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Autor: Beltrametti, M. / Sommese, Andrew J.

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A criterion for a variety to be a cone

MAURO BELTRAMETTI and ANDREW JOHN SOMMESE

Let $X \subseteq \mathbb{P}_{\mathbb{C}}$ be a normal projective Cohen–Macaulay variety and let $L = \mathcal{O}_{\mathbb{P}_{\mathbb{C}}}(1)_{|X}$. In this paper we give a criterion for X to be a cone. As a consequence we obtain the following theorem which settles affirmatively a conjecture of Conte and Murre [C–M] about 3-dimensional Gorenstein varieties with very ample anti-canonical bundle.

THEOREM. *Let X and L be as above and assume that $L^k = K_X^{-1}$ for some $k > 0$ where K_X is the dualizing sheaf of X. If the locus,* Irr (X), *of irrational singularities is not empty, then*

$$\dim \operatorname{Irr}(X) \geqq k - 1$$

with equality if and only if Irr (X) *is a linear $\mathbb{P}_{\mathbb{C}}^{k-1}$ and X is a cone with* Irr (X) *as vertex.*

In §0 we summarize background material and in §1 we obtain the above theorem as a consequence of a technical criterion for a variety to be a cone.

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§0. Background material

We work over the complex numbers $\mathbb{C}$. By *variety* we mean an irreducible and reduced quasi-projective scheme X of dimension n. We denote its structure sheaf by $\mathcal{O}_X$. For any coherent sheaf $\mathcal{F}$ on X, $h^i(\mathcal{F})$ denotes the complex dimension of $H^i(X, \mathcal{F})$. If X is normal the *dualizing sheaf* K_X is defined to be $j_* K_{\operatorname{Reg}(X)}$ where $j : \operatorname{Reg}(X) \to X$ is the inclusion of the smooth points of X and $K_{\operatorname{Reg}(X)}$ is the canonical sheaf of holomorphic n-forms.

Let $p : \bar{X} \to X$ be a resolution of singularities of X. The Leray sheaves $p_{(i)}\mathcal{O}_{\bar{X}}$ are independent of the resolution, and we shall denote them by $\mathcal{S}_i(X)$. X is normal if and only if $p_{(0)}\mathcal{O}_{\bar{X}} = \mathcal{O}_X$. We denote by Irr (X) the irrational locus of X which is the union of the supports of the sheaves $\mathcal{S}_i(X)$ for $i > 0$.

Let $\mathcal{L}$ be a line bundle on a normal variety X. $\mathcal{L}$ is said to be *numerically effective* (*nef* for short) if $\mathcal{L} \cdot C \geqq 0$ for all effective curves C on X, and $\mathcal{L}$ is said to be *big* if $c_1(\mathcal{L})^n > 0$ where $c_1(\mathcal{L})$ is the first Chern class of $\mathcal{L}$. If $\mathcal{L}$ is a nef and big line bundle on a normal projective variety X, then a convenient form (see [*S*], (0.2.1)) of the Kawamata–Viehweg vanishing theorem is

$$h^i(K_X \otimes \mathcal{L}) = 0 \quad \text{for} \quad i > \max\{0, \dim \operatorname{Irr}(X)\}. \tag{0.1}$$

If further X is Cohen–Macaulay and $\dim \operatorname{Irr}(X) = 0$, then ([*S*], (0.2.2))

$$h^0(K_X \otimes \mathcal{L}) = 0 \quad \text{implies} \quad \operatorname{Irr}(X) \quad \text{is empty}. \tag{0.2}$$

(0.3) *Basic Construction*. Let L be a line bundle on a possibly non-compact, irreducible, normal variety X. Assume that a finite dimensional vector space V of sections of L spans L off of a finite set F of X. There is a desingularization $p: \bar{X} \to X$ with a line bundle $\mathcal{L}$ on $\bar{X}$ such that:

a) $p_*\mathcal{L} \cong L \otimes \mathcal{J}_F$ where $\mathcal{J}_F$ is the image in $\mathcal{O}_X$ of $V \otimes L^{-1}$ under the natural map;

b) there is a vector space $\mathcal{V}$ of sections of $\mathcal{L}$ that spans $\mathcal{L}$ and such that the isomorphism in a) gives an isomorphism of $\mathcal{V}$ onto V.

Let us give a sketch of the construction. First blow up the ideal sheaf $\mathcal{J}_F = \operatorname{Image}(V \otimes L^{-1} \to \mathcal{O}_X)$ to obtain a modification $p': X' \to X$, a line bundle $\mathcal{L}'$ on X', and a space of sections $\mathcal{V}'$ of $\mathcal{L}'$ with properties a) and b) for p', X', $\mathcal{L}'$, $\mathcal{V}'$ instead of p, X, $\mathcal{L}$, $\mathcal{V}$. Now let $p: \bar{X} \to X$ be a desingularization obtained by composing a desingularization $q: \bar{X} \to X'$ with p'. Let $\mathcal{L} = q^*\mathcal{L}'$ and $V = q^*V'$.

(0.4) THEOREM. *Let L be a line bundle on a possibly non-compact, irreducible, normal variety X of dimension at least* 3. *Assume that a finite dimensional space V of global sections of L spans L off of a finite set $F \subset X$. Let $|V|$ denote the space of zero sets of elements of V. For a general $A \in |V|$*:

a) $\operatorname{Sing}(A - F) = A \cap \operatorname{Sing}(X - F)$ *and no component of* $\operatorname{Sing}(X - F)$ *is contained in $A - F$*;

b) *for each $i > 0$, the support of $\mathcal{S}_i(A - F)$ equals A intersected with the support of $\mathcal{S}_i(X - F)$, and no possibly embedded component of the support of $\mathcal{S}_i(X - F)$ is contained in $A - F$.*

In particular $\dim \operatorname{Irr}(X - F) \geqq \dim \operatorname{Irr}(A - F)$ *and* $\dim \operatorname{Sing}(X - F) \geqq \dim \operatorname{Sing}(A - F)$ *where the dimension of the empty set is taken to be $-\infty$. If X is Cohen–Macaulay then a general $A \in |V|$ is normal.*

Proof. Theorem (0.4.1) of [*S*] shows a) and b). To see normality let note that by the above the singular set of a general $A \in |V|$ has codimension 2 at least. Since A is Cohen–Macaulay this implies that A is normal. ■

We need the following result. For completeness we include a proof and refer also to [B], (2.6.1) and [B–S], (7.5).

(0.5) LEMMA. *Let X be a normal Cohen–Macaulay projective variety. If* Irr (X) *is finite then* $\mathcal{S}_i(X)$ *is* 0 *for* $1 \leqq i \leqq \dim X - 2$.

Proof. Since X is Cohen–Macaulay and projective we can choose X such that $H^i(L^{-1}) = 0$ for $i < \dim X$. By Theorem (0.4), choose a general element $A \in |L|$ such that A is normal, Irr (A) is empty and $\bar{A} = p^{-1}(A)$ is smooth for some desingularization $p : \bar{X} \to X$. Write $\bar{L} = p^*L$. We have

$$\begin{array}{ccccccc} \cdots \to & H^i(\bar{L}^{-1}) & \to & H^i(\mathcal{O}_{\bar{X}}) & \to & H^i(\mathcal{O}_{\bar{A}}) & \to \cdots \\ & \uparrow p^* & & \uparrow p^* & & \uparrow p_A^* & \\ \cdots \to & H^i(L^{-1}) & \to & H^i(\mathcal{O}_X) & \to & H^i(\mathcal{O}_A) & \to \cdots \end{array}$$

By the Kawamata–Viehweg vanishing theorem (0.1), $H^i(\bar{L}^{-1}) = 0$ for $i < \dim X$. Since A has only rational singularities and $\bar{A}$ is a desingularization of A, $p_{\bar{A}}^*$ is an isomorphism. Therefore from the above diagram we conclude that

$$H^i(\mathcal{O}_X) \xrightarrow{p^*} H^i(\mathcal{O}_{\bar{X}})$$

is an isomorphism for $i < \dim X - 1$ and an injection for $i = \dim X - 1$. A simple inspection of the Leray spectral sequence for p, using the assumption that the supports of $\mathcal{S}_i(X)$ are finite for $i \geqq 1$, shows that $H^0(\mathcal{S}_i(X)) = 0$ for $1 \leqq i \leqq \dim X - 2$. Since the supports of the $\mathcal{S}_i(X)$ are finite this implies that the supports of the $\mathcal{S}_i(X)$ are empty for $1 \leqq i \leqq \dim X - 2$.

(0.5.1) QUESTION. Is it true that if X is Cohen–Macaulay then for $i > 0$ the support of $\mathcal{S}_i(X)$, if not empty, is pure $(\dim X - i - 1)$-dimensional?

(0.6) Let $X \subset \mathbb{P}_{\mathbb{C}}$ be a projective variety and let $L = \mathcal{O}_{\mathbb{P}_{\mathbb{C}}}(1)_{|X}$. Let V be the subspace of $\Gamma(X, L)$ consisting of sections that vanish at a point $x \in X$. Let $p : \bar{X} \to X$ be a desingularization of X with a $\mathcal{V}$ and $\mathcal{L}$ as in the basic construction (0.3). *Then X is a cone with vertex x if and only if $\mathcal{L}$ is not big.* In particular if X is not a cone on x then $H^i(\mathcal{L}^{-1}) = 0$ for $i < \dim X$.

For further background material we refer to [S], §0.

§1. On conditions for a variety to be a cone

(1.1) THEOREM. *Let $X \subset \mathbb{P}_{\mathbf{C}}$ be a normal Cohen–Macaulay projective variety of dimension $n \geqq 3$. Let $L = \mathcal{O}_{\mathbb{P}_{\mathbf{C}}}(1)_{|X}$. Let* Irr (X) *be finite and non empty and let $N = h^0(\mathcal{S}_{n-1}(X))$. Let $x \in$* Irr (X) *and let $M = h^0(\mathcal{S}_{n-2}(A))$ for a general element A of $|L|$ that contains x. Then X is a cone on x if either*:

a) $h^0(K_X \otimes L) < M + N$, *or*

b) $h^{n-1}(\mathcal{O}_A) - h^{n-1}(\mathcal{O}_X) < M + N$ *and* $h^n(\mathcal{O}_X) = 0$.

Proof. Consider the exact sequence

$$0 \to \mathcal{O}_X(-A) \to \mathcal{O}_X \to \mathcal{O}_A \to 0$$

By the Kawamata–Viehweg vanishing theorem (0.1) we have

$$h^n(\mathcal{O}_X) + h^{n-1}(\mathcal{O}_A) - h^{n-1}(\mathcal{O}_X) = h^n(-A) = h^0(K_X \otimes L) \tag{1.1.1}$$

Thus condition b) implies condition a). Therefore it suffices to prove that X is a cone on x if condition a) holds. Let V be the subspace of $\Gamma(X, L)$ consisting of sections that vanish at x and let $\mathcal{V}$, $\bar{X}$, $\mathcal{L}$ and $p: \bar{X} \to X$ be as in the basic construction (0.3). Since Irr (X) is finite and X is Cohen–Macaulay, $p_{(j)}(\mathcal{O}_{\bar{X}})$ is 0 for $1 \leqq j \leqq n-2$ by Lemma (0.5). Thus the Leray spectral sequence gives

$$\left.\begin{array}{l} h^n(\mathcal{O}_{\bar{X}}) = h^n(\mathcal{O}_X) - a \\ h^{n-1}(\mathcal{O}_{\bar{X}}) = h^{n-1}(\mathcal{O}_X) + b \\ h^j(\mathcal{O}_X) = h^j(\mathcal{O}_{\bar{X}}) \quad \text{for} \quad j \leqq n-2 \end{array}\right\} \tag{1.1.2}$$

where $a \geqq 0$, $b \geqq 0$ and $a + b = N$.

Let A be a general element of the linear space $|V|$ of Cartier divisors associated to V. It can be assumed that A is normal and $\bar{A}$, the proper transform in $\bar{X}$ of A, belongs to $|\mathcal{V}|$ and is smooth. Assuming that X is not a cone we have $h^i(\mathcal{L}^{-1}) = 0$ for $i < n$ in view of (0.6). Thus by (1.1.2), $h^i(\mathcal{O}_A) = h^i(\mathcal{O}_{\bar{A}})$ for $i \leqq n-2$. Therefore by the Leray spectral sequence for $p_{\bar{A}}$ and the fact that Irr (A) is finite we conclude that

$$h^{n-1}(\mathcal{O}_{\bar{A}}) = h^{n-1}(\mathcal{O}_A) - M \tag{1.1.3}$$

Therefore by (1.1.1), (1.1.2), (1.1.3) we have

$$\begin{aligned} h^0(K_X \otimes L) &= h^n(\mathcal{O}_{\bar{X}}) + a + h^{n-1}(\mathcal{O}_{\bar{A}}) + M - h^{n-1}(\mathcal{O}_{\bar{X}}) + b \\ &= h^n(\mathcal{O}_{\bar{X}}) + [h^{n-1}(\mathcal{O}_{\bar{A}}) - h^{n-1}(\mathcal{O}_{\bar{X}})] + M + N \end{aligned}$$

Note that the assumption $h^0(K_X \otimes L) < M + N$ implies that the middle term must be negative. But this is absurd since $h^{n-1}(\mathscr{L}^{-1}) = 0$.

(1.2) COROLLARY. *Let X and L be as in the above theorem. Let* Irr (X) *be finite and non empty and assume that $h^0(K_X \otimes L) \leqq 1$. Then* Irr (X) *consists of one point x and X is a cone from that point.*

Proof. Since Irr (X) is finite it follows from Lemma (0.5) that $h^0(\mathscr{S}_{n-1}(X)) > 0$. A general $A \in |L|$ passing through x has Irr (A) finite by Theorem (0.4). By Elkik's theorem [E], $x \in$ Irr (A) since otherwise $x \notin$ Irr (X). Thus $h^0(\mathscr{S}_{n-2}(A)) > 0$ and

$$h^0(K_X \otimes L) \leqq 1 < h^0(\mathscr{S}_{n-2}(A)) + h^0(\mathscr{S}_{n-1}(X))$$

This implies the result by the above Theorem (1.1).

(1.3) THEOREM. *Let $X \subset \mathbb{P}_{\mathbb{C}}$ be a normal Gorenstein projective variety of dimension $n \geqq 3$ and let $L = \mathcal{O}_{\mathbb{P}_{\mathbb{C}}}(1)_{|X}$. Assume that $L^k = K_X^{-1}$ for some $k > 0$, where K_X denotes the dualizing sheaf of X. Assume that the locus of irrational singularities,* Irr (X), *is not empty. Then* dim Irr $(X) \geqq k-1$ *with equality if and only if* Irr (X) *is a linear $\mathbb{P}_{\mathbb{C}}^{k-1}$ and X is a cone with* Irr (X) *as vertex.*

Proof. We prove the above by induction on k.

If $k = 1$, then the assertion that dim Irr $(X) \geqq k - 1$ is an immediate consequence of the assumption that Irr (X) is not empty. Since $h^0(K_X \otimes L) = h^0(K_X \otimes K_X^{-1}) = h^0(\mathcal{O}_X) = 1$ the assertion follows from Corollary (1.2).

If $k > 1$, then choose a general $A \in |L|$. By (0.4), we conclude that A is a normal Gorenstein variety on which dim Irr $(X) =$ dim Irr $(A) + 1$. Since

$$K_A^{-1} = (K_X \otimes L)_A^{-1} = L_A^{k-1}$$

we conclude by the induction hypothesis that dim Irr $(A) \geqq k - 2$. This gives dim Irr $(X) =$ dim Irr $(A) + 1 \geqq k - 1$.

Further note that Irr (X) has no isolated points as components. Indeed by the argument of [S], (0.2.2) the number of isolated points in Irr (X) is bounded by $h^0(K_X \otimes L) = h^0(L^{-(k-1)}) = 0$. From this fact and the fact that Irr $(A) = A \cap$ Irr (X) is a linear $\mathbb{P}_{\mathbb{C}}^{k-2}$, it is an easy argument that Irr (X) is a linear $\mathbb{P}_{\mathbb{C}}^{k-1}$. Since any general element $A \in |L|$ is a cone on $A \cap$ Irr $(X) =$ Irr (A), elementary arguments of projective geometry show that X has to be a cone on Irr (X). ■

The following settles a conjecture of Conte and Murre positively (see [C–M], section III).

(1.4) COROLLARY. *Let* $X \subset \mathbb{P}_{\mathbb{C}}$ *be a normal Gorenstein 3-fold with* $K_X^{-1} \cong \mathcal{O}_{\mathbb{P}_{\mathbb{C}}}(1)_{|X}$. *If* Irr (X) *is finite then* Irr (X) *is a single point and* X *is the cone from this point over a Gorenstein K3-surface* A *with rational singularities. Note that* Sing (X) *is the cone over* Sing (A).

§2. Final Remarks

It follows from the results in the last section of [C–M], that if X is a normal Gorenstein 3-fold with K_X^{-1} very ample and $\dim \mathrm{Irr}(X) = 1$, then Irr (X) is a linear $\mathbb{P}^1_{\mathbb{C}}$.

Let us propose the following

QUESTIONS. Let X be a normal projective Gorenstein variety of dimension $n \geqq 3$. Assume that $L^k = K_X^{-1}$ for some $k > 0$, $L = \mathcal{O}_{\mathbb{P}_{\mathbb{C}}}(1)_{|X}$. If $\dim \mathrm{Irr}(X) = k$, is Irr (X) a linear $\mathbb{P}^k_{\mathbb{C}}$? How far can X deviate from being a cone over Irr (X) in this case?

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Dipartimento di Matematica
Via L. B. Alberti, 4
16132 Genova (Italy)

Department of Mathematics
University of Notre Dame
Notre Dame, Indiana 46556 (U.S.A.)

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