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A criterion for a variety to be a cone

MAURO BELTRAMETTI and ANDREW JOHN SOMMESE

Let $X \subseteq \mathbb{P}_{\mathbb{C}}$ be a normal projective Cohen–Macaulay variety and let $L = \mathcal{O}_{\mathbb{P}_{\mathbb{C}}}(1)|_X$. In this paper we give a criterion for X to be a cone. As a consequence we obtain the following theorem which settles affirmatively a conjecture of Conte and Murre [C–M] about 3-dimensional Gorenstein varieties with very ample anti-canonical bundle.

THEOREM. *Let X and L be as above and assume that $L^k = K_X^{-1}$ for some $k > 0$ where K_X is the dualizing sheaf of X . If the locus, $\text{Irr}(X)$, of irrational singularities is not empty, then*

$$\dim \text{Irr}(X) \geq k - 1$$

with equality if and only if $\text{Irr}(X)$ is a linear $\mathbb{P}_{\mathbb{C}}^{k-1}$ and X is a cone with $\text{Irr}(X)$ as vertex.

In §0 we summarize background material and in §1 we obtain the above theorem as a consequence of a technical criterion for a variety to be a cone.

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§0. Background material

We work over the complex numbers \mathbb{C} . By *variety* we mean an irreducible and reduced quasi-projective scheme X of dimension n . We denote its structure sheaf by \mathcal{O}_X . For any coherent sheaf \mathcal{F} on X , $h^i(\mathcal{F})$ denotes the complex dimension of $H^i(X, \mathcal{F})$. If X is normal the *dualizing sheaf* K_X is defined to be $j_* K_{\text{Reg}(X)}$ where $j: \text{Reg}(X) \rightarrow X$ is the inclusion of the smooth points of X and $K_{\text{Reg}(X)}$ is the canonical sheaf of holomorphic n -forms.

Let $p: \tilde{X} \rightarrow X$ be a resolution of singularities of X . The Leray sheaves $p_{(i)} \mathcal{O}_{\tilde{X}}$ are independent of the resolution, and we shall denote them by $\mathcal{S}_i(X)$. X is

normal if and only if $p_{(0)}\mathcal{O}_{\tilde{X}} = \mathcal{O}_X$. We denote by $\text{Irr}(X)$ the irrational locus of X which is the union of the supports of the sheaves $\mathcal{S}_i(X)$ for $i > 0$.

Let \mathcal{L} be a line bundle on a normal variety X . \mathcal{L} is said to be *numerically effective* (nef for short) if $\mathcal{L} \cdot C \geq 0$ for all effective curves C on X , and \mathcal{L} is said to be *big* if $c_1(\mathcal{L})^n > 0$ where $c_1(\mathcal{L})$ is the first Chern class of \mathcal{L} . If \mathcal{L} is a nef and big line bundle on a normal projective variety X , then a convenient form (see [S], (0.2.1)) of the Kawamata–Viehweg vanishing theorem is

$$h^i(K_X \otimes \mathcal{L}) = 0 \quad \text{for } i > \max \{0, \dim \text{Irr}(X)\}. \quad (0.1)$$

If further X is Cohen–Macaulay and $\dim \text{Irr}(X) = 0$, then ([S], (0.2.2))

$$h^0(K_X \otimes \mathcal{L}) = 0 \quad \text{implies} \quad \text{Irr}(X) \text{ is empty.} \quad (0.2)$$

(0.3) *Basic Construction.* Let L be a line bundle on a possibly non-compact, irreducible, normal variety X . Assume that a finite dimensional vector space V of sections of L spans L off of a finite set F of X . There is a desingularization $p: \tilde{X} \rightarrow X$ with a line bundle \mathcal{L} on \tilde{X} such that:

a) $p_*\mathcal{L} \cong L \otimes \mathcal{I}_F$ where \mathcal{I}_F is the image in \mathcal{O}_X of $V \otimes L^{-1}$ under the natural map;

b) there is a vector space \mathcal{V} of sections of \mathcal{L} that spans \mathcal{L} and such that the isomorphism in a) gives an isomorphism of \mathcal{V} onto V .

Let us give a sketch of the construction. First blow up the ideal sheaf $\mathcal{I}_F = \text{Image}(V \otimes L^{-1} \rightarrow \mathcal{O}_X)$ to obtain a modification $p': X' \rightarrow X$, a line bundle \mathcal{L}' on X' , and a space of sections \mathcal{V}' of \mathcal{L}' with properties a) and b) for p' , X' , \mathcal{L}' , \mathcal{V}' instead of p , X , \mathcal{L} , \mathcal{V} . Now let $p: \tilde{X} \rightarrow X$ be a desingularization obtained by composing a desingularization $q: \tilde{X} \rightarrow X'$ with p' . Let $\mathcal{L} = q^*\mathcal{L}'$ and $V = q^*\mathcal{V}'$.

(0.4) THEOREM. Let L be a line bundle on a possibly non-compact, irreducible, normal variety X of dimension at least 3. Assume that a finite dimensional space V of global sections of L spans L off of a finite set $F \subset X$. Let $|V|$ denote the space of zero sets of elements of V . For a general $A \in |V|$:

a) $\text{Sing}(A - F) = A \cap \text{Sing}(X - F)$ and no component of $\text{Sing}(X - F)$ is contained in $A - F$;

b) for each $i > 0$, the support of $\mathcal{S}_i(A - F)$ equals A intersected with the support of $\mathcal{S}_i(X - F)$, and no possibly embedded component of the support of $\mathcal{S}_i(X - F)$ is contained in $A - F$.

In particular $\dim \text{Irr}(X - F) \geq \dim \text{Irr}(A - F)$ and $\dim \text{Sing}(X - F) \geq \dim \text{Sing}(A - F)$ where the dimension of the empty set is taken to be $-\infty$. If X is Cohen–Macaulay then a general $A \in |V|$ is normal.

Proof. Theorem (0.4.1) of [S] shows a) and b). To see normality let note that by the above the singular set of a general $A \in |V|$ has codimension 2 at least. Since A is Cohen–Macaulay this implies that A is normal. ■

We need the following result. For completeness we include a proof and refer also to [B], (2.6.1) and [B–S], (7.5).

(0.5) LEMMA. *Let X be a normal Cohen–Macaulay projective variety. If $\text{Irr}(X)$ is finite then $\mathcal{S}_i(X)$ is 0 for $1 \leq i \leq \dim X - 2$.*

Proof. Since X is Cohen–Macaulay and projective we can choose X such that $H^i(L^{-1}) = 0$ for $i < \dim X$. By Theorem (0.4), choose a general element $A \in |L|$ such that A is normal, $\text{Irr}(A)$ is empty and $\bar{A} = p^{-1}(A)$ is smooth for some desingularization $p: \bar{X} \rightarrow X$. Write $\bar{L} = p^*L$. We have

$$\begin{array}{ccccccc} \cdots & \rightarrow & H^i(\bar{L}^{-1}) & \rightarrow & H^i(\mathcal{O}_{\bar{X}}) & \rightarrow & H^i(\mathcal{O}_{\bar{A}}) \rightarrow \cdots \\ & & \uparrow p^* & & \uparrow p^* & & \uparrow p_A^* \\ \cdots & \rightarrow & H^i(L^{-1}) & \rightarrow & H^i(\mathcal{O}_X) & \rightarrow & H^i(\mathcal{O}_A) \rightarrow \cdots \end{array}$$

By the Kawamata–Viehweg vanishing theorem (0.1), $H^i(\bar{L}^{-1}) = 0$ for $i < \dim X$. Since A has only rational singularities and \bar{A} is a desingularization of A , p_A^* is an isomorphism. Therefore from the above diagram we conclude that

$$H^i(\mathcal{O}_X) \xrightarrow{p^*} H^i(\mathcal{O}_{\bar{X}})$$

is an isomorphism for $i < \dim X - 1$ and an injection for $i = \dim X - 1$. A simple inspection of the Leray spectral sequence for p , using the assumption that the supports of $\mathcal{S}_i(X)$ are finite for $i \geq 1$, shows that $H^0(\mathcal{S}_i(X)) = 0$ for $1 \leq i \leq \dim X - 2$. Since the supports of the $\mathcal{S}_i(X)$ are finite this implies that the supports of the $\mathcal{S}_i(X)$ are empty for $1 \leq i \leq \dim X - 2$.

(0.5.1) QUESTION. Is it true that if X is Cohen–Macaulay then for $i > 0$ the support of $\mathcal{S}_i(X)$, if not empty, is pure $(\dim X - i - 1)$ -dimensional?

(0.6) Let $X \subset \mathbb{P}_C$ be a projective variety and let $L = \mathcal{O}_{\mathbb{P}_C}(1)|_X$. Let V be the subspace of $\Gamma(X, L)$ consisting of sections that vanish at a point $x \in X$. Let $p: \bar{X} \rightarrow X$ be a desingularization of X with a \mathcal{V} and \mathcal{L} as in the basic construction (0.3). Then X is a cone with vertex x if and only if \mathcal{L} is not big. In particular if X is not a cone on x then $H^i(\mathcal{L}^{-1}) = 0$ for $i < \dim X$.

For further background material we refer to [S], §0.

§1. On conditions for a variety to be a cone

(1.1) THEOREM. Let $X \subset \mathbb{P}_C$ be a normal Cohen–Macaulay projective variety of dimension $n \geq 3$. Let $L = \mathcal{O}_{\mathbb{P}_C}(1)|_X$. Let $\text{Irr}(X)$ be finite and non empty and let $N = h^0(\mathcal{S}_{n-1}(X))$. Let $x \in \text{Irr}(X)$ and let $M = h^0(\mathcal{S}_{n-2}(A))$ for a general element A of $|L|$ that contains x . Then X is a cone on x if either:

- a) $h^0(K_X \otimes L) < M + N$, or
- b) $h^{n-1}(\mathcal{O}_A) - h^{n-1}(\mathcal{O}_X) < M + N$ and $h^n(\mathcal{O}_X) = 0$.

Proof. Consider the exact sequence

$$0 \rightarrow \mathcal{O}_X(-A) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_A \rightarrow 0$$

By the Kawamata–Viehweg vanishing theorem (0.1) we have

$$h^n(\mathcal{O}_X) + h^{n-1}(\mathcal{O}_A) - h^{n-1}(\mathcal{O}_X) = h^n(-A) = h^0(K_X \otimes L) \quad (1.1.1)$$

Thus condition b) implies condition a). Therefore it suffices to prove that X is a cone on x if condition a) holds. Let V be the subspace of $\Gamma(X, L)$ consisting of sections that vanish at x and let \mathcal{V} , \tilde{X} , \mathcal{L} and $p: \tilde{X} \rightarrow X$ be as in the basic construction (0.3). Since $\text{Irr}(X)$ is finite and X is Cohen–Macaulay, $p_{(j)}(\mathcal{O}_{\tilde{X}})$ is 0 for $1 \leq j \leq n-2$ by Lemma (0.5). Thus the Leray spectral sequence gives

$$\left. \begin{aligned} h^n(\mathcal{O}_{\tilde{X}}) &= h^n(\mathcal{O}_X) - a \\ h^{n-1}(\mathcal{O}_{\tilde{X}}) &= h^{n-1}(\mathcal{O}_X) + b \\ h^j(\mathcal{O}_X) &= h^j(\mathcal{O}_{\tilde{X}}) \quad \text{for } j \leq n-2 \end{aligned} \right\} \quad (1.1.2)$$

where $a \geq 0$, $b \geq 0$ and $a + b = N$.

Let A be a general element of the linear space $|V|$ of Cartier divisors associated to V . It can be assumed that A is normal and \tilde{A} , the proper transform in \tilde{X} of A , belongs to $|\mathcal{V}|$ and is smooth. Assuming that X is not a cone we have $h^i(\mathcal{L}^{-1}) = 0$ for $i < n$ in view of (0.6). Thus by (1.1.2), $h^i(\mathcal{O}_A) = h^i(\mathcal{O}_{\tilde{A}})$ for $i \leq n-2$. Therefore by the Leray spectral sequence for $p_{\tilde{A}}$ and the fact that $\text{Irr}(A)$ is finite we conclude that

$$h^{n-1}(\mathcal{O}_{\tilde{A}}) = h^{n-1}(\mathcal{O}_A) - M \quad (1.1.3)$$

Therefore by (1.1.1), (1.1.2), (1.1.3) we have

$$\begin{aligned} h^0(K_X \otimes L) &= h^n(\mathcal{O}_{\tilde{X}}) + a + h^{n-1}(\mathcal{O}_{\tilde{A}}) + M - h^{n-1}(\mathcal{O}_{\tilde{X}}) + b \\ &= h^n(\mathcal{O}_{\tilde{X}}) + [h^{n-1}(\mathcal{O}_{\tilde{A}}) - h^{n-1}(\mathcal{O}_{\tilde{X}})] + M + N \end{aligned}$$

Note that the assumption $h^0(K_X \otimes L) < M + N$ implies that the middle term must be negative. But this is absurd since $h^{n-1}(\mathcal{L}^{-1}) = 0$.

(1.2) COROLLARY. *Let X and L be as in the above theorem. Let $\text{Irr}(X)$ be finite and non empty and assume that $h^0(K_X \otimes L) \leq 1$. Then $\text{Irr}(X)$ consists of one point x and X is a cone from that point.*

Proof. Since $\text{Irr}(X)$ is finite it follows from Lemma (0.5) that $h^0(\mathcal{S}_{n-1}(X)) > 0$. A general $A \in |L|$ passing through x has $\text{Irr}(A)$ finite by Theorem (0.4). By Elkik's theorem [E], $x \in \text{Irr}(A)$ since otherwise $x \notin \text{Irr}(X)$. Thus $h^0(\mathcal{S}_{n-2}(A)) > 0$ and

$$h^0(K_X \otimes L) \leq 1 < h^0(\mathcal{S}_{n-2}(A)) + h^0(\mathcal{S}_{n-1}(X))$$

This implies the result by the above Theorem (1.1).

(1.3) THEOREM. *Let $X \subset \mathbb{P}_{\mathbb{C}}$ be a normal Gorenstein projective variety of dimension $n \geq 3$ and let $L = \mathcal{O}_{\mathbb{P}_{\mathbb{C}}}(1)|_X$. Assume that $L^k = K_X^{-1}$ for some $k > 0$, where K_X denotes the dualizing sheaf of X . Assume that the locus of irrational singularities, $\text{Irr}(X)$, is not empty. Then $\dim \text{Irr}(X) \geq k - 1$ with equality if and only if $\text{Irr}(X)$ is a linear $\mathbb{P}_{\mathbb{C}}^{k-1}$ and X is a cone with $\text{Irr}(X)$ as vertex.*

Proof. We prove the above by induction on k .

If $k = 1$, then the assertion that $\dim \text{Irr}(X) \geq k - 1$ is an immediate consequence of the assumption that $\text{Irr}(X)$ is not empty. Since $h^0(K_X \otimes L) = h^0(K_X \otimes K_X^{-1}) = h^0(\mathcal{O}_X) = 1$ the assertion follows from Corollary (1.2).

If $k > 1$, then choose a general $A \in |L|$. By (0.4), we conclude that A is a normal Gorenstein variety on which $\dim \text{Irr}(X) = \dim \text{Irr}(A) + 1$. Since

$$K_A^{-1} = (K_X \otimes L)_A^{-1} = L_A^{k-1}$$

we conclude by the induction hypothesis that $\dim \text{Irr}(A) \geq k - 2$. This gives $\dim \text{Irr}(X) = \dim \text{Irr}(A) + 1 \geq k - 1$.

Further note that $\text{Irr}(X)$ has no isolated points as components. Indeed by the argument of [S], (0.2.2) the number of isolated points in $\text{Irr}(X)$ is bounded by $h^0(K_X \otimes L) = h^0(L^{-(k-1)}) = 0$. From this fact and the fact that $\text{Irr}(A) = A \cap \text{Irr}(X)$ is a linear $\mathbb{P}_{\mathbb{C}}^{k-2}$, it is an easy argument that $\text{Irr}(X)$ is a linear $\mathbb{P}_{\mathbb{C}}^{k-1}$. Since any general element $A \in |L|$ is a cone on $A \cap \text{Irr}(X) = \text{Irr}(A)$, elementary arguments of projective geometry show that X has to be a cone on $\text{Irr}(X)$. ■

The following settles a conjecture of Conte and Murre positively (see [C-M], section III).

(1.4) COROLLARY. *Let $X \subset \mathbb{P}_{\mathbb{C}}$ be a normal Gorenstein 3-fold with $K_X^{-1} \cong \mathcal{O}_{\mathbb{P}_{\mathbb{C}}}(1)|_X$. If $\text{Irr}(X)$ is finite then $\text{Irr}(X)$ is a single point and X is the cone from this point over a Gorenstein K3-surface A with rational singularities. Note that $\text{Sing}(X)$ is the cone over $\text{Sing}(A)$.*

§2. Final Remarks

It follows from the results in the last section of [C-M], that if X is a normal Gorenstein 3-fold with K_X^{-1} very ample and $\dim \text{Irr}(X) = 1$, then $\text{Irr}(X)$ is a linear $\mathbb{P}_{\mathbb{C}}^1$.

Let us propose the following

QUESTIONS. Let X be a normal projective Gorenstein variety of dimension $n \geq 3$. Assume that $L^k = K_X^{-1}$ for some $k > 0$, $L = \mathcal{O}_{\mathbb{P}_{\mathbb{C}}}(1)|_X$. If $\dim \text{Irr}(X) = k$, is $\text{Irr}(X)$ a linear $\mathbb{P}_{\mathbb{C}}^k$? How far can X deviate from being a cone over $\text{Irr}(X)$ in this case?

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