hat{G}^{ab} & \xrightarrow{H_1(r_1)} & G^{ab} & \longrightarrow & 0 \\ \downarrow & & \downarrow H_2(\alpha) & & \downarrow Id & & \downarrow H_1(r_2) & & \downarrow & & \\ H_2(\tilde{G}) & \xrightarrow{H_2(\alpha)} & H_2(G) & \xrightarrow{\delta^\alpha} & K & \longrightarrow & \tilde{G}^{ab} & \longrightarrow & G^{ab} & \longrightarrow & 0. \end{array}$$

The rows are the five-term exact sequences for the extensions  $r_1: \hat{G} \twoheadrightarrow \tilde{G}$  and

$\alpha: \tilde{G} \rightarrow G$  (see [8, Corollary VI. 8.2]). It follows that

$$\text{Ker } [H_1(r_1 \times r_2): \hat{G}^{ab} \rightarrow \tilde{G}^{ab} \times \tilde{G}^{ab}] \cong \text{Coker } (H_2(\alpha)). \tag{3}$$

Furthermore,  $\delta^{r_1} = \delta^{\alpha \circ H_2(\alpha)} = 0$ , so  $\text{Ker } (r_1) \cap [\hat{G}, \hat{G}] = 1$ , and

$$SK_1(Ar_1): SK_1(A\hat{G}) \rightarrow SK_1(A\tilde{G}) \tag{4}$$

is injective by [15, Proposition 7].

*Step 2.* Now define

$$\Gamma_{A\hat{G}}: K_1(A\hat{G}) \rightarrow H_0(\hat{G}; A\hat{G}); \quad \Gamma_{A\tilde{G}}: K_1(A\tilde{G}) \rightarrow H_0(\tilde{G}; A\tilde{G})$$

as in [20, Theorem 2.7], and recall that they are isomorphisms modulo torsion. By Theorem 1.1 in [19],

$$\Gamma_{A\hat{G}}(1 + I_1 I_2) = \text{Im } [I_1 I_2 \rightarrow H_0(\hat{G}; A\hat{G})]. \tag{5}$$

So  $\Gamma_{A\hat{G}}$  induces a homomorphism

$$\Gamma_{\mathfrak{A}}: K_1(\mathfrak{A}) \rightarrow H_0(\hat{G}; \mathfrak{A}).$$

Consider the following diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mu_A \times \hat{G}^{ab} \times SK_1(A\hat{G}) & \longrightarrow & K_1(\mathfrak{A}) & \xrightarrow{\Gamma_{\mathfrak{A}}} & H_0(\hat{G}; \mathfrak{A}) \\ & & \downarrow f & & \downarrow K_1(r_1 \times r_2) & & \downarrow H_0(r_1 \times r_2) \\ 0 & \longrightarrow & [\mu_A \times \tilde{G}^{ab} \times SK_1(A\tilde{G})]^2 & \longrightarrow & [K_1(A\tilde{G})]^2 & \xrightarrow{\Gamma_{A\tilde{G}}} & [H_0(\tilde{G}; A\tilde{G})]^2 \end{array} \tag{6}$$

The bottom row is exact since  $\text{Ker } (\Gamma_{A\tilde{G}}) = \text{tors } (K_1(A\tilde{G}))$ . The top row is exact at  $K_1(\mathfrak{A})$  since by (5),  $\text{Ker } (\Gamma_{A\hat{G}}) \rightarrow \text{Ker } (\Gamma_{\mathfrak{A}})$  is onto. By (3) and (4),  $\hat{G}^{ab} \supseteq \text{Ker } (f) \cong \text{Coker } (H_2(\alpha))$ , and this injects into  $K_1(\mathfrak{A})$  by (2). So the top row in (6) is exact, and there is an exact sequence

$$0 \rightarrow \text{Coker } (H_2(\alpha)) \rightarrow \text{Ker } (K_1(r_1 \times r_2)) \rightarrow \text{Ker } (H_0(r_1 \times r_2)). \tag{7}$$

By the Mayer–Victoris sequence for a pullback square,

$$\text{Ker } (K_1(r_1 \times r_2)) \cong \text{Coker } [K_2(A\alpha): K_2(A\tilde{G}) \rightarrow K_2(AG)]. \tag{8}$$

Step 3. The extension  $0 \rightarrow I \rightarrow \mathfrak{A} \xrightarrow{r_1} AG \rightarrow 0$  is  $\hat{G}$ -equivariantly split by the diagonal map. Thus,

$$\text{Ker} [\hat{r}_{1*}: H_0(\hat{G}; \mathfrak{A}) \rightarrow H_0(\tilde{G}; A\tilde{G})] \cong H_0(\tilde{G}; I);$$

and so

$$\begin{aligned} \text{Ker} (H_0(r_1 \times r_2)) &\cong \text{Ker} [H_0(\tilde{G}; I) \rightarrow H_0(\tilde{G}; A\tilde{G})] \\ &\cong \text{Coker} [H_1(\tilde{G}; A\tilde{G}) \rightarrow H_1(\tilde{G}; AG)] \\ &\cong H_1(G; AG) / \langle g \otimes \lambda h : \lambda \in A, [\alpha^{-1}g, \alpha^{-1}h] = 1 \rangle. \end{aligned} \tag{9}$$

Upon substituting (8) and (9) into (7), we get the exact sequence

$$0 \rightarrow \text{Coker} (H_2(\alpha)) \xrightarrow{T_\alpha} \text{Coker} (K_2(A\alpha)) \xrightarrow{\Gamma_2^*(\alpha)} H_1(G; AG) / \langle g \otimes \lambda h : [\alpha^{-1}g, \alpha^{-1}h] = 1 \rangle.$$

That  $\Gamma_2^*(\alpha)$  is the reduction of the map  $\Gamma_2^*(G)$  of [19] follows since the constructions are identical. By diagram chasing,  $T_\alpha$  is seen to be the reduction of the standard inclusion  $H_2(G) \rightarrow K_2(AG) / \{-1, G\}$ .  $\square$

In fact, in the above situation,  $\text{Im} (\Gamma_2^*(\alpha))$  can be described precisely with the help of Theorem 3.6 in [19].

Proposition 2.1 will be applied directly in Section 3, when describing  $Cl_1(\mathbb{Z}G)$  for odd  $p$ -groups  $G$ . But we first note one consequence of particular interest. The next theorem is useful when constructing maps

$$\Gamma_2: K_2(AG) \rightarrow H_1(G; AG) / \langle g \otimes \lambda g \rangle$$

for non-abelian  $p$ -groups  $G$  (compare with [21]).

**THEOREM 2.2.** *Let  $\alpha: \tilde{G} \twoheadrightarrow G$  be any surjection of  $p$ -groups such that  $\text{Ker} (\alpha) \cap [\tilde{G}, \tilde{G}] = 1$ . Then for any unramified  $p$ -ring  $A$ , the map*

$$K_2(A\alpha): K_2(A\tilde{G}) \rightarrow K_2(AG)$$

*is onto, and its kernel is generated by elements of the form  $\{g, 1 + (1 - z)^i h\}$  for  $z \in \text{Ker} (\alpha)$ ,  $i \geq 1$ , and commuting  $g, h \in G$ .*

*Proof.* Note first that

$$[\text{Ker}(\alpha), \tilde{G}] \subseteq \text{Ker}(\alpha) \cap [\tilde{G}, \tilde{G}] = 1;$$

so that  $\text{Ker}(\alpha) \subseteq Z(\tilde{G})$ . The exact sequence

$$H_2(\tilde{G}) \xrightarrow{H_2(\alpha)} H_2(G) \xrightarrow{\delta^\alpha} \text{Ker}(\alpha) \twoheadrightarrow \tilde{G}^{ab} \rightarrow G^{ab} \rightarrow 0$$

(see [8, Corollary VI. 8.2]) shows that  $H_2(\alpha)$  is onto. By hypothesis,

$$H_1(G; AG) / \langle g \otimes \lambda h : [\alpha^{-1}g, \alpha^{-1}h] = 1 \rangle = 0;$$

commuting elements in  $G$  lift to commuting elements in  $\tilde{G}$ . So  $K_2(A\alpha)$  is onto by Proposition 2.1.

Now write  $\alpha$  as a composite

$$\alpha : \tilde{G} = G_0 \xrightarrow{\alpha_1} G_1 \xrightarrow{\alpha_2} G_2 \twoheadrightarrow \cdots \xrightarrow{\alpha_n} G_n = G;$$

and so that  $\text{Ker}(\alpha_j)$  is cyclic for all  $j$ . By Theorem 1.4,

$$\text{Ker}(K_2(A\alpha_j)) = \langle \{g, 1 + (1 - z)^i h\} : z \in \text{Ker}(\alpha_j), i \geq 1, g, h \in G_{j-1}, gh = hg \rangle$$

for each  $j$ . But all such symbols lift to  $K_2(A\tilde{G})$ ; and so  $\text{Ker}(K_2(A\alpha))$  is generated as described.  $\square$

### Section 3

We can now derive algorithms for computing the groups  $Cl_1(\mathbb{Z}G)[\frac{1}{2}]$  and  $SK_1(\mathbb{Z}G)[\frac{1}{2}]$  for finite  $G$ . The key extra tool when working with odd torsion is the standard involution on  $K_n(\mathbb{Z}G)$  and  $K_n(\hat{\mathbb{Z}}_p G)$ ; for example, this is what was used in [17] to construct natural splittings

$$SK_1(\mathbb{Z}G)[\frac{1}{2}] \cong Cl_1(\mathbb{Z}G)[\frac{1}{2}] \oplus \sum_{2 < p \mid |G|} SK_1(\hat{\mathbb{Z}}_p G).$$

Recall that for any group  $G$  and any commutative ring  $R$ , an antiinvolution  $x \rightarrow \bar{x}$  on  $RG$  is defined by setting

$$\overline{\sum r_i g_i} = \sum r_i g_i^{-1} \quad (r_i \in R, g_i \in G).$$

This extends to an antiinvolution on  $GL(RG)$  – defined by setting  $\overline{(a_{ij})} = (\bar{a}_{ji})$  – and hence an involution on  $K_1(RG)$ . Similarly, an antiinvolution on  $St(RG)$  is induced by setting  $\overline{x_{ij}(a)} = x_{ji}(\bar{a})$  ( $a \in RG$ ); and this restricts to an involution on  $K_2(RG)$ .

**LEMMA 3.1.** *For any group ring  $RG$  as above, and any commuting units,  $u, v \in (RG)^*$ ,  $\overline{\{u, v\}} = \{\bar{v}, \bar{u}\}$ . In particular, for any  $g \in G$ , and  $u \in (RG)^*$  such that  $gu = ug$ ,  $\overline{\{g, u\}} = \{g, \bar{u}\}$ .*

*Proof.* Recall that  $\{u, v\} = [X, Y]$ , where  $X, Y \in St(RG)$  are arbitrary liftings of  $\text{diag}(u, u^{-1}, 1)$  and  $\text{diag}(v, 1, v^{-1})$ . Then

$$\overline{\{u, v\}} = \overline{[X, Y]} = \bar{Y}^{-1}\bar{X}^{-1}\bar{Y}\bar{X} = \{\bar{v}^{-1}, \bar{u}^{-1}\} = \{\bar{v}, \bar{u}\}.$$

The last statement follows since  $\bar{g} = g^{-1}$ .  $\square$

The importance of the involution for simplifying the computation of  $Cl_1(\mathbb{Z}G)$  follows from:

**LEMMA 3.2.** *For any odd prime  $p$  and any  $p$ -group  $G$ , the involution on  $K_2(\hat{\mathbb{Q}}_p[G])_{(p)}$  is the identity.*

*Proof.* By [22, Section 2 and 3], for any  $p$ -group  $G$  and any irreducible  $\mathbb{Q}G$ -module  $V$ , there are subgroups  $K \triangleleft H \subseteq G$  and a faithful  $\mathbb{Q}[H/K]$ -representation  $W$  such that  $V = \text{Ind}_H^G(W)$ ,  $\text{End}_{\mathbb{Q}H}(W) \cong \text{End}_{\mathbb{Q}G}(V)$ , and  $H/K$  is cyclic. Let  $A \subseteq \mathbb{Q}H$  and  $B \subseteq \mathbb{Q}G$  denote the corresponding simple summands. Then the induction map restricts to a Morita equivalence from  $A$  to  $B$ , and hence induces an isomorphism of  $K_2(\hat{\mathbb{Q}}_p \otimes_{\mathbb{Q}} A)$  to  $K_2(\hat{\mathbb{Q}}_p \otimes_{\mathbb{Q}} B)$ . Thus, if

$$S = \{(H, K) : K \triangleleft H \subseteq G, H/K \text{ cyclic}\},$$

then the map

$$\sum \text{Ind}_{H/K}^G : \sum_{(H,K) \in S} K_2(\hat{\mathbb{Q}}_p[H/K]) \rightarrow \hat{\mathbb{Q}}_p[G] \tag{1}$$

is onto. Here,  $\text{Ind}_{H/K}^G$  is the composite

$$\text{Ind}_{H/K}^G : K_2(\hat{\mathbb{Q}}_p[H/K]) \xrightarrow{\text{incl}} K_2(\hat{\mathbb{Q}}_p H) \xrightarrow{\text{Ind}_H^G} K_2(\hat{\mathbb{Q}}_p G);$$

where the first map is induced by the inclusion of  $\hat{\mathbb{Q}}_p[H/K]$  as a direct summand of  $\hat{\mathbb{Q}}_p H$ .

The  $\text{Ind}_{K/H}^G$  commute with the involution, and so by (1) it suffices to prove the lemma when  $G$  is cyclic. If  $G \cong C_{p^n}$ , write  $\hat{\mathbb{Q}}_p G \cong \prod_{i=0}^n F_i$ , where  $F_i \cong \hat{\mathbb{Q}}_p[\zeta_{p^i}]$  (a field). For each  $i$ , the involution inverts elements in  $\mu_{F_i}$ . So from the isomorphism  $K_2(F_i) \cong \mu_{F_i}$  and its naturality with respect to automorphisms of  $F_i$ , we get that  $\{\bar{u}, \bar{v}\} = -\{u, v\} = \{v, u\}$  for  $u, v \in F_i^*$ . But  $\{\bar{n}, \bar{v}\} = \{v, u\}$  by Lemma 3.1, and so the involution on  $K_2(F_i)$ , and hence on  $K_2(\hat{\mathbb{Q}}_p G)$ , is trivial.  $\square$

In fact, Lemma 3.2 also holds for 2-groups, and for arbitrary finite  $G$  if  $K_2(\hat{\mathbb{Q}}_p G)_{(p)}$  is replaced by  $C_p(\mathbb{Q}G)$  (see the definition in the introduction).

The main problem when describing  $Cl_1(\mathbb{Z}G)$  for a  $p$ -group  $G$  is computing the image of  $K_2(\hat{\mathbb{Z}}_p G)$  in  $K_2(\hat{\mathbb{Q}}_p G)$ . Lemma 3.2 shows that when  $p$  is odd, it is enough to concentrate attention on  $K_2(\hat{\mathbb{Z}}_p G)^+$ ; and (recall the formula  $\{\bar{g}, u\} = \{g, \bar{u}\}$ ) on  $K_1(\hat{\mathbb{Z}}_p G)^+$ .

**PROPOSITION 3.3.** *For any odd prime  $p$ , any unramified  $p$ -ring  $A$ , and any  $p$ -group  $G$ ,  $\Gamma_{AG}$  restricts to an isomorphism*

$$\Gamma_{AG}^+ : K_1(AG)^+ \rightarrow H_0(G; AG)^+.$$

*Proof.* By [20, Theorem 2.7], there is an exact sequence

$$0 \rightarrow G^{ab} \times SK_1(AG) \rightarrow K_1(AG) \xrightarrow{\Gamma_{AG}} H_0(G; AG) \xrightarrow{\omega} G^{ab} \rightarrow 0 \tag{1}$$

where  $\omega(\sum \lambda_i g_i) = \prod g_i^{\text{Tr}(\lambda_i)}$ . These maps all commute with the involution; and  $(G^{ab})^+ = 0$  by definition. That  $SK_1(AG)^+ = 0$  follows from the definition of the isomorphism

$$\Theta_{AG} : SK_1(AG) \rightarrow H_2(G)/H_2^{ab}(G)$$

in [15, diagram on p. 215]. So (1) restricts to an isomorphism

$$\Gamma_{AG}^+ : K_1(AG)^+ \rightarrow H_0(G; AG)^+. \quad \square$$

If  $A$  is an unramified  $p$ -ring, and  $G$  is an abelian  $p$ -group, we can now define for any  $\lambda \in A$  and  $g \in G$  a unit  $u(\lambda g) \in (AG)^{*+} \cong K_1(AG)^+$  to be the unique element such that  $\Gamma_G^+(u(\lambda g)) = \frac{1}{2}\lambda(g + g^{-1})$ . If  $G$  is an arbitrary  $p$ -group and  $g \in G$ , we let  $u(\lambda g) \in (AG)^*$  be the image of  $u(\lambda g) \in (AH)^*$ , when  $H = \langle g \rangle$ .

The results of Sections 1 and 2 can now be used to describe  $K_2(\hat{\mathbb{Z}}_p G)^+$ :

**PROPOSITION 3.4.** *For any odd prime  $p$ , any unramified  $p$ -ring  $A$ , and any  $p$ -group  $G$ ,*

$$K_2(AG)^+ = \langle \{g, u(\lambda h)\} : \lambda \in A, g, h \in G, [g, h] = 1 \rangle.$$

*Proof.* For any  $G$ , define an involution on  $H_1(G; AG)$  by setting  $\overline{g \otimes \lambda h} = g \otimes \lambda h^{-1}$ . Define

$$\Delta_G^+ : H_1(G; AG)^+ \rightarrow K_2(AG)^+$$

by setting  $\Delta_G^+(g \otimes \frac{1}{2}\lambda(h + h^{-1})) = \{g, u(\lambda h)\}$  for any  $\lambda \in A$  and commuting  $g, h \in G$ .

Fix some  $G$ , choose  $z \in Z(G)$  of order  $p$ , set  $H = G/z$ , and let  $\alpha : G \twoheadrightarrow H$  be the projection. Assume inductively that  $\Delta_H^+$  is surjective, and consider the following diagram:

$$\begin{array}{ccccccc} 0 \longrightarrow & \text{Ker}(H_1(A\alpha))^+ & \longrightarrow & H_1(G; AG)^+ & \xrightarrow{H_1(A\alpha)} & H_1(H; AH)^+ & \longrightarrow & \text{Coker}(H_1(A\alpha))^+ & \longrightarrow & 0 \\ & \downarrow f_1 & & \downarrow \Delta_G^+ & & \downarrow \Delta_H^+ & & \Gamma_2^+ \updownarrow f_2 & & \\ 0 \longrightarrow & \text{Ker}(K_2(A\alpha))^+ & \longrightarrow & K_2(AG)^+ & \xrightarrow{K_2(A\alpha)} & K_2(AH)^+ & \longrightarrow & \text{Coker}(K_2(A\alpha))^+ & \longrightarrow & 0 \end{array} \tag{1}$$

Here,  $f_1$  and  $f_2$  are induced by  $\Delta_G^+$  and  $\Delta_H^+$ , and  $\Gamma_2^+$  is the restriction of the homomorphism of Proposition 2.1. For any  $\lambda \in A$  and commuting  $g, h \in G$ ,

$$\Gamma_2^+ \circ f_2(g \otimes \frac{1}{2}\lambda(h + h^{-1})) = \Gamma_2^+(\{g, u(h)\}) = g \otimes \Gamma_{AG}(u(h)) = g \otimes \frac{1}{2}\lambda(h + h^{-1});$$

and so  $f_2$  is injective. By Theorem 1.4,

$$\begin{aligned} & \text{Ker}(K_2(A\alpha))^+ \\ &= \langle \{g, (1 - \lambda(1 - z)^i h)(1 - \lambda(1 - z^{-1})^i h^{-1})\} : \lambda \in A, i \geq 1, gh = hg \rangle; \end{aligned}$$

and so by Proposition 3.3 (applied to the  $K_1(A[Z_G(g)])^+$ ):

$$\text{Ker}(K_2(A\alpha))^+ \subseteq \langle \{g, u(\lambda h)\} : \lambda \in A, [g, h] = 1 \rangle = \text{Im}(\Delta_G^+).$$

By diagram chasing in (1),  $\Delta_G^+$  is now seen to be onto.  $\square$

It seems quite likely that the homomorphisms  $\Delta_G^+$  defined above actually induce isomorphisms

$$HC_1(AG)^+ \cong [H_1(G; AG)/\langle g \otimes \lambda g \rangle]^+ \cong K_2(AG)^+.$$

This is the case at least for abelian  $p$ -groups [21, Theorem 3.9].

It remains only to find a description of the image of any  $\{g, u(h)\}$  in  $K_2(\hat{\mathbb{Q}}_p G)$ , when  $p > 2$  and  $G$  is a  $p$ -group. Recall that  $K_2(\hat{\mathbb{Q}}_p G)$  is described in terms of norm residue symbol isomorphisms

$$(\cdot, \cdot)_F : K_2(F) \xrightarrow{\cong} \mu_F$$

defined for any finite extension  $F$  of  $\hat{\mathbb{Q}}_p$  [12, Theorem A.14].

LEMMA 3.5. *Fix an odd prime  $p$  and a  $p$ -group  $G$ ; and let  $u(g) \in (\hat{\mathbb{Z}}_p G)^*$  for  $g \in G$  be defined as above. Write*

$$\hat{\mathbb{Q}}_p G = \prod_{i=1}^k B_i; \quad B_i = M_{r_i}(F_i),$$

where for each  $i$ ,  $F_i \cong \hat{\mathbb{Q}}_p \zeta_{p^m}$  (a field) for some  $m \geq 0$  (see [22]). Let

$$\lambda_G : K_2(\hat{\mathbb{Q}}_p G) \rightarrow \prod_{i=1}^k (\mu_{F_i})_p$$

be the product of the norm residue symbol homomorphisms

$$\lambda_G^i : K_2(B_i) \cong K_2(M_{r_i}(F_i)) \cong K_2(F_i) \xrightarrow{(\cdot, \cdot)} (\mu_{F_i})_p.$$

For each  $i$ , let  $V_i$  be the irreducible  $B_i$ -representation. Then, for any commuting  $g, h \in G$ ,

$$\lambda_G(\{g, u(h)\}) = [\det_{F_i}(g, V_i^h)]_{i=1}^k. \quad (V_i^h = \{x \in V_i : hx = x\}).$$

*Proof.* Fix some  $i$ , set  $B = B_i, V = V_i, F = F_i, r = r_i$ ; and let

$$\alpha : \hat{\mathbb{Q}}_p G \twoheadrightarrow B \cong \text{End}_F(V) \cong M_r(F)$$

be the projection. Let  $m$  be such that  $F \cong \hat{\mathbb{Q}}_p \zeta_{p^m}$ . Set  $p^m = \exp(G)$ , and let

$$f : B \cong M_r(F) \rightarrow M_r(\hat{\mathbb{Q}}_p \zeta_{p^n})$$

be an inclusion. Note that taking norm residue symbols commutes ( $p$  is odd) with inclusions of cyclotomic fields: this follows, for example, from the formulas in [2].

Fix commuting  $g, h \in G$ . Then  $\langle g, h \rangle$  is an abelian group of exponent dividing  $p^n$ ; and so  $f\alpha(g)$  and  $f\alpha(h)$  are conjugate (simultaneously) to diagonal matrices:

$$f\alpha(g) \sim \text{diag}(u_1, \dots, u_r), f\alpha(h) \sim \text{diag}(v_1, \dots, v_r) \quad (u_l, v_l \in \langle \zeta_{p^n} \rangle).$$

with

$$u(h) = \sum_j \lambda_j h^j; \quad (\lambda_j \in \hat{\mathbb{Z}}_p)$$

so that

$$K_2(f\alpha)(\{g, u(h)\}) = \prod_{l=1}^r \left\{ u_l, \sum_j \lambda_j v_l^j \right\}.$$

By the formulas of Artin and Hasse [2],

$$\lambda_G^i(\{g, u(h)\}) = \prod_{l=1}^r \left( u_l, \sum_j \lambda_j v_l^j \right)_F = \prod_{l=1}^r u_l^{N_l};$$

where

$$N_l = \frac{1}{p^n} \text{Tr} \left( \log \left( \sum_j \lambda_j v_l^j \right) \right). \quad (\text{Tr}: \hat{\mathbb{Q}}_p \zeta_{p^n} \rightarrow \hat{\mathbb{Q}}_p)$$

Recall that  $\Gamma_G(u(h)) = \frac{1}{2}(h + h^{-1})$ , where  $\Gamma_G = (1 - (1/p)\Phi) \circ \log$ , and  $\Phi(\sum \lambda_i g_i) = \sum \lambda_i g_i^p$ . Thus,

$$\begin{aligned} \log(u(h)) &= \left( 1 - \frac{1}{p} \Phi \right)^{-1} \left( \frac{1}{2}(h + h^{-1}) \right) \\ &= \frac{p}{p-1} + \frac{1}{2} \left[ (h + h^{-1} - 2) + \frac{1}{p} (h^p + h^{-p} - 2) + \dots \right]. \end{aligned}$$

Hence, for  $1 \leq l \leq r$ ,

$$N_l = \frac{1}{p^n} \text{Tr} \left( \frac{p}{p-1} + \frac{1}{2} \left[ (v_l + v_l^{-1} - 2) + \frac{1}{p} (v_l^p + v_l^{-p} - 2) + \dots \right] \right)$$

$$= \begin{cases} 1 & \text{if } v_l = 1 \\ 0 & \text{if } v_l \neq 1. \end{cases} \quad (v_l \in \langle \zeta_{p^n} \rangle)$$

It follows that

$$\lambda_G^i(\{g, u(h)\}) = \prod_{v_l=1} u_l = \det_F(g, V^h). \quad \square$$

The main result can now be shown.

**THEOREM 3.6.** *Let  $p$  be an odd prime, and let  $G$  be a  $p$ -group. Write  $\mathbb{Q}G = \prod_{i=1}^k B_i$ , where each  $B_i$  is a matrix algebra over a field  $F_i$  with irreducible representation  $V_i$ . Define*

$$\psi_G : H_1(G; \mathbb{Z}G) \rightarrow \prod_{i=1}^k (\mu_{F_i})_p$$

by setting, for any commuting  $g, h \in G$ ,

$$\psi_G(g \otimes h) = [\det_{F_i}(g, V_i^h)]_{i=1}^k.$$

Then  $Cl_1(\mathbb{Z}G) \cong \text{Coker}(\psi_G)$  and

$$SK_1(\mathbb{Z}G) \cong \text{Coker}(\psi_G) \oplus (H_2(G)/H_2^{ab}(G)).$$

More precisely, there is a commutative square

$$\begin{array}{ccc} K_2(\hat{\mathbb{Q}}_p G)_{(p)} & \xrightarrow[\cong]{\lambda_G} & \prod_{i=1}^k (\mu_{F_i})_p \\ \partial \downarrow & & \downarrow \text{proj} \\ Cl_1(\mathbb{Z}G) & \xrightarrow[\cong]{\Lambda_G} & \text{Coker}(\psi_G); \end{array}$$

where  $\lambda_G$  is induced by the norm residue symbol, and  $\partial$  is the boundary map in the localization sequence.

*Proof.* By [20, Theorem 2.1 and 2.2], there is an exact sequence

$$K_2(\hat{\mathbb{Z}}_p G) \xrightarrow{\varphi_G} \text{Coker} \left[ K_2\left(\mathbb{Z}\left[\frac{1}{p}\right][G]\right) \rightarrow K_2(\hat{\mathbb{Q}}_p G) \right] \xrightarrow{\partial} Cl_1(\mathbb{Z}G) \rightarrow 0$$

and an isomorphism

$$\text{Coker} \left[ K_2\left(\mathbb{Z}\left[\frac{1}{p}\right][G]\right) \rightarrow K_2(\hat{\mathbb{Q}}_p G) \right] \cong K_2(\hat{\mathbb{Q}}_p G)_{(p)} \xrightarrow{\lambda_G} \prod_{i=1}^k (\mu_{F_i})_p.$$

(note that  $\mathbb{Z}[1/p][G]$  is a maximal order). Consider the diagram

$$\begin{array}{ccccc} H_1(G; \mathbb{Z}G)^+ & \xrightarrow{\psi_G^+} & \prod_{i=1}^k (\mu_{F_i})_p & \xrightarrow{\text{proj}} & \text{Coker}(\psi_G) \longrightarrow 0 \\ & & \uparrow \lambda_G & & \uparrow \Lambda_G \\ & \downarrow \Delta_G^+ & (1) & \cong & (2) \\ K_2(\hat{\mathbb{Z}}_p G)^+ & \xrightarrow{\varphi_G^+} & K_2(\hat{\mathbb{Q}}_p G)_{(p)} & \xrightarrow{\partial} & Cl_1(\mathbb{Z}G) \longrightarrow 0. \end{array}$$

By Lemma 3.2,  $\text{Im}(\varphi_G^+) = \text{Im}(\varphi_G)$ ; and  $\text{Im}(\psi_G^+) = \text{Im}(\psi_G)$  since  $\psi_G(g \otimes h) = \psi_G(g \otimes h^{-1})$  by definition. So the rows above are exact. The map  $\Delta_G^+$ , defined by setting  $\Delta_G^+(g \otimes h) = \{g, u(h)\}$ , is onto by Proposition 3.4, and (1) commutes by Lemma 3.5. So there is a unique isomorphism

$$\Lambda_G : Cl_1(\mathbb{Z}G) \xrightarrow{\cong} \text{Coker}(\psi_G)$$

which makes (2) commute.

The exact sequence

$$0 \rightarrow Cl_1(\mathbb{Z}G) \rightarrow SK_1(\mathbb{Z}G) \rightarrow SK_1(\hat{\mathbb{Z}}_p G) \rightarrow 0$$

is naturally split by [17, Theorem 4.8], and

$$SK_1(\hat{\mathbb{Z}}_p G) \cong H_2(G)/H_2^{ab}(G) \cong H_2(G)/\langle g \wedge h : g, h \in G, gh = hg \rangle$$

by [15, Theorem 3]. So

$$SK_1(\mathbb{Z}G) \cong \text{Coker}(\psi_G) \oplus (H_2(G)/H_2^{ab}(G)). \quad \square$$

In [17, Theorem 4.8], the computation of  $Cl_1(\mathbb{Z}G)_{(p)}$  for odd  $p$  and arbitrary

finite  $G$  was reduced to the case of a  $p$ -group. More precisely, if  $C_1, \dots, C_k$  are conjugacy class representatives for cyclic subgroups in  $G$  of order prime to  $p$ , and  $N_i = N_G(C_i)$ ,  $Z_i = Z_G(C_i)$ , and  $\mathfrak{B}(Z_i)$  is the set of  $p$ -subgroups, then

$$Cl_1(\mathbb{Z}G)_{(p)} \cong \sum_{i=1}^k H_0\left(N_i/Z_i; \varinjlim_{H \in \mathfrak{B}(Z_i)} Cl_1(\mathbb{Z}H)\right). \tag{3.7}$$

Here, the limits are taken with respect to inclusion and conjugation among subgroups.

This direct sum decomposition is somewhat awkward, and hence a more direct description of  $Cl_1(\mathbb{Z}G)_{(p)}$  seems also desirable. In fact, one can define homomorphisms

$$\psi_G : H_1(G; \mathbb{Z}G) \rightarrow \prod_{i=1}^k (\mu_{F_i}), \quad \left( \mathbb{Q}G \cong \prod_{i=1}^k B_i, F_i = Z(B_i) \right)$$

for arbitrary finite  $G$ , such that  $Cl_1(\mathbb{Z}G)[\frac{1}{2}] \cong \text{Coker}(\psi_G)[\frac{1}{2}]$ . But alone the definition of  $\psi_G$  become quite complicated as soon as we start working with non- $p$ -groups; and the most efficient way of describing  $Cl_1(\mathbb{Z}G)[\frac{1}{2}]$  for concrete  $G$  does seem to be by means of (3.7) above, together with Theorem 3.6. Some techniques for calculating with the help of (3.7) are presented in [17, Section 5].

### Section 4

Theorem 3.6 reduces the calculation of  $Cl_1(\mathbb{Z}G)$ , for an odd order  $p$ -group  $G$ , to a straightforward combinatorial algorithm. We now give some examples to illustrate how this works in practice. Examples of calculations for abelian  $G$  are presented in [1]; and for non-abelian  $G$  of order  $p^3$ ,  $Cl_1(\mathbb{Z}G)$  is calculated in [19, Theorem 7.5] using a weaker form of the theorem. So here we take some non-abelian groups of order  $p^4$  to give a sample of some of the techniques which can be used. Throughout this section,  $p$  denotes a fixed odd prime.

Note first that for any  $p$ -group  $G$  and any commuting  $g, h \in G$ ,

$$\psi_G(g \otimes g) = 0; \quad \psi_G(g \otimes h^n) = \psi_G(g \otimes h) \quad (\text{if } p \nmid n);$$

and

$$\psi_G(aga^{-1} \otimes aha^{-1}) = \psi_G(g \otimes h) \quad (\text{any } a \in G).$$

Thus, when describing  $\text{Im}(\psi_G)$ , it suffices to consider  $\psi_G(g \otimes h)$  as  $h$  runs through a set of  $\mathbb{Q}$ -conjugacy class representatives in  $G$ , and  $g$  a set of generators for  $Z_G(h)/h$ .

An irreducible representation  $V$  of  $G$  will be described by listing eigenvalues for the actions of various group elements on  $V$  – or, when necessary, by describing the irreducible components of  $V|_H$  for some appropriate  $H \subseteq G$ .

Finally, note that when  $|G| = p^4$ , then  $SK_1(\hat{\mathbb{Z}}_p G) = 0$  by [15, Proposition 23]. So  $SK_1(\mathbb{Z}G) = Cl_1(\mathbb{Z}G)$  in this case.

**PROPOSITION 4.1.** *Assume  $G \cong H \times C_p$ , where  $H$  is non-abelian,  $|H| = p^3$ , and  $\exp(H) = p$ . Then*

$$SK_1(\mathbb{Z}G) \cong (\mathbb{Z}/p)^{(p^2+3p-6)/2}.$$

*Proof.* Fix generators  $a, b \in H$  and  $c \in C_p$ ; and set  $z = [a, b]$ . Then  $Z(G) = \langle z, c \rangle$ , and for any  $g \in G \setminus Z(G)$ ,  $Z_G(g) = \langle Z(G), g \rangle$ . Set  $\zeta = \zeta_p$ , and note that

$$\mathbb{Q}[G] \cong \mathbb{Q} \times \prod^{p^2+p+1} \mathbb{Q}[\zeta] \times \prod^p M_p(\mathbb{Q}[\zeta]).$$

The following table describes  $\psi = \psi_G$ . Here,  $(H^{ab})^*$  denotes the set of irreducible complex characters of  $H^{ab}$ , and  $(*)$  for eigenvalues means that all powers of  $\zeta$  occur.

Representation	$U \cong \mathbb{Q}\zeta$	$V_m \cong \mathbb{Q}\zeta$	$W_\chi \cong \mathbb{Q}\zeta$	$X_m \cong (\mathbb{Q}\zeta)^p$
Indexed by	—	$0 \leq m < p$	$\chi \in (H^{ab})^*$	$0 \leq m < p$
E'val $(a, b, c, z)$	$(\zeta, 1, 1, 1)$	$(\zeta^m, \zeta, 1, 1)$	$(\chi(a), \chi(b), \zeta, 1)$	$(*, *, \zeta^m, \zeta)$
$\psi(a \otimes cz^{-i})$	$\zeta$	$\zeta^m$	1	1
$\psi(b \otimes cz^{-i})$	1	$\zeta$	1	1
$\psi(a \otimes (1 - c))$	1	1	$\chi(a)$	1
$\psi(b \otimes (1 - c))$	1	1	$\chi(b)$	1
$\psi(c \otimes 1)$	1	1	$\zeta$	1
$\psi(G \otimes (1 - z))$	1	1	1	1
$\psi(z \otimes gc^{-i})$	1	1	1	$\zeta$
$\psi(c \otimes gc^{-i})$	1	1	$\zeta$ (if $\chi(g) = \zeta^i$ )	$\zeta^m$
$(g \in H \setminus \langle z \rangle)$			1 (if $\chi(g) \neq \zeta^i$ )	
$\psi(c \otimes \xi)$	1	1	1	$\zeta^m$

Here, in the last line,  $\xi = a(1 + b + \dots + b^{p-1}) - b(c + c^2 + \dots + c^{p-1})$ . By inspection,

$$SK_1(\mathbb{Z}G) \cong \text{Coker}(\psi) \cong (\mathbb{Z}/p)^{p-1} \oplus (\mathbb{Z}/p[C_p \times C_p]/I) \oplus (\mathbb{Z}/p)^{p-2}, \tag{1}$$

where  $I \subseteq \mathbb{Z}/p[C_p \times C_p]$  is the ideal generated by elements  $(\sum_{g \in K} g)$  for subgroups  $K \subseteq C_p \times C_p$  of order  $p$ .

Write  $C_p \times C_p = \langle g \rangle \times \langle h \rangle$ , and let  $J = \langle 1 - g, 1 - h \rangle \subseteq \mathbb{Z}/p[C_p \times C_p]$  denote the Jacobson radical. Then

$$I = \langle (1 - g)^{p-1}, (1 - h)^{p-1}; (1 - g^i h)^{p-1}; 1 \leq i \leq p - 1 \rangle.$$

Furthermore, for any  $1 \leq i \leq p - 1$ :

$$(1 - g^i h) = 1 - [1 - (1 - g)]^i [1 - (1 - h)] \equiv i(1 - g) + (1 - h) \pmod{J^2}$$

and so

$$\begin{aligned} (1 - g^i h)^{p-1} &\equiv \sum_{k=0}^{p-1} \binom{p-1}{k} i^k (1 - g)^k (1 - h)^{p-1-k} \\ &= \sum_{k=0}^{p-1} (-i)^k (1 - g)^k (1 - h)^{p-1-k} \pmod{J^p}. \end{aligned}$$

The determinant of  $[(-i)^k]_{i,k=1}^{p-2}$  is invertible over  $\mathbb{Z}/p$  (a van der Monde determinant), and so

$$I + J^p = \langle (1 - g)^k (1 - h)^{p-1-k}; 0 \leq k \leq p - 1 \rangle = J^{p-1}.$$

But  $J^{2p-1} = 0$ , and hence this implies that  $I = J^{p-1}$ . So as a group,

$$\begin{aligned} \mathbb{Z}/p[C_p \times C_p]/I &\cong (\mathbb{Z}/p)^{1/2p(p-1)} \text{ with basis} \\ \{(1 - g)^i (1 - h)^j; i, j \geq 0, i + j < p - 1\}. \end{aligned}$$

The result now follows from (1).  $\square$

In the above example, the fact that  $[G, G]$  was central helped to keep the description of  $\psi_G$  simple. The next example illustrates additional complexities which can arise when this is no longer the case. First, a lemma is needed.

**LEMMA 4.2.** *Let  $G$  be cyclic of order  $p^n (n \geq 1)$  with generator  $g \in G$ .*

Then, for any

$$0 \neq \alpha = \sum_{i=0}^{p^n-1} a_i g^i \in \mathbb{Z}/p[G] \quad (a_i \in \mathbb{Z}/p);$$

$\mathbb{Z}/p[G]/(\alpha) \cong (\mathbb{Z}/p)^k$  (as groups), where

$$k = \min \left\{ m \geq 0: \sum_{i=0}^{p^n-1} \binom{i}{m} a_i \neq 0 \text{ in } \mathbb{Z}/p \right\}.$$

*Proof.* By direct calculation,

$$\begin{aligned} \alpha &= \sum_{i=0}^{p^n-1} a_i g^i = \sum_{i=0}^{p^n-1} a_i (1 + (g-1))^i = \sum_{i=0}^{p^n-1} a_i \sum_{m=0}^i \binom{i}{m} (g-1)^m \\ &= \sum_{m=0}^{p^n-1} \left( \sum_{i=0}^{p^n-1} \binom{i}{m} a_i \right) (g-1)^m. \end{aligned} \tag{1}$$

Recall that  $\mathbb{Z}/p[G]$  is a local ring with maximal ideal generated by  $(g-1)$ . So if  $k$  is defined as above, then  $\alpha = (g-1)^k u$  for some unit  $u$  in  $\mathbb{Z}/p[G]$ , and

$$rk[\mathbb{Z}/p[G]/(\alpha)] = rk[\mathbb{Z}/p[G]/(g-1)^k] = k. \quad \square$$

**PROPOSITION 4.3.** Set  $H = \langle a \rangle \times \langle b \rangle \times \langle c \rangle = C_p^3$ ,  $K = \langle x \rangle \cong C_p$ , and let  $G$  be any extension of the form

$$1 \rightarrow H \rightarrow G \rightarrow K \rightarrow 1$$

such that

$$xax^{-1} = ab, \quad xbx^{-1} = bc, \quad xcx^{-1} = c.$$

Then

$$SK_1(\mathbb{Z}G) \cong Cl_1(\mathbb{Z}G) \cong (\mathbb{Z}/p)^{3(p-1)/2}.$$

*Proof.* The action of  $x$  on  $\mathbb{Q}H$  fixes  $\mathbb{Q}[H/\langle b, c \rangle]$ , and permutes the other  $p^2 + p$  summands freely. Thus,

$$\mathbb{Q}G \cong \mathbb{Q}[G^{ab}] \times \prod_{i=1}^{p+1} M_p(\mathbb{Q}[\zeta]);$$

where  $\zeta = \zeta_p$ . The following table presents  $\psi_G$ , where the nonabelian representations are described by their restrictions to  $H$ :

Representation	$U \cong \mathbb{Q}\zeta$	$V_m \cong \mathbb{Q}\zeta$	$W \cong (\mathbb{Q}\zeta)^p$	$X_m \cong (\mathbb{Q}\zeta)^p$
Indexed by	—	$0 \leq m < p$	—	$0 \leq m < p$
E'val $(a, b, c; x)$	$(1, 1, 1; \zeta)$	$(\zeta, 1, 1; \zeta^m)$	$(\zeta^r, \zeta, 1)$ $(1 \leq r \leq p)$	$(\zeta^{m+1/2r(r-1)}, \zeta^r, \zeta)$ $(1 \leq r \leq p)$
$\psi(a \otimes c)$	1	$\zeta$	1	1
$\psi(x \otimes c)$	$\zeta$	$\zeta^m$	1	1
$\psi(a \otimes (1 - c))$	1	1	1	$\zeta^T$
$\psi(x \otimes (1 - c))$	1	1	1	$\zeta^u (x^p = c^u)$
$\psi(a \otimes (b - c))$	1	1	1	$\zeta^m$
$\psi(c \otimes b)$	1	1	1	$\zeta$
$\psi(b \otimes ac^{-i})$	1	1	$\zeta$	$\zeta^{R(i-m)}$
$\psi(c \otimes ac^{-i})$	1	1	1	$\zeta^{S(i-m)}$
$\psi(c \otimes gx)(g \in H)$	1	1	1	$\begin{cases} \zeta & ((gx)^p = 1) \\ 1 & ((gx)^p \neq 1). \end{cases}$

Here,  $T = \sum_{r=1}^p \frac{1}{2}r(r-1)$ ;

$$R(i) = \sum \{r: 1 \leq r \leq p, \frac{1}{2}r(r-1) \equiv i \pmod{p}\};$$

$$S(i) = \#\{r: 1 \leq r \leq p, \frac{1}{2}r(r-1) \equiv i \pmod{p}\}.$$

Note that solutions to  $\frac{1}{2}r(r-1) \equiv i$  come in pairs  $\{r, p+1-r\}$  (unless  $r = (p+1)/2$ ). This shows that for all  $i$ ,

$$R(i) = \frac{p+1}{2} S(i) \equiv \frac{1}{2}S(i) \pmod{p}.$$

Identify  $\prod_{X_m} \langle \zeta \rangle$  with  $\mathbb{Z}/p[C_p]$ , by identifying  $X_m$  with  $g^m$  for some generator  $g$  of  $C_p$ . Then

$$SK_1(\mathbb{Z}G) \cong \text{Coker}(\psi_G)$$

$$\cong (\mathbb{Z}/p)^{p-1} \oplus \left( \mathbb{Z}/p[C_p] / \left\langle \sum_m g^m, \sum_m mg^m, \sum_m S(i-m)g^m \text{ (any } i) \right\rangle \right)$$

$$\cong (\mathbb{Z}/p)^{p-1} \oplus \mathbb{Z}/p[C_p]/I,$$

where  $I$  is the ideal generated by

$$\alpha = \sum_m S(m)g^{-m} = \sum_{k=1}^p g^{-1/2k(k-1)}.$$

By Lemma 4.2, we will be done upon showing that

$$\sum_{k=1}^p \binom{\frac{1}{2}k(k-1)}{n} \begin{cases} \equiv 0 \\ \not\equiv 0 \end{cases} \pmod{p} \quad \begin{cases} \text{for } 0 \leq n < \frac{p-1}{2} \\ \text{for } n = \frac{p-1}{2} \end{cases} \tag{1}$$

But the sum is a polynomial in  $k$  (over  $\mathbb{Z}/p$ ) of degree exactly  $2n$ ; and (1) follows since

$$p-1 = \min \left\{ m \geq 0 : \sum_{k=1}^p k^m \not\equiv 0 \pmod{p} \right\}. \quad \square$$

The groups covered above turn out to be the most difficult cases for computing  $SK_1(\mathbb{Z}G)$  when  $|G|=p^4$ . In fact, all other groups of order  $p^4$  are covered by the following proposition (this can easily be checked directly, but also follows from the classification in [9, section III.12]).

**PROPOSITION 4.4.** *Assume that  $G$  is non-abelian of order  $p^4$ , and that there is a subgroup  $H \triangleleft G$  such that  $H \cong C_{p^3}$  or  $H \cong C_{p^2} \times C_p$ . Then*

$$\begin{aligned} SK_1(\mathbb{Z}G) = Cl_1(\mathbb{Z}G) &\cong (\mathbb{Z}/p)^{p-1} && \text{if } G^{ab} \cong C_p \times C_p \\ &\cong (\mathbb{Z}/p)^{2(p-1)} && \text{if } G^{ab} \cong C_{p^2} \times C_p \\ &\cong (\mathbb{Z}/p)^{(p+2)(p-1)/2} && \text{if } G^{ab} \cong C_p \times C_p \times C_p. \end{aligned}$$

*Proof.* Write

$$\mathbb{Q}G = \mathbb{Q}[G^{ab}] \times M \quad \text{and} \quad \mathbb{Q}H = \mathbb{Q}[H/[G, G]] \times M';$$

where  $M$  is a product of rank  $p$  matrix algebras over fields. Then the inclusion  $M' \subseteq M$  is a sum of inclusions of the form

$$\prod_{r=1}^p \mathbb{Q}\zeta_{p^r} \subseteq M_p(\mathbb{Q}\zeta_{p^r}); \quad \mathbb{Q}\zeta_{p^{r+1}} \subseteq M_p(\mathbb{Q}\zeta_{p^r}) \quad (r = 1, 2).$$

In particular,  $K_2(\hat{\mathbb{Q}}_p \otimes_{\mathbb{Q}} M')_{(p)}$  surjects onto  $K_2(\hat{\mathbb{Q}}_p \otimes_{\mathbb{Q}} M)_{(p)}$ . Since  $Cl_1(\mathbb{Z}H) = 0$  [10, Theorems 4.4.1 and 5.1.1], this shows that

$$K_2(\hat{\mathbb{Q}}_p \otimes_{\mathbb{Q}} M)_{(p)} \subseteq \text{Im} [\varphi_G : K_2(\hat{\mathbb{Z}}_p G) \rightarrow K_2(\hat{\mathbb{Q}}_p G)_{(p)}].$$

In other words, if  $\mathbb{Q}[G^{ab}] \cong \prod_{i=1}^k F_i$ , then

$$SK_1(\mathbb{Z}G) = Cl_1(\mathbb{Z}G) \cong \text{Coker} \left[ \text{proj} \circ \psi_G : H_1(G; \mathbb{Z}G) \rightarrow \prod_{i=1}^k (\mu_{F_i})_p \right].$$

If  $G^{ab} \cong C_p \times C_p$ , with basis  $\{a, b\}$ , then  $\text{Im}(\text{proj} \circ \psi_G)$  is generated by the images of  $a \otimes 1$  and  $b \otimes 1$ , and so  $SK_1(\mathbb{Z}G)$  has rank  $(p+1) - 2 = p - 1$ . If  $G^{ab} \cong C_p^3$ , then there are generators  $a, b, c$  such that  $c \in Z(G)$ , and the computation follows from the table in the proof of Proposition 5.1. The proof when  $G^{ab} \cong C_{p^2} \times C_p$  is similar.  $\square$

It is interesting to note that for each of these classes of  $p$ -groups, the rank of  $Cl_1(\mathbb{Z}G)$  is a polynomial in  $p$ . This has already been remarked in the case of abelian  $p$ -groups (see [1, Conjecture 5.8]); but is harder to formulate as a precise conjecture in the non-abelian case.

## Section 5

As another application of Theorem 1.4, we now study the relationship between the complex Artin cokernel

$$A_{\mathbb{C}}(G) = \text{Coker} \left[ \sum \{R_{\mathbb{C}}(H) : H \subseteq G \text{ cyclic}\} \xrightarrow{\text{Ind}} R_{\mathbb{C}}(G) \right]$$

of a finite group  $G$ , and  $Cl_1(RG)$  for large  $R$ .

First, epimorphisms

$$I_{RG} : A_{\mathbb{C}}(G) \twoheadrightarrow Cl_1(RG)$$

are constructed, for  $G$  any finite group and  $R$  the ring of integers in any number field  $K \subseteq \mathbb{C}$  (the identification of  $K$  as a subfield of  $\mathbb{C}$  is needed when defining  $I_{RG}$ ). The  $I_{RG}$  are shown to be natural with respect to homomorphisms and transfer maps, and then shown to be isomorphisms for sufficiently large  $R$ .

The following lemma on norm residue symbols will be needed.

LEMMA 5.1. Fix a prime  $p$ , fix extensions  $E \supseteq F \supseteq \hat{\mathbb{Q}}_p$ , and let  $\hat{\mu} \subseteq E^*$  and  $\mu \subseteq F^* \cap \hat{\mu}$  be groups of roots of unity. Then the diagram

$$\begin{array}{ccc} K_2(E) & \xrightarrow{(\cdot)_{\hat{\mu}}} & \hat{\mu} \\ \downarrow \text{trf}_F^E & & \downarrow [\hat{\mu}: \mu] \\ K_2(F) & \xrightarrow{(\cdot)_{\mu}} & \mu \end{array} \quad (1)$$

commutes; where  $(\cdot)_{\hat{\mu}}$  and  $(\cdot)_{\mu}$  are the norm residue symbol homomorphisms.

*Proof.* Set  $n = |\hat{\mu}|$  and  $m = |\mu|$ . Fix  $u \in F^*$  and  $v \in E^*$ , and let  $E(\alpha)/E$  be an extension such that  $\alpha^n = u$ . The diagram

$$\begin{array}{ccc} E^* & \xrightarrow{\hat{s}} & \text{Gal}(E(\alpha)/E) \\ \downarrow N_{E/F} & & \downarrow \text{res} \\ F^* & \xrightarrow{s} & \text{Gal}(F(\alpha^{n/m})/F) \end{array}$$

commutes by [23, Section XI.3]; where  $\hat{s}$  and  $s$  are the reciprocity maps and  $\text{res}$  is induced by restriction. By [23, Proposition XIV.6],

$$\begin{aligned} (u, N_{E/F}(v))_{\mu} &= s(N_{E/F}(v))(\alpha^{n/m})/\alpha^{n/m} \\ &= [\hat{s}(v)(\alpha)/\alpha]^{n/m} = ((u, v)_{\hat{\mu}})^{n/m}. \end{aligned} \quad (2)$$

Since  $\text{trf}_F^E(\{u, c\}) = \{u, N_{E/F}(v)\}$  for  $u \in F^*$  and  $v \in E^*$ , this shows that (1) commutes on the subgroup  $\{F^*, E^*\} \subseteq K_2(E)$ . Furthermore,

$$\text{trf}_F^E(\{F^*, E^*\}) = \{F^*, N_{E/F}(E^*)\} = K_2(F):$$

the last equality is shown in [14, Lemma] when  $\text{Gal}(E/F)$  is cyclic, and follows from [6, Chapter VI, §2.2] ( $N_{E/F}$  is onto) when  $\text{Gal}(E/F)$  is non-abelian simple. Since  $K_2(E) \cong \mu_E$  and  $K_2(F) \cong \mu_F$  are cyclic [12, Theorem A.14], it follows that  $\{F^*, E^*\} \supseteq K_2(E)_{(p)}$  for any prime  $p \mid |K_2(F)|$ , and hence any  $p \mid |\mu|$ . So (1) commutes.  $\square$

Now fix a finite group  $G$ , and let  $K \subseteq \mathbb{C}$  be any splitting field for  $G$ : i.e.,  $KG$  is a product of matrix algebras over  $K$ . As in [20, Section 2], we define for each prime  $p$ :

$$\begin{aligned} C_p(KG) &= \text{Coker} \left[ K_2\left(\mathfrak{M}\left[\frac{1}{p}\right]\right) \rightarrow K_2(\hat{K}_p G) \right] \quad (\hat{K}_p = \mathbb{Q}_p \otimes_{\mathbb{Q}} K) \\ &\cong \text{Coker} [K_2(\mathfrak{M}) \rightarrow K_2(\mathfrak{M}_p)] \quad (\mathfrak{M}_p = \hat{\mathbb{Z}}_p \otimes_{\mathbb{Z}} \mathfrak{M}) \end{aligned}$$

where  $\mathfrak{M} \subseteq KG$  is any maximal order. Then  $C_p(KG)$  is a  $p$ -group for all  $p$  (since  $K_2(\mathfrak{M}_p)$  is a  $p$ -group). Finally, set

$$C(KG) = \sum_p C_p(KG).$$

Write  $KG = \prod_{i=1}^k B_i$ , where  $B_i \cong \text{End}_K(V_i)$  for each  $i$ , and  $V_1, \dots, V_k$  are the irreducible  $KG$ -modules. By results going back to Bass, Milnor, and Serre [5],  $C(KG) = 0$  if  $K$  has a real embedding. If  $K$  is purely imaginary, then there is an isomorphism

$$\lambda_{KG}: C(KG) \cong \prod_{i=1}^k (\mu_K)$$

such that for any prime  $\mathfrak{p} \subseteq R$ , and any units  $u \in K^*$  and  $v \in (\hat{K}_{\mathfrak{p}}[G])^*$ ,

$$\lambda_{KG}(\{u, v\}_{\mathfrak{p}}) = [(u, \det_K(v, V_i))]_{i=1}^k$$

Here,  $\{u, v\}$  denotes the image of

$$\{u, v\} \in K_2(\hat{K}_{\mathfrak{p}}[G]) \rightarrow C(KG);$$

and

$$(\cdot)_{\mathfrak{p}}: (\hat{K}_{\mathfrak{p}})^* \times (\hat{K}_{\mathfrak{p}})^* \rightarrow \mu_K$$

denotes the norm residue symbol with values in  $\mu_K$ . See [20, Theorem 2.2] for more details.

Thus, when  $K \subseteq \mathbb{C}$  is a splitting field for  $G$  and has no real embedding and  $KG \cong \prod_{i=1}^k B_i$  as above, an isomorphism  $\tilde{I}_{KG}$  from  $R_{\mathbb{C}}(G)$  to  $C(KG)$  can be defined as the composite

$$\tilde{I}_{KG}: R_{\mathbb{C}}(G) \cong \prod_{i=1}^k \mathbb{Z} \xrightarrow{\prod [1 \rightarrow \exp(2\pi i/m)]} \prod_{i=1}^k \mu_K \xrightarrow{\lambda_{KG}^{-1}} C(KG) \quad (m = |\mu_K|).$$

In other words, for each  $1 \leq i \leq k$ , we set

$$I_{KG}([V_i]) = \lambda_{KG}^{-1}([\exp(2\pi i/m)]_i);$$

where  $[V_i] \in R_{\mathbb{C}}(G)$  denotes the class of  $\mathbb{C} \otimes_K V_i$ .

If  $K$  is a splitting field for  $G$  but has a real embedding, we set  $\tilde{I}_{KG} = 0$  ( $C(KG) = 0$ ). If  $K \subseteq \mathbb{C}$  is a number field which does not split  $G$ , set  $n = \exp(G)$  and  $L = K(\zeta_n)$ , and define

$$\tilde{I}_{KG} = \text{trf}_{KG}^{LG} \circ \tilde{I}_{LG} : R_{\mathbb{C}}(G) \rightarrow C(LG) \rightarrow C(KG).$$

(Note that  $L$  is a splitting field for  $G$  by [5, Theorem 4.1.1].) This definition of the  $\tilde{I}_{KG}$  seems rather artificial; but the following proposition shows that these maps do have all desired naturality properties.

**PROPOSITION 5.2.** *For any number field  $K \subseteq \mathbb{C}$  and any finite group  $G$ ,  $\tilde{I}_{KG}$  is surjective. The  $\tilde{I}_{KG}$  are natural in that for any homomorphism  $\alpha : \tilde{G} \rightarrow G$  of finite groups, for any  $H \subseteq G$ , and for any pair  $K \subseteq L \subseteq \mathbb{C}$  of number fields, the following diagrams all commute:*

$$\begin{array}{ccccc} R_{\mathbb{C}}(\tilde{G}) & \xrightarrow{R_{\mathbb{C}}(\alpha)} & R_{\mathbb{C}}(G) & \xrightarrow{\text{Res}_H^G} & R_{\mathbb{C}}(H) \\ \downarrow \tilde{I}_{K\tilde{G}} & (2) & \downarrow \tilde{I}_{KG} & (3) & \downarrow \tilde{I}_{KH} \\ C(K\tilde{G}) & \xrightarrow{C(K\alpha)} & C(KG) & \xrightarrow{\text{trf}_{KH}^{KG}} & C(KH) \end{array}$$

$$\begin{array}{ccc} R_{\mathbb{C}}(G) & & \\ \swarrow (1) & & \searrow \\ C(LG) & \xrightarrow{\text{trf}_{KG}^{LG}} & C(KG) \end{array}$$

*Proof.* The proposition will be proven in four steps. For finite  $G$  and arbitrary  $K \subseteq \mathbb{C}$ , we regard  $K_0(KG) = R_K(G)$  as a subring of  $R_{\mathbb{C}}(G)$  in the usual fashion (identifying  $[V] \in R_K(G)$  with  $[C \otimes_K V] \in R_{\mathbb{C}}(G)$ ).

*Step 1.* By construction,  $\tilde{I}_{KG}$  is surjective if  $K$  splits  $G$ . To see that  $\tilde{I}_{KG}$  is surjective in general, we must show for any  $G$ , and any number fields  $K \subseteq L$ , that the transfer map

$$\text{trf}_{KG}^{LG} : K_2(\hat{L}_p G) \rightarrow K_2(\hat{K}_p G)$$

is onto for each prime  $p$  ( $\hat{L}_p = \hat{\mathbb{Q}}_p \otimes_{\mathbb{Q}} L$ , etc.).

Write  $\hat{K}_p G \cong \prod_{i=1}^k M_{n_i}(D_i)$ , where the  $D_i$  are division algebras. For each  $i$ , set  $F_i = Z(D_i)$ , the center, and let  $E_i \subseteq D_i$  be a maximal subfield. By [3, Corollary 4.15],  $K_2(D_i)$  is generated by symbols  $\{F_i^*, D_i^*\}$ ; and hence  $K_2(E_i) \rightarrow K_2(D_i)$  is onto by [14, Proposition].

Consider the following square, for each  $1 \leq i \leq k$ :

$$\begin{array}{ccc} K_1(L \otimes_K E_i) & \xrightarrow{\text{incl}} & K_2(L \otimes_K D_i) \\ \downarrow i' & & \downarrow i' \\ K_2(E_i) & \longrightarrow & K_2(D_i). \end{array}$$

Here  $t_i$  and  $t'_i$  are the transfer maps. The square commutes since the two sides are induced by tensoring with the bimodules

$$D_i \otimes_{E_i} (L \otimes_K E_i) \cong L \otimes_K D_i.$$

The map  $t'_i$  is the product of the transfer homomorphisms for the field summands of  $L \otimes_K E_i$ , each of which is onto by [12, Corollary A.15]. So  $t_i$  is also onto. But  $\text{trf}_{KG}^{LG}$  is isomorphic to the sum of the  $t_i$ , and is hence surjective.

*Step 2.* Fix  $K$  and  $G$  such that  $K$  is a totally imaginary splitting field for  $G$ . In particular,  $K_0(KG) = R_C(G)$ . For any finite dimensional (left)  $KG$ -module  $V$ , define

$$f_V : C(K) \rightarrow C(KG)$$

to be the homomorphism induced by the functor

$$V \otimes_K : K\text{-mod} \rightarrow KG\text{-mod}.$$

If  $V$  is irreducible, then  $f_V$  is just the Morita equivalence identifying  $C(B)$  with  $C(K)$ , where  $B \subseteq KG$  is the simple summand with irreducible representation  $V$ . So by definition,

$$\tilde{I}_{KG}([V]) = f_V(\lambda_K^{-1}(\exp(2\pi i/m))); \quad (m = |\mu_K|) \tag{4}$$

where  $\lambda_K : C(K) \xrightarrow{\cong} \mu_K$  is induced by the norm residue symbol. Both sides of (4) are additive ( $f_{V \oplus W} = f_V + f_W$ ), so (4) holds for arbitrary  $V$ .

*Step 3.* We can now show the commutativity of triangle (1) above: that  $\tilde{I}_{KG} = \text{trf}_{KG}^{LG} \circ \tilde{I}_{LG}$  for any  $G$  and any number fields  $K \subseteq L \subseteq \mathbb{C}$ . It suffices to do this when  $K$  and  $L$  both are totally imaginary splitting fields for  $G$ . In particular,  $K_0(KG) = R_C(G)$ .

By (4), for any finite dimensional  $KG$ -module  $V$ ,

$$\begin{aligned} \tilde{I}_{KG}([V]) &= f_V(\lambda_K^{-1}(\exp(2\pi i/m))), & (m = |\mu_K|) \\ \text{trf}_{KG}^{LG}(\tilde{I}_{LG}([V])) &= \text{trf}_{KG}^{LG} \circ f_{L \otimes_K V}(\lambda_L^{-1}(\exp(2\pi i/n))); & (n = |\mu_L|) \end{aligned}$$

and it remains to check the commutativity of the following diagram:

$$\begin{array}{ccccc} \mu_L & \xrightarrow[\cong]{\lambda_L^{-1}} & C(L) & \xrightarrow{f_{L \otimes V}} & C(LG) \\ \downarrow n/m & (5) & \downarrow \text{trf}_K^L & (6) & \downarrow \text{trf}_{KG}^L \\ \mu_K & \xrightarrow{\lambda_K^{-1}} & C(K) & \xrightarrow{f_V} & C(KG). \end{array}$$

But (5) commutes by Lemma 5.1, while (6) commutes since the two composites are induced by tensoring with the bimodules

$$KG^{LG} \otimes_{LG} (L \otimes_K V)_L =_{KG} V \otimes_K L_L.$$

*Step 4.* Now fix a homomorphism  $\alpha: \tilde{G} \rightarrow G$  of finite groups, and a subgroup  $H \subseteq G$ . We must show that (2) and (3) commute for any number field  $K \subseteq \mathbb{C}$ . If  $L \supseteq K$  is any pair of number fields, then the squares

$$\begin{array}{ccccc} C(L\tilde{G}) & \xrightarrow{C(L\alpha)} & C(LG) & \xrightarrow{\text{trf}_{LH}^L} & C(LH) \\ \downarrow \text{trf}_{K\tilde{G}}^L & & \downarrow \text{trf}_{KG}^L & & \downarrow \text{trf}_{KH}^L \\ C(K\tilde{G}) & \xrightarrow{C(K\alpha)} & C(KG) & \xrightarrow{\text{trf}_{KH}^{KG}} & C(KH) \end{array}$$

commute (just compare bimodules). So by (1) (Step 3), it suffices to prove the commutativity of (2) and (3) when  $K$  is a splitting field for  $\tilde{G}$ ,  $G$  and  $H$  (and totally imaginary).

Fix such a  $K$ ; in particular,  $K_0(K\tilde{G}) = R_C(\tilde{G})$  and  $K_0(KG) = R_C(G)$ . Fix finite dimensional modules  $V$  over  $K\tilde{G}$  and  $W$  over  $KG$ . Set

$$x = \lambda_K^{-1}(\exp(2\pi i/|\mu_K|)) \in C(K).$$

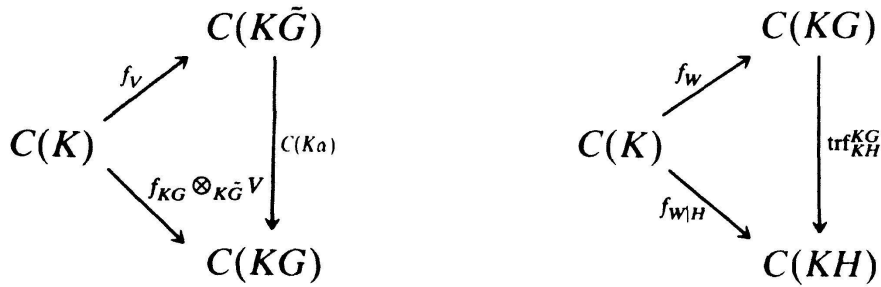
Then, by (4),

$$\begin{aligned} \tilde{I}_{KG}(R_C(\alpha)([V])) &= f_{KG \otimes_{K\tilde{G}} V}(x), \\ C(K\alpha) \circ \tilde{I}_{K\tilde{G}}([V]) &= C(K\alpha) \circ f_V(x); \\ \tilde{I}_{KH}(\text{Res}_H^G([W])) &= f_{W|H}(x), \end{aligned}$$

and

$$\text{trf}_{KH}^{KG} \circ \tilde{I}_{KG}([W]) = \text{trf}_{KH}^{KG} \circ f_W(x).$$

So we will be done upon showing that the following triangles commute:



But they are induced by the following pairs of isomorphic bimodules:

$${}_{KG}(KG \otimes_{K\tilde{G}} V)_K \cong {}_{KG}KG \otimes_{K\tilde{G}} V_K \quad \text{and} \quad {}_{KH}W_K \cong {}_{KH}KG \otimes_{KG} W_K;$$

and we are done.  $\square$

Again fix a finite group  $G$  and a number field  $K \subseteq \mathbb{C}$ , and let  $R \subseteq K$  be the ring of integers. Then  $Cl_1(RG)$  is described by a localization sequence

$$\sum_p K_2(\hat{R}_p G) \rightarrow C(KG) \xrightarrow{\partial_{RG}} Cl_1(RG) \rightarrow 0$$

(see [20, Theorem 2.1] for details). We now consider the composite

$$R_C(G) \xrightarrow{\tilde{I}_{KG}} C(KG) \xrightarrow{\partial_{RG}} Cl_1(RG).$$

Both maps are natural with respect to induction from subgroups of  $G$ . Hence, since  $Cl_1(RH) = SK_1(RH) = 0$  for any cyclic  $H \subseteq G$  by [1, Theorem 3.3],  $\partial_{RG} \circ \tilde{I}_{KG}$  vanishes on any element of  $R_C(G)$  induced up from a cyclic subgroup. Thus,  $\partial_{RG} \circ \tilde{I}_{KG}$  factors through a homomorphism

$$I_{RG} : A_C(G) \rightarrow Cl_1(RG),$$

where  $A_C(G)$  is the Artin cokernel.

**THEOREM 5.3.** *For any finite group  $G$ , and any number field  $K \subseteq \mathbb{C}$  with ring of integers  $R$ ,*

$$I_{RG} : A_C(G) \rightarrow Cl_1(RG)$$

*is surjective. The  $I_{RG}$  are natural in that for any homomorphism  $\alpha : \tilde{G} \rightarrow G$  of finite groups, any  $H \subseteq G$ , and any pair  $R \subseteq S$  of rings of integers in number fields, the*

following diagrams all commute:

$$\begin{array}{ccccc}
 & A_{\mathbb{C}}(G) & & A_{\mathbb{C}}(\tilde{G}) & \xrightarrow{A_{\mathbb{C}}(\alpha)} & A_{\mathbb{C}}(G) & \xrightarrow{\text{Res}_{H}^G} & A_{\mathbb{C}}(H) \\
 & \swarrow I_{SG} & & \downarrow I_{R\tilde{G}} & & \downarrow I_{RG} & & \downarrow I_{RH} \\
 & Cl_1(SG) & \xrightarrow{\text{trf}_{RG}^{SG}} & Cl_1(R\tilde{G}) & \xrightarrow{Cl_1(A\alpha)} & Cl_1(RG) & \xrightarrow{\text{trf}_{RH}^{RG}} & Cl_1(RH)
 \end{array}$$

*Proof.* For any  $R$  and  $G$ ,  $I_{RG}$  is surjective since  $\tilde{I}_{KG}$  and  $\partial_{RG}$  both are surjective. The naturality properties follow from the corresponding properties for the  $\tilde{I}_{KG}$  (Proposition 5.2), and the naturality of the boundary maps  $\partial_{RG}$  in the localization sequences.  $\square$

Now that the  $I_{RG}$  have been constructed, we can finally apply Theorem 1.4 to show that they are isomorphisms for sufficiently large  $R$ . For any finite  $G$ ,  $a_{\mathbb{C}}(G)$  will denote the complex Artin exponent: the order of  $1 \in R_{\mathbb{C}}(G)$  in  $A_{\mathbb{C}}(G)$ . By Frobenius reciprocity,

$$a_{\mathbb{C}}(G) = \exp(A_{\mathbb{C}}(G));$$

i.e.,  $a_{\mathbb{C}}(G) \cdot x$  is induced from cyclic subgroups for any  $x \in R_{\mathbb{C}}(G)$ . By the Artin induction theorem [7, Theorem 39.1],  $a_{\mathbb{C}}(G) \mid |G|$ .

**THEOREM 5.4.** *Let  $G$  be any finite group, and set  $n = a_{\mathbb{C}}(G) \cdot \exp(G)$ . Let  $K$  be any number field such that  $\zeta_n \in K$ , and let  $R \subseteq K$  be the ring of integers. Then  $I_{RG}$  is an isomorphism:  $Cl_1(RG) \cong A_{\mathbb{C}}(G)$ .*

*Proof.* This will be shown first for  $p$ -groups, then for  $p$ -elementary groups, and finally for arbitrary finite groups.

*Step 1.* Let  $G$  be a  $p$ -group, and set  $p^k = a_{\mathbb{C}}(G)$ ,  $p^m = \exp(G)$ , and  $q = p^{k+m}$ . By Theorem 5.3(1), it will suffice to show that  $I_{RG}$  is an isomorphism when  $K = \mathbb{Q}\zeta_q$  and  $R = \mathbb{Z}\zeta_q$ .

Let  $C_q$  be a (multiplicative) cyclic group of order  $q$  with generator  $z$ . Consider the pullback square

$$\begin{array}{ccc}
 \hat{\mathbb{Z}}_p[C_q \times G] & \xrightarrow{\alpha} & \hat{\mathbb{Z}}_p \zeta_q[G] \\
 \downarrow & & \downarrow \\
 \mathbb{Z}_p[(C_q/z^{p^{n+m-1}}) \times G] & \xrightarrow{\beta} & \mathbb{Z}/p[(C_q/z^{p^{n+m-1}}) \times G];
 \end{array} \tag{1}$$

where  $\alpha$  is induced by:  $\alpha(z) = \zeta_q$ . Then  $K_2(\beta)$  is onto by [17, Lemma 1.7] if  $p > 2$ ; or if  $p = 2$  since the only torsion in  $K_1(\hat{\mathbb{Z}}_2[(C_q/z^{p^{n+m-1}}) \times G], 2)$  is  $(-1)$  (see [15, Proposition 2]). So by the Mayer–Vietoris sequence for (1),  $K_2(\alpha)$  is onto.

Now consider the following commutative diagram:

$$\begin{array}{ccccccc}
 K_2(\hat{\mathbb{Z}}_p[C_q \times G]) & \longrightarrow & C(\mathbb{Q}[C_q \times G])_{(p)} & R_c(G) \otimes \mathbb{Z}/q & & & \\
 (2) \quad \downarrow K_2(\alpha) & & \alpha_* \downarrow & \swarrow \bar{I}_{Kc} & & \downarrow \partial \circ \bar{I}_{Kc} & (2) \\
 K_2(\hat{\mathbb{Z}}_p \zeta_q[G]) & \xrightarrow{\varphi_{RG}} & C_p(\mathbb{Q}\zeta_q[G]) & \xrightarrow{\partial} & Cl_1(RG)_{(p)} & \longrightarrow & 0
 \end{array}$$

where the bottom row is exact [20, Theorem 2.1]. By Theorem 1.4,

$$\begin{aligned}
 K_2(\hat{\mathbb{Z}}_p[C_q \times G]) &= K_2(\hat{\mathbb{Z}}_p G) \oplus K_2(\hat{\mathbb{Z}}_p[C_q \times G], (1-z)) \\
 &= K_2(\hat{\mathbb{Z}}_p G) \oplus \langle \{h, 1 - (1-z)^i g\}, \{z, 1 - (1-z)^i g\} : \\
 &\hspace{15em} h \in G, g \in C_q \times G, hg = gh, i \geq 1 \rangle.
 \end{aligned}$$

It follows that

$$K_2(\hat{\mathbb{Z}}_p \zeta_p[G]) = K_2(\hat{\mathbb{Z}}_p G) + X + Y;$$

where with  $\zeta = \zeta_q$ :

$$X = \langle \{h, 1 - (1-\zeta)^i \zeta^j g\} : h, g \in G, hg = gh, i \geq 1, j \in \mathbb{Z} \rangle$$

and

$$Y = \langle \{\zeta, 1 - (1-\zeta)^i \zeta^j g\} : g \in G, i \geq 1, j \in \mathbb{Z} \rangle.$$

Recall that  $p^m = \exp(G)$ . Then

$$\exp(X) \mid p^m \quad \text{and} \quad \exp(\varphi_{RG}(K_2(\hat{\mathbb{Z}}_p G))) \mid \exp(C(\mathbb{Q}G)_{(p)}) \mid p^m.$$

Furthermore, by definition,

$$\varphi_{RG}(Y) \subseteq \text{Im} \left[ \sum \{C(\mathbb{Q}\zeta_q[H]) : H \subseteq G \text{ cyclic}\} \rightarrow C(\mathbb{Q}\zeta_q[G]) \right].$$

So by diagram (2) (recalling that  $q = p^{k+m}$ ):

$$\begin{aligned}
 \text{Ker}(\partial \circ \bar{I}_{KG}) &\subseteq p^k R_c(G) + \text{Im} \left[ \sum \{R_c(H) : H \subseteq G \text{ cyclic}\} \rightarrow R_c(G) \right] \\
 &= p^k R_c(G) + \text{Ker} [R_c(G) \rightarrow A_c(G)].
 \end{aligned}$$

Since  $p^k = a_{\mathbb{C}}(G) = \exp(A_{\mathbb{C}}(G))$ ,

$$\text{Ker } [I_{RG}: A_{\mathbb{C}}(G) \rightarrow Cl_1(RG)] \subseteq p^k A_{\mathbb{C}}(G) = 0;$$

and so  $I_{RG}$  is an isomorphism.

*Step 2.* Now assume that  $G$  is  $p$ -elementary:  $G \cong C_m \times H$  where  $p \nmid m$  and  $H$  is a  $p$ -group. Set  $n = a_{\mathbb{C}}(G) \cdot \exp(G)$ , fix a number field  $K \subseteq \mathbb{C}$  containing  $\zeta_n$ , and let  $R$  be the ring of integers of  $K$ . Then

$$\begin{aligned} A_{\mathbb{C}}(G) &\cong \text{Coker} \left[ \sum \{R_{\mathbb{C}}(C_m) \otimes R_{\mathbb{C}}(H_0) : H_0 \subseteq H \text{ cyclic}\} \rightarrow R_{\mathbb{C}}(C_m) \otimes R_{\mathbb{C}}(H) \right] \\ &\cong R_{\mathbb{C}}(C_m) \otimes A_{\mathbb{C}}(H) \cong \prod_{H \subseteq G}^m A_{\mathbb{C}}(H). \end{aligned}$$

On the other hand, the identification  $K[G] \cong \prod^m K[H]$  (each factor corresponding to a character of  $C_m$ ) induces an inclusion  $RG \subseteq \prod^m R[H]$  of index prime to  $p$ ; and hence an isomorphism

$$Cl_1(RG)_{(p)} \cong \prod_{H \subseteq G}^m Cl_1(RH)_{(p)} \cong \prod_{H \subseteq G}^m A_{\mathbb{C}}(H) \cong A_{\mathbb{C}}(G).$$

(see [17, Proposition 1.2]). Since  $I_{RG}$  is onto, it must be an isomorphism.

*Step 3.* Now let  $G$  be an arbitrary finite group, set  $n = a_{\mathbb{C}}(G) \cdot \exp(G)$ , and let  $R$  be any ring of integers containing  $\zeta$ . Let  $\mathcal{E}$  be the set of elementary subgroups of  $G$ . For any  $H \in \mathcal{E}$ ,  $\exp(H) \mid \exp(G)$  and  $a_{\mathbb{C}}(H) \mid a_{\mathbb{C}}(G)$ , so  $I_{RH}$  is an isomorphism by Step 2. Consider the following square, which commutes by Theorem 5.3:

$$\begin{array}{ccc} A_{\mathbb{C}}(G) & \xrightarrow{I_{RG}} & Cl_1(RG) \\ \downarrow \Sigma \text{Res}_H^G & & \downarrow \Sigma \text{trf}_H^G \\ \sum_{H \in \mathcal{E}} A_{\mathbb{C}}(H) & \xrightarrow[\cong]{\Sigma I_{RH}} & \sum_{H \in \mathcal{E}} Cl_1(RH) \end{array}$$

In the language of [10],  $A_{\mathbb{C}}(-)$  is a module over the Frobenius functor  $R_{\mathbb{C}}(-)$ , and hence is detected by restriction to elementary subgroups. So  $\Sigma \text{Res}_H^G$  is injective in the above square, and  $I_{RG}$  is an isomorphism.  $\square$

By [4, Theorem XI.4.7], for any finite  $G$ ,

$$a_{\mathbb{C}}(G) = \prod_{p \mid |G|} a_{\mathbb{C}}(G_p),$$

where  $G_p$  is a  $p$ -Sylow subgroup. Thus, the description of

$$a_{\mathbb{C}}(G) = \exp(A_{\mathbb{C}}(G)) = \max_R (\exp(Cl_1(RG)))$$

reduces immediately to the  $p$ -group case.

If  $G$  is a non-cyclic  $p$ -group, then there is a surjection  $G \twoheadrightarrow C_p \times C_p$  and an induced surjection of  $A_{\mathbb{C}}(G)$  onto  $A_{\mathbb{C}}(C_p \times C_p)$ . This last group is easily checked to be non-zero (see [1, Lemma 5.5] for details). Thus, for any finite  $G$ ,  $A_{\mathbb{C}}(G)$  is  $p$ -torsion free if and only if  $G_p$  is cyclic,  $A_{\mathbb{C}}(G) = 0$  if and only if  $G$  is metacyclic, and these in turn imply similar statements about the  $Cl_1(RG)$  (and  $SK_1(RG)$ ). In fact, for fixed  $p$  and  $R$  such that  $\zeta_p \in R$  (or  $\zeta_4 \in R$  if  $p = 2$ ), and any  $G$ ,  $Cl_1(RG)_{(p)} = 0$  if and only if  $G_p$  is cyclic (see [1, Theorem 3.5]).

A general description of  $a_{\mathbb{C}}(G)$  has been given by Gluck [27]. The formula is much more complicated than that for the rational Artin exponent  $a_{\mathbb{Q}}(G)$  given by Lam [11]. If  $G$  is non-cyclic, and abelian or of exponent  $p$ , then  $a_{\mathbb{C}}(G) = a_{\mathbb{Q}}(G) = (1/p) |G|$ . On the other hand, if  $G$  is a semidihedral 2-group, then  $a_{\mathbb{C}}(G) = 2$  ( $a_{\mathbb{Q}}(G) = 4$ ); and if  $p$  is odd and  $G$  a non-abelian group of order  $p^3$  and exponent  $p^2$ , then  $a_{\mathbb{C}}(G) = p$  ( $a_{\mathbb{Q}}(G) = p^2$ ).

To end, we note that Theorem 5.3 allows a new interpretation of the following result in [13] (Theorem 1).

**COROLLARY 5.5.** *Let  $G$  be a finite group, and let  $R$  be the ring of integers in some number field. Then  $Cl_1(RG)$  is generated by induction from elementary subgroups of  $G$ .*

*Proof.* By the Brauer induction theorem,  $R_{\mathbb{C}}(G)$ , and hence  $A_{\mathbb{C}}(G)$  are generated in induction from elementary subgroups of  $G$ . The result follows since  $I_{RG}: A_{\mathbb{C}}(G) \rightarrow Cl_1(RG)$  is natural and surjective.  $\square$

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Department of Mathematics  
 Aarhus University  
 Ny Munkegade  
 8000 Aarhus, Denmark

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