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Free $\mathbb{Q}/\mathbb{Z}$ Actions

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This paper begins a study of the type of object that a manifold with a free $\mathbb{Q}/\mathbb{Z}$ action is. This group is a dense subgroup of S^1 and is locally finite, and hence is a natural place for studying the relation between actions of continuous and discrete groups. Here we shall mainly focus on the aspects of the theory that stem from the locally finite nature of the group; connections with S^1 actions will be the topic of another paper. The type of result we shall be concerned with is the following:

THEOREM. *Let M be a closed even dimensional simply connected manifold. Suppose that for each n, the cyclic group $\mathbb{Z}_n$ acts freely on M. Then the group $\mathbb{Q}/\mathbb{Z}$ also does.*

Note that we do not assert that the $\mathbb{Q}/\mathbb{Z}$ action restricts to any of the original $\mathbb{Z}_n$ actions. That may not be true. It is also possible to study other locally finite groups, such as $\mathbb{Z}[1/p]/\mathbb{Z}$. (This group is the "$p$-Sylow subgroup" of $\mathbb{Q}/\mathbb{Z}$.) The natural analogue of the above theorem holds for this group. In odd dimensions we produce an obstruction to the existence of $\mathbb{Q}/\mathbb{Z}$ actions. This we apply to prove:

COROLLARY. *If Σ is a simply connected $\mathbb{Z}_p$ homology S^{2k+1}, $k \geq 2$, then Σ admits a free PL $\mathbb{Z}[1/p]/\mathbb{Z}$ action. If Σ is smooth and p is odd the action can be taken smooth iff an invariant (related to the smooth structure of Σ, and which can be killed by taking the connected sum with an appropriate homotopy sphere) in $\pi_{2k+1}(G/O)_{(p)}$ lies in the image of $[\mathbb{C}P^k : G/O]_{(p)}$.*

This corollary is clearly saying something about an S^1 action, since $\mathbb{C}P^k = S^{2k+1}/S^1$. Observe that $[\mathbb{C}P^k : G/O]_{(p)}$ is the localization of the normal invariants of $\mathbb{C}P^k$, and S^1 actions correspond to those elements with a vanishing surgery obstruction. This only begins the story of the connections between $\mathbb{Q}/\mathbb{Z}$ and S^1. As I mentioned above, another paper will develop these ideas further.

This paper is organized as follows. The first section will deal with some homotopy theory necessary for the present work, and prove the analogues of these theorems for finite dimensional CW complexes. The next section studies the relevant finiteness obstruction, proves theorems for Hilbert cube manifolds. The obstruction lies in a $\lim^1$ group of Whitehead groups. (It is interesting that here finiteness is largely a K_1 phenomenon.) Section three is devoted to finite dimensional manifolds. The following section gives a method for computing the obstruction from the type of homotopy data given in section one, and, for example, proves the above results. The final section is devoted to remarks and problems.

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§1. Constructing homotopy towers

It is clear that in setting up the basic homotopy theory it is important to have the notion of a homotopy type possessing free $\mathbb{Z}_n$ actions for each n. For various reasons, we shall assume that all spaces are simply connected. (Of course, we do not mean to include spaces that arise by construction, such as quotient spaces and the like.) One difficulty we wish to avoid is the possibility of allowing products with the universal contractible $\mathbb{Z}_n$-space, since this would permit construction of actions on all homotopy types. This we do by positing that *all spaces are finite dimensional* or, occasionally, Hilbert cube manifolds (which are "morally" finite dimensional in view of triangulation [C]). The second difficulty is more serious: should we mean that there is a fixed space of the given homotopy type that has all the $\mathbb{Z}_n$ actions, or should we mean that for each n there is a space of the given homotopy type that has a free $\mathbb{Z}_n$ action? In the Hilbert cube context there is clearly no difference between these notions since simple homotopy equivalent Hilbert cube manifolds are homeomorphic [C] (and since we are in a simply connected context, simplicity of the homotopy equivalence is guaranteed). We now show that this is also the case for finite (dimensional) complexes.

THEOREM 1. *Let X be a simply connected finite complex. If for each n there is a finite dimensional complex X_n homotopy equivalent to X admitting a free $\mathbb{Z}_n$ action, then there is a finite dimensional complex X^* homotopy equivalent to X admitting free $\mathbb{Z}_n$ actions for each n. If each X_n can be chosen to be a finite complex, then so can X^*.*

The relation to $\mathbb{Q}/\mathbb{Z}$ actions is given by the following result:

THEOREM 2. *Let X be a simply connected finite complex. If for each n there is a finite dimensional complex X_n homotopy equivalent to X admitting a free $\mathbb{Z}_n$ action, then there is a finite dimensional complex X^* homotopy equivalent to X admitting a free $\mathbb{Q}/\mathbb{Z}$ action.*

A $\mathbb{Q}/\mathbb{Z}$ action is understood to be with the discrete topology on $\mathbb{Q}/\mathbb{Z}$. Such an action is free if the fixed sets of all elements other than the identity are empty. We shall take up the issue of finiteness for Theorem 2 and the compact Hilbert cube manifold case of it in the next section.

We require a lemma:

LEMMA 1. *If X is a k-dimensional simply connected CW complex, and Π is a finite group acting freely on a finite dimensional complex X^* homotopy equivalent to X, then X/Π is homotopy equivalent to a complex K of dimension at most $k+1$. If X^* is finite, then K can also be taken finite.*

(If X is a Poincaré space, then one can see that X^*/Π is in fact a k-dimensional Poincaré space by a rather different argument. This would also suffice for the applications to manifolds.)

Proof (Jim Davis). We use Wall's finiteness obstruction theory [Wa1] to see that the result is equivalent to showing that the cellular chain complex of X^*/Π is chain equivalent to a $k+1$ dimensional one. Consider the dual chain complex C^{n-*} ($n = \dim X$). It has no homology for $n - * \leq n - k$ since it is chain equivalent as a $\mathbb{Z}$-chain complex to the dual complex of X and X is k-dimensional. Since all the chain groups are projective (in fact, free) we can inductively "roll up" the complex, up to chain equivalence, to obtain one with vanishing groups in low dimensions. (That is, inductively assume that the complex has vanishing groups through some dimension; since it is acyclic through some larger dimension, projectivity implies that the final nontrivial map splits, and one can remove the lowest nonzero group and part of the preceding one.) Now dualize to get the chain equivalence to a $k+1$ dimensional chain complex.

The second statement also follows from Wall's theory. □

Proof of Theorem 1. We shall first give our proof for the second case where the X_n's are assumed to be finite complexes. We shall establish the stronger statement that the actions on the constructed finite complex can be chosen equivariantly homotopy equivalent to the actions on the X_n's. Since all of the X_n's are homotopy equivalent to X, we can, by virtue of the lemma, assume that they are all of some uniform dimension, at most $\dim X+1$. One can embed the quotients in a high dimensional (say, $2\cdot\dim X+3$) Euclidean space. Consider the regular neighborhoods of these complexes. Standard PL topology shows that the universal covers of these complexes are all PL homeomorphic to the regular neighborhood of X. Consequently, this is a finite complex homotopy equivalent to X admitting free actions of $\mathbb{Z}_n$ for all n.

Even if the X_n's are not all finite, the quotients are easily seen to be finitely dominated, so that one can cross with a S^1 and then be in the first case, and then unwrap. □

To prove Theorem 2, we require a few lemmas:

LEMMA 2. *In the above situation, we can assume that the actions of* $\mathbb{Z}_n$ *on* $H_*(X;\mathbb{Z})$ *are trivial.*

Proof. $\operatorname{Aut} H_*(X;\mathbb{Z})$ is virtually torsion free since X is finite. As a result the homomorphisms $\mathbb{Z}_n\to\operatorname{Aut} H_*(X;\mathbb{Z})$ have arbitrarily large kernels as n ranges over all of the natural numbers. Consequently, by restricting the actions to these subgroups, we obtain enough homologically trivial actions □

LEMMA 3. *If the actions of* $\mathbb{Z}_n$ *on* X_n *are homologically trivial, then the quotient spaces are nilpotent complexes.*

This is well known. A proof can be found in [We1]. Note that all of the Sullivan's theorems [Su] on rational homotopy theory hold for the category of nilpotent complexes, not just the simply connected case studied there (with the same proofs, since Postnikov systems exist for nilpotent spaces).

LEMMA 4. $X_n/\mathbb{Z}_n$ *has the same rational homotopy type as* X.

Proof. The cohomology of the quotient is the invariant cohomology, away from primes dividing n, so the projection is an isomorphism by homological triviality. □

LEMMA 5. *There is a bound on the torsion in the homology of* $X_n/\mathbb{Z}_n$ *in terms of that of* X *and* n.

Proof. This follows from the Serre spectral sequence and Lemma 1. □

We now recall a very well known fact:

LEMMA 6 (Koenig's Lemma). *Let T be a based infinite tree with all vertices having finite valence, then there are non-self-intersecting paths from the base point of infinite length.*

Proof. We describe the path. In a tree there is a unique path from the base point to any other vertex, and all edges are part of such a path. This orients all edges. Of the finitely many edges emanating from the basepoint, at least one leads to infinitely many other points. Now move in that direction and remove the part of the tree that is not accessible to this point in an orientation preserving way. Then start again with this tree and this point as basepoint. □

Proof of Theorem 2. Consider the following tree: Vertices will be certain homotopy types of spaces, and will be connected by edges if certain homotopy conditions are fulfilled. The basepoint is the homotopy type of X. All other vertices are the homotopy types of the finitely dominated nilpotent complexes with $\pi_1 = \mathbb{Z}_{n!}$ and the universal cover of the homotopy type of X. A pair of vertices v and w is connected by an edge if for some k, $\pi_1(X(v)) = \mathbb{Z}_{k!}$ and $\pi_1(X(w)) = \mathbb{Z}_{(k\pm 1)!}$, and there is a map from one to the other that induces the inclusion on π_1 and is an isomorphism of higher homotopy groups.

We assert that Koenig's lemma is applicable. All that must be checked is finite valence. This is true since a vertex is only connected to other vertices with fundamental group slightly larger or smaller, and by Lemmas 3–5, and Sullivan's theorem [Su] on finiteness of integral homotopy types of given rational type and torsion bound, there are only finitely many of these. This produces for us a path, $X = v_1 \to v_2 \to \cdots \to v_n \to \cdots$ where each v_i corresponds to a space with $\pi_1 = \mathbb{Z}_{i!}$ and the arrows can be interpreted as maps homotopic to covering projection. By Lemma 1 we can assume that the dimensions of these spaces are uniformly bounded. Let $\mathscr{L}$ be the infinite mapping telescope of these maps. $\mathscr{L}$ is one dimension higher and $\pi_1(\mathscr{L}) = \mathbb{Q}/\mathbb{Z}$. Finally, X is homotopy equivalent to $\tilde{\mathscr{L}}$, the universal cover of $\mathscr{L}$, which has, by covering space theory, a free proper $\mathbb{Q}/\mathbb{Z}$ action and is the space required by the theorem. □

It is an amusing exercise to draw a picture of $\tilde{\mathscr{L}}$ and the $\mathbb{Q}/\mathbb{Z}$ action on it.

§2. Finitising $\mathbb{Q}/\mathbb{Z}$ actions

In this section we study the homotopy theory of free $\mathbb{Q}/\mathbb{Z}$ actions and their finiteness properties. An annoyance for our purposes is the fact that equivariant maps that are unequivariant homotopy equivalences of free $\mathbb{Q}/\mathbb{Z}$ spaces need not be invertible. (The domain action can be properly discontinuous and the range action can have dense orbits.) There are two equivalent ways around this problem: to formally invert such maps or to deal with unequivariant homotopy classes that as equivariant as possible.

PROPOSITION 1. *For a pair of free $\mathbb{Q}/\mathbb{Z}$ spaces, X and Y, the following are equivalent*: *a*) *there is a map $f: X \to Y$ that is a homotopy equivalence and for each n is homotopic to an f_n that commutes with the given $\mathbb{Z}_n$ actions.* *b*) *there is a chain of free $\mathbb{Q}/\mathbb{Z}$ spaces X_i and equivariant maps that are homotopy equivalences $X \leftarrow X_1 \to X_2 \leftarrow \cdots \to Y$.*

Proof. By considering the actions of $\mathbb{Z}_n$ and covering space theory it readily follows that b) implies a). For the reverse direction one can consider the construction $\tilde{\mathcal{L}}$ for the given collection of $\mathbb{Z}_n$ actions. $\tilde{\mathcal{L}}(X) \to X$ is an equivariant map that is a homotopy equivalence, and the action is properly discontinuous on $\tilde{\mathcal{L}}(X)$. The constructions of §1 yield a family of maps from the partial quotients that are compatible with respect to covering maps, so that one can construct a map, using the homotopy extension principle, $\tilde{\mathcal{L}}(X) \to \tilde{\mathcal{L}}(Y)$ which is a chain of length 2. $\square$

We shall say that two free $\mathbb{Q}/\mathbb{Z}$ spaces are *h-equivalent* if either of these equivalent conditions hold.

Now we can formulate our next theorem:

THEOREM 3. *A finitely dominated simply connected complex with a free $\mathbb{Q}/\mathbb{Z}$ action is h-equivalent to a finite complex iff for each n, the $\mathbb{Z}_n$ action is equivariantly homotopy equivalent to an action on a finite complex and an obstruction in* $\lim^1$ *Wh* $(\mathbb{Z}_n)$ *vanishes.*

Here $\mathrm{Wh}(\mathbb{Z}_n)$ is viewed as an inverse system with an arrow given by each divisibility relation and the induced map on groups being induced by transfer. This system has as a cofinal system:

$$\mathrm{Wh}(\mathbb{Z}_1) \leftarrow \mathrm{Wh}(\mathbb{Z}_2) \leftarrow \mathrm{Wh}(\mathbb{Z}_6) \leftarrow \mathrm{Wh}(\mathbb{Z}_{24}) \leftarrow \cdots$$

For a general linearly ordered inverse limit system of abelian groups:

$$\mathbf{A} := A_1 \leftarrow A_2 \leftarrow A_3 \leftarrow A_4 \leftarrow \cdots$$

one can form the inverse limit, which we shall denote $\lim^0$, and the first derived functor $\lim^1$. A convenient way to describe these is as follows: Consider ΠA_i, the *product* of the A_i's. Let d: $\Pi A_i \to \Pi A_i$ be the map that shifts A_{i+1} to A_i. Then there is an exact sequence (this can be taken as a definition):

$$0 \to \lim^0(\mathbf{A}) \to \Pi A_i \xrightarrow{(1-d)} \Pi A_i \to \lim^1(\mathbf{A}) \to 0,$$

i.e., $\lim^0 = \ker(1-d)$ and $\lim^1 = \operatorname{cok}(1-d)$. $\lim^1$ is a measure of the failure of the Mittag–Leffler condition, i.e., measures the failure of the images of successive stages of the sequence to stabilize. In particular, if infinitely many of the A_i are finite, $\lim^1 = 0$.

Remarks. One can redo this theorem for other locally finite groups besides $\mathbb{Q}/\mathbb{Z}$. S. Cappell and J. Shaneson have shown (in unpublished work) that the $\lim^1$ relevant for finitising $\mathbb{Z}[1/2]/\mathbb{Z}$ actions is a sum of infinitely many copies of $\hat{\mathbb{Z}}_2/\mathbb{Z}$, where $\hat{\mathbb{Z}}_2$ is the additive group of 2-adic integers. There are also nonsimply connected versions using the inverse limit system $\pi_1(X/\mathbb{Z}_n)$. We do not pursue this here since the proof is a routine rewriting of the one given here, and the methods of section one only produce $\mathbb{Q}/\mathbb{Z}$ actions for nilpotent spaces, and this is rarely the case for nonsimply connected spaces. (It does seem possible, however, that an appropriate notion of *relative nilpotence* is possible so that for an appropriate class of actions on nonsimply connected spaces, one would be able to redo §1.) Finally we note that the condition of finiteness of the $\mathbb{Z}_n$ actions is equivalent to an obstruction in $\lim^0(\tilde{K}_0(\mathbb{Z}_n))$ vanishing by Wall's finiteness theory [W1].

Definition of the obstruction: Let K_i be *finite* complexes with free $\mathbb{Z}_{i!}$ actions that are equivariantly homotopy equivalent to the action of $\mathbb{Z}_{i!}$ on X. Such exist by hypothesis. Moreover, there is a canonical homotopy equivalence k_i: $K_i/\mathbb{Z}_{i!} \to K_{i+1}/\mathbb{Z}_{i!}$. We shall call the k_i's the *bonding maps* of the homotopy tower. Consider the product of the torsions of the bonding maps, $\Pi\tau(k_i) \in \Pi\mathrm{Wh}(\mathbb{Z}_{i!})$. This is *not* well defined, since one can, for example, change any number of components using the realization theorem for Whitehead torsion. Nonetheless, the image of this element in $\lim^1$, which we'll denote by $o(X)$, is well defined and is the obstruction in the statement of Theorem 3.

PROPOSITION 2. *$o(X)$ is independent of the choices of finite complexes used in its definition, and only depends on the h-type of X. Moreover, if X is h-equivalent to a finite free $\mathbb{Q}/\mathbb{Z}$ complex, $o(X) = 0$.*

Proof. The second statement follows from the first, since one could then choose all the h_i's to be homeomorphisms, whose torsions vanish. To prove the first, consider two choices $\mathbf{K}$ and $\mathbf{L}$ of approximating sequences of finite complexes with k_i and l_i their bonding maps. Let $f_i: K_i/\mathbb{Z}_{i!} \to L_i/\mathbb{Z}_{i!}$ be the given homotopy equivalences. Then we have the commutativity $f_i k_i = l_i \tilde{f}_{i+1}$. This leads to the relation, in the obvious notation, that:

$$\tau(\mathbf{l}) - \tau(\mathbf{k}) = (1-d)\tau(\mathbf{f}).$$

In particular modulo the image of $1-d$ the torsion is independent of sequence, which is precisely what it means to define an element of $\lim^1$. □

Proof of Theorem 3. Proposition 2 justifies the necessity statement. To prove sufficiency, we adopt the strategy of proof of Theorem 1. Suppose $o(X) = 0$. This means that for any finite approximating tower the torsion of the bonding maps lies in $\operatorname{im}(1-d)$. Using a preimage and the realization theorem for torsions, we can find an approximating tower, say $\mathbf{K}$, with bonding maps simple homotopy equivalences. It is no loss of generality to assume that all the complexes of $\mathbf{K}$ are of some uniformly bounded dimension. Embed in high dimensional Euclidean space and take regular neighborhoods. Standard *PL* topological arguments (spines, simple homotopy theory, expansions and collapses, and their relationship to shelling) allow one to homotop the bonding maps to *PL* homeomorphisms. This shows that the action of $\mathbb{Z}_{(i+1)!}$ on the common universal cover induced from K_{i+1} is conjugate upon restriction to $\mathbb{Z}_{i!}$, by a homeomorphism k_i, to the action induced from K_i. By conjugating the generator of $\mathbb{Z}_{(i+1)!}$ by k_i^{-1} one can inductively extend the action of $\mathbb{Z}_{i!}$ to $\mathbb{Z}_{(i+1)!}$, which produces the desired $\mathbb{Q}/\mathbb{Z}$ action. □

We leave it to the reader to deduce the relevant corollaries about $\mathbb{Q}/\mathbb{Z}$ actions on Hilbert cube manifolds.

§3. Applications to closed manifolds

We now prove:

THEOREM 4. *A simply connected manifold M which possesses homotopy compatible free $\mathbb{Z}_n$ actions for all n admits a free h-equivalent $\mathbb{Q}/\mathbb{Z}$ action iff* $o(M) = 0$.

The hypothesis means that the $X/\mathbb{Z}_n$'s have the usual homotopy equivalence properties. Certainly the condition is necessary. Before proving sufficiency, we note a consequence mentioned in the introduction:

COROLLARY 1. *Let M be a closed even dimensional simply connected manifold. Suppose that for each n, the cyclic group $\mathbb{Z}_n$ acts freely on M. Then the group $\mathbb{Q}/\mathbb{Z}$ also does.*

Proof of Corollary 1. Theorem 2 provides a collection of homotopy compatible actions. Theorem 4 will apply once we show that $o(M) = 0$. It follows from the duality conditions on the torsions of homotopy equivalences between closed manifolds (see proof of claim 1 below) and the even dimensionality of M that the torsions of all the bonding maps are zero, so the result follows. $\square$

Proof of Sufficiency. In dimension three one can show using invariant knots that no simply connected manifolds other than S^3 have free cyclic group actions arbitrary orders. In dimension four, it is an easy Lefshetz fixed point theorem argument to show that no simply connected closed manifold at all has free cyclic group actions of arbitrary orders. We thus assume that $\dim M \geq 5$. Suppose $o(X) = 0$, so that by Theorem 3, one has a simple homotopy tower of finite complexes.

CLAIM 1. *All of the spaces in this tower can be taken to be simple Poincaré spaces.*

Proof. Since each space is a finite complex and finitely covered by a Poincaré space, they are each finite Poincaré spaces. The torsion τ of the Poincaré duality map of a k-dimensional finite Poincaré space satisfies (Milnor's) duality $\tau^* = (-1)^k \tau \in \mathrm{Wh}(\pi_1)$. All of the relevant Poincaré spaces have cyclic fundamental groups so their Whitehead groups are torsion free abelian groups and the involution * is trivial. Consequently, in odd dimensions, simplicity is automatic. In even dimensions, all of spaces given by the proof of Theorem 2 are quotients of M by free actions, so that they are automatically simple duality spaces. We now have to consider the torsions of the bonding maps. These satisfy $\tau^* = -\tau$, so their torsion is automatically zero. $\square$

Now we will try to use surgery theory. Notice that each space has a normal invariant that agrees with the normal invariant $M \to \tilde{K}_n$. Moreover, there are only finitely many of these (since M and the K_i have the same rational homotopy type). Applying Koenig's lemma again, we can construct a tower of normal invariants for the K_i's that are compatible with respect to covers, that is, in this tower, all the Spivak normal fibrations can be lifted to smooth vector bundles and these lifts can be chosen compatible with covering projections.

CLAIM 2. *We can complete surgery on such a tower of normal invariants.*

Proof of claim. If $\dim M \equiv 1 \bmod 4$, then $L^s(\mathbb{Z}_n) = 0$, and there is no difficulty. If $\dim M$ is even, then all of the targets are closed manifolds, so that only the signature and arf invariants can arise as obstructions. Transfer calculations show that only zero obstructions can come from a tower of normal invariants. Finally if $\dim M \equiv 3 \bmod 4$, there is a codimension one arf invariant (an element of $\mathbb{Z}_2$) that might occur. However the trick of taking the connected sum along an S^1 of the original problem with $S^1 \times$ Kervaire problem changes the normal invariants of the tower in a transfer invariant way, and after this process, we can complete the surgery. □

We now have a tower of manifolds M_n, with M_n simple homotopy equivalent to $K_n/\mathbb{Z}_{n!}$, $M_1 = M$, and M_n normally cobordant to $\tilde{M}_{n+1}$. If we are in an even dimension, the vanishing of the odd dimensional L-groups (except for the $\mathbb{Z}_2$ in L_3 that we've discussed) shows that each M_n can be taken homeomorphic to $\tilde{M}_{n+1}$ This produces, as in the proof of Theorem 3, a $\mathbb{Q}/\mathbb{Z}$ action on M. If the dimension is odd, then the obstructions to getting these manifolds homeomorphic lie in the reduced group $\tilde{L}^s_{\text{even}}(\mathbb{Z}_n)$. An argument similar to the proof of Theorem 3, using the action of the L-groups on structure sets to replace the realization of Whitehead torsions, shows that the obstruction to obtaining a "normally cobordant" $\mathbb{Q}/\mathbb{Z}$ action lies in $\lim^1 \tilde{L}^s_{\text{even}}(\mathbb{Z}_n)$. This $\lim^1 = 0$ since all the bonding maps can be readily checked to be onto, so the Mittag–Leffler condition holds. □

COROLLARY 2. *Every simply connected four-manifold has a semi-free topological locally linear $\mathbb{Z}_{(6)}/\mathbb{Z}$ action on it.*

Proof. Edmonds [E] has shown that all such manifolds have semi-free topological locally linear $\mathbb{Z}_k$ actions on them for all odd k not divisible by 3. One applies a relative version of Corollary 1 to obtain the result. □

In §5, we shall discuss noncompact manifolds and some simplifications that occur in their theory.

§4. Computation of the obstruction

Now we shall try to give some information on computing the obstruction in cases where the bonding maps are not simple homotopy equivalences for trivial reasons.

Let R be a commutative ring, and $K_1(R)$ be $H_1(GL(R))$, its first algebraic K-group. Via the determinant we obtain a splitting

$$K_1(R) = R^{\cdot} \oplus SK_1(R),$$

where $R^{\cdot}$ denotes the units of R. If S is a multiplicative subset of R, then we have the Bass localization sequence,

$$K_1(R) \to K_1(S^{-1}R) \to K_1(R, S) \xrightarrow{\partial} \tilde{K}_0(R) \to \tilde{K}_0(S^{-1}R),$$

where $K_1(R, S)$ is the Grothendieck group of S-torsion R-modules of finite projective dimension, and the boundary ∂ given by Euler characteristic of the projective resolution in K_0. Now, if $R = \mathbb{Z}\pi$ for π finite, and $S = \mathbb{Z}\text{-}\{0\}$, the kernel of the first map is $SK_1(R)$ which is also the torsion of the Whitehead group. (The last map in the exact sequence is trivial.) For π cyclic, the theorem of [BMS] implies that SK_1 vanishes. This leads to an exact sequence:

$$0 \to \mathrm{Wh}\,(\mathbb{Z}\pi) \to \mathrm{Wh}\,(\mathcal{I}) \to \mathrm{Ker}\,\partial_\pi/\mathrm{im}\,\mathbb{Q}^{\cdot} \to 0,$$

where $\mathrm{Wh}\,(\mathcal{I}) = K_1(\mathcal{I})/\pm\pi$ and $\mathcal{I}$ is the augmentation ideal of $\mathbb{Q}\pi$. Now let us replace each term in the exact sequence by the inverse limit systems obtained by letting π range over cyclic groups and the maps being induced by restriction (geometrically, transfer). Any short exact sequence of inverse limit systems:

$$0 \to \mathbf{A} \to \mathbf{B} \to \mathbf{C} \to 0$$

gives rise to an exact sequence of derived functors (see §2, and view a derived functor as a homology group of a complex):

$$0 \to \lim^0 \mathbf{A} \to \lim^0 \mathbf{B} \to \lim^0 \mathbf{C} \to \lim^1 \mathbf{A} \to \lim^1 \mathbf{B} \to \lim^1 \mathbf{C} \to 0.$$

Denote the connecting homomorphism $\lim^0 \mathbf{C} \to \lim^1 \mathbf{A}$ by δ. Now let $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ be $\mathrm{Wh}\,(\mathbb{Z}\pi)$, $\mathrm{Wh}\,(\mathcal{I})$, and $\mathrm{Ker}\,\partial_\pi/\mathrm{im}\,\mathbb{Q}^{\cdot}$ respectively.

From a tower of finite complexes one obtains an element of $\lim^0$ $\mathrm{Ker}\,\partial_\pi/\mathrm{im}\,\mathbb{Q}^{\cdot}$. This is because each space of the tower has a homologically trivial action on it (i.e. is a special space in the sense of Milnor's paper [M]) and one can take the homotopy invariant part of the Reidemeister torsion $R\tau$. These are natural with respect to transfer and hence define an element of $\lim^0 \operatorname{Ker} \partial_\pi / \operatorname{im} \mathbb{Q}^{\cdot}$ as advertised.

THEOREM 5. $o(X) = \delta(R\tau(X/\mathbb{Z}_n))$.

Proof. This is a standard sort of fact. (See e.g. [DW].) The image of $o(X)$ in $\lim^1 \operatorname{Wh}(\mathscr{I})$ vanishes since the torsion of the bonding maps is precisely $(1-d)$ applied to the (usual, i.e. nonhomotopy invariant) Reidemeister torsion of the torsion by standard formulae. As a result, the difficulty lies in the deviation of these elements from lying in the image of $\operatorname{Wh}(\mathbb{Z}\pi)$, i.e. it lies in the cokernel of the natural map, i.e. in the inverse limit of the homotopy invariant part of the Reidemeister torsion. An easy analysis of the snake map δ shows that it is essentially this correspondence. $\square$

The advantage of this description is that $R\tau(X/\mathbb{Z}_n)$ can be directly computed in many cases. The homology groups of $X/\mathbb{Z}_n$ with $\mathscr{I}$ coefficients are exactly the sorts of modules used in the definition of $K_1(R, S)$. We shall now apply this to a certain natural class of examples of towers.

In [CW] a general method for transferring homologically trivial group actions from one manifold to another is developed. The setting is this: One has a free π action on a manifold M and a map $f: N \to M$ which is a $\mathbb{Z}_{(P)}$ homology equivalence where P is the order of π. The problem is to construct a free π action on N so that f can be homotoped to an equivariant map. For π an odd order group the result is that this is possible iff the local normal invariant of f lies in the image of $[M/\pi: F/\mathrm{Cat}_{(P)}]$. Now we study the question for $\mathbb{Q}/\mathbb{Z}$, or, for wider range of application, $\mathbb{Z}_{(Q)}/\mathbb{Z}$ actions.

THEOREM 6. *Let M and N be simply connected manifolds, $\mathbb{Z}_{(Q)}/\mathbb{Z}$ acting freely on M, $f: N \to M$ a $\mathbb{Z}_{(P)}$ homology equivalence, P the set of primes complementary to Q. Then there is a free $\mathbb{Z}_{(Q)}/\mathbb{Z}$ on N such that $f_{(P)}$ is a (p-local) h-equivalence iff $v(f) \in [M: F/\mathrm{Cat}_{(P)}]$ lies in the image of $[\mathscr{L}(M): F/\mathrm{Cat}_{(P)}]$.*

Proof. Everything is the same as in [CW] till we reach the obstruction $o(X)$. That can be computed using Theorem 5 by the same method used there for computing the Wall finiteness obstruction for a propagation. Thus we only have to compute for (**k**) i.e. the trivial module with k elements viewed as an element of all of the relative K-groups, where k is the alternating product of the orders of the homology groups of f. This can be computed directly, or by using the obvious

$\mathbb{Z}_{(Q)}/\mathbb{Z}$ actions on S^1 and the k-th power map, which is a geometric example where the obstruction is manifestly zero. □

Remark. In another paper we will show that if the $\mathbb{Z}_{(Q)}/\mathbb{Z}$ action on N is part of an S^1 action then $\mathscr{L}(N)$ and N/S^1 have the same P-adic completions, which greatly facilitates calculation. On the other hand, $\mathscr{L}(N)$ and N have the same rational homotopy type.

If we apply Theorem 6 to a simply connected odd dimensional $\mathbb{Z}_{(p)}$ homology sphere and the degree one map to the sphere we obtain the corollary mentioned in the introduction. (For the finite subgroups, these actions were first constructed by Peter Loffler [L] and, later using propagation, by the author [We1].)

§5. Final remarks

Our first remarks concern the study of noncompact manifolds. We shall study simply connected manifolds with finitely many tame simply connected ends (i.e. if the dimension is greater than five, the interiors of compact simply connected manifolds with boundary with all boundary components simply connected). The main point is that formally much of the set up is the same because one has a simple homotopy theory [Si] and a surgery theory [Ma][Ta] for noncompact manifolds with tame ends, but that many of the obstruction groups are easier to analyse because the relevant groups for the finite subgroups are finite, which leads to vanishing $\lim^1$ groups. We state the results:

THEOREM (1–3)′. *Let* $\mathbf{X}$ *be a noncompact locally finite finite dimensional simply connected CW complex with finitely many simply connected ends. Suppose that for each* n, *there is an* X_n *of the same (sort and) proper homotopy type as* $\mathbf{X}$ *admitting a free* $\mathbb{Z}_n$ *action, then there is such an* X *homotopy equivalent to* $\mathbf{X}$ *admitting a free* $\mathbb{Q}/\mathbb{Z}$ *action.*

Proof. Same as for Theorems 1–3 but making use of Siebenmann's exact sequence [Si] for the proper Whitehead groups to see they're copies of K_0, and, hence, finite. Thus the $\lim^1$ group vanishes, so there is no obstruction. □

THEOREM 4′. *Let* M *be a simply connected noncompact manifold with finitely many tame simply connected ends, then if* M *has free* $\mathbb{Z}_n$ *actions for each* n, *then* M *has a free* $\mathbb{Q}/\mathbb{Z}$ *action.*

Proof. Same as Theorem 4 but with different surgery calculations. The new ingredient, given the relation between proper simple surgery and L^h is the following:

LEMMA. $\lim^1 L^h_{\text{even}}(\mathbb{Z}_k) = 0$.

Proof. We use the exact Rothenberg sequence:

$$0 \to L^s_{\text{even}}(\mathbb{Z}_k) \to L^h_{\text{even}}(\mathbb{Z}_k) \to H^{\text{even}}(\mathbb{Z}_2; \mathrm{Wh}\,(\mathbb{Z}_k)) \to 0,$$

and the associated exact sequence of inverse limit systems to deduce the result from the vanishing of the inverse limit systems corresponding to $L^s_{\text{even}}(\mathbb{Z}_k)$ (verified in §3) and $H^{\text{even}}(\mathbb{Z}_2; \mathrm{Wh}\,(\mathbb{Z}_k))$ (because of the finiteness of these groups) using the exact sequence of §4. □

The analogue of the homology propagation Theorem 6 is left to the reader.

Now we prove an "inductive" result:

THEOREM 7. *Let M be a simply connected noncompact manifold with finitely many tame simply connected ends, then if for each prime p, M has free $\mathbb{Z}[1/p]/\mathbb{Z}$ actions, then M has a free $\mathbb{Q}/\mathbb{Z}$ action.*

Proof. This follows from Theorem 4′ and the following finite group result; results similar to this have also been noticed by Amir Assadi:

THEOREM 8. *Let M be a simply connected noncompact manifold with finitely many tame simply connected ends, then if M admits free $\mathbb{Z}$-homologically trivial actions of $\mathbb{Z}_m$ and $\mathbb{Z}_n$ for relatively prime m and n, then M admits a free homologically $\mathbb{Z}_{mn}$ action.*

Proof. We Zabrodsky mix: that is take the (proper) homotopy pullback of $M/\mathbb{Z}_{m(m)}$ and $M/\mathbb{Z}_n[1/m]$ over $M_{(0)}$ (see [Wel][CW][DW] for more details of this and what follows). This is a proper Poincaré space in the appropriate sense. It has a normal invariant that agrees with the normal invariants on $\mathbb{Z}_n$ and $\mathbb{Z}_m$. The obstruction to doing surgery is determined by L^p obstructions at the ends. When these are in even groups, multisignature detects, and can be computed to be zero; when in odd groups, the two-sylow subgroup detects, and the geometric hypothesis implies the obstruction is zero.

This produces an action on a manifold properly homotopy equivalent and normally cobordant to M, but an additional transfer argument enables one to get the action on M. □

We close with several obviously open questions:

PROBLEM 1. Is there an example where $o(X) \neq 0$?

PROBLEM 2. Which of the positive noncompact results extend to compact manifolds?

PROBLEM 3. Are there natural examples of towers of actions? It seems possible that algebraic geometry via p-adically equivalent nonisomorphic varieties should lead to these.

PROBLEM 4. How necessary was the restriction to free actions in all of the theory?

PROBLEM 5. How critical is simple connectivity?

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