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SK_1 of finite group rings: V

ROBERT OLIVER

We continue here the study of

$$SK_1(\mathbb{Z}G) = \text{Ker} [K_1(\mathbb{Z}G) \rightarrow K_1(\mathbb{Q}G)]$$

for finite G : the group shown by Wall [26] to be precisely the torsion subgroup of $\text{Wh}(G)$. In earlier papers in this series, $SK_1(\mathbb{Z}G)$ has been studied via the extension

$$0 \rightarrow Cl_1(\mathbb{Z}G) \rightarrow SK_1(\mathbb{Z}G) \rightarrow \sum_{p \mid |G|} SK_1(\hat{\mathbb{Z}}_p G) \rightarrow 0; \quad (0.1)$$

where $Cl_1(\mathbb{Z}G) \subseteq SK_1(\mathbb{Z}G)$ is the subgroup of elements described via K_2 in localization sequences.

This paper contains the last step in deriving a combinatorial algorithm for describing the odd torsion in $SK_1(\mathbb{Z}G)$. By [17, Theorem 4.8], $SK_1(\mathbb{Z}G)[\frac{1}{2}]$ splits naturally as a sum

$$SK_1(\mathbb{Z}G)[\frac{1}{2}] \cong Cl_1(\mathbb{Z}G)[\frac{1}{2}] \oplus \sum_{p>2} SK_1(\hat{\mathbb{Z}}_p G).$$

The groups $SK_1(\hat{\mathbb{Z}}_p G)$ (also for $p = 2$) are described by [15, Theorem 3] and [16, Theorem 2], in terms of $H_2(Z_i)$ for certain subgroups $Z_i \subseteq G$. On the other hand, in [17], the problem of describing $Cl_1(\mathbb{Z}G)_{(p)}$ for any odd prime p and any finite G is reduced to the case where G is a p -group (see [17, Theorem 4.8], and the discussion at the end of Section 3 below).

The following theorem is the central result of this paper, and gives a relatively simple way of describing $Cl_1(\mathbb{Z}G)$ when G is a p -group (and p odd). Note that if G is any group, and G acts on $\mathbb{Z}G$ by conjugation, then for any set $S \subseteq G$ of conjugacy class representatives,

$$H_1(G; \mathbb{Z}G) \cong \sum_{h \in S} H_1(Z_G(h)) \otimes \mathbb{Z}(h).$$

(If $X \subseteq G$ is any conjugacy class, and $h \in X$, then $\mathbb{Z}(X) \cong \text{Ind}_{\mathbb{Z}_G(h)}^G(\mathbb{Z})$ as $\mathbb{Z}G$ -modules.) Thus, $H_1(G; \mathbb{Z}G)$ is generated by elements $g \otimes h$ for commuting $g, h \in G$.

THEOREM 3.6. *Fix an odd prime p and a p -group G . Write $\mathbb{Q}G = \prod_{i=1}^k B_i$, where each B_i is simple with center F_i and irreducible representation V_i . For each i , let $(\mu_{F_i})_p$ be the group of p -th power roots of unity. Define*

$$\psi_G: H_1(G; \mathbb{Z}G) \rightarrow \prod_{i=1}^k (\mu_{F_i})_p,$$

where G acts on $\mathbb{Z}G$ by conjugation, by setting

$$\psi_G(g \otimes h) = [\det_{F_i}(g, V_i^h)]_i \quad (g, h \in G, gh = hg, V_i^h = \{x \in V_i: hx = x\}).$$

Then $Cl_1(\mathbb{Z}G) \cong \text{Coker}(\psi_G)$.

Examples of computations of $Cl_1(\mathbb{Z}G)$ using Theorem 3.6 for non-abelian G are given in Section 4. For abelian G , the isomorphism $SK_1(\mathbb{Z}G) \cong \text{Coker}(\psi_G)$ is proven in [1, Theorem 1.8], and some examples of calculations of $SK_1(\mathbb{Z}G)$ using that are given in Section 5 of the same paper.

Theorem 3.6 (and the other theorems referred to above) are stated, for simplicity, as describing the components of $SK_1(\mathbb{Z}G)$ as abstract groups only. But the proofs also contain enough information so that one can take a specific element in $SK_1(\mathbb{Z}G)$ (e.g., a specific element in $\text{Coker}(\psi_G)$ as described above), and represent it by a matrix. The opposite problem, taking a specific matrix over $\mathbb{Z}G$ and deciding how it sits in $SK_1(\mathbb{Z}G)$ (if it does) is harder in general; the study in [20] of the Whitehead transfer homomorphism for oriented S^1 -fiber bundles gives one example where this can be done.

In general, for any finite group G , $Cl_1(\mathbb{Z}G)$ is described by localization exact sequences

$$K_2^{\text{top}}(\hat{\mathbb{Z}}_p G) \xrightarrow{\varphi} C_p(\mathbb{Q}G) \xrightarrow{\partial} Cl_1(\mathbb{Z}G)_{(p)} \rightarrow 0$$

for each prime p ; where for any maximal order $\mathfrak{M} \subseteq \mathbb{Q}G$:

$$\begin{aligned} C_p(\mathbb{Q}G) &\cong \varprojlim_n \text{Coker} [K_2(\mathfrak{M}) \rightarrow K_2(\mathfrak{M}/p^n \mathfrak{M})] \cong \varprojlim_n Cl_1(\mathfrak{M}; p^n \mathfrak{M}) \\ &\cong \text{Coker} \left[K_2 \left(\mathfrak{M} \left[\frac{1}{p} \right] \right) \rightarrow K_2^{\text{top}}(\hat{\mathbb{Q}}_p G) \right]_{(p)}. \end{aligned}$$

The $C_p(\mathbb{Q}G)$ are described by the work of Bak and Rehmann on the congruence subgroup problem [3]. The remaining problem is then to find a set of generators for $K_2^{\text{top}}(\hat{\mathbb{Z}}_p G)$, or at least for its image in $C_p(\mathbb{Q}G)$. In the case of an odd prime p and a p -group G , the formula

$$Cl_1(\mathbb{Z}G) \cong \text{Coker} \left[\psi_G : H_1(G; \mathbb{Z}G) \rightarrow \prod_{i=1}^k (\mu_{F_i})_p \right]$$

can be explained by noting that norm residue symbols define an isomorphism of $C_p(\mathbb{Q}G)$ with $\prod (\mu_{F_i})_p$, and that $H_1(G; \mathbb{Z}G) (\cong H_1(G; \hat{\mathbb{Z}}_p G))$ and $K_2(\hat{\mathbb{Z}}_p G)$ both are closely related to the cyclic homology group $HC_1(\hat{\mathbb{Z}}_p G)$ (see [21]).

The key new result here about generators for $K_2(\hat{\mathbb{Z}}_p G)$ is:

THEOREM 1.4. *Let p be any prime, and fix a p -group G and an element $z \in Z(G)$. Then*

$$\begin{aligned} \text{Ker} [K_2^{\text{top}}(\hat{\mathbb{Z}}_p G) \rightarrow K_2^{\text{top}}(\hat{\mathbb{Z}}_p [G/z])] \\ = \langle \{g, 1 + \lambda(1 - z)^i h\} : \lambda \in \hat{\mathbb{Z}}_p, i \geq 1, g, h \in G, gh = hg \rangle. \end{aligned}$$

Since $\text{Coker} [K_2^{\text{top}}(\hat{\mathbb{Z}}_p G) \rightarrow K_2^{\text{top}}(\hat{\mathbb{Z}}_p [G/z])]$ is also known in the above situation (see Proposition 2.1 below), it should in principle now be possible to inductively construct a set of generators for $K_2^{\text{top}}(\hat{\mathbb{Z}}_p G)$. Unfortunately, it's not always easy to explicitly lift elements from $K_2^{\text{top}}(\hat{\mathbb{Z}}_p [G/z])$ to $K_2^{\text{top}}(\hat{\mathbb{Z}}_p G)$, even where they are known to lift. But such an inductive procedure does work to give generators for $K_2^{\text{top}}(\hat{\mathbb{Z}}_p G)^+$ when p is odd, and this suffices when computing $Cl_1(\mathbb{Z}G)$.

Another consequence of Theorem 1.4 involves a comparison of $Cl_1(RG)$ – when G is any finite group and R the ring of integers is some number field $K \subseteq \mathbb{C}$ – with the “complex Artin cokernel”

$$A_{\mathbb{C}}(G) = \text{Coker} \left[\sum \{R_{\mathbb{C}}(H) : H \subseteq G \text{ cyclic}\} \xrightarrow{\Sigma \text{Ind}} R_{\mathbb{C}}(G) \right].$$

Natural epimorphisms $I_{RG} : A_{\mathbb{C}}(G) \twoheadrightarrow Cl_1(RG)$ are constructed, for such R and G , via localization sequences. Theorem 1.4 can then be applied to show that for any G , I_{RG} is an isomorphism for R large enough. Thus, $A_{\mathbb{C}}(G)$ represents the “largest possible” $Cl_1(RG)$ when G is fixed and R varies. This is the second unexpected appearance of Artin cokernels when studying $K_n(RG)$: it was shown in [18] that $D(\mathbb{Z}G)^+ \cong A_{\mathbb{Q}}(G)$ when G is a p -group and p any odd regular prime.

The obvious remaining question is: what about 2-power torsion in $SK_1(\mathbb{Z}G)$? Unlike the case of odd torsion, this cannot be completely reduced to studying $Cl_1(\mathbb{Z}G)$ for 2-groups G , but the results in [17] show that the main problem is with 2-groups. If G is a p -group (for any p) and $[G, G]$ is central and cyclic, then we can show that $K_2^{\text{top}}(\hat{\mathbb{Z}}_p G)$ is generated by $\{-1, -1\}$ and symbols $\{g, u\}$ for $g \in G$ and $u \in (\hat{\mathbb{Z}}_p[Z_G(g)])^*$; and when $p = 2$ this suffices to get a description of $Cl_1(\mathbb{Z}G)$. But there *are* 2-groups G for which $K_2^{\text{top}}(\mathbb{Z}_2 G)$ is not generated by such symbols, and there may not be any simple algorithm for describing $Cl_1(\mathbb{Z}G)$ in general. The best conjecture we have been able to make so far gives upper and lower bounds for $Cl_1(\mathbb{Z}G)$, bounds which differ by exponent two. The question of whether the inclusion $Cl_1(\mathbb{Z}G)_{(2)} \subseteq SK_1(\mathbb{Z}G)_{(2)}$ ever fails to split is also still open.

The paper is organized as follows. Section 1 and 2 deal with the problems of finding generators for $\text{Ker}(K_2(\hat{\mathbb{Z}}_p \alpha))$, and of detecting $\text{Coker}(K_2(\hat{\mathbb{Z}}_p \alpha))$, respectively, when α is a surjection of p -groups whose kernel is central and cyclic. This is applied in Section 3 to prove that $Cl_1(\mathbb{Z}G) \cong \text{Coker}(\psi_G)$ when G is an odd p -group; and ways of using that to compute the odd torsion in $Cl_1(\mathbb{Z}G)$ for arbitrary finite G are discussed. Examples are given in Section 4 to illustrate how Theorem 3.6 works in practice for computing $Cl_1(\mathbb{Z}G)$. Finally, in Section 5, the relationship between $Cl_1(RG)$ and the complex Artin cokernel is studied, and the isomorphism $Cl_1(RG) \cong A_c(G)$ proven for large R .

As for notation, C_n always denotes a (multiplicative) cyclic group of order n , and ζ_n a primitive n -th root of unity. If F is any field, then μ_F denotes the group of roots of unity in F , and $(\mu_F)_p$ the group of p -th power roots of unity.

If R is a $\hat{\mathbb{Q}}_p$ -algebra or a $\hat{\mathbb{Z}}_p$ -order (e.g., $R = \hat{\mathbb{Q}}_p G$ or $\hat{\mathbb{Z}}_p G$), then $K_2(R)$ *always denotes the topological* K_2 . The precise definition of these groups, and their occurrence in localization sequences, is described in [20]: in Theorem 2.1 and the preceding discussion (see also [3]). Here we just note that if R is a $\hat{\mathbb{Z}}_p$ -order, then

$$K_2^{\text{top}}(R) \cong \varprojlim_n K_2(R/p^n R).$$

Section 1

If R is a ring, and $I \subseteq R$ is a 2-sided ideal, we define here

$$K_2(R, I) = \text{Ker}[K_2(R) \rightarrow K_2(R/I)].$$

A braid diagram analogous to that in [12, Remark 6.6] shows that for any ideals

$\bar{I} \subseteq i \subseteq R$, there is an exact sequence

$$0 \rightarrow K_2(R, \bar{I}) \rightarrow K_2(R, I) \rightarrow K_2(R/\bar{I}, I/\bar{I}) \xrightarrow{\partial} K_1(R, \bar{I}) \rightarrow K_1(R, I) \rightarrow \dots$$

The main result of this section is to describe a set of generators for $K_2(\hat{\mathbb{Z}}_p G, (1-z))$; when p is any prime, G is any p -group, and $z \in Z(G)$. Three lemmas will first be needed.

LEMMA 1.1. *Fix a prime p , and a finite ring R of p -power order. Let $J \subseteq R$ be the Jacobson radical, and let $\{\alpha_1, \dots, \alpha_k\} \subseteq J$ be any set of elements such that $\{p, \alpha_1, \dots, \alpha_k\}$ generates J (as an ideal). Then for any ideal $I \subseteq J$ of R such that $IJ = JI = 0$, and such that $I \subseteq \langle \alpha_1, \dots, \alpha_k \rangle_R$ if $p = 2$, $K_2(R, I)$ is generated by symbols of the form*

$$\{1 - \alpha_i, 1 - x\} : 1 \leq i \leq k, \quad x \in I. \quad (1)$$

Proof. We use the notation and relations for pointed bracket symbols in [25, Proposition 96–97]. By [17, Proposition 2.3], $K_2(R, I)$ is generated by symbols of the form

$$\{1 - \alpha, 1 + x\} = \langle \alpha, 1 + x \rangle = \langle \alpha, x \rangle$$

for $\alpha \in J$ and $x \in I$ ($\alpha x = x\alpha = 0$). Write $\alpha = pr_0 + \alpha_1 r_1 + \dots + \alpha_k r_k$; so that

$$\begin{aligned} \langle \alpha, x \rangle &= \langle pr_0, x \rangle + \sum_{i=1}^k \langle \alpha_i r_i, x \rangle = \langle p, r_0 x \rangle + \sum_{i=1}^k \langle \alpha_i, r_i x \rangle \\ &= \sum_{i=1}^k \{1 - \alpha_i, 1 + r_i x\} + \langle p, r_0 x \rangle + p \langle -1, r_0 x \rangle \\ &= \sum_{i=1}^k \{1 - \alpha_i, 1 + r_i x\} + \langle p, r_0 x \rangle + \left\langle -p + \binom{p}{2} r_0 x, r_0 x \right\rangle \quad (x^2 = 0) \\ &= \sum_{i=1}^k \{1 - \alpha_i, 1 + r_i x\} + \left\langle \binom{p}{2} r_0 x, r_0 x \right\rangle. \quad (px \in JI = 0). \end{aligned}$$

If p is odd, then $\binom{p}{2} x = 0$, and we are done. If $p = 2$ and $I \subseteq \langle \alpha_1, \dots, \alpha_k \rangle_R$, then the same procedure shows that for any $x \in I$, $\langle x, x \rangle$ is a sum of symbols of the form in (1). \square

The following technical relation between symbols will be needed in the calculations.

LEMMA 1.2. *Let R be any ring. Fix $a, u \in R^*$ and $n \geq 2$ such that*

$$[a^n, u] = 1 = [a^i u a^{-i}, a^j u a^{-j}]$$

for any i, j . Then

$$\begin{aligned} \{a, u(a u a^{-1})(a^2 u a^{-2}) \cdots (a^{n-1} u a^{1-n})\} \\ = \{a^n, u\} + (n-1)\{u, u\} + \sum_{i=1}^{n-1} \{a^i u a^{-i}, u\}. \end{aligned}$$

Proof. In $St(R)$, set $x = h_{12}(u)$, $y = h_{13}(a)$, and

$$T = (y x y^{-1})(y^2 x y^{-2}) \cdots (y^{n-1} x y^{1-n}).$$

Then

$$\begin{aligned} \{a, u(a u a^{-1}) \cdots (a^{n-1} u a^{1-n})\} &= [y, x T] \\ &= (y x y^{-1})(y T y^{-1}) T^{-1} x^{-1} = T(y^n x y^{-n}) T^{-1} x^{-1} = [T, y^n x y^{-n}][y^n, x] \\ &= (\text{diag}(a u a^{-1} \cdot a^2 u a^{-2} \cdots a^{n-1} u a^{1-n}, u^{1-n}) * \text{diag}(u, u^{-1})) + \{a^n, u\} \\ &= \sum_{i=1}^{n-1} \{a^i u a^{-i}, u\} + \{u^{1-n}, u^{-1}\} + \{a^n, u\}. \end{aligned}$$

Here, for commuting matrices $M, N \in E(R)$, $M^* N \in K_2(R)$ denotes the commutator $[\tilde{M}, \tilde{N}]$ of liftings to $\tilde{M}, \tilde{N} \in St(R)$. \square

The third lemma will be needed when constructing filtrations of group rings by ideals. By a p -ring is meant the ring of integers in any finite extension of \mathbb{Q}_p .

LEMMA 1.3. *Fix a prime p , a p -group G , and some $z \in Z(G)$. Let $p^n = |z|$. Then, for any p -ring A , there are isomorphisms*

$$f_k: A/p^n[G/z] \xrightarrow{\cong} \frac{(1-z)^k A G}{(1-z)^{k+1} A G} \quad (k \geq 1)$$

and

$$f'_k: A/p[G/z] \xrightarrow{\cong} \frac{(1-z)^k A/p[G]}{(1-z)^{k+1} A/p[G]}; \quad (1 \leq k \leq p^n - 1)$$

both induced by sending ξ to $(1-z)^k \xi$ for $\xi \in AG$.

Proof. Note first that for any $\xi \in AG$, and any $k \geq 1$,

$$(1-z)^k p^n \xi \equiv (1-z)^k (1+z+z^2+\cdots+z^{p^n-1}) \xi = 0 \pmod{(1-z)^{k+1} AG}. \quad (1)$$

Thus, $(1-z)^k AG / (1-z)^{k+1} AG$ has exponent at most p^n for $k \geq 1$; and is in particular finite. So the map

$$(1-z)^k: (1-z)AG \xrightarrow{\cong} (1-z)^{k+1} AG$$

is an isomorphism: it is clearly onto, and the groups are free A -modules of the same rank.

Thus, for $\xi \in AG$ and $k \geq 1$, if $(1-z)^k \xi = (1-z)^{k+1} \eta$ for some $\eta \in AG$, then $(1-z)(\xi - (1-z)\eta) = 0$, and so

$$\xi \in (1-z)\eta + (1+z+\cdots+z^{p^n-1})AG \subseteq (1-z)AG + p^n AG.$$

Together with (1), this shows that $(1-z)^k \xi \in (1-z)^{k+1} AG$ if and only if $\xi \in p^n AG + (1-z)AG$. So f_k is well defined and an isomorphism.

If $1 \leq k \leq p^n - 1$, and $\xi' \in A/p[G]$ is such that $(1-z)^k \xi' \in (1-z)^{k+1} A/p[G]$ then

$$(1+z+\cdots+z^{p^n-1})\xi' = (1-z)^{p^n-1} \xi' \in (1-z)^{p^n} A/p[G] = 0;$$

and so $\xi' \in (1-z)A/p[G]$. The converse is clear, and so f'_k is a well defined isomorphism. \square

The main result of this section can now be shown:

THEOREM 1.4. *Fix a prime p , an unramified p -ring A , a p -group G , and an element $z \in Z(G)$. Then*

$$K_2(AG, (1-z)AG) = \text{Ker} [K_2(AG) \rightarrow K_2(A[G/z])]$$

is a finite group, and is generated by symbols of the form

$$\{g, 1 - \lambda(1 - z)^i h\} : g, h \in G, [g, h] = 1, \lambda \in A, i \geq 1.$$

Proof. Let $H_0 = \langle z \rangle$, and fix a series of subgroups

$$H_0 \subseteq H_1 \subseteq \cdots \subseteq H_n = G$$

such that for each $i = 1, \dots, n$,

$$H_i \triangleleft G \quad \text{and} \quad [H_i : H_{i-1}] = p.$$

For each i , fix $z_i \in H_i \setminus H_{i-1}$. Note that in G/H_{i-1} , z_i is central of order p .

Let $p^m = |z|$. Define

$$S = \{(k; r, i_0, \dots, i_k) : 0 \leq k \leq n, i_0 \geq 1, 0 \leq r \leq m-1, 0 \leq i_1, \dots, i_k \leq p-1\}.$$

For each $\sigma = (k; r, i_0, \dots, i_k) \in S$, set $k(\sigma) = k$, and

$$X(\sigma) = p^r(1-z)^{i_0}(1-z_1)^{i_1} \cdots (1-z_k)^{i_k} \in AG.$$

Define ideals $I'(\sigma) \subseteq I(\sigma) \subseteq AG$ by setting

$$I'(\sigma) = \langle (1-z)^{i_0+1}, p^{r+1}(1-z)^{i_0}; p^r(1-z)^{i_0}(1-z_1)^{i_1} \cdots (1-z_j)^{i_j+1}; 1 \leq j \leq k \rangle$$

and

$$I(\sigma) = I'(\sigma) + \langle X(\sigma) \rangle.$$

The idea now is to use S as a bookkeeping system for filtering the ideal $(1-z)AG$ into “pieces” small enough so that the theorem can be proven starting with Lemma 1.1. The following diagram gives a visual overview of this

filtration in the case where $p = 3$, $m = 2$, and $n = 2$ (i.e., $|z| = 9$ and $|G| = 81$):

	$k(\sigma) = 0$	$k(\sigma) = 1$	$k(\sigma) = 2$	
$(1 - z)AG$	(0; 0, 1)	(1; 0, 1, 0)	(2; 0, 1, 0, 0)	(1)
			(2; 0, 1, 0, 1)	
			(2; 0, 1, 0, 2)	
		(1; 0, 1, 1)	(2; 0, 1, 1, 0)	
			(2; 0, 1, 1, 1)	
			(2; 0, 1, 1, 2)	
		(1; 0, 1, 2)	(2; 0, 1, 2, 0)	
			(2; 0, 1, 2, 1)	
			(2; 0, 1, 2, 2)	
$(1 - z)^2AG$	(0; 1, 1)	(1; 1, 1, 0)	(2; 1, 1, 0, 0)	
			(2; 1, 1, 0, 1)	
			(2; 1, 1, 0, 2)	
		(1; 1, 1, 1)	(2; 1, 1, 1, 0)	
			(2; 1, 1, 1, 1)	
			(2; 1, 1, 1, 2)	
		(1; 1, 1, 2)	(2; 1, 1, 2, 0)	
			(2; 1, 1, 2, 1)	
			(2; 1, 1, 2, 2)	
	(0; 0, 2)	(1; 0, 2, 0)	(2; 0, 2, 0, 0)	
			(2; 0, 2, 0, 1)	
			(2; 0, 2, 0, 2)	
		(1; 0, 2, 1)	(2; 0, 2, 1, 0)	
			(2; 0, 2, 1, 1)	
			(2; 0, 2, 1, 2)	

The horizontal lines represent ideals in AG , ordered sequentially with the largest at the top. Each box represents some element $\sigma \in S$; the horizontal line at the top of the box represents $I(\sigma)$, while the line at the bottom represents $I'(\sigma)$. That the $I(\sigma)$ and $I'(\sigma)$ actually do correspond with this picture will be shown in Step 2A below.

Step 1. We now show that for any $\sigma \in S$, there is an isomorphism

$$f_\sigma: A/p[G/H_{k(\sigma)}] \xrightarrow{\cong} I(\sigma)/I'(\sigma) \quad (2)$$

defined by setting $f_\sigma([\xi]) = [X(\sigma) \cdot \xi]$ for $\xi \in AG$. This will be proven by induction on $k = k(G)$. If $k = 0$, so $\sigma = (0; r, i)$ for some $i \geq 1$ and $0 \leq r \leq m-1$, then

$$(1-z)^i AG / (1-z)^{i+1} AG \cong A/p^m[G/Z] = A/p^m[G/H_0]$$

by Lemma 1.3; and so

$$I(\sigma)/I'(\sigma) = \frac{p^r(1-z)^i AG + (1-z)^{i+1} AG}{p^{r+1}(1-z)^i AG + (1-z)^{i+1} AG} \cong \frac{p^r A/p^m[G/H_0]}{p^{r+1} A/p^m[G/H_0]} \cong A/p[G/H_0].$$

Now assume $k \geq 1$, and write $\sigma = (k; r, i_0, \dots, i_k)$. Set

$$\hat{\sigma} = (k-1; r, i_0, \dots, i_{k-1}) \in S.$$

By induction, we can assume that $I(\hat{\sigma})/I'(\hat{\sigma}) \cong A/p[G/H_{k-1}]$. By definition

$$I'(\sigma) = I'(\hat{\sigma}) + X(\hat{\sigma})(1-z_k)^{i_k+1} AG,$$

$$I(\sigma) = I'(\hat{\sigma}) + X(\hat{\sigma})(1-z_k)^{i_k} AG,$$

$$I(\hat{\sigma}) = I'(\hat{\sigma}) + X(\hat{\sigma}) AG.$$

Thus, $I(\hat{\sigma}) \supseteq I(\sigma) \supseteq I'(\sigma) \supseteq I'(\hat{\sigma})$; and by Lemma 1.3:

$$I(\sigma)/I'(\sigma) \cong \frac{(1-z_k)^{i_k} A/p[G/H_{k-1}]}{(1-z_k)^{i_k+1} A/p[G/H_{k-1}]} \cong A/p[G/H_k].$$

(Recall that $H_k = \langle H_{k-1}, z_k \rangle$, and that $0 \leq i_k \leq p-1$.)

Step 2. We next show that for any $\sigma \in S$,

$$K_2(AG/I'(\sigma), I(\sigma)/I'(\sigma)) = \langle \{g, 1 - X(\sigma)\lambda h\} : [g, h] \in H_{k(\sigma)}, \lambda \in A \rangle. \quad (3)$$

This will be proven by downwards induction on $k = k(\sigma)$.

Note first that AG is a local ring with Jacobson radical

$$J(AG) = \langle p, 1 - g : g \in G \rangle. \quad (4)$$

If $\sigma \in S$ and $k(\sigma) = n$, then $H_n = G$, and so $I(\sigma)/I'(\sigma) \cong A/p$ by Step 1. In particular,

$$(I(\sigma)/I'(\sigma)) \cdot J(AG/I'(\sigma)) = 0 = J(AG/I'(\sigma)) \cdot (I(\sigma)/I'(\sigma)).$$

So (3) follows in this case from Lemma 1.1 (applied using $\{1 - g : g \in G\}$ for the α_i 's).

Now fix some $\sigma = (k; r, i_0, \dots, i_k) \in S$, where $k < n$. For each $0 \leq i \leq p-1$, set

$$\sigma_i = (k+1; r, i_0, \dots, i_k, i) \in S.$$

Assume inductively that (3) holds for the σ_i .

Step 2A. We now show that the $I(\sigma_i) \supseteq I'(\sigma_i)$ and $I(\sigma) \supseteq I'(\sigma)$ have the relations implied by diagram (1) above. By definition, $I(\sigma_0) = I(\sigma)$ ($X(\sigma_0) = X(\sigma)$). For any $0 \leq i \leq p-2$,

$$I'(\sigma_i) = I'(\sigma) + X(\sigma)(1 - z_{k+1})^{i+1}AG = I(\sigma_{i+1}). \quad (5)$$

Furthermore,

$$I'(\sigma_{p-1}) = I'(\sigma) + X(\sigma)(1 - z_{k+1})^pAG = I'(\sigma)$$

by (2): since $X(\sigma)(1 - z_{k+1})^p = f((1 - z_{k+1})^p)$ and

$$(1 - z_{k+1})^p = (1 - z_{k+1}^p) = 0 \in A/p[G/H_k]. \quad (z_{k+1}^p \in H_k).$$

We thus have a filtration

$$I(\sigma) = I(\sigma_0) \supseteq I(\sigma_1) \supseteq \dots \supseteq I(\sigma_{p-1}) \supseteq I'(\sigma_{p-1}) = I'(\sigma); \quad (6)$$

and $I(\sigma_i) = I'(\sigma_{i-1})$ for $1 \leq i \leq p-1$.

Step 2B. For shortness in notation, we now write $K_1(I)$, $K_2(I)$ for $K_1(R, I)$, $K_2(R, I)$: R is always a quotient ring of AG . We are assuming that (3) holds for

the σ_i ; i.e., that

$$K_2(I(\sigma_i)/I'(\sigma_i)) = \langle \{g, 1 - X(\sigma_i)\lambda h\} : [g, h] \in H_{k+1}, \lambda \in A \rangle \quad (7)$$

for each $0 \leq i \leq p-1$. Let $\{\lambda_1, \dots, \lambda_s\}$ be a $\hat{\mathbb{Z}}_p$ -basis for A . Let $h_1, \dots, h_t \in G$ be conjugacy class representatives (mod H_{k+1}) for those elements such that $[g_l, h_l] \in z_{k+1}H_k$ for some $g_l \in G$; fix also such g_l . Then (7) takes the form

$$K_2(I(\sigma_i)/I'(\sigma_i)) = M_i + \langle \{g_l, 1 - X(\sigma_i)\lambda_j h_l\} : 1 \leq j \leq s, 1 \leq l \leq t \rangle; \quad (8)$$

where

$$M_i = \langle \{g, 1 - \lambda X(\sigma_i)h\} : [g, h] \in H_k, \lambda \in A \rangle. \quad (9)$$

Step 2C. Now assume that $i < p-1$; and consider the relative exact sequence

$$K_2(I(\sigma_i)/I'(\sigma_{i+1})) \rightarrow K_2(I(\sigma_i)/I'(\sigma_i)) \xrightarrow{\partial} K_1(I(\sigma_{i+1})/I'(\sigma_{i+1}))$$

(recall that $I'(\sigma_i) = I(\sigma_{i+1})$). By (2) (and [24, Corollary 2.6]):

$$K_1(I(\sigma_{i+1})/I'(\sigma_{i+1})) \cong H_0(G; A/p[G/H_{k+1}]), \quad (10)$$

where G acts by conjugation. Furthermore, for $1 \leq j \leq s$, $1 \leq l \leq t$,

$$\begin{aligned} \partial(\{g_l, 1 - X(\sigma_i)\lambda_j h_l\}) &= [g_l, 1 - X(\sigma_i)\lambda_j h_l] = 1 - X(\sigma_i)\lambda_j(g_l h_l g_l^{-1} - h_l) \\ &= 1 + X(\sigma_i)(1 - z_{k+1})\lambda_j h_l = 1 + X(\sigma_{i+1})\lambda_j h_l \pmod{I'(\sigma_{i+1})} \end{aligned}$$

(recall that $[g_l, h_l] \in z_{k+1}H_k$). By (10), these elements are all independent in $K_1(I(\sigma_{i+1})/I'(\sigma_{i+1}))$. So by (8) and (9),

$$\begin{aligned} \text{Im}[K_2(I(\sigma_i)/I'(\sigma)) \rightarrow K_2(I(\sigma_i)/I'(\sigma_i))] \\ &= \text{Im}[K_2(I(\sigma_i)/I'(\sigma_{i+1})) \rightarrow K_2(I(\sigma_i)/I'(\sigma_i))] \\ &= M_i = \langle \{g, 1 - \lambda X(\sigma_i)h\} : [g, h] \in H_k, \lambda \in A \rangle; \quad (11) \end{aligned}$$

all elements in M_i lift (using (2)) to $K_2(I(\sigma_i)/I'(\sigma)) \subseteq K_2(I(\sigma)/I'(\sigma))$.

Step 2D. By (8) and (11) (and (6)),

$$K_2(I(\sigma)/I'(\sigma)) = M + \langle \{g, 1 - \lambda X(\sigma)(1 - z_{k+1})^{p-1}h\} : [g, h] \in z_{k+1}H_k, \lambda \in A \rangle \quad (12)$$

where

$$\begin{aligned} M &= \langle \{g, 1 - \lambda X(\sigma)(1 - z_{k+1})^i h\} : 0 \leq i \leq p-1, [g, h] \in H_k, \lambda \in A \rangle \\ &= \langle \{g, 1 - \lambda X(\sigma)h\} : [g, h] \in H_k, \lambda \in A \rangle. \end{aligned}$$

(Note that $X(\sigma)^2 = 0$ in $I(\sigma)/I'(\sigma)$.) We want to show that $K_2(I(\sigma)/I'(\sigma)) = M$. Fix $\lambda \in A$ and $g, h \in G$ such that $[g, h] \in z_{k+1}H_k$, and set $u = 1 - X(\sigma)\lambda h$. Then

$$1 - X(\sigma)\lambda(1 - z_{k+1})^{p-1}h = \prod_{i=0}^{p-1} (1 - X(\sigma)\lambda z_{k+1}^i h) = \prod_{i=0}^{p-1} g^i u g^{-i} \in AG/I'(\sigma)$$

by (2) ($I(\sigma)/I'(\sigma) \cong A/p[G/H_k]$). So by Lemma 1.2,

$$\begin{aligned} \{g, 1 - x(\sigma)\lambda(1 - z_{k+1})^{p-1}h\} &= \{g, u \cdot gug^{-1} \cdots g^{p-1}ug^{1-p}\} \\ &= \{g^p, u\} + (p-1)\{u, u\} + \sum_{j=1}^{p-1} \{g^j u g^{-j}, u\}. \end{aligned}$$

By definition, $\{g^p, u\} \in M$. For any $0 \leq j \leq p-1$:

$$\begin{aligned} \{g^j u g^{-j}, u\} &= \{1 - X(\sigma)\lambda z_{k+1}^j h, 1 - X(\sigma)\lambda h\} \\ &= \left\{ 1 - (1 - z), 1 - X(\sigma) \frac{X(\sigma)}{1 - z} \lambda^2 z_{k+1}^j h^2 \right\} \in M \end{aligned}$$

(see [17, Lemma 2.2] for the last step). So from (12) we now get that $K_2(I(\sigma)/I'(\sigma)) = M$; and this finishes the proof of (3).

Step 3. Now fix some $i \geq 1$. For any $0 \leq r \leq m-1$, (3) applied to $\sigma = (0; r, i)$ says that

$$\begin{aligned} K_2(p^r(1-z)^i AG / \langle p^{r+1}(1-z)^i, (1-z)^{i+1} \rangle) \\ = \langle \{g, 1 - \lambda p^r(1-z)^i h\} : [g, h] \in \langle z \rangle, \lambda \in A \rangle. \end{aligned}$$

For any such g , h , and λ , note that (in AG)

$$[g, 1 - \lambda p^r(1 - z)^i h] \equiv 0; 1 - \lambda p^r(1 - z)^i h \equiv (1 - \lambda(1 - z)^i h)^{p^r} \pmod{(1 - z)^{i+1}AG}.$$

It follows that

$$K_2((1 - z)^i AG / (1 - z)^{i+1} AG) = \langle \{g, 1 - \lambda(1 - z)^i h\} : [g, h] \in \langle z \rangle, \lambda \in A \rangle. \quad (13)$$

Step 4. The rest of the proof is analogous to Step 2B and 2C. Let $\lambda_1, \dots, \lambda_s$ be a $\hat{\mathbb{Z}}_p$ -basis for A , and let $h_1, \dots, h_t \in G$ be conjugacy class representatives for G/z . For $1 \leq l \leq t$, choose $g_l \in G$ so that $[g_l, h_l] = z^{q_l}$, and $1 \leq q_l \leq p^m = |z|$ is minimal. Then by (13),

$$\begin{aligned} K_2((1 - z)^i AG / (1 - z)^{i+1} AG) \\ = N_i + \langle \{g_l, 1 - \lambda_j(1 - z)^i h_l\} : 1 \leq l \leq t, 1 \leq j \leq s \rangle, \end{aligned} \quad (14)$$

where

$$N_i = \langle \{g, 1 - \lambda(1 - z)^i h\} : [g, h] = 1, \lambda \in A \rangle.$$

Consider the exact sequence

$$K_2\left(\frac{(1 - z)^i AG}{(1 - z)^{i+2} AG}\right) \rightarrow K_2\left(\frac{(1 - z)^i AG}{(1 - z)^{i+1} AG}\right) \xrightarrow{\partial} K_1\left(\frac{(1 - z)^{i+1} AG}{(1 - z)^{i+2} AG}\right). \quad (15)$$

For any j, l :

$$\partial(\{g_l, 1 - \lambda_j(1 - z)^i h_l\}) = [g_l, 1 - \lambda_j(1 - z)^i h_l] = 1 + q_l \lambda_j (1 - z)^{i+1} h_l.$$

By Lemma 1.3, these elements are independent in

$$K_1((1 - z)^{i+1} AG / (1 - z)^{i+2} AG) \cong H_0(G; A/p^m[G])$$

and have order p^m/q_l (q_l is a power of p). Furthermore, for each j and l , $[g_l^{p^m/q_l}, h_l] = 1$, and so

$$p^m/q_l \cdot \{g_l, 1 - \lambda_j(1 - z)^i h_l\} \in N_i.$$

So by (14), and the exactness of (15),

$$\text{Im} \left[K_2 \left(\frac{(1-z)^i AG}{(1-z)^{i+2} AG} \right) \rightarrow K_2 \left(\frac{(1-z)^i AG}{(1-z)^{i+1} AG} \right) \right] = \text{Ker} (\partial) = N_i.$$

Every element of N_i lifts to $K_2((1-z)^i AG) \subseteq K_2((1-z)AG)$. Thus, for any $i \geq 1$,

$$K_2((1-z)^i AG) = K_2((1-z)^{i+1} AG) + \langle \{g, 1 - \lambda(1-z)^i h\} : gh = hg, \lambda \in A \rangle. \quad (16)$$

By induction, for any $N > 1$,

$$K_2((1-z)AG) = K_2((1-z)^N AG) + \langle \{g, 1 - \lambda(1-z)^i h\} : gh = hg, \lambda \in A, 1 \leq i < N \rangle. \quad (17)$$

Let $p^k = \exp(G)$, and recall that $|z| = p^m$. Then $p(1-z) \mid (1-z)^{p^m}$, and so

$$1 + (1-z)^{(k+1)p^m} AG \subseteq 1 + p^{k+1}(1-z)AG \subseteq \{(1 + (1-z)\xi)^{p^k} : \xi \in AG\}.$$

Thus, for any commuting $h, g \in G$, any $\lambda \in A$, and any $i \geq (k+1)p^m$:

$$\{g, 1 - \lambda(1-z)^i h\} = \{g, (1 - (1-z)\xi)^{p^k}\} = \{g^{p^k}, 1 - (1-z)\xi\} = 0. \quad (\text{some } \xi \in AG).$$

By (16), for any $N > (k+1)p^m$, $K_2((1-z)^{(k+1)p^m} AG) = K_2((1-z)^N AG)$; and so

$$K_2((1-z)^{(k+1)p^m}) = \varprojlim_N K_2((1-z)^{(k+1)p^m} AG / (1-z)^N AG) = 0 \quad (18)$$

Equation (17) now takes the form

$$K_2((1-z)AG) = \langle \{g, 1 - \lambda(1-z)^i h\} : gh = hg, \lambda \in A, 1 \leq i < (k+1)p^m \rangle.$$

Furthermore, it suffices to take λ belonging to some $\hat{\mathbb{Z}}_p$ -basis for A . This shows that $K_2((1-z)AG)$ is generated by a finite set of elements of finite order, and is hence finite. \square

With some more work, one can in fact show that $K_2(AG, (1-z)AG)$ is

generated by symbols $\{g, 1 - \lambda(1 - z)h\}$, where $gh = hg$ in G and λ lies in any fixed $\hat{\mathbb{Z}}_p$ -basis for A .

One easy consequence of Theorem 1.4 is:

THEOREM 1.5. *For any prime p , any unramified p -ring A , and any p -group G , $K_2(AG)$ is finite.*

Proof. Fix some $1 \neq z \in Z(G)$. Then $K_2(AG, (1 - z)AG)$ is finite by Theorem 1.4. We may assume inductively that $K_2(A[G/z])$ is finite; and so $K_2(AG)$ is also finite. \square

In fact, using the results in [17], this can be extended to arbitrary finite G . Whether it is true for arbitrary \mathbb{Z}_p -orders, we do not know.

Section 2

Theorem 1.4 gives a set of generators for $\text{Ker}(K_2(A\alpha))$, when $\alpha: \tilde{G} \twoheadrightarrow G$ is a central extension of p -groups with cyclic kernel. In this section, we study $\text{Coker}(K_2(A\alpha))$ when $\text{Ker}(\alpha) \subseteq Z(\tilde{G})$. This problem was studied in [19]: $\text{Coker}(K_2(A\alpha))$ is described there for an arbitrary surjection α , but only up to a mysterious contribution by $H_3(G)$. What we show here is that the $H_3(G)$ contribution vanishes when α is a central extension.

PROPOSITION 2.1. *Let p be any prime, let A be an unramified p -ring, and let $\alpha: \tilde{G} \twoheadrightarrow G$ be any central extension of p -groups (i.e., $\text{Ker}(\alpha) \subseteq Z(\tilde{G})$). Then there is an exact sequence*

$$0 \rightarrow \text{Coker}(H_2(\alpha)) \xrightarrow{T_\alpha} \text{Coker}[K_2(A\alpha): K_2(A\tilde{G}) \rightarrow K_2(AG)] \\ \xrightarrow{\Gamma_2^*(\alpha)} H_1(G; AG) / \langle g \otimes \lambda h : [\alpha^{-1}g, \alpha^{-1}h] = 1 \rangle.$$

Here, T_α is included by the usual inclusion $H_2(G) \rightarrow K_2(AG) / \{-1, G\}$, and $\Gamma_2^*(\alpha)$ is induced by the homomorphism

$$\Gamma_2^*(G): K_2(AG) \rightarrow H_1(G; AG) / \langle g \otimes \lambda g^n : g \in G, \lambda \in A, n \in \mathbb{Z} \rangle$$

of [19, Theorem 3.6]. In particular, for any $g \in G$, $H = Z_G(g)$, and any

$$u \in (AH)^*,$$

$$\Gamma_2^*(\alpha)(\{g, u\}) = g \otimes \Gamma_H(u) \in H_1(G; AG) / \langle g \otimes \lambda h : [\alpha^{-1}g, \alpha^{-1}h] = 1 \rangle.$$

Proof. Define the group \hat{G} and the order \mathfrak{A} to be the pullbacks:

$$\begin{array}{ccc} \hat{G} & \xrightarrow{r_1} & \tilde{G} \\ \downarrow r_2 & & \downarrow \alpha \\ \tilde{G} & \xrightarrow{\alpha} & G \end{array} \quad \begin{array}{ccc} \mathfrak{A} & \xrightarrow{\hat{r}_1} & A\tilde{G} \\ \downarrow \hat{r}_2 & & \downarrow A\alpha \\ A\tilde{G} & \xrightarrow{A\alpha} & AG. \end{array}$$

Set

$$I_1 = \text{Ker}[A\hat{G} \xrightarrow{Ar_1} A\tilde{G}], \quad I_2 = \text{Ker}[A\hat{G} \xrightarrow{Ar_2} A\tilde{G}], \quad I = \text{Ker}[A\tilde{G} \xrightarrow{A\alpha} AG].$$

Then $\mathfrak{A} \cong A\hat{G}/(I_1 \cap I_2)$; and so by Lemma 2.4 in [16],

$$\mathfrak{A} \cong A\hat{G}/I_1 I_2.$$

Step 1. By [26, Theorem 4.1],

$$\text{tors}(K_1(A\hat{G})) \cong \mu_A \times \hat{G}^{ab} \times SK_1(A\hat{G}); \quad (1)$$

where μ_A denotes the group of roots of unity in A . We first claim that

$$\hat{G}^{ab} \hookrightarrow K_1(A\hat{G}/I_1 I_2) \cong K_1(\mathfrak{A}) \quad (2)$$

is injective. To see this, let $I(A\hat{G})$ denote the augmentation ideal of $A\hat{G}$. Then $I(A\hat{G})^2 \supseteq I_1 I_2$, and by [19, Proposition 2.2]:

$$A\hat{G}/I(A\hat{G})^2 \cong A \times (A \otimes \hat{G}^{ab}).$$

The isomorphism identifies $g \in \hat{G}^{ab}$ with $(1, 1 \otimes g)$, and so $\hat{G}^{ab} \subseteq K_1(A\hat{G}/I(A\hat{G})^2)$.

Now set $K = \text{Ker}(\alpha) \cong \text{Ker}(r_1)$, and consider the following diagram:

$$\begin{array}{ccccccc} H_2(\hat{G}) & \longrightarrow & H_2(\tilde{G}) & \xrightarrow{\delta_{r_1}} & K & \longrightarrow & \hat{G}^{ab} \xrightarrow{H_1(r_1)} G^{ab} \longrightarrow 0 \\ \downarrow & & \downarrow H_2(\alpha) & & \downarrow Id & & \downarrow H_1(r_2) \downarrow \\ H_2(\tilde{G}) & \xrightarrow{H_2(\alpha)} & H_2(G) & \xrightarrow{\delta^\alpha} & K & \longrightarrow & \tilde{G}^{ab} \longrightarrow G^{ab} \longrightarrow 0. \end{array}$$

The rows are the five-term exact sequences for the extensions $r_1: \hat{G} \twoheadrightarrow \tilde{G}$ and

$\alpha: \tilde{G} \rightarrow G$ (see [8, Corollary VI. 8.2]). It follows that

$$\text{Ker } [H_1(r_1 \times r_2): \hat{G}^{ab} \rightarrow \tilde{G}^{ab} \times \tilde{G}^{ab}] \cong \text{Coker } (H_2(\alpha)). \quad (3)$$

Furthermore, $\delta^{r_1} = \delta^\alpha \circ H_2(\alpha) = 0$, so $\text{Ker } (r_1) \cap [\hat{G}, \hat{G}] = 1$, and

$$SK_1(Ar_1): SK_1(A\hat{G}) \rightarrow SK_1(A\tilde{G}) \quad (4)$$

is injective by [15, Proposition 7].

Step 2. Now define

$$\Gamma_{AG}: K_1(A\hat{G}) \rightarrow H_0(\hat{G}; A\hat{G}); \quad \Gamma_{A\tilde{G}}: K_1(A\tilde{G}) \rightarrow H_0(\tilde{G}; A\tilde{G})$$

as in [20, Theorem 2.7], and recall that they are isomorphisms modulo torsion. By Theorem 1.1 in [19],

$$\Gamma_{A\hat{G}}(1 + I_1 I_2) = \text{Im } [I_1 I_2 \rightarrow H_0(\hat{G}; A\hat{G})]. \quad (5)$$

So $\Gamma_{A\hat{G}}$ induces a homomorphism

$$\Gamma_{\mathfrak{A}}: K_1(\mathfrak{A}) \rightarrow H_0(\hat{G}; \mathfrak{A}).$$

Consider the following diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mu_A \times \hat{G}^{ab} \times SK_1(A\hat{G}) & \longrightarrow & K_1(\mathfrak{A}) & \xrightarrow{\Gamma_{\mathfrak{A}}} & H_0(\hat{G}; \mathfrak{A}) \\ & & \downarrow f & & \downarrow K_1(r_1 \times r_2) & & \downarrow H_0(r_1 \times r_2) \\ 0 & \longrightarrow & [\mu_A \times \tilde{G}^{ab} \times SK_1(A\tilde{G})]^2 & \longrightarrow & [K_1(A\tilde{G})]^2 & \xrightarrow{\Gamma_{A\tilde{G}}} & [H_0(\tilde{G}; A\tilde{G})]^2 \end{array} \quad (6)$$

The bottom row is exact since $\text{Ker } (\Gamma_{A\tilde{G}}) = \text{tors } (K_1(A\tilde{G}))$. The top row is exact at $K_1(\mathfrak{A})$ since by (5), $\text{Ker } (\Gamma_{A\hat{G}}) \rightarrow \text{Ker } (\Gamma_{\mathfrak{A}})$ is onto. By (3) and (4), $\hat{G}^{ab} \supseteq \text{Ker } (f) \cong \text{Coker } (H_2(\alpha))$, and this injects into $K_1(\mathfrak{A})$ by (2). So the top row in (6) is exact, and there is an exact sequence

$$0 \rightarrow \text{Coker } (H_2(\alpha)) \rightarrow \text{Ker } (K_1(r_1 \times r_2)) \rightarrow \text{Ker } (H_0(r_1 \times r_2)). \quad (7)$$

By the Mayer–Victoris sequence for a pullback square,

$$\text{Ker } (K_1(r_1 \times r_2)) \cong \text{Coker } [K_2(A\alpha): K_2(A\tilde{G}) \rightarrow K_2(AG)]. \quad (8)$$

Step 3. The extension $0 \rightarrow I \rightarrow \mathfrak{A} \xrightarrow{\tau_1} AG \rightarrow 0$ is \hat{G} -equivariantly split by the diagonal map. Thus,

$$\text{Ker} [\hat{r}_{1*}: H_0(\hat{G}; \mathfrak{A}) \rightarrow H_0(\tilde{G}; A\tilde{G})] \cong H_0(\tilde{G}; I);$$

and so

$$\begin{aligned} \text{Ker} (H_0(r_1 \times r_2)) &\cong \text{Ker} [H_0(\tilde{G}; I) \rightarrow H_0(\tilde{G}; A\tilde{G})] \\ &\cong \text{Coker} [H_1(\tilde{G}; A\tilde{G}) \rightarrow H_1(\tilde{G}; AG)] \\ &\cong H_1(G; AG) / \langle g \otimes \lambda h : \lambda \in A, [\alpha^{-1}g, \alpha^{-1}h] = 1 \rangle. \end{aligned} \quad (9)$$

Upon substituting (8) and (9) into (7), we get the exact sequence

$$0 \rightarrow \text{Coker} (H_2(\alpha)) \xrightarrow{T_\alpha} \text{Coker} (K_2(A\alpha)) \xrightarrow{\Gamma_2^*(\alpha)} H_1(G; AG) / \langle g \otimes \lambda h : [\alpha^{-1}g, \alpha^{-1}h] = 1 \rangle.$$

That $\Gamma_2^*(\alpha)$ is the reduction of the map $\Gamma_2^*(G)$ of [19] follows since the constructions are identical. By diagram chasing, T_α is seen to be the reduction of the standard inclusion $H_2(G) \rightarrow K_2(AG) / \{-1, G\}$. \square

In fact, in the above situation, $\text{Im} (\Gamma_2^*(\alpha))$ can be described precisely with the help of Theorem 3.6 in [19].

Proposition 2.1 will be applied directly in Section 3, when describing $Cl_1(\mathbb{Z}G)$ for odd p -groups G . But we first note one consequence of particular interest. The next theorem is useful when constructing maps

$$\Gamma_2: K_2(AG) \rightarrow H_1(G; AG) / \langle g \otimes \lambda g \rangle$$

for non-abelian p -groups G (compare with [21]).

THEOREM 2.2. *Let $\alpha: \tilde{G} \twoheadrightarrow G$ be any surjection of p -groups such that $\text{Ker} (\alpha) \cap [\tilde{G}, \tilde{G}] = 1$. Then for any unramified p -ring A , the map*

$$K_2(A\alpha): K_2(A\tilde{G}) \rightarrow K_2(AG)$$

is onto, and its kernel is generated by elements of the form $\{g, 1 + (1 - z)^i h\}$ for $z \in \text{Ker} (\alpha)$, $i \geq 1$, and commuting $g, h \in G$.

Proof. Note first that

$$[\text{Ker}(\alpha), \tilde{G}] \subseteq \text{Ker}(\alpha) \cap [\tilde{G}, \tilde{G}] = 1;$$

so that $\text{Ker}(\alpha) \subseteq Z(\tilde{G})$. The exact sequence

$$H_2(\tilde{G}) \xrightarrow{H_2(\alpha)} H_2(G) \xrightarrow{\delta^\alpha} \text{Ker}(\alpha) \twoheadrightarrow \tilde{G}^{ab} \rightarrow G^{ab} \rightarrow 0$$

(see [8, Corollary VI. 8.2]) shows that $H_2(\alpha)$ is onto. By hypothesis,

$$H_1(G; AG) / \langle g \otimes \lambda h : [\alpha^{-1}g, \alpha^{-1}h] = 1 \rangle = 0;$$

commuting elements in G lift to commuting elements in \tilde{G} . So $K_2(A\alpha)$ is onto by Proposition 2.1.

Now write α as a composite

$$\alpha: \tilde{G} = G_0 \xrightarrow{\alpha_1} G_1 \xrightarrow{\alpha_2} G_2 \twoheadrightarrow \cdots \xrightarrow{\alpha_n} G_n = G;$$

and so that $\text{Ker}(\alpha_j)$ is cyclic for all j . By Theorem 1.4,

$$\text{Ker}(K_2(A\alpha_j)) = \langle \{g, 1 + (1 - z)^i h\} : z \in \text{Ker}(\alpha_j), i \geq 1, g, h \in G_{j-1}, gh = hg \rangle$$

for each j . But all such symbols lift to $K_2(A\tilde{G})$; and so $\text{Ker}(K_2(A\alpha))$ is generated as described. \square

Section 3

We can now derive algorithms for computing the groups $Cl_1(\mathbb{Z}G)[\frac{1}{2}]$ and $SK_1(\mathbb{Z}G)[\frac{1}{2}]$ for finite G . The key extra tool when working with odd torsion is the standard involution on $K_n(\mathbb{Z}G)$ and $K_n(\hat{\mathbb{Z}}_p G)$; for example, this is what was used in [17] to construct natural splittings

$$SK_1(\mathbb{Z}G)[\frac{1}{2}] \cong Cl_1(\mathbb{Z}G)[\frac{1}{2}] \oplus \sum_{2 < p \mid |G|} SK_1(\hat{\mathbb{Z}}_p G).$$

Recall that for any group G and any commutative ring R , an antiinvolution $x \rightarrow \bar{x}$ on RG is defined by setting

$$\overline{\sum r_i g_i} = \sum r_i g_i^{-1} \quad (r_i \in R, g_i \in G).$$

This extends to an antiinvolution on $GL(RG)$ – defined by setting $\overline{(a_{ij})} = (\bar{a}_{ji})$ – and hence an involution on $K_1(RG)$. Similarly, an antiinvolution on $St(RG)$ is induced by setting $\overline{x_{ij}(a)} = x_{ji}(\bar{a})$ ($a \in RG$); and this restricts to an involution on $K_2(RG)$.

LEMMA 3.1. *For any group ring RG as above, and any commuting units, $u, v \in (RG)^*$, $\overline{\{u, v\}} = \{\bar{v}, \bar{u}\}$. In particular, for any $g \in G$, and $u \in (RG)^*$ such that $gu = ug$, $\overline{\{g, u\}} = \{g, \bar{u}\}$.*

Proof. Recall that $\{u, v\} = [X, Y]$, where $X, Y \in St(RG)$ are arbitrary liftings of $\text{diag}(u, u^{-1}, 1)$ and $\text{diag}(v, 1, v^{-1})$. Then

$$\overline{\{u, v\}} = \overline{[X, Y]} = \bar{Y}^{-1} \bar{X}^{-1} \bar{Y} \bar{X} = \{\bar{v}^{-1}, \bar{u}^{-1}\} = \{\bar{v}, \bar{u}\}.$$

The last statement follows since $\bar{g} = g^{-1}$. \square

The importance of the involution for simplifying the computation of $Cl_1(\mathbb{Z}G)$ follows from:

LEMMA 3.2. *For any odd prime p and any p -group G , the involution on $K_2(\hat{\mathbb{Q}}_p[G])_{(p)}$ is the identity.*

Proof. By [22, Section 2 and 3], for any p -group G and any irreducible $\mathbb{Q}G$ -module V , there are subgroups $K \triangleleft H \subseteq G$ and a faithful $\mathbb{Q}[H/K]$ -representation W such that $V = \text{Ind}_H^G(W)$, $\text{End}_{\mathbb{Q}H}(W) \cong \text{End}_{\mathbb{Q}G}(V)$, and H/K is cyclic. Let $A \subseteq \mathbb{Q}H$ and $B \subseteq \mathbb{Q}G$ denote the corresponding simple summands. Then the induction map restricts to a Morita equivalence from A to B , and hence induces an isomorphism of $K_2(\hat{\mathbb{Q}}_p \otimes_{\mathbb{Q}} A)$ to $K_2(\hat{\mathbb{Q}}_p \otimes_{\mathbb{Q}} B)$. Thus, if

$$S = \{(H, K) : K \triangleleft H \subseteq G, H/K \text{ cyclic}\},$$

then the map

$$\sum \text{Ind}_{H/K}^G : \sum_{(H,K) \in S} K_2(\hat{\mathbb{Q}}_p[H/K]) \rightarrow \hat{\mathbb{Q}}_p[G] \quad (1)$$

is onto. Here, $\text{Ing}_{H/K}^G$ is the composite

$$\text{Ind}_{H/K}^G : K_2(\hat{\mathbb{Q}}_p[H/K]) \xrightarrow{\text{incl}} K_2(\hat{\mathbb{Q}}_p H) \xrightarrow{\text{Ind}_H^G} K_2(\hat{\mathbb{Q}}_p G);$$

where the first map is induced by the inclusion of $\hat{\mathbb{Q}}_p[H/K]$ as a direct summand of $\hat{\mathbb{Q}}_p H$.

The $\text{Ind}_{K/H}^G$ commute with the involution, and so by (1) it suffices to prove the lemma when G is cyclic. If $G \cong C_{p^n}$, write $\hat{\mathbb{Q}}_p G \cong \prod_{i=0}^n F_i$, where $F_i \cong \hat{\mathbb{Q}}_p[\zeta_{p^i}]$ (a field). For each i , the involution inverts elements in μ_{F_i} . So from the isomorphism $K_2(F_i) \cong \mu_{F_i}$ and its naturality with respect to automorphisms of F_i , we get that $\{\bar{u}, \bar{v}\} = -\{u, v\} = \{v, u\}$ for $u, v \in F_i^*$. But $\{\bar{n}, \bar{v}\} = \{v, u\}$ by Lemma 3.1, and so the involution on $K_2(F_i)$, and hence on $K_2(\hat{\mathbb{Q}}_p G)$, is trivial. \square

In fact, Lemma 3.2 also holds for 2-groups, and for arbitrary finite G if $K_2(\hat{\mathbb{Q}}_p G)_{(p)}$ is replaced by $C_p(\mathbb{Q}G)$ (see the definition in the introduction).

The main problem when describing $Cl_1(\mathbb{Z}G)$ for a p -group G is computing the image of $K_2(\hat{\mathbb{Z}}_p G)$ in $K_2(\hat{\mathbb{Q}}_p G)$. Lemma 3.2 shows that when p is odd, it is enough to concentrate attention on $K_2(\hat{\mathbb{Z}}_p G)^+$; and (recall the formula $\{\bar{g}, u\} = \{g, \bar{u}\}$) on $K_1(\hat{\mathbb{Z}}_p G)^+$.

PROPOSITION 3.3. *For any odd prime p , any unramified p -ring A , and any p -group G , Γ_{AG} restricts to an isomorphism*

$$\Gamma_{AG}^+ : K_1(AG)^+ \rightarrow H_0(G; AG)^+.$$

Proof. By [20, Theorem 2.7], there is an exact sequence

$$0 \rightarrow G^{ab} \times SK_1(AG) \rightarrow K_1(AG) \xrightarrow{\Gamma_{AG}} H_0(G; AG) \xrightarrow{\omega} G^{ab} \rightarrow 0 \quad (1)$$

where $\omega(\sum \lambda_i g_i) = \prod g_i^{\text{Tr}(\lambda_i)}$. These maps all commute with the involution; and $(G^{ab})^+ = 0$ by definition. That $SK_1(AG)^+ = 0$ follows from the definition of the isomorphism

$$\Theta_{AG} : SK_1(AG) \rightarrow H_2(G)/H_2^{ab}(G)$$

in [15, diagram on p. 215]. So (1) restricts to an isomorphism

$$\Gamma_{AG}^+ : K_1(AG)^+ \rightarrow H_0(G; AG)^+. \quad \square$$

If A is an unramified p -ring, and G is an abelian p -group, we can now define for any $\lambda \in A$ and $g \in G$ a unit $u(\lambda g) \in (AG)^{**} \cong K_1(AG)^+$ to be the unique element such that $\Gamma_G^+(u(\lambda g)) = \frac{1}{2}\lambda(g + g^{-1})$. If G is an arbitrary p -group and $g \in G$, we let $u(\lambda g) \in (AG)^*$ be the image of $u(\lambda g) \in (AH)^*$, when $H = \langle g \rangle$.

The results of Sections 1 and 2 can now be used to describe $K_2(\hat{\mathbb{Z}}_p G)^+$:

PROPOSITION 3.4. *For any odd prime p , any unramified p -ring A , and any p -group G ,*

$$K_2(AG)^+ = \langle \{g, u(\lambda h)\} : \lambda \in A, g, h \in G, [g, h] = 1 \rangle.$$

Proof. For any G , define an involution on $H_1(G; AG)$ by setting $\overline{g \otimes \lambda h} = g \otimes \lambda h^{-1}$. Define

$$\Delta_G^+ : H_1(G; AG)^+ \rightarrow K_2(AG)^+$$

by setting $\Delta_G^+(g \otimes \frac{1}{2}\lambda(h + h^{-1})) = \{g, u(\lambda h)\}$ for any $\lambda \in A$ and commuting $g, h \in G$.

Fix some G , choose $z \in Z(G)$ of order p , set $H = G/z$, and let $\alpha : G \twoheadrightarrow H$ be the projection. Assume inductively that Δ_H^+ is surjective, and consider the following diagram:

$$\begin{array}{ccccccc} 0 \longrightarrow & \text{Ker}(H_1(A\alpha))^+ & \longrightarrow & H_1(G; AG)^+ & \xrightarrow{H_1(A\alpha)} & H_1(H; AH)^+ & \longrightarrow \text{Coker}(H_1(A\alpha))^+ \longrightarrow 0 \\ & \downarrow f_1 & & \downarrow \Delta_G^+ & & \downarrow \Delta_H^+ & \uparrow \Gamma_2^+ \downarrow f_2 \\ 0 \longrightarrow & \text{Ker}(K_2(A\alpha))^+ & \longrightarrow & K_2(AG)^+ & \xrightarrow{K_2(A\alpha)} & K_2(AH)^+ & \longrightarrow \text{Coker}(K_2(A\alpha))^+ \longrightarrow 0 \end{array} \quad (1)$$

Here, f_1 and f_2 are induced by Δ_G^+ and Δ_H^+ , and Γ_2^+ is the restriction of the homomorphism of Proposition 2.1. For any $\lambda \in A$ and commuting $g, h \in G$,

$$\Gamma_2^+ \circ f_2(g \otimes \frac{1}{2}\lambda(h + h^{-1})) = \Gamma_2^+(\{g, u(h)\}) = g \otimes \Gamma_{AG}(u(h)) = g \otimes \frac{1}{2}\lambda(h + h^{-1});$$

and so f_2 is injective. By Theorem 1.4,

$$\begin{aligned} & \text{Ker}(K_2(A\alpha))^+ \\ &= \langle \{g, (1 - \lambda(1 - z)^i h)(1 - \lambda(1 - z^{-1})^i h^{-1})\} : \lambda \in A, i \geq 1, gh = hg \rangle; \end{aligned}$$

and so by Proposition 3.3 (applied to the $K_1(A[Z_G(g)])^+$):

$$\text{Ker}(K_2(A\alpha))^+ \subseteq \langle \{g, u(\lambda h)\} : \lambda \in A, [g, h] = 1 \rangle = \text{Im}(\Delta_G^+).$$

By diagram chasing in (1), Δ_G^+ is now seen to be onto. \square

It seems quite likely that the homomorphisms Δ_G^+ defined above actually induce isomorphisms

$$HC_1(AG)^+ \cong [H_1(G; AG)/\langle g \otimes \lambda g \rangle]^+ \cong K_2(AG)^+.$$

This is the case at least for abelian p -groups [21, Theorem 3.9].

It remains only to find a description of the image of any $\{g, u(h)\}$ in $K_2(\hat{\mathbb{Q}}_p G)$, when $p > 2$ and G is a p -group. Recall that $K_2(\hat{\mathbb{Q}}_p G)$ is described in terms of norm residue symbol isomorphisms

$$(\cdot, \cdot)_F: K_2(F) \xrightarrow{\cong} \mu_F$$

defined for any finite extension F of $\hat{\mathbb{Q}}_p$ [12, Theorem A.14].

LEMMA 3.5. *Fix an odd prime p and a p -group G ; and let $u(g) \in (\hat{\mathbb{Z}}_p G)^*$ for $g \in G$ be defined as above. Write*

$$\hat{\mathbb{Q}}_p G = \prod_{i=1}^k B_i; \quad B_i = M_{r_i}(F_i),$$

where for each i , $F_i \cong \hat{\mathbb{Q}}_p \zeta_{p^m}$ (a field) for some $m \geq 0$ (see [22]). Let

$$\lambda_G: K_2(\hat{\mathbb{Q}}_p G) \rightarrow \prod_{i=1}^k (\mu_{F_i})_p$$

be the product of the norm residue symbol homomorphisms

$$\lambda_G^i: K_2(B_i) \cong K_2(M_{r_i}(F_i)) \cong K_2(F_i) \xrightarrow{(\cdot, \cdot)} (\mu_{F_i})_p.$$

For each i , let V_i be the irreducible B_i -representation. Then, for any commuting $g, h \in G$,

$$\lambda_G(\{g, u(h)\}) = [\det_{F_i}(g, V_i^h)]_{i=1}^k. \quad (V_i^h = \{x \in V_i: hx = x\}).$$

Proof. Fix some i , set $B = B_i$, $V = V_i$, $F = F_i$, $r = r_i$; and let

$$\alpha: \hat{\mathbb{Q}}_p G \rightarrow B \cong \text{End}_F(V) \cong M_r(F)$$

be the projection. Let m be such that $F \cong \hat{\mathbb{Q}}_p \zeta_{p^m}$. Set $p^m = \exp(G)$, and let

$$f: B \cong M_r(F) \rightarrow M_r(\hat{\mathbb{Q}}_p \zeta_{p^n})$$

be an inclusion. Note that taking norm residue symbols commutes (p is odd) with inclusions of cyclotomic fields: this follows, for example, from the formulas in [2].

Fix commuting $g, h \in G$. Then $\langle g, h \rangle$ is an abelian group of exponent dividing p^n ; and so $f\alpha(g)$ and $f\alpha(h)$ are conjugate (simultaneously) to diagonal matrices:

$$f\alpha(g) \sim \text{diag}(u_1, \dots, u_r), f\alpha(h) \sim \text{diag}(v_1, \dots, v_r) \quad (u_l, v_l \in \langle \zeta_{p^n} \rangle).$$

with

$$u(h) = \sum_j \lambda_j h^j; \quad (\lambda_j \in \hat{\mathbb{Z}}_p)$$

so that

$$K_2(f\alpha)(\{g, u(h)\}) = \prod_{l=1}^r \left\{ u_l, \sum_j \lambda_j v_l^j \right\}.$$

By the formulas of Artin and Hasse [2],

$$\lambda_G^i(\{g, u(h)\}) = \prod_{l=1}^r \left(u_l, \sum_j \lambda_j v_l^j \right)_F = \prod_{l=1}^r u_l^{N_l};$$

where

$$N_l = \frac{1}{p^n} \text{Tr} \left(\log \left(\sum_j \lambda_j v_l^j \right) \right). \quad (\text{Tr}: \hat{\mathbb{Q}}_p \zeta_{p^n} \rightarrow \hat{\mathbb{Q}}_p)$$

Recall that $\Gamma_G(u(h)) = \frac{1}{2}(h + h^{-1})$, where $\Gamma_G = (1 - (1/p)\Phi) \circ \log$, and $\Phi(\sum \lambda_i g_i) = \sum \lambda_i g_i^p$. Thus,

$$\begin{aligned} \log(u(h)) &= \left(1 - \frac{1}{p} \Phi \right)^{-1} \left(\frac{1}{2}(h + h^{-1}) \right) \\ &= \frac{p}{p-1} + \frac{1}{2} \left[(h + h^{-1} - 2) + \frac{1}{p} (h^p + h^{-p} - 2) + \dots \right]. \end{aligned}$$

Hence, for $1 \leq l \leq r$,

$$N_l = \frac{1}{p^n} \operatorname{Tr} \left(\frac{p}{p-1} + \frac{1}{2} \left[(v_l + v_l^{-1} - 2) + \frac{1}{p} (v_l^p + v_l^{-p} - 2) + \cdots \right] \right) \\ = \begin{cases} 1 & \text{if } v_l = 1 \\ 0 & \text{if } v_l \neq 1. \end{cases} \quad (v_l \in \langle \zeta_{p^n} \rangle)$$

It follows that

$$\lambda_G^i(\{g, u(h)\}) = \prod_{v_l=1} u_l = \det_F(g, V^h). \quad \square$$

The main result can now be shown.

THEOREM 3.6. *Let p be an odd prime, and let G be a p -group. Write $\mathbb{Q}G = \prod_{i=1}^k B_i$, where each B_i is a matrix algebra over a field F_i with irreducible representation V_i . Define*

$$\psi_G : H_1(G; \mathbb{Z}G) \rightarrow \prod_{i=1}^k (\mu_{F_i})_p$$

by setting, for any commuting $g, h \in G$,

$$\psi_G(g \otimes h) = [\det_{F_i}(g, V_i^h)]_{i=1}^k.$$

Then $Cl_1(\mathbb{Z}G) \cong \operatorname{Coker}(\psi_G)$ and

$$SK_1(\mathbb{Z}G) \cong \operatorname{Coker}(\psi_G) \oplus (H_2(G)/H_2^{ab}(G)).$$

More precisely, there is a commutative square

$$\begin{array}{ccc} K_2(\hat{\mathbb{Q}}_p G)_{(p)} & \xrightarrow[\cong]{\lambda_G} & \prod_{i=1}^k (\mu_{F_i})_p \\ \partial \downarrow & & \downarrow \text{proj} \\ Cl_1(\mathbb{Z}G) & \xrightarrow[\cong]{\Lambda_G} & \operatorname{Coker}(\psi_G); \end{array}$$

where λ_G is induced by the norm residue symbol, and ∂ is the boundary map in the localization sequence.

Proof. By [20, Theorem 2.1 and 2.2], there is an exact sequence

$$K_2(\hat{\mathbb{Z}}_p G) \xrightarrow{\varphi_G} \text{Coker} \left[K_2\left(\mathbb{Z}\left[\frac{1}{p}\right][G]\right) \rightarrow K_2(\hat{\mathbb{Q}}_p G) \right] \xrightarrow{\partial} Cl_1(\mathbb{Z}G) \rightarrow 0$$

and an isomorphism

$$\text{Coker} \left[K_2\left(\mathbb{Z}\left[\frac{1}{p}\right][G]\right) \rightarrow K_2(\hat{\mathbb{Q}}_p G) \right] \cong K_2(\hat{\mathbb{Q}}_p G)_{(p)} \xrightarrow{\lambda_G} \prod_{i=1}^k (\mu_{F_i})_p.$$

(note that $\mathbb{Z}[1/p][G]$ is a maximal order). Consider the diagram

$$\begin{array}{ccccccc} H_1(G; \mathbb{Z}G)^+ & \xrightarrow{\psi_G^+} & \prod_{i=1}^k (\mu_{F_i})_p & \xrightarrow{\text{proj}} & \text{Coker}(\psi_G) & \longrightarrow & 0 \\ \downarrow \Delta_G^+ & (1) & \cong \uparrow \lambda_G & (2) & \uparrow \Lambda_G & & \\ K_2(\hat{\mathbb{Z}}_p G)^+ & \xrightarrow{\varphi_G^+} & K_2(\hat{\mathbb{Q}}_p G)_{(p)} & \xrightarrow{\partial} & Cl_1(\mathbb{Z}G) & \longrightarrow & 0. \end{array}$$

By Lemma 3.2, $\text{Im}(\varphi_G^+) = \text{Im}(\varphi_G)$; and $\text{Im}(\psi_G^+) = \text{Im}(\psi_G)$ since $\psi_G(g \otimes h) = \psi_G(g \otimes h^{-1})$ by definition. So the rows above are exact. The map Δ_G^+ , defined by setting $\Delta_G^+(g \otimes h) = \{g, u(h)\}$, is onto by Proposition 3.4, and (1) commutes by Lemma 3.5. So there is a unique isomorphism

$$\Lambda_G: Cl_1(\mathbb{Z}G) \xrightarrow{\cong} \text{Coker}(\psi_G)$$

which makes (2) commute.

The exact sequence

$$0 \rightarrow Cl_1(\mathbb{Z}G) \rightarrow SK_1(\mathbb{Z}G) \rightarrow SK_1(\hat{\mathbb{Z}}_p G) \rightarrow 0$$

is naturally split by [17, Theorem 4.8], and

$$SK_1(\hat{\mathbb{Z}}_p G) \cong H_2(G)/H_2^{ab}(G) \cong H_2(G)/\langle g \wedge h : g, h \in G, gh = hg \rangle$$

by [15, Theorem 3]. So

$$SK_1(\mathbb{Z}G) \cong \text{Coker}(\psi_G) \oplus (H_2(G)/H_2^{ab}(G)). \quad \square$$

In [17, Theorem 4.8], the computation of $Cl_1(\mathbb{Z}G)_{(p)}$ for odd p and arbitrary

finite G was reduced to the case of a p -group. More precisely, if C_1, \dots, C_k are conjugacy class representatives for cyclic subgroups in G of order prime to p , and $N_i = N_G(C_i)$, $Z_i = Z_G(C_i)$, and $\mathfrak{P}(Z_i)$ is the set of p -subgroups, then

$$Cl_1(\mathbb{Z}G)_{(p)} \cong \sum_{i=1}^k H_0\left(N_i/Z_i; \varinjlim_{H \in \mathfrak{P}(Z_i)} Cl_1(\mathbb{Z}H)\right). \quad (3.7)$$

Here, the limits are taken with respect to inclusion and conjugation among subgroups.

This direct sum decomposition is somewhat awkward, and hence a more direct description of $Cl_1(\mathbb{Z}G)_{(p)}$ seems also desirable. In fact, one can define homomorphisms

$$\psi_G: H_1(G; \mathbb{Z}G) \rightarrow \prod_{i=1}^k (\mu_{F_i}), \quad \left(\mathbb{Q}G \cong \prod_{i=1}^k B_i, F_i = Z(B_i) \right)$$

for arbitrary finite G , such that $Cl_1(\mathbb{Z}G)[\frac{1}{2}] \cong \text{Coker}(\psi_G)[\frac{1}{2}]$. But alone the definition of ψ_G become quite complicated as soon as we start working with non- p -groups; and the most efficient way of describing $Cl_1(\mathbb{Z}G)[\frac{1}{2}]$ for concrete G does seem to be by means of (3.7) above, together with Theorem 3.6. Some techniques for calculating with the help of (3.7) are presented in [17, Section 5].

Section 4

Theorem 3.6 reduces the calculation of $Cl_1(\mathbb{Z}G)$, for an odd order p -group G , to a straightforward combinatorial algorithm. We now give some examples to illustrate how this works in practice. Examples of calculations for abelian G are presented in [1]; and for non-abelian G of order p^3 , $Cl_1(\mathbb{Z}G)$ is calculated in [19, Theorem 7.5] using a weaker form of the theorem. So here we take some non-abelian groups of order p^4 to give a sample of some of the techniques which can be used. Throughout this section, p denotes a fixed odd prime.

Note first that for any p -group G and any commuting $g, h \in G$,

$$\psi_G(g \otimes g) = 0; \quad \psi_G(g \otimes h^n) = \psi_G(g \otimes h) \quad (\text{if } p \nmid n);$$

and

$$\psi_G(aga^{-1} \otimes aha^{-1}) = \psi_G(g \otimes h) \quad (\text{any } a \in G).$$

Thus, when describing $\text{Im}(\psi_G)$, it suffices to consider $\psi_G(g \otimes h)$ as h runs through a set of \mathbb{Q} -conjugacy class representatives in G , and g a set of generators for $Z_G(h)/h$.

An irreducible representation V of G will be described by listing eigenvalues for the actions of various group elements on V – or, when necessary, by describing the irreducible components of $V|H$ for some appropriate $H \subseteq G$.

Finally, note that when $|G| = p^4$, then $SK_1(\hat{\mathbb{Z}}_p G) = 0$ by [15, Proposition 23]. So $SK_1(\mathbb{Z}G) = Cl_1(\mathbb{Z}G)$ in this case.

PROPOSITION 4.1. *Assume $G \cong H \times C_p$, where H is non-abelian, $|H| = p^3$, and $\exp(H) = p$. Then*

$$SK_1(\mathbb{Z}G) \cong (\mathbb{Z}/p)^{(p^2+3p-6)/2}.$$

Proof. Fix generators $a, b \in H$ and $c \in C_p$; and set $z = [a, b]$. Then $Z(G) = \langle z, c \rangle$, and for any $g \in G \setminus Z(G)$, $Z_G(g) = \langle Z(G), g \rangle$. Set $\zeta = \zeta_p$, and note that

$$\mathbb{Q}[G] \cong \mathbb{Q} \times \prod^{p^2+p+1} \mathbb{Q}[\zeta] \times \prod^p M_p(\mathbb{Q}[\zeta]).$$

The following table describes $\psi = \psi_G$. Here, $(H^{ab})^*$ denotes the set of irreducible complex characters of H^{ab} , and $(*)$ for eigenvalues means that all powers of ζ occur.

Representation Indexed by E' val (a, b, c, z)	$U \cong \mathbb{Q}\zeta$ — $(\zeta, 1, 1, 1)$	$V_m \cong \mathbb{Q}\zeta$ $0 \leq m < p$ $(\zeta^m, \zeta, 1, 1)$	$W_\chi \cong \mathbb{Q}\zeta$ $\chi \in (H^{ab})^*$ $(\chi(a), \chi(b), \zeta, 1)$	$X_m \cong (\mathbb{Q}\zeta)^p$ $0 \leq m < p$ $(*, *, \zeta^m, \zeta)$
$\psi(a \otimes cz^{-i})$	ζ	ζ^m	1	1
$\psi(b \otimes cz^{-i})$	1	ζ	1	1
$\psi(a \otimes (1 - c))$	1	1	$\chi(a)$	1
$\psi(b \otimes (1 - c))$	1	1	$\chi(b)$	1
$\psi(c \otimes 1)$	1	1	ζ	1
$\psi(G \otimes (1 - z))$	1	1	1	1
$\psi(z \otimes gc^{-i})$	1	1	1	ζ
$\psi(c \otimes gc^{-i})$	1	1	ζ (if $\chi(g) = \zeta^i$) 1 (if $\chi(g) \neq \zeta^i$)	ζ^m
$(g \in H \setminus \langle z \rangle)$				
$\psi(c \otimes \xi)$	1	1	1	ζ^m

Here, in the last line, $\xi = a(1 + b + \cdots + b^{p-1}) - b(c + c^2 + \cdots + c^{p-1})$. By inspection,

$$SK_1(\mathbb{Z}G) \cong \text{Coker}(\psi) \cong (\mathbb{Z}/p)^{p-1} \oplus (\mathbb{Z}/p[C_p \times C_p]/I) \oplus (\mathbb{Z}/p)^{p-2}, \quad (1)$$

where $I \subseteq \mathbb{Z}/p[C_p \times C_p]$ is the ideal generated by elements $(\sum_{g \in K} g)$ for subgroups $K \subseteq C_p \times C_p$ of order p .

Write $C_p \times C_p = \langle g \rangle \times \langle h \rangle$, and let $J = \langle 1 - g, 1 - h \rangle \subseteq \mathbb{Z}/p[C_p \times C_p]$ denote the Jacobson radical. Then

$$I = \langle (1 - g)^{p-1}, (1 - h)^{p-1}; (1 - g^i h)^{p-1}; 1 \leq i \leq p-1 \rangle.$$

Furthermore, for any $1 \leq i \leq p-1$:

$$(1 - g^i h) = 1 - [1 - (1 - g)]^i [1 - (1 - h)] \equiv i(1 - g) + (1 - h) \pmod{J^2}$$

and so

$$\begin{aligned} (1 - g^i h)^{p-1} &\equiv \sum_{k=0}^{p-1} \binom{p-1}{k} i^k (1 - g)^k (1 - h)^{p-1-k} \\ &= \sum_{k=0}^{p-1} (-i)^k (1 - g)^k (1 - h)^{p-1-k} \pmod{J^p}. \end{aligned}$$

The determinant of $[(-i)^k]_{i,k=1}^{p-2}$ is invertible over \mathbb{Z}/p (a van der Monde determinant), and so

$$I + J^p = \langle (1 - g)^k (1 - h)^{p-1-k}; 0 \leq k \leq p-1 \rangle = J^{p-1}.$$

But $J^{2p-1} = 0$, and hence this implies that $I = J^{p-1}$. So as a group,

$$\begin{aligned} \mathbb{Z}/p[C_p \times C_p]/I &\cong (\mathbb{Z}/p)^{1/2p(p-1)} \text{ with basis} \\ \{(1 - g)^i (1 - h)^j; i, j \geq 0, i + j < p-1\}. \end{aligned}$$

The result now follows from (1). \square

In the above example, the fact that $[G, G]$ was central helped to keep the description of ψ_G simple. The next example illustrates additional complexities which can arise when this is no longer the case. First, a lemma is needed.

LEMMA 4.2. *Let G be cyclic of order p^n ($n \geq 1$) with generator $g \in G$.*

Then, for any

$$0 \neq \alpha = \sum_{i=0}^{p^n-1} a_i g^i \in \mathbb{Z}/p[G] \quad (a_i \in \mathbb{Z}/p);$$

$\mathbb{Z}/p[G]/(\alpha) \cong (\mathbb{Z}/p)^k$ (as groups), where

$$k = \min \left\{ m \geq 0: \sum_{i=0}^{p^n-1} \binom{i}{m} a_i \neq 0 \text{ in } \mathbb{Z}/p \right\}.$$

Proof. By direct calculation,

$$\begin{aligned} \alpha &= \sum_{i=0}^{p^n-1} a_i g^i = \sum_{i=0}^{p^n-1} a_i (1 + (g-1))^i = \sum_{i=0}^{p^n-1} a_i \sum_{m=0}^i \binom{i}{m} (g-1)^m \\ &= \sum_{m=0}^{p^n-1} \left(\sum_{i=0}^{p^n-1} \binom{i}{m} a_i \right) (g-1)^m. \end{aligned} \quad (1)$$

Recall that $\mathbb{Z}/p[G]$ is a local ring with maximal ideal generated by $(g-1)$. So if k is defined as above, then $\alpha = (g-1)^k u$ for some unit u in $\mathbb{Z}/p[G]$, and

$$rk[\mathbb{Z}/p[G]/(\alpha)] = rk[\mathbb{Z}/p[G]/(g-1)^k] = k. \quad \square$$

PROPOSITION 4.3. Set $H = \langle a \rangle \times \langle b \rangle \times \langle c \rangle = C_p^3$, $K = \langle x \rangle \cong C_p$, and let G be any extension of the form

$$1 \rightarrow H \rightarrow G \rightarrow K \rightarrow 1$$

such that

$$xax^{-1} = ab, \quad xbx^{-1} = bc, \quad xcx^{-1} = c.$$

Then

$$SK_1(\mathbb{Z}G) \cong Cl_1(\mathbb{Z}G) \cong (\mathbb{Z}/p)^{3(p-1)/2}.$$

Proof. The action of x on $\mathbb{Q}H$ fixes $\mathbb{Q}[H/\langle b, c \rangle]$, and permutes the other $p^2 + p$ summands freely. Thus,

$$\mathbb{Q}G \cong \mathbb{Q}[G^{ab}] \times \prod_{i=1}^{p+1} M_p(\mathbb{Q}[\zeta]);$$

where $\zeta = \zeta_p$. The following table presents ψ_G , where the nonabelian representations are described by their restrictions to H :

Representation	$U \cong \mathbb{Q}\zeta$	$V_m \cong \mathbb{Q}\zeta$	$W \cong (\mathbb{Q}\zeta)^p$	$X_m \cong (\mathbb{Q}\zeta)^p$
Indexed by	—	$0 \leq m < p$	—	$0 \leq m < p$
E'val $(a, b, c; x)$	$(1, 1, 1; \zeta)$	$(\zeta, 1, 1; \zeta^m)$	$(\zeta^r, \zeta, 1)$ $(1 \leq r \leq p)$	$(\zeta^{m+1/2r(r-1)}, \zeta^r, \zeta)$ $(1 \leq r \leq p)$
$\psi(a \otimes c)$	1	ζ	1	1
$\psi(x \otimes c)$	ζ	ζ^m	1	1
$\psi(a \otimes (1 - c))$	1	1	1	ζ^T
$\psi(x \otimes (1 - c))$	1	1	1	$\zeta^u (x^p = c^u)$
$\psi(a \otimes (b - c))$	1	1	1	ζ^m
$\psi(c \otimes b)$	1	1	1	ζ
$\psi(b \otimes ac^{-i})$	1	1	ζ	$\zeta^{R(i-m)}$
$\psi(c \otimes ac^{-i})$	1	1	1	$\zeta^{S(i-m)}$
$\psi(c \otimes gx)(g \in H)$	1	1	1	$\begin{cases} \zeta & ((gx)^p = 1) \\ 1 & ((gx)^p \neq 1). \end{cases}$

Here, $T = \sum_{r=1}^p \frac{1}{2}r(r-1)$;

$$R(i) = \sum \{r: 1 \leq r \leq p, \frac{1}{2}r(r-1) \equiv i \pmod{p}\};$$

$$S(i) = \# \{r: 1 \leq r \leq p, \frac{1}{2}r(r-1) \equiv i \pmod{p}\}.$$

Note that solutions to $\frac{1}{2}r(r-1) \equiv i$ come in pairs $\{r, p+1-r\}$ (unless $r = (p+1)/2$). This shows that for all i ,

$$R(i) = \frac{p+1}{2} S(i) \equiv \frac{1}{2}S(i) \pmod{p}.$$

Identify $\prod_{X_m} \langle \zeta \rangle$ with $\mathbb{Z}/p[C_p]$, by identifying X_m with g^m for some generator g of C_p . Then

$$SK_1(\mathbb{Z}G) \cong \text{Coker}(\psi_G)$$

$$\cong (\mathbb{Z}/p)^{p-1} \oplus \left(\mathbb{Z}/p[C_p] / \left\langle \sum_m g^m, \sum_m mg^m, \sum_m S(i-m)g^m \text{ (any } i) \right\rangle \right)$$

$$\cong (\mathbb{Z}/p)^{p-1} \oplus \mathbb{Z}/p[C_p]/I,$$

where I is the ideal generated by

$$\alpha = \sum_m S(m)g^{-m} = \sum_{k=1}^p g^{-1/2k(k-1)}.$$

By Lemma 4.2, we will be done upon showing that

$$\sum_{k=1}^p \binom{\frac{1}{2}k(k-1)}{n} \begin{cases} \equiv 0 \\ \not\equiv 0 \end{cases} \pmod{p} \quad \begin{cases} \text{for } 0 \leq n < \frac{p-1}{2} \\ \text{for } n = \frac{p-1}{2} \end{cases} \quad (1)$$

But the sum is a polynomial in k (over \mathbb{Z}/p) of degree exactly $2n$; and (1) follows since

$$p-1 = \min \left\{ m \geq 0 : \sum_{k=1}^p k^m \not\equiv 0 \pmod{p} \right\}. \quad \square$$

The groups covered above turn out to be the most difficult cases for computing $SK_1(\mathbb{Z}G)$ when $|G| = p^4$. In fact, all other groups of order p^4 are covered by the following proposition (this can easily be checked directly, but also follows from the classification in [9, section III.12]).

PROPOSITION 4.4. *Assume that G is non-abelian of order p^4 , and that there is a subgroup $H \triangleleft G$ such that $H \cong C_{p^3}$ or $H \cong C_{p^2} \times C_p$. Then*

$$\begin{aligned} SK_1(\mathbb{Z}G) = Cl_1(\mathbb{Z}G) &\cong (\mathbb{Z}/p)^{p-1} && \text{if } G^{ab} \cong C_p \times C_p \\ &\cong (\mathbb{Z}/p)^{2(p-1)} && \text{if } G^{ab} \cong C_{p^2} \times C_p \\ &\cong (\mathbb{Z}/p)^{(p+2)(p-1)/2} && \text{if } G^{ab} \cong C_p \times C_p \times C_p. \end{aligned}$$

Proof. Write

$$\mathbb{Q}G = \mathbb{Q}[G^{ab}] \times M \quad \text{and} \quad \mathbb{Q}H = \mathbb{Q}[H/[G, G]] \times M';$$

where M is a product of rank p matrix algebras over fields. Then the inclusion $M' \subseteq M$ is a sum of inclusions of the form

$$\prod_{r=1}^p \mathbb{Q}\zeta_{p^r} \subseteq M_p(\mathbb{Q}\zeta_{p^r}); \quad \mathbb{Q}\zeta_{p^{r+1}} \subseteq M_p(\mathbb{Q}\zeta_{p^r}) \quad (r = 1, 2).$$

In particular, $K_2(\hat{\mathbb{Q}}_p \otimes_{\mathbb{Q}} M')_{(p)}$ surjects onto $K_2(\hat{\mathbb{Q}}_p \otimes_{\mathbb{Q}} M)_{(p)}$. Since $Cl_1(\mathbb{Z}H) = 0$ [10, Theorems 4.4.1 and 5.1.1], this shows that

$$K_2(\hat{\mathbb{Q}}_p \otimes_{\mathbb{Q}} M)_{(p)} \subseteq \text{Im} [\varphi_G : K_2(\hat{\mathbb{Z}}_p G) \rightarrow K_2(\hat{\mathbb{Q}}_p G)_{(p)}].$$

In other words, if $\mathbb{Q}[G^{ab}] \cong \prod_{i=1}^k F_i$, then

$$SK_1(\mathbb{Z}G) = Cl_1(\mathbb{Z}G) \cong \text{Coker} \left[\text{proj} \circ \psi_G : H_1(G; \mathbb{Z}G) \rightarrow \prod_{i=1}^k (\mu_{F_i})_p \right].$$

If $G^{ab} \cong C_p \times C_p$, with basis $\{a, b\}$, then $\text{Im}(\text{proj} \circ \psi_G)$ is generated by the images of $a \otimes 1$ and $b \otimes 1$, and so $SK_1(\mathbb{Z}G)$ has rank $(p+1) - 2 = p - 1$. If $G^{ab} \cong C_{p^3}$, then there are generators a, b, c such that $c \in Z(G)$, and the computation follows from the table in the proof of Proposition 5.1. The proof when $G^{ab} \cong C_{p^2} \times C_p$ is similar. \square

It is interesting to note that for each of these classes of p -groups, the rank of $Cl_1(\mathbb{Z}G)$ is a polynomial in p . This has already been remarked in the case of abelian p -groups (see [1, Conjecture 5.8]); but is harder to formulate as a precise conjecture in the non-abelian case.

Section 5

As another application of Theorem 1.4, we now study the relationship between the complex Artin cokernel

$$A_{\mathbb{C}}(G) = \text{Coker} \left[\sum \{R_{\mathbb{C}}(H) : H \subseteq G \text{ cyclic}\} \xrightarrow{\text{Ind}} R_{\mathbb{C}}(G) \right]$$

of a finite group G , and $Cl_1(RG)$ for large R .

First, epimorphisms

$$I_{RG} : A_{\mathbb{C}}(G) \twoheadrightarrow Cl_1(RG)$$

are constructed, for G any finite group and R the ring of integers in any number field $K \subseteq \mathbb{C}$ (the identification of K as a subfield of \mathbb{C} is needed when defining I_{RG}). The I_{RG} are shown to be natural with respect to homomorphisms and transfer maps, and then shown to be isomorphisms for sufficiently large R .

The following lemma on norm residue symbols will be needed.

LEMMA 5.1. Fix a prime p , fix extensions $E \supseteq F \supseteq \hat{\mathbb{Q}}_p$, and let $\hat{\mu} \subseteq E^*$ and $\mu \subseteq F^* \cap \hat{\mu}$ be groups of roots of unity. Then the diagram

$$\begin{array}{ccc} K_2(E) & \xrightarrow{(\cdot)_{\hat{\mu}}} & \hat{\mu} \\ \downarrow \text{trf}_F^E & & \downarrow [\hat{\mu}: \mu] \\ K_2(F) & \xrightarrow{(\cdot)_{\mu}} & \mu \end{array} \quad (1)$$

commutes; where $(\cdot)_{\hat{\mu}}$ and $(\cdot)_{\mu}$ are the norm residue symbol homomorphisms.

Proof. Set $n = |\hat{\mu}|$ and $m = |\mu|$. Fix $u \in F^*$ and $v \in E^*$, and let $E(\alpha)/E$ be an extension such that $\alpha^n = u$. The diagram

$$\begin{array}{ccc} E^* & \xrightarrow{\hat{s}} & \text{Gal}(E(\alpha)/E) \\ \downarrow N_{E/F} & & \downarrow \text{res} \\ F^* & \xrightarrow{s} & \text{Gal}(F(\alpha^{n/m})/F) \end{array}$$

commutes by [23, Section XI.3]; where \hat{s} and s are the reciprocity maps and res is induced by restriction. By [23, Proposition XIV.6],

$$\begin{aligned} (u, N_{E/F}(v))_{\mu} &= s(N_{E/F}(v))(\alpha^{n/m})/\alpha^{n/m} \\ &= [\hat{s}(v)(\alpha)/\alpha]^{n/m} = ((u, v)_{\hat{\mu}})^{n/m}. \end{aligned} \quad (2)$$

Since $\text{trf}_F^E(\{u, c\}) = \{u, N_{E/F}(v)\}$ for $u \in F^*$ and $v \in E^*$, this shows that (1) commutes on the subgroup $\{F^*, E^*\} \subseteq K_2(E)$. Furthermore,

$$\text{trf}_F^E(\{F^*, E^*\}) = \{F^*, N_{E/F}(E^*)\} = K_2(F):$$

the last equality is shown in [14, Lemma] when $\text{Gal}(E/F)$ is cyclic, and follows from [6, Chapter VI, §2.2] ($N_{E/F}$ is onto) when $\text{Gal}(E/F)$ is non-abelian simple. Since $K_2(E) \cong \mu_E$ and $K_2(F) \cong \mu_F$ are cyclic [12, Theorem A.14], it follows that $\{F^*, E^*\} \supseteq K_2(E)_{(p)}$ for any prime $p \mid |K_2(F)|$, and hence any $p \mid |\mu|$. So (1) commutes. \square

Now fix a finite group G , and let $K \subseteq \mathbb{C}$ be any splitting field for G : i.e., KG is a product of matrix algebras over K . As in [20, Section 2], we define for each prime p :

$$\begin{aligned} C_p(KG) &= \text{Coker} \left[K_2\left(\mathfrak{M}\left[\frac{1}{p}\right]\right) \rightarrow K_2(\hat{K}_p G) \right] \quad (\hat{K}_p = \mathbb{Q}_p \otimes_{\mathbb{Q}} K) \\ &\cong \text{Coker} [K_2(\mathfrak{M}) \rightarrow K_2(\mathfrak{M}_p)] \quad (\mathfrak{M}_p = \hat{\mathbb{Z}}_p \otimes_{\mathbb{Z}} \mathfrak{M}) \end{aligned}$$

where $\mathfrak{M} \subseteq KG$ is any maximal order. Then $C_p(KG)$ is a p -group for all p (since $K_2(\mathfrak{M}_p)$ is a p -group). Finally, set

$$C(KG) = \sum_p C_p(KG).$$

Write $KG = \prod_{i=1}^k B_i$, where $B_i \cong \text{End}_K(V_i)$ for each i , and V_1, \dots, V_k are the irreducible KG -modules. By results going back to Bass, Milnor, and Serre [5], $C(KG) = 0$ if K has a real embedding. If K is purely imaginary, then there is an isomorphism

$$\lambda_{KG}: C(KG) \xrightarrow{\cong} \prod_{i=1}^k (\mu_K)$$

such that for any prime $\mathfrak{p} \subseteq R$, and any units $u \in K^*$ and $v \in (\hat{K}_{\mathfrak{p}}[G])^*$,

$$\lambda_{KG}(\{u, v\}_{\mathfrak{p}}) = [(u, \det_K(v, V_i))]_{i=1}^k$$

Here, $\{u, v\}$ denotes the image of

$$\{u, v\} \in K_2(\hat{K}_{\mathfrak{p}}[G]) \rightarrow C(KG);$$

and

$$(\cdot, \cdot)_{\mathfrak{p}}: (\hat{K}_{\mathfrak{p}})^* \times (\hat{K}_{\mathfrak{p}})^* \rightarrow \mu_K$$

denotes the norm residue symbol with values in μ_K . See [20, Theorem 2.2] for more details.

Thus, when $K \subseteq \mathbb{C}$ is a splitting field for G and has no real embedding and $KG \cong \prod_{i=1}^k B_i$ as above, an isomorphism \tilde{I}_{KG} from $R_{\mathbb{C}}(G)$ to $C(KG)$ can be defined as the composite

$$\tilde{I}_{KG}: R_{\mathbb{C}}(G) \cong \prod_{i=1}^k \mathbb{Z} \xrightarrow{\prod [1 \mapsto \exp(2\pi i/m)]} \prod_{i=1}^k \mu_K \xrightarrow[\cong]{\lambda_{KG}^{-1}} C(KG) \quad (m = |\mu_K|).$$

In other words, for each $1 \leq i \leq k$, we set

$$I_{KG}([V_i]) = \lambda_{KG}^{-1}([\exp(2\pi i/m)]_i);$$

where $[V_i] \in R_{\mathbb{C}}(G)$ denotes the class of $\mathbb{C} \otimes_K V_i$.

If K is a splitting field for G but has a real embedding, we set $\tilde{I}_{KG} = 0$ ($C(KG) = 0$). If $K \subseteq \mathbb{C}$ is a number field which does not split G , set $n = \exp(G)$ and $L = K(\zeta_n)$, and define

$$\tilde{I}_{KG} = \text{trf}_{KG}^{LG} \circ \tilde{I}_{LG} : R_{\mathbb{C}}(G) \rightarrow C(LG) \rightarrow C(KG).$$

(Note that L is a splitting field for G by [5, Theorem 4.1.1].) This definition of the \tilde{I}_{KG} seems rather artificial; but the following proposition shows that these maps do have all desired naturality properties.

PROPOSITION 5.2. *For any number field $K \subseteq \mathbb{C}$ and any finite group G , \tilde{I}_{KG} is surjective. The \tilde{I}_{KG} are natural in that for any homomorphism $\alpha: \tilde{G} \rightarrow G$ of finite groups, for any $H \subseteq G$, and for any pair $K \subseteq L \subseteq \mathbb{C}$ of number fields, the following diagrams all commute:*

$$\begin{array}{ccccc} R_{\mathbb{C}}(G) & & R_{\mathbb{C}}(\tilde{G}) & \xrightarrow{R_{\mathbb{C}}(\alpha)} & R_{\mathbb{C}}(G) & \xrightarrow{\text{Res}_H^G} & R_{\mathbb{C}}(H) \\ & \swarrow (1) \searrow & \downarrow \tilde{I}_{KG} & (2) & \downarrow \tilde{I}_{KG} & (3) & \downarrow \tilde{I}_{KH} \\ C(LG) & \xrightarrow{\text{trf}_{KG}^{LG}} & C(KG) & \xrightarrow{C(K\alpha)} & C(KG) & \xrightarrow{\text{trf}_{KH}^{KG}} & C(KH) \end{array}$$

Proof. The proposition will be proven in four steps. For finite G and arbitrary $K \subseteq \mathbb{C}$, we regard $K_0(KG) = R_K(G)$ as a subring of $R_{\mathbb{C}}(G)$ in the usual fashion (identifying $[V] \in R_K(G)$ with $[C \otimes_K V] \in R_{\mathbb{C}}(G)$).

Step 1. By construction, \tilde{I}_{KG} is surjective if K splits G . To see that \tilde{I}_{KG} is surjective in general, we must show for any G , and any number fields $K \subseteq L$, that the transfer map

$$\text{trf}_{KG}^{LG} : K_2(\hat{L}_p G) \rightarrow K_2(\hat{K}_p G)$$

is onto for each prime p ($\hat{L}_p = \hat{\mathbb{Q}}_p \otimes_{\mathbb{Q}} L$, etc.).

Write $\hat{K}_p G \cong \prod_{i=1}^k M_{n_i}(D_i)$, where the D_i are division algebras. For each i , set $F_i = Z(D_i)$, the center, and let $E_i \subseteq D_i$ be a maximal subfield. By [3, Corollary 4.15], $K_2(D_i)$ is generated by symbols $\{F_i^*, D_i^*\}$; and hence $K_2(E_i) \rightarrow K_2(D_i)$ is onto by [14, Proposition].

Consider the following square, for each $1 \leq i \leq k$:

$$\begin{array}{ccc} K_1(L \otimes_K E_i) & \xrightarrow{\text{incl}} & K_2(L \otimes_K D_i) \\ \downarrow i' & & \downarrow i_i \\ K_2(E_i) & \longrightarrow & K_2(D_i). \end{array}$$

Here t_i and t'_i are the transfer maps. The square commutes since the two sides are induced by tensoring with the bimodules

$$D_i \otimes_{E_i} (L \otimes_K E_i) \cong L \otimes_K D_i.$$

The map t'_i is the product of the transfer homomorphisms for the field summands of $L \otimes_K E_i$, each of which is onto by [12, Corollary A.15]. So t_i is also onto. But trf_{KG}^{LG} is isomorphic to the sum of the t_i , and is hence surjective.

Step 2. Fix K and G such that K is a totally imaginary splitting field for G . In particular, $K_0(KG) = R_C(G)$. For any finite dimensional (left) KG -module V , define

$$f_V: C(K) \rightarrow C(KG)$$

to be the homomorphism induced by the functor

$$V \otimes_K: K\text{-mod} \rightarrow KG\text{-mod}.$$

If V is irreducible, then f_V is just the Morita equivalence identifying $C(B)$ with $C(K)$, where $B \subseteq KG$ is the simple summand with irreducible representation V . So by definition,

$$\tilde{I}_{KG}([V]) = f_V(\lambda_K^{-1}(\exp(2\pi i/m))); \quad (m = |\mu_K|) \quad (4)$$

where $\lambda_K: C(K) \xrightarrow{\cong} \mu_K$ is induced by the norm residue symbol. Both sides of (4) are additive ($f_{V \oplus W} = f_V + f_W$), so (4) holds for arbitrary V .

Step 3. We can now show the commutativity of triangle (1) above: that $\tilde{I}_{KG} = \text{trf}_{KG}^{LG} \circ \tilde{I}_{LG}$ for any G and any number fields $K \subseteq L \subseteq \mathbb{C}$. It suffices to do this when K and L both are totally imaginary splitting fields for G . In particular, $K_0(KG) = R_C(G)$.

By (4), for any finite dimensional KG -module V ,

$$\begin{aligned} \tilde{I}_{KG}([V]) &= f_V(\lambda_K^{-1}(\exp(2\pi i/m))), \quad (m = |\mu_K|) \\ \text{trf}_{KG}^{LG}(\tilde{I}_{LG}([V])) &= \text{trf}_{KG}^{LG} \circ f_{L \otimes_K V}(\lambda_L^{-1}(\exp(2\pi i/n))); \quad (n = |\mu_L|) \end{aligned}$$

and it remains to check the commutativity of the following diagram:

$$\begin{array}{ccccc} \mu_L & \xrightarrow[\cong]{\lambda_L^{-1}} & C(L) & \xrightarrow{f_L \otimes v} & C(LG) \\ \downarrow n/m & (5) & \downarrow \text{trf}_K^L & (6) & \downarrow \text{trf}_{KG}^{LG} \\ \mu_K & \xrightarrow{\lambda_K^{-1}} & C(K) & \xrightarrow{f_V} & C(KG). \end{array}$$

But (5) commutes by Lemma 5.1, while (6) commutes since the two composites are induced by tensoring with the bimodules

$$KG^{LG} \otimes_{LG} (L \otimes_K V)_L = {}_K G V \otimes_K L_L.$$

Step 4. Now fix a homomorphism $\alpha: \tilde{G} \rightarrow G$ of finite groups, and a subgroup $H \subseteq G$. We must show that (2) and (3) commute for any number field $K \leq \mathbb{C}$. If $L \supseteq K$ is any pair of number fields, then the squares

$$\begin{array}{ccccc} C(L\tilde{G}) & \xrightarrow{C(L\alpha)} & C(LG) & \xrightarrow{\text{trf}_{LH}^{LG}} & C(LH) \\ \downarrow \text{trf}_{K\tilde{G}}^L & & \downarrow \text{trf}_{KG}^{LG} & & \downarrow \text{trf}_{KH}^{LH} \\ C(K\tilde{G}) & \xrightarrow{C(K\alpha)} & C(KG) & \xrightarrow{\text{trf}_{KH}^{KG}} & C(KH) \end{array}$$

commute (just compare bimodules). So by (1) (Step 3), it suffices to prove the commutativity of (2) and (3) when K is a splitting field for \tilde{G} , G and H (and totally imaginary).

Fix such a K ; in particular, $K_0(K\tilde{G}) = R_{\mathbb{C}}(\tilde{G})$ and $K_0(KG) = R_{\mathbb{C}}(G)$. Fix finite dimensional modules V over $K\tilde{G}$ and W over KG . Set

$$x = \lambda_K^{-1}(\exp(2\pi i/|\mu_K|)) \in C(K).$$

Then, by (4),

$$\begin{aligned} \tilde{I}_{KG}(R_{\mathbb{C}}(\alpha)([V])) &= f_{KG \otimes_{K\tilde{G}} V}(x), \\ C(K\alpha) \circ \tilde{I}_{K\tilde{G}}([V]) &= C(K\alpha) \circ f_V(x); \\ \tilde{I}_{KH}(\text{Res}_H^G([W])) &= f_{W|H}(x), \end{aligned}$$

and

$$\text{trf}_{KH}^{KG} \circ \tilde{I}_{KG}([W]) = \text{trf}_{KH}^{KG} \circ f_W(x).$$

So we will be done upon showing that the following triangles commute:

$$\begin{array}{ccc}
 & C(K\tilde{G}) & \\
 f_V \nearrow & \downarrow C(K\alpha) & \\
 C(K) & & \\
 f_{KG} \otimes_{K\tilde{G}} V \searrow & & \\
 & C(KG) &
 \end{array}
 \qquad
 \begin{array}{ccc}
 & C(KG) & \\
 f_W \nearrow & \downarrow \text{trf}_{KH}^{KG} & \\
 C(K) & & \\
 f_{W|H} \searrow & & \\
 & C(KH) &
 \end{array}$$

But they are induced by the following pairs of isomorphic bimodules:

$${}_{KG}(KG \otimes_{K\tilde{G}} V)_K \cong {}_{KG}KG \otimes_{K\tilde{G}} V_K \quad \text{and} \quad {}_{KH}W_K \cong {}_{KH}KG \otimes_{KG} W_K;$$

and we are done. \square

Again fix a finite group G and a number field $K \subseteq \mathbb{C}$, and let $R \subseteq K$ be the ring of integers. Then $Cl_1(RG)$ is described by a localization sequence

$$\sum_p K_2(\hat{R}_p G) \rightarrow C(KG) \xrightarrow{\partial_{RG}} Cl_1(RG) \rightarrow 0$$

(see [20, Theorem 2.1] for details). We now consider the composite

$$R_C(G) \xrightarrow{\tilde{I}_{KG}} C(KG) \xrightarrow{\partial_{RG}} Cl_1(RG).$$

Both maps are natural with respect to induction from subgroups of G . Hence, since $Cl_1(RH) = SK_1(RH) = 0$ for any cyclic $H \subseteq G$ by [1, Theorem 3.3], $\partial_{RG} \circ \tilde{I}_{KG}$ vanishes on any element of $R_C(G)$ induced up from a cyclic subgroup. Thus, $\partial_{RG} \circ \tilde{I}_{KG}$ factors through a homomorphism

$$I_{RG}: A_C(G) \rightarrow Cl_1(RG),$$

where $A_C(G)$ is the Artin cokernel.

THEOREM 5.3. *For any finite group G , and any number field $K \subseteq \mathbb{C}$ with ring of integers R ,*

$$I_{RG}: A_C(G) \rightarrow Cl_1(RG)$$

is surjective. The I_{RG} are natural in that for any homomorphism $\alpha: \tilde{G} \rightarrow G$ of finite groups, any $H \subseteq G$, and any pair $R \subseteq S$ of rings of integers in number fields, the

following diagrams all commute:

$$\begin{array}{ccccc}
 & A_{\mathbb{C}}(G) & & A_{\mathbb{C}}(\tilde{G}) & \xrightarrow{A_{\mathbb{C}}(\alpha)} & A_{\mathbb{C}}(G) & \xrightarrow{\text{Res}_{H}^G} & A_{\mathbb{C}}(H) \\
 & \swarrow I_{SG} & \searrow I_{RG} & \downarrow I_{RG} & & \downarrow I_{RG} & & \downarrow I_{RH} \\
 Cl_1(SG) & \xrightarrow{\text{trf}_{RG}^{SG}} & Cl_1(RG) & Cl_1(R\tilde{G}) & \xrightarrow{Cl_1(A\alpha)} & Cl_1(RG) & \xrightarrow{\text{trf}_{RH}^{RG}} & Cl_1(RH).
 \end{array}$$

Proof. For any R and G , I_{RG} is surjective since \tilde{I}_{KG} and ∂_{RG} both are surjective. The naturality properties follow from the corresponding properties for the \tilde{I}_{KG} (Proposition 5.2), and the naturality of the boundary maps ∂_{RG} in the localization sequences. \square

Now that the I_{RG} have been constructed, we can finally apply Theorem 1.4 to show that they are isomorphisms for sufficiently large R . For any finite G , $a_{\mathbb{C}}(G)$ will denote the complex Artin exponent: the order of $1 \in R_{\mathbb{C}}(G)$ in $A_{\mathbb{C}}(G)$. By Frobenius reciprocity,

$$a_{\mathbb{C}}(G) = \exp(A_{\mathbb{C}}(G));$$

i.e., $a_{\mathbb{C}}(G) \cdot x$ is induced from cyclic subgroups for any $x \in R_{\mathbb{C}}(G)$. By the Artin induction theorem [7, Theorem 39.1], $a_{\mathbb{C}}(G) \mid |G|$.

THEOREM 5.4. *Let G be any finite group, and set $n = a_{\mathbb{C}}(G) \cdot \exp(G)$. Let K be any number field such that $\zeta_n \in K$, and let $R \subseteq K$ be the ring of integers. Then I_{RG} is an isomorphism: $Cl_1(RG) \cong A_{\mathbb{C}}(G)$.*

Proof. This will be shown first for p -groups, then for p -elementary groups, and finally for arbitrary finite groups.

Step 1. Let G be a p -group, and set $p^k = a_{\mathbb{C}}(G)$, $p^m = \exp(G)$, and $q = p^{k+m}$. By Theorem 5.3(1), it will suffice to show that I_{RG} is an isomorphism when $K = \mathbb{Q}\zeta_q$ and $R = \mathbb{Z}\zeta_q$.

Let C_q be a (multiplicative) cyclic group of order q with generator z . Consider the pullback square

$$\begin{array}{ccc}
 \hat{\mathbb{Z}}_p[C_q \times G] & \xrightarrow{\alpha} & \hat{\mathbb{Z}}_p\zeta_q[G] \\
 \downarrow & & \downarrow \\
 \mathbb{Z}_p[(C_q/z^{p^{n+m-1}}) \times G] & \xrightarrow{\beta} & \mathbb{Z}/p[(C_q/z^{p^{n+m-1}}) \times G];
 \end{array} \tag{1}$$

where α is induced by: $\alpha(z) = \zeta_q$. Then $K_2(\beta)$ is onto by [17, Lemma 1.7] if $p > 2$; or if $p = 2$ since the only torsion in $K_1(\hat{\mathbb{Z}}_2[(C_q/z^{p^{n+m-1}}) \times G], 2)$ is (-1) (see [15, Proposition 2]). So by the Mayer–Vietoris sequence for (1), $K_2(\alpha)$ is onto.

Now consider the following commutative diagram:

$$(2) \quad \begin{array}{ccccccc} K_2(\hat{\mathbb{Z}}_p[C_q \times G]) & \longrightarrow & C(\mathbb{Q}[C_q \times G])_{(p)} & R_c(G) \otimes \mathbb{Z}/q & & & \\ \downarrow K_2(\alpha) & & \alpha_* \downarrow & \swarrow \tilde{I}_{K\alpha} & \downarrow \partial \circ \tilde{I}_{K\alpha} & & \\ K_2(\hat{\mathbb{Z}}_p \zeta_q[G]) & \xrightarrow{\varphi_{RG}} & C_p(\mathbb{Q}\zeta_q[G]) & \xrightarrow{\partial} & Cl_1(RG)_{(p)} & \longrightarrow & 0 \end{array} \quad (2)$$

where the bottom row is exact [20, Theorem 2.1]. By Theorem 1.4,

$$\begin{aligned} K_2(\hat{\mathbb{Z}}_p[C_q \times G]) &= K_2(\hat{\mathbb{Z}}_p G) \oplus K_2(\hat{\mathbb{Z}}_p[C_q \times G], (1-z)) \\ &= K_2(\hat{\mathbb{Z}}_p G) \oplus \langle \{h, 1 - (1-z)^i g\}, \{z, 1 - (1-z)^i g\} : \\ &\quad h \in G, g \in C_q \times G, hg = gh, i \geq 1 \rangle. \end{aligned}$$

It follows that

$$K_2(\hat{\mathbb{Z}}_p \zeta_q[G]) = K_2(\hat{\mathbb{Z}}_p G) + X + Y;$$

where with $\zeta = \zeta_q$:

$$X = \langle \{h, 1 - (1-\zeta)^i \zeta^j g\} : h, g \in G, hg = gh, i \geq 1, j \in \mathbb{Z} \rangle$$

and

$$Y = \langle \{\zeta, 1 - (1-\zeta)^i \zeta^j g\} : g \in G, i \geq 1, j \in \mathbb{Z} \rangle.$$

Recall that $p^m = \exp(G)$. Then

$$\exp(X) \mid p^m \quad \text{and} \quad \exp(\varphi_{RG}(K_2(\hat{\mathbb{Z}}_p G))) \mid \exp(C(\mathbb{Q}G)_{(p)}) \mid p^m.$$

Furthermore, by definition,

$$\varphi_{RG}(Y) \subseteq \text{Im} \left[\sum \{C(\mathbb{Q}\zeta_q[H]) : H \subseteq G \text{ cyclic}\} \rightarrow C(\mathbb{Q}\zeta_q[G]) \right].$$

So by diagram (2) (recalling that $q = p^{k+m}$):

$$\begin{aligned} \text{Ker}(\partial \circ \tilde{I}_{KG}) &\subseteq p^k R_c(G) + \text{Im} \left[\sum \{R_c(H) : H \subseteq G \text{ cyclic}\} \rightarrow R_c(G) \right] \\ &= p^k R_c(G) + \text{Ker}[R_c(G) \rightarrow A_c(G)]. \end{aligned}$$

Since $p^k = a_{\mathbb{C}}(G) = \exp(A_{\mathbb{C}}(G))$,

$$\text{Ker}[I_{RG}: A_{\mathbb{C}}(G) \rightarrow Cl_1(RG)] \subseteq p^k A_{\mathbb{C}}(G) = 0;$$

and so I_{RG} is an isomorphism.

Step 2. Now assume that G is p -elementary: $G \cong C_m \times H$ where $p \nmid m$ and H is a p -group. Set $n = a_{\mathbb{C}}(G) \cdot \exp(G)$, fix a number field $K \subseteq \mathbb{C}$ containing ξ_n , and let R be the ring of integers of K . Then

$$\begin{aligned} A_{\mathbb{C}}(G) &\cong \text{Coker} \left[\sum \{R_{\mathbb{C}}(C_m) \otimes R_{\mathbb{C}}(H_0) : H_0 \subseteq H \text{ cyclic}\} \rightarrow R_{\mathbb{C}}(C_m) \otimes R_{\mathbb{C}}(H) \right] \\ &\cong R_{\mathbb{C}}(C_m) \otimes A_{\mathbb{C}}(H) \cong \prod_{i=1}^m A_{\mathbb{C}}(H). \end{aligned}$$

On the other hand, the identification $K[G] \cong \prod^m K[H]$ (each factor corresponding to a character of C_m) induces an inclusion $RG \subseteq \prod^m R[H]$ of index prime to p ; and hence an isomorphism

$$Cl_1(RG)_{(p)} \cong \prod_{i=1}^m Cl_1(RH)_{(p)} \cong \prod_{i=1}^m A_{\mathbb{C}}(H) \cong A_{\mathbb{C}}(G).$$

(see [17, Proposition 1.2]). Since I_{RG} is onto, it must be an isomorphism.

Step 3. Now let G be an arbitrary finite group, set $n = a_{\mathbb{C}}(G) \cdot \exp(G)$, and let R be any ring of integers containing ξ . Let \mathcal{E} be the set of elementary subgroups of G . For any $H \in \mathcal{E}$, $\exp(H) \mid \exp(G)$ and $a_{\mathbb{C}}(H) \mid a_{\mathbb{C}}(G)$, so I_{RH} is an isomorphism by Step 2. Consider the following square, which commutes by Theorem 5.3:

$$\begin{array}{ccc} A_{\mathbb{C}}(G) & \xrightarrow{I_{RG}} & Cl_1(RG) \\ \downarrow \Sigma \text{Res}_H^G & & \downarrow \Sigma \text{trf}_H^G \\ \sum_{H \in \mathcal{E}} A_{\mathbb{C}}(H) & \xrightarrow[\cong]{\Sigma I_{RH}} & \sum_{H \in \mathcal{E}} Cl_1(RH) \end{array}$$

In the language of [10], $A_{\mathbb{C}}(-)$ is a module over the Frobenius functor $R_{\mathbb{C}}(-)$, and hence is detected by restriction to elementary subgroups. So ΣRes_H^G is injective in the above square, and I_{RG} is an isomorphism. \square

By [4, Theorem XI.4.7], for any finite G ,

$$a_{\mathbb{C}}(G) = \prod_{p \mid |G|} a_{\mathbb{C}}(G_p),$$

where G_p is a p -Sylow subgroup. Thus, the description of

$$a_{\mathbb{C}}(G) = \exp(A_{\mathbb{C}}(G)) = \max_R (\exp(Cl_1(RG)))$$

reduces immediately to the p -group case.

If G is a non-cyclic p -group, then there is a surjection $G \twoheadrightarrow C_p \times C_p$ and an induced surjection of $A_{\mathbb{C}}(G)$ onto $A_{\mathbb{C}}(C_p \times C_p)$. This last group is easily checked to be non-zero (see [1, Lemma 5.5] for details). Thus, for any finite G , $A_{\mathbb{C}}(G)$ is p -torsion free if and only if G_p is cyclic, $A_{\mathbb{C}}(G) = 0$ if and only if G is metacyclic, and these in turn imply similar statements about the $Cl_1(RG)$ (and $SK_1(RG)$). In fact, for fixed p and R such that $\xi_p \in R$ (or $\xi_4 \in R$ if $p = 2$), and any G , $Cl_1(RG)_{(p)} = 0$ if and only if G_p is cyclic (see [1, Theorem 3.5]).

A general description of $a_{\mathbb{C}}(G)$ has been given by Gluck [27]. The formula is much more complicated than that for the rational Artin exponent $a_{\mathbb{Q}}(G)$ given by Lam [11]. If G is non-cyclic, and abelian or of exponent p , then $a_{\mathbb{C}}(G) = a_{\mathbb{Q}}(G) = (1/p) |G|$. On the other hand, if G is a semidihedral 2-group, then $a_{\mathbb{C}}(G) = 2$ ($a_{\mathbb{Q}}(G) = 4$); and if p is odd and G a non-abelian group of order p^3 and exponent p^2 , then $a_{\mathbb{C}}(G) = p$ ($a_{\mathbb{Q}}(G) = p^2$).

To end, we note that Theorem 5.3 allows a new interpretation of the following result in [13] (Theorem 1).

COROLLARY 5.5. *Let G be a finite group, and let R be the ring of integers in some number field. Then $Cl_1(RG)$ is generated by induction from elementary subgroups of G .*

Proof. By the Brauer induction theorem, $R_{\mathbb{C}}(G)$, and hence $A_{\mathbb{C}}(G)$ are generated in induction from elementary subgroups of G . The result follows since $I_{RG}: A_{\mathbb{C}}(G) \rightarrow Cl_1(RG)$ is natural and surjective. \square

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