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The cyclic groups and the free loop space

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Let $\Lambda(X)$ denote the space of unbased (or free) loops on a space X, $\Lambda(X) = \mathrm{Maps}\,(S^1, X)$. Notice that the circle group S^1 acts on $\Lambda(X)$ by rotating the loops. The topology of $\Lambda(X)$ as an S^1-equivariant space has been of interest to topologists and geometers for over sixty years, primarily because of its applications to the problem of finding closed geodesics on a Riemannian manifold. In particular the topology of the homotopy orbit space

$$\Theta(X) = ES^1 \times_{S^1} \Lambda(X)$$

where ES^1 is a contractible, free S^1-equivariant space, has been used extensively to study this problem. Also in recent years, with the invention of A. Connes' "Cyclic Homology theory" [Con] and its generalization by J. Loday and D. Quillen [L–Q], there has been much work trying to understand the relationship between $\Theta(X)$, algebraic K-theory, and, via the theory of Waldhausen, the space of pseudo-isotopies of a manifold [B, G, W].

In this paper we study the homotopy type of the space $\Theta(\Sigma X) = ES^1 \times_{S^1} \Lambda(\Sigma X)$, where "$\Sigma$" denotes reduced suspension. In particular, if we let $\bar{\Theta}(Y)$ denote the quotient

$$\bar{\Theta}(Y) = \Theta(Y)/\Theta(\text{point}) = ES^1 \times_{S^1} \Lambda(Y)/BS^1 = ES^1_+ \wedge_{S^1} \Lambda(Y),$$

we then describe a simple combinatorial model $Z(X)$ for $\bar{\Theta}(\Sigma X)$. We then use this model to prove that stably, $\bar{\Theta}(\Sigma X)$ splits as a wedge of spaces of the form $EZ_{n+} \wedge_{Z_n} X^{(n)}$, where the subscript "+" denotes a disjoint basepoint, $X^{(n)}$ denotes the n-fold smash product of X with itself, and Z_n is the cyclic group of order n, which acts on $X^{(n)}$ by cyclically permuting coordinates.

We now state our theorems more precisely. Let $F(R^\infty, n)$ be the configuration

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space

$$F(R^\infty, n) = \{(t_1, \ldots, t_n) \in R^\infty \times \cdots \times R^\infty : t_i \neq t_j \quad \text{if} \quad i \neq j\}.$$

$F(R^\infty, n)$ is a contractible space that is acted upon freely by the symmetric group Σ_n and hence by the cyclic subgroup Z_n. We define the space $Z(X)$ as follows.

$$Z(X) = \coprod_n F(R^\infty, n) \times_{Z_n} X^n / \sim$$

where the equivalence relation "$\sim$" is generated by setting

$$(t_1, \ldots, t_n) \times_{Z_n} (x_1, \ldots, x_{n-1}, *) \sim (t_1, \ldots, t_{n-1}) \times_{Z_{n-1}} (x_1, \ldots, x_{n-1})$$

where $* \in X$ is the basepoint. In the statements of the following theorems we assume that X is a connected, based space of the homotopy type of a based C.W. complex.

THEOREM A. *There is a homology equivalence*

$$h: Z(X) \to \bar{\Theta}(\Sigma X) = ES^1_+ \wedge_{S^1} \Lambda(\Sigma X).$$

Remark. Let $C(X) = \coprod_n F(R^\infty, n) \times_{\Sigma_n} X^n / \sim$. Thus $C(X)$ is the same combinatorial construction as $Z(X)$ except with the symmetric groups replacing the cyclic groups. Recall that $C(X)$ is the Dyer–Lashof model for the homotopy type of $Q(X) = \Omega^\infty \Sigma^\infty X$ (see [M].) Theorem A can therefore be thought of as an analogue of the Dyer–Lashof construction for the cyclic groups. More will be said about this analogy later.

Now observe that $Z(X)$ is a naturally filtered space, with $F_m(Z(X)) = \coprod_{n=1}^m F(\mathbb{R}^\infty, n) \times_{Z_n} X^n / \sim$. Notice that the subquotients

$$F_m(Z(X)) / F_{m-1}(Z(X)) = F(R^\infty, m)_+ \wedge_{Z_m} X^{(m)} = EZ_{m_+} \wedge_{Z_m} X^{(m)}.$$

Our next theorem states that stably $Z(X)$ splits as a wedge of these subquotients:

THEOREM B. *There is a splitting of suspension spectra*

$$\Sigma^\infty Z(X) \simeq \bigvee_{n \geq 1} \Sigma^\infty (EZ_{n_+} \wedge_{Z_n} X^{(n)}).$$

Combining Theorems A and B we then have the following.

COROLLARY C. *There is a splitting of infinite loop spaces*

$$Q\Theta(\Sigma X) = Q(ES^1 \times_{S^1} \Lambda(\Sigma X)) \simeq \prod_{n\geq 1} Q(EZ_{n_+} \wedge_{Z_n} X^{(n)}) \times QCP^\infty,$$

where

$$Q(Y) = \Omega^\infty \Sigma^\infty Y.$$

Corollary C has an interesting application to Waldhausen's notion of algebraic K-theory of spaces. Let $A(X)$ be the space defined by Waldhausen in [W]. The homotopy groups of $A(X)$ are the K-groups of the space X. If X is a manifold these groups contain the homotopy groups of the stable pseudo-isotopy space, $P(X) = \varinjlim_k P(X \times I^k)$ where $P(Y) = \mathrm{Diff}\,(Y \times I, \partial Y \times I \cup Y \times \{0\})$, as a direct summand. By combining results of Dwyer, Hsiang, and Staffeldt [DHS] and Goodwillie [G], Burghelea [B] proved that there is an isomorphism between the rational homology groups of $Q\bar{\Theta}(X)$ and $\Omega\bar{A}(X)$, where $\bar{A}(X)$ is the homotopy fiber of the projection map $A(X) \to A(\text{point})$. Using the fact that the rational homotopy type of an infinite loop space is determined by its homology groups, Corollary C then implies the following:

COROLLARY D. *There is a rational homotopy equivalence*

$$\Omega A \Sigma X \simeq \Omega A(\text{point}) \times \prod_{n>1} Q(EZ_{n_+} \wedge_{Z_n} X^{(n)}).$$

We remark that Goodwillie has conjectured that the rational equivalence of Corollary D is in fact an integral homotopy equivalence.

The organization of this paper is as follows. In Section 1 we describe the map $h: Z(X) \to \bar{\Theta}(\Sigma X)$ of Theorem A. In Section 2 we prove Theorem B. The point of proving Theorem B before we complete the proof of Theorem A is that Theorem B allows us to make an easy calculation of the homology of $Z(X)$. In Section 3 we prove Theorem A by computing the homomorphism h induces in homology. In Section 4 we investigate in more detail the relation between Theorem A and the Dyer–Lashof approximation of $Q(X)$.

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introducing them to the cyclic homology theory, and for sharing his insights in many hours of vigorous mathematical conversation.

§1. The map $h: Z(X) \to \bar{\Theta}(\Sigma X)$

The purpose of this section is to define the map $h: Z(X) \to \bar{\Theta}(\Sigma X)$ used to prove Theorem A. In Section 3 we will prove that h is a homotopy equivalence.

Let $\alpha_1: X \to \Omega\Sigma X$ be the adjoint of the identity map of ΣX. Thus for each $x \in X$ the formula for $\alpha_1(x): S^1 \to \Sigma X$ is given by $\alpha_1(x)(t) = t \wedge x \in S^1 \wedge X = \Sigma X$. For each n we define the map

$$\alpha_n: X^n \to \Omega\Sigma X$$

as follows.

Let $\pi_n: S^1 \to \bigvee_n S^1$ be the pinch map defined by identifying the n^{th} roots of unity to the basepoint. Then for each $(x_1, \ldots, x_n) \in X^n$ define a map $\alpha_n(x_1, \ldots, x_n)$: $S^1 \to \Sigma X$ to be the composition

$$\alpha_n(x_1, \ldots, x_n): S^1 \xrightarrow{\pi_n} \bigvee_n S^1 \xrightarrow{\alpha_1(x_1) \vee \cdots \vee \alpha_1(x_n)} \Sigma X. \tag{1.0}$$

This defines a map $\alpha_n: X^n \to \Omega\Sigma X \hookrightarrow \Lambda\Sigma X$. Notice that X^n is acted upon by the cyclic group Z_n by cyclically permuting the coordinates and $\Lambda(\Sigma X)$ is acted upon by S^1 by rotation of loops. Consider the inclusion homomorphism $i_n: Z_n \to S^1$ defined by sending the generator to $e^{2\pi i/n}$. One can easily check that $\alpha_n: X^n \to \Lambda(\Sigma X)$ is equivariant with respect to the homomorphism i_n. Thus α_n extends in a unique manner (up to homotopy) to an equivariant map

$$F(R^\infty, n) \times X^n \to ES^1 \times \Lambda\Sigma X.$$

and hence to a map of the orbit spaces

$$h_n: F(R^\infty, n) \times_{Z_n} X^n \to ES^1 \times_{S^1} \Lambda\Sigma X.$$

THEOREM 1.1. *There exists a map $h: Z(X) \to ES^1_+ \wedge_{S^1} \Lambda(\Sigma X)$ so that each composition*

$$F(R^\infty, n) \times_{Z_n} X^n \to Z(X) \xrightarrow{h} ES^1_+ \wedge_{S^1} \Lambda(\Sigma X)$$

is homotopic to h_n.

Proof of 1.1. The proof of this theorem will necessarily be somewhat technical for the following reasons. The maps h_n have so far only been defined up to homotopy, but in order to insure they be sufficiently compatible to induce a map on $Z(X)$ we need to construct a particular combinatorial model for each of the maps h_n and the space ES^1. (Actually we define combinatorially a free S^1-space $A(S^1)$ and an equivariant map $A(S^1) \to ES^1$.) Unfortunately after doing this the maps h_n still will not preserve the equivalence relations necessary to induce a map $Z(X) \to E(S^1) \times_{S^1} \Lambda(\Sigma X)$. However, they will preserve these relations up to homotopy. In order to keep track of the homotopies we will make use of the notion of "whiskered basepoints". We begin by constructing a space $A(S^1)$ which is S^1-free and maps $j_n: F(R^\infty, n) \to A(S^1)$ that are equivariant with respect to the homomorphisms $i_n: Z_n \hookrightarrow S^1$.

We identify S^1 with the group R/Z. Given m points $t_1, \ldots, t_m$ in R/Z we call a map $g: S^1 \to R^\infty$ *affine with respect to* $t_1, \ldots, t_m$ if the restrictions of g to the arcs between t_i and t_{i+1} for $i = 1, \ldots, m-1$ and to the arc between t_m and t_1, are all affine maps. Thus a map that is affine with respect to $t_1, \ldots, t_m$ is completely determined by its values on these points. The t_i's will be referred to as the vertices of the map g.

Define the space $\bar{A}(S^1)$ to be the space of such affine maps together with their vertices. More precisely,

$$\bar{A}(S^1) = \{(g; t_1, \ldots, t_m) : \text{each } t_i \in R/Z \text{ and } g: S^1 \to R^\infty \\ \text{is affine with respect to } t_1 \cdots t_m\}/\sim$$

where the equivalence relation "$\sim$" is given by $(g; t_1, \ldots, t_m) \sim (f; s_1, \ldots, s_n)$ if $g = f: S^1 \to R^\infty$ and the tuples $t_1 \cdots t_m$ and $s_1 \cdots s_n$ represent the same set of distinct points in R/Z. So for example $(g; 0, 1/3, 2/3) \sim (g; 2/3, 0, 1/3)$ and $(g; 1/4, 1/2, 1/2, 3/4) \sim (g; 1/4, 1/2, 3/4)$.

Let $\alpha \in S^1 = R/Z$, and $(g; t_1, \ldots, t_m) \in \bar{A}(S^1)$. Define $\alpha(g; t_1 \cdots t_m)$ to be $(\alpha g; \alpha + t_1, \ldots, \alpha + t_m) \in \bar{A}(S^1)$, where $\alpha g(x) = g(x - \alpha)$. Notice that αg is the unique map that is affine with respect to $\alpha + t_1, \ldots, \alpha + t_m$ and has the property that $\alpha g(\alpha + t_j i) = g(t_j)$.

This defines an S^1-action on $\bar{A}(S^1)$. Observe that the only fixed points under this action are points of the form $(\varepsilon; 0, 1/n, 2/n, \ldots, n - 1/n)$ where $n > 1$ and $\varepsilon: S^1 \to R^\infty$ is a constant map. Note that such a point is fixed by $\alpha = 1/n \in R/Z = S^1$.

Define $A(S^1)$ to be the complement of this fixed point set in $\bar{A}(S^1)$. Notice that $A(S^1)$ is an S^1-invariant subspace of $\bar{A}(S^1)$ and hence is a free S^1-space. Thus if ES^1 is any contractible free S^1-space, we can construct an S^1-equivariant map $e: A(S^1) \to E(S^1)$, and hence an induced map

$$e_*: A(S^1)_+ \wedge_{S^1} \Lambda(\Sigma X) \to ES^1_+ \wedge_{S^1} \Lambda(\Sigma X).$$

Our map $h: Z(X) \to ES_+ \wedge_{S^1} \Lambda(\Sigma X)$ will factor through this map e_*. First we show how each map h_n factors through e_*.

Consider the map $j_n: F(R^\infty, n) \to A(S^1)$ defined as follows. Let $\vec{S} = (s_0, \ldots, s_{n-1}) \in F(R^\infty, n-1)$. Let $j_n(\vec{S}): S^1 \to R^\infty$ be the affine map with vertices $0, 1/n, 2/n, \ldots, n-1/n$ determined by $j_n(\vec{S})(i/n) = s_i \in R^\infty$.

We then define

$$j_n: F(R^\infty, n) \to A(S^1)$$

by $j_n(s_0, \ldots, s_{n-1}) = (j_n(\vec{S}); 0, 1/n, 2/n, \ldots, n-1/n)$.

It is apparent that j_n is equivariant with respect to the inclusion $i_n: Z_n \to S^1$. Thus we can form the map

$$\bar{h}_n = j_n \times \alpha_n: F(R^\infty, n) \times_{Z_n} X^n \to A(S^1) \times_{S^1} \Lambda(\Sigma X)$$

and by definition, the composition $e_* \circ \bar{h}_n: F(R^\infty, n) \times_{Z_n} X^n \to A(S^1) \times_{S^1} \Lambda(\Sigma X) \to E(S^1) \times_{S^1} \Lambda(\Sigma X)$ is homotopic to h_n.

In order to produce a map $h: Z(X) \to E(S^1) \times_{S^1} \Lambda(\Sigma X)$ we need to extend the maps $\bar{h}_n$ in a particular way to the space $EZ_n \times_{Z_n} (X')^n$, where X' is a space containing X as a strong deformation retract. These extended maps will respect the necessary equivalence relations to produce a map $Z(X') \to A(S^1) \times_{S^1} \Lambda(\Sigma X)$. The retraction $X' \to X$ will induce a homotopy equivalence $Z(X') \to Z(X)$ by which we construct the map $h: Z(X) \to ES^1_+ \wedge_{S^1} \Lambda(\Sigma X)$.

We define X' to be X with a "whiskered basepoint." That is, if 1 is the unit interval $[0, 1]$ we define

$$X' = X \amalg I/* \sim 0$$

where $* \in X$ is the basepoint. The basepoint of X' is taken to be $1 \in I \subset X'$. By identifying I to a point we get a natural basepoint preserving retraction

$$r: X' \to X$$

which is a homotopy equivalence since $* \in X$ is a nondegenerate basepoint. r induces a homotopy equivalence $r_*: Z(X') \to Z(X)$. (Observe that $Z(\ )$ is a homotopy functor because each of the functors $F(R^\infty, n) \times_{Z_n} (\ )^n$ is.) Thus it is sufficient to construct a map

$$h': Z(X') \to A(S^1)_+ \wedge_{S^1} \Lambda(\Sigma X)$$

so that the compositions

$$F(R^\infty, n) \times_{Z_n} X^n \hookrightarrow F(R^\infty, n) \times_{Z_n} (X')^n \to Z(X') \xrightarrow{h'} \Lambda(S^1)_+ \wedge_{S^1} \Lambda(\Sigma X)$$

equal the maps $\bar{h}_n$ constructed above.

Our next step therefore is to extend $\bar{h}_n = j_n \times \alpha_n$ to a map $h'_n : F(R^\infty, n) \times_{Z_n} (X')^n \to \Lambda(S^1)_+ \wedge_{S^1} \Lambda(\Sigma X)$. We first extend α_n to a map $\alpha'_n : (X')^n \to \Lambda(\Sigma X)$.

To do this consider the map

$$\Phi : (X')^n \to I^n$$

defined by $\Phi(x_0, \ldots, x_{n-1}) = (\tau_0, \ldots, \tau_{n-1})$, where

$$\tau_i = \begin{cases} 0 & \text{if } x_i \in X \subset X \amalg I/* \sim 0 = X' \\ x_i & \text{if } x_i \in I \subset X'. \end{cases}$$

Φ is clearly a well defined continuous map. Notice that the restriction of Φ to $X^n \subset (X')^n$ is the constant map at $(0, 0, \ldots, 0) \in I^n$. Now let $l = l(x_0, \ldots, x_{n-1}) = \tau_0 + \cdots + \tau_{n-1}$ and consider the n points $z_0, \ldots, z_{n-1}$ in R/Z, defined as follows. Let $z_0 = 0 \in R/Z$, and inductively define z_{j+1} to be $z_j + r_j$ where

$$r_j = \begin{cases} \dfrac{1 - \tau_j}{n - l} & \text{if } l < n \\ 0, & \text{if } l = n \end{cases}$$

(Note: $l = n$ iff each x_i is the basepoint $1 \in I \subset X'$.) Observe that if $(x_0, \ldots, x_{n-1}) \in X^n \subset (X')^n$, then $(z_0, \ldots, z_{n-1}) = (0, 1/n, 2/n, \ldots, n - 1/n)$. Moreover, note that if any $x_j = 1 \in I \subset X'$, then $\tau_j = 1$ and hence $z_{j+1} = z_j$.

Let S^1_r be a circle of circumference r and let

$$\pi(x_0, \ldots, x_{n-1}) : S^1 = R/Z \to S^1_{r_0} \vee \cdots \vee S^1_{r_{n-1}}$$

be the map defined by identifying the points $z_0, \ldots, z_{n-1} \in R/Z$ to the basepoint.

Now define the map $i_r : S^1_r \to S^1_1$ to be the linear stretching map, induced by the map of intervals $[0, r] \to [0, 1]$ given by multiplication by $1/r$ if $r > 0$, and defined to be the inclusion of the basepoint if $r = 0$. Finally, define the map $\alpha'_n(x_0 \cdots x_{n-1}) : S^1 \to \Sigma X$ to be the composition

$$\alpha'_n(x_0, \ldots, x_{n-1}) : S^1 \xrightarrow[\pi(x_0, \ldots, x_{n-1})]{} S^1_{r_0} \vee \cdots \vee S^1_{r_{n-1}} \xrightarrow[\vee i_{r_j}]{} \bigvee_n S^1$$

$$\xrightarrow[\alpha_1(r(x_0)) \vee \cdots \vee \alpha_1(r(x_{n-1}))]{} \Sigma X \quad (1.2)$$

where $r: X' \to X$ is the retraction defined earlier, and where $\alpha_1: X \to \Omega\Sigma X$ is the adjoint of the identity. A check of the definitions of the maps in formula (1.2) shows that it induces a well defined continuous map

$$\alpha'_n: (X')^n \to \Lambda\Sigma X.$$

Moreover by comparing it to formula (1.0) one sees that α'_n extends $\alpha_n: X^n \to \Lambda(\Sigma X)$.

We now extend α'_n to a continuous map $h'_n: F(R^\infty, n) \times_{Z_n} (X')^n \to \Lambda(S^1)_+ \wedge_{S^1} \Lambda(\Sigma X)$ that extends $\bar{h}_n$.

Consider a point $(\vec{s}, \vec{x}) = (s_0, \ldots, s_{n-1}) \times (x_0 \cdots x_{n-1}) \in F(R^\infty, n) \times_{Z_n} (X')^n$. Let $z_0 \cdots z_{n-1}$ be the n points in $R/Z = S^1$ defined above. (So $z_{j+1} = z_j + r_j$.) Define $j_n(\vec{s}, \vec{x}): S^1 \to R^\infty$ to be the affine map with vertices at $z_0, \ldots, z_{n-1}$ with the property that

$$j_n(\vec{s}, \vec{x})(z_j) = j_n(\vec{s})(z_j),$$

where $j_n(\vec{s}): S^1 \to R^\infty$ is the map described earlier. Notice that $j_n(\vec{s}, \vec{x})$ and $j_n(\vec{s})$ are *not* equal maps since they are affine maps with respect to different sets of vertices. (The vertices of $j_n(\vec{s}, \vec{x})$ are $z_0, \ldots, z_{n-1}$ and the vertices of $j_n(\vec{s})$ are $0, 1/n, \ldots, n-1/n$.) Indeed $j_n(\vec{s}, \vec{x})$ and $j_n(\vec{s})$ are equal if and only if $(z_0 \cdots z_{n-1}) = (0, 1/n, \ldots, n-1/n)$.

We now define

$$\tilde{h}_n: F(R^\infty, n) \times (X')^n \to \Lambda(S^1) \times \Lambda(\Sigma X) \tag{1.5}$$

by $\tilde{h}_n(s_0, \ldots, s_{n-1}) \times (x_0, \ldots, x_{n-1}) = (j_n(\vec{s}, \vec{x}); z_0, \ldots, z_{n-1}) \times \alpha'_n(x_0 \cdots x_{n-1})$.

Observe that $\tilde{h}_n$ is *not* equivariant with respect to the inclusion $i_n: Z_n \hookrightarrow S^1$. However we claim that $\tilde{h}_n$ *does* induce a map on orbit spaces

$$h'_n: F(R^\infty, n) \times_{Z_n} (X')^n \to \Lambda(S^1)_+ \wedge_{S^1} \Lambda(\Sigma X)$$

(which necessarily extends $\bar{h}_n$ since $\tilde{h}_n$ extends $j_n \times \alpha_n$.) To see this it is enough to observe that if $t \in Z_n$ represents the generator, so that $i_n(t) = 1/n \in R/Z = S^1$, then

$$\tilde{h}_n(t(\vec{s}, \vec{x})) = \beta(\vec{x}) \cdot \tilde{h}_n(\vec{s}, \vec{x})$$

where $\beta(\vec{x}) \in S^1 = R/Z$ is given by $\beta(\vec{x}) = 1 - z_{n-1}$, where z_{n-1} is as above.

Now notice that by the definitions of $j_n(\vec{s}, \vec{x}): S^1 \to R^\infty$ and $\alpha'_n: (X')^n \to$

$\Lambda(\Sigma X)$, if the j^{th} coordinate x_j of $(x_0 \cdots x_{n-1}) \in (X')^n$ is the basepoint (i.e., $x_j = 1 \in 1 \subset X \amalg I/* \sim 0 = X'$), then $h_n'(s_0 \cdots s_{n-1} \times x_0 \cdots x_{n-1}) = (j_n(\vec{s}, \vec{x}); z_0, \ldots, z_{n-1}) \times \alpha_n'(x_0, \ldots, x_{n-1}) \in A(S^1)_+ \wedge_{S^1} \Lambda(\Sigma X)$, is equal to $h_{n-1}'(s_0 \cdots s_{j-1}, s_{j+1}, \ldots, s_{n-1} \times x_0, \ldots, x_{j-1}, x_{j+1} \cdots x_{n-1})$. This is essentially because if $x_j = 1$ then unless $x_0 = x_1 = \cdots = x_{n-1} = 1$, we would have $z_{j+1} = z_j$. If, however, $x_0 = x_1 = \cdots = x_{n-1} = 1$, then $\alpha_n'(1, \ldots, 1)$ is the basepoint in $\Lambda(\Sigma X)$ and hence the images of both $h_n'(s_0 \cdots s_{n-1} \times x_0 \cdots x_{n-1})$ and $h_{n-1}'(s_0 \cdots s_{j-1}, s_{j+1}, \ldots, s_{n-1} \times x_0 \cdots x_{j-1}, x_{j+1}, \ldots, x_{n-1})$ are the basepoint in $A(S^1)_+ \wedge_{S^1} \Lambda(\Sigma X)$.

Now this is precisely the relation needed so that the disjoint union of maps

$$\coprod_n h_n' : \coprod_{n>1} F(R^\infty, n) \times_{Z_n} (X')^n \to A(S^1)_+ \wedge_{S^1} \Lambda(\Sigma X)$$

factors through a map

$$h' : Z(X') \to A(S^1_+ \wedge_{S^1} \Lambda(\Sigma X).$$

Now as argued above, since each h_n' extends $\bar{h}_n$, the existence of h' implies the truth of Theorem 1.1.

§2. The stable splitting of $Z(X)$; the proof of Theorem B

The goal of this section is to prove Theorem B, which we now restate:

THEOREM B. *There is a splitting of suspension spectra*

$$\Sigma^\infty Z(X) \simeq \bigvee_{n>1} \Sigma^\infty(EZ_{n_+} \wedge_{Z_n} X^{(n)}).$$

Now recall that $Z(X)$ is a naturally filtered space, with m^{th} filtration given by

$$F_m(Z(X)) = \coprod_{n=1}^{m} F(R^\infty, n) \times_{Z_n} X^n/\sim \hookrightarrow Z(X)$$

Notice that the subquotients are given by

$$F_m(Z(X))/F_{m-1}(Z(X)) = F(R^\infty, m)_+ \wedge_{Z_m} X^{(m)} = EZ_{m_+} \wedge_{Z_m} X^{(m)}.$$

Thus Theorem B will follow from the following splitting theorem.

THEOREM 2.1. *For each m there exists stable "retraction" map $r_m : \Sigma^\infty F_m(Z(X)) \to \Sigma^\infty F_{m-1}(Z(X))$ with the property that the composition*

$$\Sigma^\infty F_{m-1}(Z(X)) \xrightarrow{i} \Sigma^\infty F_m(Z(X)) \xrightarrow{r_m} \Sigma^\infty F_{m-1}(Z(X))$$

is homotopic to the identity. Here i is the inclusion of the $m-1^{\text{st}}$ filtration into the m^{th} filtration.

Proof. We begin by recalling now $F_m(Z(X))$ is built out of $F_{m-1}(Z(X))$.

Let $\bar{X}^m \hookrightarrow X^m$ be the subspace consisting of m-tuples $(x_1, \ldots, x_m)$ with at least one of the x_i's equal to the basepoint $* \in X$. $\bar{X}^m$ is a Z_m-invariant subspace of X^m and so we may define the map

$$f_m : F(R^\infty, m) \times_{Z_m} \bar{X}^m \to F(R^\infty, m) \times_{Z_m} X^m$$

to be induced by the inclusion $\bar{X}^m \hookrightarrow X^m$. Now define a map $g_m : F(R^\infty, m) \times_{Z_m} \bar{X}^m \to F_{m-1}(Z(X))$ as follows.

Consider a point $(t_1, \ldots, t_m) \times_{Z_m} (x_1, \ldots, x_m) \in F(R^\infty, m) \times_{Z_m} \bar{X}^m$. Suppose that k of the points in the n-tuple $(x_1, \ldots, x_n)$ are the basepoint. Say $x_i = \cdots = x_{i_k} = * \in X$. (By the definition of $\bar{X}^m$, $k \geq 1$.) Consider the projection map that sends $(t_1, \ldots, t_n) \times_{Z_n} (x_1, \ldots, x_n)$ to the point in $F(R^\infty, m-k) \times_{Z_{m-k}} X^{m-k}$ given by deleting the $i_1, \ldots, i_k$ coordinates in both $(t_1, \ldots, t_m)$ and in $(x_1, \ldots, x_m)$. By the identifications in the definition of $F_{m-1}(Z(X))$ one sees that these projections induce a well defined, continuous map

$$g_m : F(R^\infty, m) \times_{Z_m} \bar{X}^m \to F_{m-1}(Z(X)).$$

Notice furthermore that $F_m(Z(X))$ is the equalizer (or strict pushout) of the maps $f_m : F(R^\infty, m) \times_{Z_m} \bar{X}^m \to F(R^\infty, m) \times_{Z_m} X^m$ and $g_m : F(R^\infty, m) \times_{Z_m} \bar{X}^m \to F_{m-1}(Z(X))$. That is, $F_m(Z(X)) = F_{m-1}(Z(X)) \amalg F(R^\infty, m) \times_{Z_m} X^m/\sim$ where for $z \in F(R^\infty, m) \times_{Z_m} \bar{X}^m$ we set $f_m(z) = g_m(z)$.

Now since the basepoint $* \in X$ is nondegenerate, the inclusion $\bar{X}^m \in X^m$ is a Z_m-equivariant cofibration, thus $f_m : F(R^\infty, m) \times_{Z_m} \bar{X}^m \to F(R^\infty, m) \times_{Z_m} X^m$ is a cofibration.

This implies that $F_m(Z(X))$ is homotopy equivalent to the *homotopy equalizer*. $\bar{F}_m(Z(X))$ of the maps f_m and g_m.

That is, $\bar{F}_m(Z(X))$ is the double mapping cylinder,

$$\bar{F}_m(Z(X)) = (F_{m-1}(Z(X)) \amalg F(R^\infty, m) \times_{Z_m} X^m) \amalg (F(R^\infty, m) \times_{Z_m} \bar{X}^m) \times I/\sim$$

where for $(z, t) \subset (F(R^\infty, m) \times_{Z_m} \bar{X}^m) \times I$ we identify $(z, 0) \sim f_m(z)$ and $(z, 1) \sim g_m(z)$. Thus the following diagram is a homotopy pushout diagram:

$$\begin{array}{ccc} F(R^\infty, m) \times_{Z_m} \bar{X}^m & \xrightarrow{f_m} & F(R^\infty, m) \times_{Z_m} X^m \\ {\scriptstyle g_m}\downarrow & & \downarrow \\ F_{m-1}(Z(X)) & \longrightarrow & \bar{F}_m(Z(X)) \simeq F_m(Z(X)) \end{array} \tag{2.2}$$

where the two unnamed maps are the obvious inclusions.

Now the suspension functor $\Sigma^\infty(\_\_)$ that associates to a space its suspension spectrum, is a functor that preserves homotopy pushout squares. Hence the following is a homotopy pushout diagram of spectra.

$$\begin{array}{ccc} \Sigma^\infty(F(R^\infty, m) \times_{Z_m} \bar{X}^m) & \xrightarrow{f_m} & \Sigma^\infty(F(R^\infty, m) \times_{Z_m} X^m) \\ {\scriptstyle g_m}\downarrow & & \downarrow \\ \Sigma^\infty(F_{m-1}(Z(X)) & \longrightarrow & \Sigma^\infty(F_m(Z(X)) \end{array} \tag{2.3}$$

Thus by standard properties of homotopy pushout diagrams, to prove that there is a retraction map $r_m : \Sigma^\infty F_m(Z(X)) \to \Sigma^\infty(F_{m-1}(Z(X))$ it is sufficient to prove that there is a retraction

$$\rho_m : \Sigma^\infty(F(R^\infty, m) \times_{Z_m} X^m) \to \Sigma^\infty(F(R^\infty, m) \times_{Z_m} \bar{X}^m)$$

of the inclusion f_m. That is, we have reduced the proof of Theorem 2.1 (and hence Theorem B) to the following.

LEMMA 2.4. *There exists a stable map $\rho_m : \Sigma^\infty(F(R^\infty, m) \times_{Z_m} X^m) \to \Sigma^\infty(F(R^\infty, m) \times_{Z_m} \bar{X}^m)$ so that the composition $\rho_m \circ f_m$ is homotopic to the identity.*

Note. This lemma also implies that the cofiber of $f_m = EZ_{m+} \wedge_{Z_m} X^{(m)}$ is a stable wedge summand of $EZ_m \times_{Z_m} X^m$. Indeed the reader can verify that the proof below actually proves that $EG_+ \wedge_G X^{(m)}$ is a stable wedge summand of $EG \times_G X^m$, where G is any subgroup of Σ_m acting by permuting coordinates.

Proof of 2.4. We study the stable homotopy type of $F(R^\infty, m) \times_{Z_m} X^m$ in some detail. First notice that the homeomorphism $X_+ \wedge X_+ = (X \times X)_+$ generalizes to give a homeomorphism

$$F(R^\infty, m)_+ \wedge_{Z_m} X_+^{(m)} \simeq (F(R^\infty, m) \wedge_{Z_m} X^m)_+.$$

Thus we have an equivalence of spectra

$$\begin{aligned}\Sigma^\infty(F(R^\infty, m)\times_{Z_m} X^m)_+ &\simeq \Sigma^\infty(F(R^\infty, m)_+ \wedge_{Z_m} X_+^{(m)})\\ &\simeq F(R^\infty, m)_+ \wedge_{Z_m} \Sigma^\infty(X_+)^{(m)} \end{aligned}\tag{2.5}$$

where the second equivalence is a special case of the well known observation that an extended power of a suspension spectrum is equivalent to the suspension spectrum of the extended power of the corresponding space. This follows directly from the definition of extended power spectra for which we refer the reader to [Br].

Now the spaces X_+ and $X \vee S^0$ are homeomorphic, although their basepoints are in different components. Once we suspend them they are (based) homotopy equivalent and hence the resulting equivalence of spectra

$$\Sigma^\infty(X_+) \simeq \Sigma^\infty(X \vee S^0)$$

induces an equivalence of spectra

$$\begin{aligned}F(R^\infty, m)_+ \wedge_{Z_m} \Sigma^\infty(X_+)^{(m)} &\simeq F(R^\infty, m)_+ \wedge_{Z_m} \Sigma^\infty(X \vee S^0)^{(m)}\\ &\simeq \Sigma^\infty(F(R^\infty, m) \wedge_{Z_m} (X \vee S^0)^{(m)})\end{aligned}\tag{2.6}$$

But this last spectrum clearly splits as

$$\Sigma^\infty(F(R^\infty, m)_+ \wedge_{Z_m} (X \vee S^0)^{(m)}) \simeq S^0 \vee \Sigma^\infty F(R^\infty, m)_+ \wedge_{Z_m} X^{(m)} \vee \Theta \tag{2.7}$$

where Θ is used to denote the remaining wedge component.

Now let

$$\tilde{p}_m : \Sigma^\infty(F(R^\infty, m)\times_{Z_m} X^m) \to \Theta$$

be the composite $\Sigma^\infty(F(R^\infty, m)\times_{Z_m} X^m) \hookrightarrow \Sigma^\infty(F(R^\infty, m)\times_{Z_m} X^m)_+ \xrightarrow{R_m} \Theta$ where R_m is induced by the equivalences in 2.5 and 2.6 and by the obvious projection onto the component given in the splitting in 2.7.

Now it is clear to see what effect the equivalence in (2.5)–(2.7) have on homology groups. Using this it is an easy exercise that we leave to the reader to check that the composition

$$\Sigma^\infty(F(R^\infty, m)\times_{Z_m} \bar{X}^m) \xrightarrow{f_m} \Sigma^\infty(F(R^\infty, m)\times_{Z_m} X^m) \xrightarrow{\tilde{p}_m} \Theta$$

induces an isomorphism in homology, and is therefore an equivalence of spectra. The existence of the retraction $\rho_m : \Sigma^\infty(F(R^\infty, m) \times_{Z_m} X^m) \to \Sigma^\infty(F(R^\infty, m) \times_{Z_m} \bar{X}^m)$ of f_m necessary to prove 2.4 is now clear.

Remark. Notice the similarity between the above proof of the splitting of $Z(X)$ (Theorem B) and the second author's proof of the Snaith splitting of $\Omega^n\Sigma^n X$ (and in particular of the Dyer–Lashof model $C(X)$) given in [Coh]. In [Coh], however, much use was made of the existence of natural homomorphisms between the symmetric groups, Σ_{m-1} and Σ_m. Since no analogous homomorphisms exist between the cyclic groups, the above modifications of the argument were necessary. We leave it to the reader to check that the above argument also works to give a proof of Snaith's splittings.

We end this section by using Theorem B to make a calculation of $H_*(Z(X))$. We first need the following calculational result.

LEMMA 2.8. *Let G be a subgroup of the symmetric group of Σ_n. Let G act on X^n by permuting coordinates. We then have*

$$H_*(EG \times_G X^n) \cong H_*(G; \{H_* X^n\})$$

Proof. This is a classical result of Steenrod.

COROLLARY 2.9. $\bar{H}_*(Z(X)) = \bigoplus_{n \geq 1} \bar{H}_*(Z_n; \{\bar{H}_*(X^{(n)})\})$ *where* $\bar{H}_*(\_)$ *denotes reduced homology.*

Proof. Consider the cofibration sequence

$$EZ_n \times_{Z_n} \bar{X}^n \underset{f_n}{\hookrightarrow} EZ_n \times_{Z_n} X^n \to EZ_n^+ \wedge_{Z_n} X^{(n)}.$$

Since the inclusion $H_*(\bar{X}^n) \hookrightarrow H_*(X^n)$ is a monomorphism, then by 2.8, the Serre spectral sequence of the fibration $\bar{X}^n \to EZ_n \times_{Z_n} \bar{X}^n \to BZ_n$ also collapses, and $f_{n_*} : H_*(EZ_n \times_{Z_n} \bar{X}^n) \to H_*(EZ_n \times_{Z_n} X^n)$ is therefore a monomorphism. Thus $\bar{H}_*(EZ_n^+ \wedge_{Z_n} X^{(n)}) = \operatorname{coker} f_{n_*} = \bar{H}_*(Z_n; \{\bar{H}_*(X^{(n)})\})$. 2.9 now follows from the splitting in Theorem B.

§3. Proof of Theorem A

In this section we prove Theorem A by proving that any map $h : Z(X) \to ES_+^1 \wedge_{S^1} \Lambda(\Sigma X)$ satisfying Theorem 1.1 induces an isomorphism in homology.

Observe that since the identification map

$$\coprod_n EZ_n \times_{Z_n} X^n \to Z(X)$$

induces a surjection in homology (2.8 and 2.9), then all maps $h: Z(X) \to ES^1_+ \wedge_{S^1} \Lambda(\Sigma X)$ satisfying 1.1 (i.e., that extend the map $\coprod h_n: \coprod EZ_n \times_{Z_n} X^n \to ES^1_+ \wedge_{S^1} \Lambda(\Sigma X)$) induce the same homomorphism in homology. Thus we are essentially reduced to computing the homomorphism,

$$h_{n*}: H_*(EZ_n \times_{Z_n} X^n) \to \tilde{H}_*(ES^1_+ \wedge_{S^1} \Lambda(\Sigma X)).$$

Our first step is to prove that $Z(X)$ and $ES^1_+ \wedge_{S^1} \Lambda(\Sigma X)$ have the same homology groups.

PROPOSITION 3.1. $\tilde{H}_*(ES^1_+ \wedge_{S^1} \Lambda(\Sigma X)) \cong \oplus_{n>1} \tilde{H}_*(Z_n; \{H_* X^{(n)}\})$.

Proof. This proposition will follow by combining several calculations of Loday and Quillen [L–Q] of certain "cyclic homology" groups, with results of Burghelea [B] and Goodwillie [G] concerning how these calculations are related to free (unbased) loop spaces. We therefore begin by collecting some basic facts about cyclic homology theory.

Let A be an associative (differential graded) algbra with identity over a commutative ring k. As in [L–Q] we write A^n for the n-fold tensor product $A^{\otimes n}$ of A over k. Define

$b': A^{n+1} \to A^n$ by the formula

$$b'(a_0, \ldots, a_n) = \sum_{i=0}^{n-1} (-1)^i (a_0, \ldots, a_i a_{i+1}, \ldots, a_n)$$

and define $b(a_0, \ldots, a_n) = b'(a_0, \ldots, a_n) + (-1)^n (a_n a_0, a_1, \ldots, a_n)$.

The complex $CH'_*(A): \cdots \xrightarrow{b'} A^{n+1} \xrightarrow{b'} A^n \xrightarrow{b'} \cdots A$ is the acyclic *Hochschild resolution* of A as a module over $A \otimes A^{op}$. By considering the tensor product complex $CH'_*(A) \otimes_{A \otimes A^{op}} A$ we get a chain complex $CH_*(A): \cdots \to A^n \xrightarrow{b} A^{n-1} \to \cdots \xrightarrow{b} A$ called the *Hochschild complex* for A. (Note: this is a bigraded complex if A is a D.G.A. Its homology, called the *Hoschild homology* of A, we denote by $HH_*(A)$.

Loday and Quillen have studied the following double complex, the analogue of which when A is a differential graded algebra was studied by Burghelea, Staffeldt, and Goodwillie [B, S, G].

$CC_*(A)$:

$$\begin{tikzcd}
\vdots \arrow[d, "b"] & \vdots \arrow[d, "b'"] & \vdots \arrow[d, "b"] & \\
A^n \arrow[d, "b"] & A^n \arrow[l, "1+(-1)^n t"'] \arrow[d, "b'"] & A^n \arrow[l, "N"'] \arrow[d, "b"] & \cdots \arrow[l, "1+(-1)^n t"'] \\
A^{n-1} \arrow[d, "b"] & A^{n-1} \arrow[l, "1+(-1)^{n-1} t"'] \arrow[d, "b'"] & A^{n-1} \arrow[l, "N"'] \arrow[d, "b"] & \cdots \arrow[l, "1+(-1)^{n-1} t"'] \\
\vdots & \vdots & \vdots & \\
A & A \arrow[l, "1-t"] & A \arrow[l, "N"] & \cdots \arrow[l]
\end{tikzcd}$$

So the even degree columns are the Hochschild complexes and the odd degree columns are the acyclic Hochschild resolutions. In the n^{th} row the symbol t is the automorphism of A^n given by

$$t(a_1, \ldots, a_n) = (a_n, a_1, \ldots, a_{n-1})$$

and $N = 1 + t + \cdots + t^{n-1}$ is the corresponding norm operator on A^n. Notice that the n^{th} row is a complex whose homology is $H_*(Z_n; \{A^n\})$ where Z_n acts on A^n via the operator t.

In [L–Q] it was shown that $CC_*(A)$ is in fact a double complex whose homology, which we denote $HC_*(A)$ is called the *cyclic homology* of A. In the case when the ring k is a field of characteristic zero it is shown that this definition agrees with that of Connes in [Con]. If A is an augmented algbra with augmentation ideal $\bar{A}$, so that $A = \bar{A} \oplus k$ as algebras, the reduced cyclic homology, $\bar{H}C_*(A)$ is defined by $\bar{H}C_*(A) = HC_*(\bar{A})$.

Several calculations of $HH_*(A)$ and $HC_*(A)$ were done in [L–Q]. The ones that concern us here are the following:

PROPOSITION 3.2. $HC_q(k) = H_q(CP^\infty; k)$.

PROPOSITION 3.3. *Let V be a k-module and let $A = T(V) = \oplus_{m\geq 0} V^m$ be the tensor algebra on V. We then have*

$$\text{a. } HH_q(A) = \begin{cases} \bigoplus_{m\geq 0} V^m/1-t & \text{if } q=0 \\ \bigoplus_{m\geq 1} (V^m)^t & \text{if } q=1 \\ 0 & \text{if } q>1 \end{cases}$$

where t is the cyclic permutation of V^m defined above, and where $(V^m)^t$ and $V^m/1-t$ are the invariants and coinvariants of t respectively.

$$\text{b. } \tilde{H}C_n(T(V)) = \bigoplus_{m>0} H_n(Z_m; \{V^m\})$$

Remark. In part a the isomorphism $\oplus_m (V^m)^t \to HH_1(A)$ is induced by the map $V^m = V^{m-1} \otimes V^1 \hookrightarrow A \otimes A = CH_1(A)$ (see [L–Q, §5])

Burghelea in [B] and Goodwillie in [G] described in detail certain algebraic relationships between the free loop space $\Lambda(X)$ and Hochschild and cyclic homology theory. Their studies involve the differential graded algebra $\mathcal{S}_*(M(X); k)$, the singular chain complex on the Moore-loop space of X, $M(X)$, with coefficients in a field k. This is an algebra via the usual multiplication of Moore loops. (N.B. the Moore loop space $M(X)$ is used instead of the usual based loop space ΩX because the singular chains $\mathcal{S}_*(M(X))$ is strictly associative, where $\mathcal{S}_*(\Omega X)$ is only chain homotopy associative.)

In [G, §V] Goodwillie constructed a map of chain complexes from the Hochschild complex of $\mathcal{S}_*(M(X))$ to the singular complex of $\Lambda(X)$,

$$\psi : CH_*(\mathcal{S}_*(M(X)) \to \mathcal{S}_*(\Lambda(X))$$

defined via the compositions

$$\mathcal{S}_*(M(X))^{n+1} \xrightarrow[\phi]{} \mathcal{S}_*(M(X)^{n+1}) \xrightarrow[\bar{\lambda}]{} \mathcal{S}_*(\Lambda(X)^{\Delta^n}) \xrightarrow[\mu]{} \mathcal{S}_{*+n}(\Lambda(X))$$

where ϕ is the shuffle map (that induces the Eilenberg–Zilber isomorphism), Δ^n is the standard n-simplex, $\Lambda(X)^{\Delta^n}$ is the space of maps from Δ^n to $\Lambda(X)$, μ is the standard juxtaposition map, and $\bar{\lambda}$ is induced by the map

$$\lambda : M(X)^{n+1} \to \Lambda(X)^{\Delta^n}$$

defined by

$$\lambda(f_0 \cdots f_n)(s_0 \cdots s_n)(t) = (f_0 \cdot \cdots \cdot f_n)\Big(t(\Sigma\, |f_i|) - \sum_{i<j} s_i\, |f_j|\Big).$$

Here $f_i \in M(X)$, $f_0 \cdot \cdots \cdot f_n \in M(X)$ is their product, $(s_0 \cdots s_n) \in \Delta^n$, $|f|$ denotes the length of a Moore loop, and t is an element of S^1 parameterized by the interval $[0, \Sigma\, |f_i|]$. (see [G] for details of these maps. Note our notation is somewhat different than his.) In [G] the following was proved.

PROPOSITION 3.4. *The map* $\psi: CH_*(\mathcal{S}_*(M(X)) \to \mathcal{S}_*(\Lambda(X))$ *induces an isomorphism in homology,*

$$\psi_*: HH_*(\mathcal{S}_*(M(X); k)) \xrightarrow{\cong} H_*(\Lambda(X); k).$$

Now in [G] Goodwillie proved that ψ is actually an isomorphism of cyclic objects in the sense of Connes [Con] and used this to prove the following.

PROPOSITION 3.5. *There is an isomorphism*

$$\psi_*^c: \bar{H}C_*(\mathcal{S}_*(M(X); k) \to H_*(ES^1_+ \wedge_{S^1} \Lambda(X); k).$$

Remark. The isomorphism ψ_*^c in 3.5 is induced by ψ in an explicit way described in detail in [G].

From now on we will assume that all chain complexes will be over a field k and all homology will be taken with k-coefficients. Now consider the map of singular chains (over k)

$$\alpha_*: \mathcal{S}_*(X) \to \mathcal{S}_*(\Omega\Sigma X) \to \mathcal{S}_*(M(\Sigma X))$$

induced by the map $\alpha_1: X \to \Omega\Sigma X$ described above.

Let $\bar{\mathcal{S}}_*(X)$ denote the reduced chains on X and let $T(\bar{\mathcal{S}}_*(X))$ denote the tensor algebra. Consider the chain map

$$\mu_*: T(\bar{\mathcal{S}}_*(X)) \to \mathcal{S}_*(M(\Sigma X))$$

induced by the compositions

$$\bar{\mathcal{S}}_*(X)^n \subset \mathcal{S}_*(X^n) \xrightarrow[\phi]{} \mathcal{S}_*(X^n) \xrightarrow[\alpha_1^n]{} \mathcal{S}_*(M(\Sigma X)^n) \to \mathcal{S}_*(M(\Sigma X))$$

where ϕ is the shuffle map and the last map is induced by the multiplication in $M(\Sigma X)$. The following is well known.

LEMMA 3.6. *$\mu_*: T(\bar{\mathcal{S}}_*(X)) \to \mathcal{S}_*(M\Sigma X)$ is a chain homotopy equivalence of differential graded algebras.*

Finally, let $r_*: \bar{\mathcal{S}}_*(X) \to \bar{H}_*(X)$ be any chain homotopy equivalence, where $\bar{H}_*(X)$ is viewed as a chain complex with trivial boundary map. r_* exists because we are working over a field k. Then we get an induced chain homotopy equivalence of D.G.A.'s.

$$r_*: T(\bar{\mathcal{S}}_*(X)) \to T(\bar{H}_*(X)). \tag{3.7}$$

Now since both Hochschild homology and cyclic homology are invariants of the chain homotopy type of a D.G.A., then, by combining 3.3a, 3.4, 3.6, and 3.7 we obtain a calculation of $H_*(\Lambda\Sigma X)$:

PROPOSITION 3.8. *There exist isomorphisms*

$$H_*(\Lambda(\Sigma X)) \xleftarrow[\psi]{\cong} HH_*(\mathcal{S}_*(M(\Sigma X)) \xrightarrow[r_*\circ\mu_*^{-1}]{\cong} HH_*(T(\bar{H}_*(X)))$$
$$\cong \bigoplus_{m\geq 0} \bar{H}_*(X)^m/1-t \oplus \bigoplus_{m\geq 1} (\bar{H}_*(X)^m)^t.$$

Similarly, by combining 3.3b, 3.5, 3.6, and 3.7 we obtain the following calculation of $H_*(ES^1_+ \wedge_{S^1} \Lambda(\Sigma(X))$, which shows, by comparison to 2.9 that it is isomorphic to $H_*(Z(X))$.

PROPOSITION 3.9. *There exist isomorphisms*

$$\bar{H}_*(ES^1_+ \wedge_{S^1} \Lambda(\Sigma X)) \xleftarrow[\psi_*^c]{\cong} \bar{H}C_*(\mathcal{S}_*(M\Sigma X)) \xrightarrow[r_*\oplus\mu_*^{-1}]{\cong} \bar{H}C_*(T(\bar{H}_*(X))$$
$$\cong \bigoplus_{m\geq 1} \bar{H}_*(Z_m; \{\bar{H}_*(X)^m\})$$

We now proceed to show that the map $h: Z(X) \to ES^1_+ \wedge_{S^1} \Lambda(\Sigma X)$ constructed in the last section induces a homology equivalence. Now as observed in the beginning of this section, the projection map

$$p: \coprod_n EZ_n \times_{Z_n} X^n \to Z(X)$$

induces a surjection in homology and so the map $h_*:H_*(Z(X))\to H_*(ES^1_+\wedge_{S^1}\Lambda(\Sigma X))$ is determined by the fact that it extends each $h_{n_*}:H_*(EZ_n\times_{Z_n}X^n)\to H_*(ES^1_+\wedge_{S^1}\Lambda(\Sigma X))$. We notice furthermore that by 2.8 and 2.9 the kernel of the

$$p_*:\oplus_n H_*(Z_n;(H_*(X^n)))=\oplus H_*(EZ_n\times_{Z_n}X^n)\to H_*(Z(X))$$
$$=\oplus_n H_*(Z_n;\bar{H}_*(X)^n)$$

is given by the sum of the kernels of the reduction maps

$$\rho_{n_*}:H_*(Z_n;\{H_*(X)^n\})\to H_*(Z_n;\{\bar{H}_*(X)^n\})$$

induced by the natural reduction $H_*(X)\to\bar{H}_*(X)$. Hence in view of 2.9 and 3.9, to prove that $h_*:H_*(Z(X))\to H_*(ES^1_+\wedge_{S^1}\Lambda(\Sigma X))$ is an isomorphism, it is sufficient to prove the following.

THEOREM 3.10. *The kernel of* $\oplus_n h_{n_*}:\oplus_n H_*(Z_n;\{H_*(X)^n\})=H_*(EZ_n\times_{Z_n}X^n)\to H_*(ES^1_+\wedge_{S^1}\Lambda(\Sigma X))$ *is equal to the kernel of the reduction map* $\oplus_n H_*(Z_n;\{H_*(X)^n\}\to\oplus_n H_*(Z_n;\{\bar{H}_*X\}^n\}$.

Proof. Consider the circle bundle over $ES^1\times_{S^1}\Lambda(\Sigma X)$ classified by the projection map $\pi:ES^1\times_{S^1}\Lambda(\Sigma X)\to ES^1\times_{S^1}*=BS^1$. The total space of this bundle is $ES^1\times\Lambda(\Sigma X)$ which is homotopy equivalent to $\Lambda(\Sigma X)$.

Similarly, consider S^1-bundle over $EZ_n\times_{Z_n}X^n$ classified by the composite

$$EZ_n\times_{Z_n}X^n\xrightarrow[h_n]{}ES^1\times_{S^1}\Lambda(\Sigma X)\to BS^1,$$

which, by the definition of h_n is homotopic to the composition

$$EZ_n\times_{Z_n}X^n\to EZ_n\times_{Z_n}*=BZ_n\xrightarrow[Bi_n]{}BS^1$$

where Bi_n is induced by the homomorphism $i_n:Z_n\hookrightarrow S^1$. The total space of this bundle is easily seen to be $S^1\times_{Z_n}X^n$. Moreover, h_n induces a map of associated circle bundles, which on the total space level can be seen to be homotopic to the map

$$\bar{h}_n:S^1\times_{Z_n}X^n\to\Lambda(\Sigma X)$$

defined to be the composition

$$\bar{h}_n:S^1\times_{Z_n}X^n\xrightarrow[1\times\alpha_n]{}S^1\times_{Z_n}\Lambda(\Sigma X)\xrightarrow[\mu]{}\Lambda(\Sigma X)\tag{3.11}$$

where μ is given by the S^1 action on $\Lambda(\Sigma X)$.

The main step in proving 3.10 will be calculating the homomorphism $\tilde{h}_{n_*}: H_*(S^1 \times_{Z_n} X^n) \to H_*(\Lambda(\Sigma X))$.

One can observe that for dimensional considerations, the Serre spectral sequence for the homology of the fibration sequence

$$X^n \to S^1 \times_{Z_n} X^n \to S^1$$

collapses at the E_2-level. Thus we have

$$H_*(S^1 \times_{Z_n} X^n) \cong H_*(S^1; \{H_*(X^n)\})$$

where $Z = \pi_1(S^1)$ acts on $H_*(X^n)$ through the projection $Z \to Z_n$ by cyclically permuting coordinates.

To compute this group, consider the following resolution of Z as a module over the group ring $Z[Z] = Z[x, x^{-1}]$.

$$0 \to Z[x, x^{-1}] \xrightarrow[\partial_1]{} Z[x, x^{-1}] \xrightarrow[\epsilon]{} Z$$

where $\epsilon(m \cdot x^n) = mZ$ and where $\partial_1(m \cdot x^n) = m(x^n - x^{n+1}) \epsilon Z[x, x^{-1}]$. Taking the tensor product of this resolution (over $Z[Z]$) with $H_*(X^n)$, one sees that we get the following chain complex $C_*(S^1; \{H_*(X^n)\})$, whose homology is $H_*(S^1; \{H_*(X^n)\} \cong H_*(S^1 \times_{Z_n} X^n)$.

$$C_*(S^1; H_*(X^n)): 0 \to H_*(X^n) \xrightarrow[1-t_*]{} H_*(X^n) \to 0 \tag{3.12}$$

where $t \in Z/n$ is the generator which acts by cyclically permuting coordinates. We therefore have the following

COROLLARY 3.13. *$H_*(S^1 \times_{Z_n} X^n) = H_*(S^1; \{H_*(X^n)\})$ and*

$$H_q(S^1; \{H_*(X^n)\}) \cong \begin{cases} H_*(X^n)/1 - t_* & \text{if } q = 0 \\ H_*(X^n)^t & \text{if } q = 1 \\ 0 & \text{otherwise} \end{cases}$$

Compare this calculation with the calculation of $H_*(\Lambda\Sigma X)$ given in 3.8. By comparing the Gysin sequences for the $S^1 = SO(2)$ bundles $ES^1 \times (S^1 \times_{Z_n} X^n) \to EZ_n \times_{Z_n} X^n$ and $ES^1 \times (\Lambda\Sigma X) \to ES^1 \times_{S^1} \Lambda\Sigma X$ and by using the calculation of

$H_*(ES^1 \times_{S^1} \Lambda\Sigma X)$ given in 3.9, it is easily seen that Theorem 3.10 is a consequence of the following.

PROPOSITION 3.14. *The kernel of* $\bar{h}_n : H_*(S^1 \times_{Z_n} X^n) \to H_*(\Lambda\Sigma X)$ *is equal to the kernel to the reduction map*

$$\begin{array}{ccc} H_*(S^1; (H_*(X)^n\}) & \longrightarrow & H_*(S^1; \{\bar{H}_*(X)^n)\}) \\ \downarrow{\cong} & & \downarrow{\cong} \\ H_*(X^n)/1 - t_* \oplus H_*(X^n) & \longrightarrow & \bar{H}_*(X)^n/1 - t_* \oplus (\bar{H}_*(X)^n)^t \end{array}$$

Proof of 3.14. Consider the bigraded chain complex

$$C(S^1; \mathscr{S}(X^n)) : 0 \to \mathscr{S}_*(X^n) \xrightarrow[1-t_*]{} \mathscr{S}(X^n) \to 0$$

The homology of this double complex is, by the above arguments, equal to $H_*(S^1; H_*(X^n))$. Consider the map of (double) chain complexes

$$\gamma_* : C_*(S^1; \mathscr{S}_*(X^n)) \to CH_*(T(\bar{\mathscr{S}}_*(X)))$$

defined as follows:

$$\gamma_0 : C_0(S^1; \mathscr{S}_*(X^n)) \to CH_0(T(\bar{\mathscr{S}}_*(X)))$$

is defined by the map

$$\mathscr{S}_*(X^n) \xrightarrow[c]{} \mathscr{S}_*(X)^n \xrightarrow[\rho]{} \bar{\mathscr{S}}_*(X)^n \subset T(\bar{\mathscr{S}}_*(X))$$

where c is the usual component map and ρ is the reduction. Define $\gamma_1 : C_1(S^1; \mathscr{S}_*(X^n)) \to CH_1(T(\bar{\mathscr{S}}_*(X))$ by the composition

$$\begin{aligned} \gamma_1 : \mathscr{S}_*(X^n) \xrightarrow[c]{} \mathscr{S}_*(X)^n = \mathscr{S}_*(X)^{n-1} \otimes \mathscr{S}_*(X) &\xrightarrow[\rho\otimes\rho]{} \bar{\mathscr{S}}_*(X)^{n-1} \otimes \bar{\mathscr{S}}_*(X) \\ &\subset T(\bar{\mathscr{S}}_*(X)) \otimes T(\bar{S}_*(X)) = CH_1(T\bar{\mathscr{S}}_*(X)) \end{aligned}$$

Notice that by the formula given for the Hochschild differential $b : T(\bar{\mathscr{S}}_*(X))^2 \to T(\mathscr{S}_*(X))$, we have that for $(v_1, \ldots, v_{n-1}) \otimes v_n \in \bar{\mathscr{S}}_*(X)^{n-1} \otimes \bar{\mathscr{S}}_*(X) \subset T(\bar{\mathscr{S}}_*(X)) \otimes T(\bar{\mathscr{S}}_*(X))$, then

$$\begin{aligned} b((v_1, \ldots, v_{n-1}) \otimes v_n) = (v_1, \ldots, v_n) - (v_n, v_1, v_2 \cdots v_{n-1}) \\ \subset \bar{\mathscr{S}}_*(X)^n \subset T(\bar{\mathscr{S}}_*(X)). \end{aligned}$$

Thus by the above formula for γ_0 and γ_1 we have that

$$b\circ\gamma_1=\gamma_0\circ(1-t_*):C_1(S^1;\mathscr{S}_*(X^n))\to CH_0(T(\bar{\mathscr{S}}_*(X)))$$

Thus γ_* is a map of chain complexes.

LEMMA 3.15. *The map $\bar{h}_{n_*}:H_*(S^1\times_{Z_n}X^n)\to H_*(\Lambda\Sigma X)$ is given by the composition*

$$H_*(S^1\times_{Z_n}X^n)=H_*(S^1;\{H_*(X^n)\}))\xrightarrow{\gamma_*}HH_*(T(\bar{\mathscr{S}}_*(X)))\underset{\cong}{\xrightarrow{\bar{\psi}}}H_*(\Lambda\Sigma X),$$

where γ_* is induced by the chain map described above, and $\bar{\psi}$ is the composite

$$\bar{\psi}_*:HH_*(T\bar{\mathscr{S}}_*(X))\xrightarrow[r_*]{}HH_*(T(\bar{H}_*(X)))\xrightarrow[\mu_*]{}HH_*(\mathscr{S}_*(M(\Sigma X)))\xrightarrow[\psi]{}H_*(\Lambda\Sigma X).$$

Notice that with respect to the isomorphism $H_*(S^1\times_{Z_n}X^n)\cong H_*(X^n)/1-t_*\oplus H_*(X^n)^{t_*}$ given in (3.13) and the isomorphism $H_*(\Lambda(\Sigma X))\cong\oplus\bar{H}_*(X)^m/1-t\oplus\oplus_{m>1}(\bar{H}_*(X)^m)^t$ of (3.8), then the remark following 3.3 and 3.15 imply that $\bar{h}_{n_*}$: $H_*(S^1\times_{Z_n}X^n)\to H_*(\Lambda(\Sigma X))$ is given by the composition

$$H_*(X^n)/1-t_*-\oplus H_*(X^n)^t\xrightarrow[\rho]{}\bar{H}_*(X)^n/1-t\oplus(\bar{H}_*(X)^n)^t$$
$$\subset\bigoplus_{m>0}\bar{H}_*(X)^m/1-t\oplus\bigoplus_{m>1}(\bar{H}_*(X)^m)^t.$$

Hence proposition 3.14 (and hence Theorem 3.10) would clearly follow from 3.15.

Proof of 3.15. Consider the map of chain complexes

$$F_*:C_*(S^1;\mathscr{S}_*(X^n))\to\mathscr{S}_*(S^1\times_{Z_n}X^n)$$

defined as follows.

$$F_0:\mathscr{S}_*(X^n)\hookrightarrow\mathscr{S}_*(S^1\times_{Z_n}X^n)$$

is induced by the inclusion $X^n\to 0\times X^n\hookrightarrow S^1\times_{Z_n}X^n$.

$$F_1:C_1(S^1;\mathscr{S}_*(X^n))\to\mathscr{S}_*(S^1\times_{Z_n}X^n)$$

is defined by the composition

$$F_1 : \mathcal{S}_*(X^n) \xrightarrow[g]{} \mathcal{S}_1(S^1) \otimes_{Z_n} \mathcal{S}_*(X^n) \xrightarrow[\phi]{} \mathcal{S}_{*-1}(S^1 \times_{Z_n} X^n)$$

where $g(\sigma) = e \otimes \sigma$, where the 1-simplex $e : \Delta^1 = I \to S^1$ is given by $e(t) = e^{2\pi i t}$. In the above composition, ϕ is given by the shuffle map.

The chain map F_* is clearly a chain homotopy equivalence.

Now consider the following diagram

$$\begin{tikzcd}
C_*(S^1; \mathcal{S}_*(X^n)) \arrow[r, "F_*"] \arrow[d, "\gamma_*"] & \mathcal{S}_*(S^1 \times_{Z_n} X^n) \arrow[r, "\bar{h}_{n*}"] & \mathcal{S}_*(\Lambda \Sigma X) \\
CH_*(T(\bar{\mathcal{S}}_*(X))) \arrow[rr, "\mu_*"] & & CH_*(M(\Sigma X)) \arrow[u, "\psi"]
\end{tikzcd}$$

A tedious but straightforward exercise using the explicit definitions of the chain maps given above, shows that this diagram commutes up to sign. Lemma 3.15 now follows.

Now as argued above, Lemma 3.15 was the last step in proving that

$$h : Z(X) \to ES^1_+ \wedge_{S^1} \Lambda(\Sigma X)$$

induces a homology isomorphism with coefficients in any field, and hence an integral homology isomorphism.

Remark. It is easily checked that if X is 1-connected then so are both $Z(X)$ and $ES^1_+ \wedge_{S^1} \Lambda \Sigma X$ and hence in this case h is a homotopy equivalence.

§4. $Z(X)$ and the Dyer–Lashof construction

As before, let $C(X)$ be the Dyer–Lashof construction,

$$C(X) = \coprod_{n>1} F(R^\infty, n) \times_{\Sigma_n} X^n / \sim$$

where, as in the definition of $Z(X)$, the equivalence relation "$\sim$" is generated by $(t_1, \ldots, t_n) \times_{\Sigma_n} (x_1, \ldots, x_{n-1}{}^*) \sim (t_1, \ldots, t_{n-1}) \times_{\Sigma^{n-1}} (x_1, \ldots, x_{n-1})$. If X is a connected C.W. complex there is a homotopy equivalence

$$g : C(X) \to Q(X) = \Omega^\infty \Sigma^\infty(X).$$

This theorem and its generalizations to the finite loop spaces $\Omega^n\Sigma^n(X)$ are proved in May's book [M].

Notice that the inclusion homomorphisms $Z_n \hookrightarrow \Sigma_n$ induce maps $p_n: F(R^\infty, n) \times_{Z_n} X^n \to F(R^\infty, n) \times_{\Sigma_n} X^n$ which in turn induce a map

$$p: Z(X) \to C(X).$$

By means of the homotopy equivalence $h: Z(X) \to ES^1_+ \wedge_{S^1} \Lambda(\Sigma X)$ and $g: C(X) \to Q(X)$ we get an induced map, which, by abuse of notation we also call

$$p: ES^1_+ \wedge_{S^1} \Lambda(\Sigma X) \to Q(X).$$

The purpose of this section is to analyze this map.

To ease notation, let $B(X)$ denote $ES^1_+ \wedge_{S^1} \Lambda(\Sigma X)$. Notice that $B(*) = *$ and so if C is any contractible space and $H: \hat{C} \to C$ is a contraction ($\hat{C}$ denotes the cone on C) then H induces a contraction

$$B(H): \widehat{B(C)} \to B(C).$$

Let $u: B(\hat{X}) \to B(\Sigma X)$ and $l: B(\bar{X}) \to B(\Sigma X)$ denote the maps induced by the inclusions of $\hat{X}$ in ΣX as the upper and lower cones, respectively. We then have the following homotopy pull-back diagram:

$$\begin{tikzcd} \Omega B(\Sigma X) \arrow[r] \arrow[d] & B(\hat{X}) \arrow[d, "u"] \\ B(\hat{X}) \arrow[r, "l"'] & B(\Sigma X) \end{tikzcd}$$

By the pull-back property, the inclusions $B(X) \to B(\hat{X})$ induced by the inclusion $X \in \hat{X}$ lifts to a map

$$E: B(X) \to \Omega B(\Sigma X).$$

Iterating this procedure we get a stable version, $B^s(X) = \varinjlim_k \Omega^k B \Sigma^k(X)$. See [W] for a more detailed version of this stabilization process.

Note that

$$B^s(X) = \varinjlim_k \Omega^k ES^1_+ \wedge_{S^1} \Lambda(\Sigma^{k+1} X).$$

Now consider the map

$$i_k: \Omega^k\Sigma^k X \to \Omega^k ES^1_+ \wedge_{S^1} \Lambda(\Sigma^{k+1}(X))$$

defined to be the k-fold loops of the composite map

$$\Sigma^k X \underset{\alpha_1}{\hookrightarrow} \Omega\Sigma(\Sigma^k X) = \Omega\Sigma^{k+1}X \subset \Lambda(\Sigma^{k+1}(X)) \hookrightarrow ES^1_+ \wedge_{S^1} \Lambda(\Sigma^{k+1}(X)).$$

Notice that i factors up to homotopy as the composition

$$i_k: \Omega^k\Sigma^k X = \Omega^k F_1(Z(\Sigma^k X)) \hookrightarrow \Omega^k Z(\Sigma^k(X)) \xrightarrow[\Omega^k h]{} \Omega^k ES^1_+ \wedge_{S^1} \Lambda(\Sigma^{k+1}X). \quad (4.0)$$

Now the inclusion $F_1(Z(\Sigma^k X)) \hookrightarrow Z(\Sigma^k X)$ is easily seen to be $2k$ connected, essentially since $F_2/F_1 = F(R^\infty, 2)^+ \wedge_{Z_2} (\Sigma^k X)^{(2)}$ is $2k+1$-connected. Thus the map i_k is k-connected. Hence when we pass to the limit, we have the following.

LEMMA 4.1. *The induced map $i: QX \to B^s(X)$ is a homotopy equivalence.*

The following will give us our description of $p: B(X) = ES^1_+ \wedge_{S^1} \Lambda(\Sigma X) \to QX$.

THEOREM 4.2. *The following diagram homotopy commutes.*

$$\begin{array}{ccccc}
Z(X) & \xrightarrow[\simeq]{h} & ES^1_+ \wedge_{S^1} \Lambda(\Sigma X) & = & B(X) \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle E} \\
C(X) & \xrightarrow[g]{\simeq} & QX & \xrightarrow[i]{\simeq} & B^s(X)
\end{array}$$

Remark. The point of this theorem is to say that with respect to the homotopy equivalences i, g and h above, the natural map $p: Z(X) \to C(X)$ induced by the inclusions $Z_n \hookrightarrow \Sigma_n$, is given by the stabilization map

$$E: B(X) \to B^s(X).$$

Proof of 4.2. The left hand square commutes by the definition of $p: ES^1_+ \wedge_{S^1} \Lambda(\Sigma X) \to Q(X)$. Thus, since h and g are equivalences its enough to show the outside rectangle homotopy commutes.

Let $Z^s(X) = \varinjlim_k \Omega^k Z\Sigma^k(X)$, then by the observations made above, the map

$i_1: QX \hookrightarrow \Omega^\infty F_1 \Sigma^\infty X \hookrightarrow \Omega^\infty Z \Sigma^\infty X = Z^s(X)$ is an equivalence. Now similar to what was done above, let $C^s(X) = \varinjlim_k \Omega^k C(\Sigma^k X)$ and $Q^s X = \varinjlim_k \Omega^k Q \Sigma^k X = QX$.

Consider the following diagram.

$$\begin{tikzcd}
& B(X) \arrow[dl, "h"'] \arrow[dr] \arrow[d, phantom, "1"] & \\
Z(X) \arrow[r, "E"] \arrow[d, "p"'] \arrow[dr, phantom, "3"] & Z^s(X) \arrow[r, "h^s"] \arrow[d, "p^s"] \arrow[d, phantom, shift left=4ex, "4"] & B^s(X) \arrow[dl, phantom, shift right=2ex, "5"] \\
C(X) \arrow[r, "E"'] \arrow[rr, bend right, "g"'] & C^s(X) \arrow[r, "g^s"'] \arrow[d, phantom, "2"] & Q^s(X) = QX \arrow[u, "i"'] \arrow[ul, "i_1"'] \\
& \ &
\end{tikzcd}$$

where the superscript "s" on the maps h^s, g^s, p^s denote the maps induced by h, g, and p respectively on stabilized functors.

Note that quadrilaterals 1, 2, and 3 in the above diagram homotopy commute by the naturality of the stabilization functor E. The fact that triangle 5 homotopy commutes was verified above (4.0). In triangle 4 we claim that $i_1: QX \to Z^s(X)$ and $g^s \circ p^s: Z^s(X) \to QX$ are homotopy inverse to each other. To see this, note that $g^s \circ p^s \circ i_j: QX \to QX$ is a map of infinite loop spaces that is homotopic to the stabilization $X \subset QX$ when restricted to X. Thus $g^s \circ p^s \circ i_1$ is homotopic to the identity.

These observations put together imply that the outside of the above diagram homotopy commutes, but this is the same as the outside of the diagram in the statement of 4.2. This proves Theorem 4.2.

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