Zeitschrift: Commentarii Mathematici Helvetici

Herausgeber: Schweizerische Mathematische Gesellschaft

Band: 62 (1987)

Artikel: On the derived category of a finite-dimensional algebra.

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DOI: https://doi.org/10.5169/seals-47352

Comment. Math. Helvetici 62 (1987) 339–389 0010-2571/87/030339-51$01.50 + 0.20/0

On the derived category of a finite-dimensional algebra

DIETER HAPPEL

Let A be a finite-dimensional associative algebra with 1 over a field k (which we suppose to be algebraically closed throughout this article). The thread of this work is the investigation of the derived category $D^b(A)$ of bounded complexes over the category $\operatorname{mod} A$ of finite-dimensional left A-modules.

The construction of $D^b(\mathscr{A})$ for an arbitrary abelian category $\mathscr{A}$ goes back to the inspiration of Grothendieck. The formulation in terms of triangulated categories was developed by Verdier [V].

Our main results describe $D^b(A)$ if A has finite global dimension. In section 1 we show that $D^b(A)$ is suitable for studying tilting processes. Indeed, we prove that for a tilting triple $(A, {}_AM_B, B)$ (compare 1.7) the derived categories $D^b(A)$ and $D^b(B)$ are equivalent as triangulated categories. Since our intuition is geometric it is useful to determine the quiver of an additive category (compare 3.7). In section 4 we compute the quiver of $D^b(A)$ for a hereditary finite-dimensional k-algebra A.

If $\vec{\Delta}$ is a Dynkin-quiver (i.e. the underlying graph of $\vec{\Delta}$ is a Dynkin diagram of type $\mathbb{A}_n$, $\mathbb{D}_n$, $\mathbb{E}_6$, $\mathbb{E}_7$ or $\mathbb{E}_8$) we derive a description of the finite-dimensional k-algebras A such that $D^b(A)$ and $D^b(k\vec{\Delta})$ are equivalent as triangulated categories (compare section 5).

In section 10 we associate with A an infinite-dimensional k-algebra $\hat{A}$ without 1 called the repetitive algebra. If follows from general considerations on Frobenius categories (section 9) that the stable category $\underline{\operatorname{mod}}\,\hat{A}$ (10.1) is a triangulated category. Our main theorem asserts that $\underline{\operatorname{mod}}\,\hat{A}$ and $D^b(A)$ are equivalent as triangulated categories if A has finite global dimension.

These results were announced at the Conference on Representations of Algebras in Ottawa 1984.

My special thanks go to C. M. Ringel who introduced me to representation theory. His ideas written or unwritten influenced this work quite considerably. I am indebted to P. Gabriel for his valuable efforts during the preparation of this manuscript. Also I thank him for pointing out a false argument in the proof of Theorem 10.10.

0. Notation and terminology

In this preliminary section we present the main notions used throughout this work and give some guidance to basis texts we need to refer to.

0.1 Given any category $\mathcal{K}$ *the composition of morphisms* $f:X\to Y$ and $g:Y\to Z$ in $\mathcal{K}$ is denoted by fg.

We usually adopt the categorical language of [ML]. In particular, our additive categories have finite direct sums. Unless otherwise stated, we assume that they are *Krull–Schmidt categories* (see [Ri6]).

Let $\mathfrak{a}$ be an additive category. A *path* in $\mathfrak{a}$ is a sequence of indecomposable objects X_i $(0 \leqslant i \leqslant r)$ and non-zero morphisms $f_i: X_i \to X_{i+1}$ $(0 \leqslant i < r)$ lying in the radical $\mathcal{R}\operatorname{Hom}(X_i, X_{i+1})$ [Ri6]. If $r>0$ and $X_0 = X_r$, the path is called a *cycle.* We call $\mathfrak{a}$ *directed* if it does not contain any cycle.

0.2 A *differential complex* or simply a *complex* $X^{\cdot} = (X^i, d_X^i)_{i\in\mathbb{Z}}$ over $\mathfrak{a}$ is by definition a collection of objects X^i and morphisms $d^i = d_X^i : X^i \to X^{i+1}$ such that $d^i d^{i+1} = 0$. A complex $X^{\cdot} = (X^i, d^i)$ is *bounded below* if $X^i = 0$ for all but finitely many $i<0$ and *bounded above* if $X^i = 0$ for all but finitely many $i>0$. It is *bounded* if it is bounded above and below. A complex $X^{\cdot} = (X^i, d^i)$ is a *stalk complex* if there exists i_0 such that $X^{i_0} \neq 0$ and $X^i = 0$ for all $i \neq i_0$. The object X^{i_0} is then called the *stalk.*

Suppose that $X^i = 0$ for $i<r$ and $s<i$ and $X^r \neq 0 \neq X^s$. Then the *width* $w(X^{\cdot})$ of $X^{\cdot}$ is by definition equal to $s-r+1$. If $s \leqslant 0$, s is called the *deviation* of $X^{\cdot}$ and is denoted by $d(X^{\cdot})$.

Denote by $C(\mathfrak{a})$ the category of complexes over $\mathfrak{a}$, by $C^+(\mathfrak{a})$ (resp. $C^-(\mathfrak{a})$, resp. $C^b(\mathfrak{a})$) the full subcategories of complexes bounded below (resp. above, resp. above and below). If $X^{\cdot} = (X^i, d_X^i)$ and $Y^{\cdot} = (Y^i, d_Y^i)$ are two complexes, a morphism $f^{\cdot}: X^{\cdot} \to Y^{\cdot}$ is a sequence of morphisms $f^i : X^i \to Y^i$ of $\mathfrak{a}$ such that $d_X^i f^{i+1} = f^i d_Y^i$ for all $i \in \mathbb{Z}$. These morphisms are composed in an obvious way.

There is a full embedding of $\mathfrak{a}$ into $C(\mathfrak{a})$ which sends each object X of $\mathfrak{a}$ into the stalk complex $X^{\cdot} = (X^i, d^i)$ with $X^0 = X$. We will identify this complex with X.

The *shift functor* T is defined by $(TX^{\cdot})^i = X^{i+1}$, $(d_{TX})^i = -(d_X)^{i+1}$ and $(Tf^{\cdot})^i = f^{i+1}$ if $f^{\cdot}$ is a morphism of $C(\mathfrak{a})$. It is an automorphism of $C(\mathfrak{a})$. We denote the inverse by T^-.

The *mapping cone* $C_f^{\cdot}$ of a morphism $f^{\cdot}: X^{\cdot} \to Y^{\cdot}$ is the complex $C_f^{\cdot} = ((TX^{\cdot})^i \oplus Y^i, d_{C_f}^i)$ with "differential"

$$X^{i+1} \oplus Y^i \xrightarrow{\begin{pmatrix} -d_X^{i+1} & f^{i+1} \\ 0 & d_Y^i \end{pmatrix}} X^{i+2} \oplus Y^{i+1}$$

For instance if $Z^{\cdot} \in C(\mathfrak{a})$ satisfies $Z^i = 0$ for $i < 0$, and if $Z'^{\cdot}$ is the associated *truncated complex* ($Z'^i = 0$ for $i \leqslant 0$ and $d^i_{Z'} = d^i_Z$ for $i \geqslant 1$), d^0_Z induces a morphism from $T^- Z^0$ to $Z'^{\cdot}$ whose mapping cone is $Z^{\cdot}$.

If $\mathfrak{a}$ is a full subcategory of an abelian category $\mathscr{A}$ then the *cohomology objects* $H^i(X^{\cdot})$ are defined for $X^{\cdot} \in C^b(\mathfrak{a})$. And a morphism $u^{\cdot}: X^{\cdot} \to Y^{\cdot}$ of $C^b(\mathfrak{a})$ is called a *quasi-isomorphism* if the induced morphisms $H^i(u^{\cdot}): H^i(X^{\cdot}) \to H(Y^{\cdot})$ are isomorphisms for all i.

0.3 Let $\mathscr{C}$ be an additive category and T an automorphism of $\mathscr{C}$, which will be called the *translation functor*. A *sextuple* (X, Y, Z, u, v, w) in $\mathscr{C}$ is given by objects X, Y, Z and morphisms $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX$. A *morphism of sextuples* from (X, Y, Z, u, v, w) to (X', Y', Z', u', v', w') is a triple (f, g, h) of morphisms such that the following diagram commutes:

$$\begin{array}{ccccccc}
X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & TX \\
{\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle h}\downarrow & & {\scriptstyle Tf}\downarrow \\
X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & TX'
\end{array}$$

Following Verdier [V], we call a set $\mathscr{T}$ of sextuples in $\mathscr{C}$ a *triangulation* of $\mathscr{C}$ if the following conditions are satisfied. The elements of $\mathscr{T}$ are then called *triangles*.

(TR1) Every sextuple isomorphic to a triangle is a triangle. Every morphism $u: X \to Y$ can be embedded into a triangle (X, Y, Z, u, v, w). The sextuple $(X, X, 0, 1_X, 0, 0)$ is a triangle.

(TR2) (X, Y, Z, u, v, w) is a triangle if and only if $(Y, Z, TX, v, w, -Tu)$ is a triangle.

(TR3) Given two triangles (X, Y, Z, u, v, w) and (X', Y', Z', u', v', w'), and morphisms $f: X \to X'$, $g: Y \to Y'$ such that $fu' = ug$, there exists a morphism (f, g, h) from the first triangle to the second.

(TR4) (The octahedral axiom). Consider triangles (X, Y, Z', u, i, i'), (Y, Z, X', v, j, j') and (X, Z, Y', uv, k, k'). Then there exists morphisms $f: Z' \to Y'$, $g: Y' \to X'$ such that the following diagram commutes and the third row is a triangle.

$$\begin{array}{ccccccccc}
T^-Y' & \xrightarrow{T^-k'} & X & \xrightarrow{1_X} & X & & & & \\
\downarrow{\scriptstyle T^-g} & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle uv} & & & & \\
T^-X' & \xrightarrow{T^-j'} & Y & \xrightarrow{v} & Z & \xrightarrow{j} & X' & \xrightarrow{j'} & TY \\
& & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle 1_{X'}} & & \downarrow{\scriptstyle Ti} \\
& & Z' & \xrightarrow{f} & Y' & \xrightarrow{g} & X' & \xrightarrow{j'Ti} & TZ' \\
& & \downarrow{\scriptstyle i'} & & \downarrow{\scriptstyle k'} & & & & \\
& & TX & \xrightarrow{1_{TX}} & TX & & & &
\end{array}$$

(Compare with [V]; our "non-octohedral" presentation is best suited for section 3.)

The additive category $\mathscr{C}$ together with a translation functor T and a triangulation $\mathscr{T}$ is called a *triangulated category.*

Let $\mathscr{C}$, $\mathscr{C}'$ be triangulated categories. An additive functor $F:\mathscr{C}\rightarrow\mathscr{C}'$ is called *exact* if it commutes up to isomorphism with the translation functors and sends triangles to triangles.

If an exact functor $F:\mathscr{C}\rightarrow\mathscr{C}'$ is an equivalence of categories, we call it a *triangle-equivalence.* $\mathscr{C}$ and $\mathscr{C}'$ are then called *triangle-equivalent.* For the basic properties of triangulated categories we refer the reader to [V], [Ha] and [BBD].

0.4 Examples of triangulated categories are the *homotopy categories* $K(\mathfrak{a})$, $K^+(\mathfrak{a})$, $K^-(\mathfrak{a})$ and $K^b(\mathfrak{a})$ associated with the categories of complexes defined in 0.2 or the *derived categories* $D(\mathscr{A})$, $D^+(\mathscr{A})$, $D^-(\mathscr{A})$, $D^b(\mathscr{A})$ if $\mathscr{A}$ is an abelian category [Ha, chapter 1]. The *localization functor* from $K^b(\mathscr{A})$ to $D^b(\mathscr{A})$ will be denoted by Q^b. Note that $\mathscr{A}$ becomes a full subcategory of $D^b(\mathscr{A})$ by sending each object of $\mathscr{A}$ into the corresponding stalk complex.

0.5 We will mainly deal with finite-dimensional algebras (associative with 1) over k. By $\operatorname{mod} A$ we denote the category of finite-dimensional left A-modules. Its derived category (of bounded complexes) is denoted by $D^b(A)$. Certain full subcategories of $\operatorname{mod} A$ are of interest to us. By ${}_A\mathscr{P}$, ${}_A\mathscr{I}$ we denote the full subcategories of $\operatorname{mod} A$ having as objects the projective A-modules and the injective A-modules respectively. For an A-module M we denote by $\operatorname{add} M$ the full subcategory of $\operatorname{mod} A$ having as objects the direct sums of summands of M.

For the basic properties in representation theory we refer the reader to [G2] and [Ri6].

0.6 In section 10 we will consider infinite-dimensional k-algebras (without 1). The information on covering techniques needed in sections 5, 7, 10 can be found in [BG] and [G3].

1. Invariance under tilting functors

Let A be a finite-dimensional k-algebra which we suppose to be of finite global dimension throughout this section. Let M be an A-module. Then we obtain a natural functor $\varphi: K^b(\operatorname{add} M)\rightarrow D^b(A)$ which is the composition of the embedding functor $K^b(\operatorname{add} M)$ into $K^b(\operatorname{mod} A)$ and the localization functor $Q^b: K^b(\operatorname{mod} A)\rightarrow D^b(A)$.

1.1 LEMMA. *If* $\mathrm{Ext}_A^i(M, M)=0$ *for all* $i>0$, *then* φ *is full and faithful.*

Proof. Let $M_1^{\cdot}, M_2^{\cdot} \in K^b(\mathrm{add} M)$. Applying T if necessary, we may assume that $M_2^i=0$ for $i<0$ and $M_2^0=M_2\neq 0$. We proceed by double induction on the widths of $M_1^{\cdot}$ and $M_2^{\cdot}$. If $w(M_1^{\cdot})=w(M_2^{\cdot})=1$, then there exists $i\in\mathbb{Z}$ such that $M_1^{\cdot}=T^iM_1$ for some $M_1\in \mathrm{add}M$. If $i=0$, then $\mathrm{Hom}_{K^b(\mathrm{add}\,M)}(M_1^{\cdot}, M_2^{\cdot})=\mathrm{Hom}_{D^b(A)}(M_1^{\cdot}, M_2^{\cdot})$. Otherwise $\mathrm{Hom}_{K^b(\mathrm{add}\,M)}(M_1^{\cdot}, M_2^{\cdot})=0$ and $\mathrm{Hom}_{D^b(A)}(M_1^{\cdot}, M_2^{\cdot})=0$ for $i>0$ and $=\mathrm{Ext}_A^{-i}(M_1, M_2)$ for $i<0$, and the assertion follows by assumption.

If $w(M_1^{\cdot})=1$ and $w(M_2^{\cdot})=r$, then we consider the triangle $T^-M_2^0\to M_2^{\prime\cdot}\to M_2^{\cdot}\to M_2^0$ where $M_2^{\prime\cdot}$ is the truncated complex (0.2). We apply the cohomological functors $\mathrm{Hom}_{K^b(\mathrm{add}\,M)}(M_1^{\cdot}, -)$ and $\mathrm{Hom}_{D^b(A)}(M_1^{\cdot}, -)$ to this triangle. Using induction and the 5-lemma we infer that $\mathrm{Hom}_{K^b(\mathrm{add}\,M)}(M_1^{\cdot}, M_2^{\cdot}) \xrightarrow{\sim} \mathrm{Hom}_{D^b(A)}(M_1^{\cdot}, M_2^{\cdot})$ under φ.

The remaining part of the proof is dual.

1.2 We say that an A-module X has *finite M-codimension* (M-codim$(X)<\infty$) if there exists an exact sequence $0\to X\to M^0\to M^1\to\cdots\to M^s\to 0$ with $M^i\in \mathrm{add}M$ for $0\leq i\leq s$.

LEMMA. *Let M be an A-module such that* $\mathrm{Ext}_A^i(M, M)=0$ *for* $i>0$ *and suppose that* ${}_AA$ *has finite M-codimension. Then* proj. dim $M\leq r$ *implies that there is an exact sequence* $0\to {}_AA\to M^0\to M^1\cdots\to M^{s-1}\to M^s\to 0$ *such that* $s\leq r$.

Proof. By assumption there exists an exact sequence $0\to {}_AA\to M^0\to M^1\to \cdots\to M^{s-1}\xrightarrow{d^{s-1}} M^s\to 0$. We choose such an exact sequence with s minimal. Assume $s>r$ and set $K^{s-1}=\ker d^{s-1}$. It follows that $\mathrm{Ext}_A^1(M, K^{s-1})=0$. Therefore d^{s-1} is a retraction. This contradicts the minimality of s. So $s\leq r$.

1.3 LEMMA. *Let M be an A-module such that* $\mathrm{Ext}_A^i(M, M)=0$ *for* $i>0$. *Let P be an indecomposable projective A-module. If* M-codim$({}_AA)<\infty$, *then* M-codim$(P)<\infty$.

Proof. Let $B=\mathrm{End}\,M$. Then proj. dim $M_B<\infty$. In fact, apply $\mathrm{Hom}_A(-, M)$ to the exact sequence $0\to {}_AA\to M^0\to M^1\to\cdots\to M^s\to 0$. This give a finite projective resolution of M_B. Let P be an indecomposable projective A-module. Then $P=Ae$ for some primitive idempotent $e\in A$. Let $0\to Q_t\to\cdots\to Q_0\to eM\to 0$ be a projective resolution of eM considered as right B-module. This implies that M-codim$(P)<\infty$.

1.4 For later reference we include here also

LEMMA. *Let A and B be finite dimensional k-algebras such that $D^b(A)$ and $D^b(B)$ are triangle-equivalent. Then A is of finite global dimension if and only if so is B.*

Proof. Suppose that $\text{gl.dim}\, A < \infty$ and let F be a triangle-equivalence from $D^b(B)$ to $D^b(A)$. Let S_i, S_j be simple B-modules. It is enough to show that there exists an $r_0 \in \mathbb{N}$, independent of S_i, S_j, such that $\text{Ext}_B^r(S_i, S_j) = 0$ for all $r \geq r_0$. As $\text{Ext}_B^r(S_i, S_j) = \text{Hom}_{D^b(B)}(S_i, T^r S_j) \xrightarrow{\sim} \text{Hom}_{D^b(A)}(F(S_i), T^r F(S_j))$ there exists $r_{ij} \in \mathbb{N}$ with $\text{Hom}_{D^b(A)}(F(S_i), T^r(S_j))$ for $r \geq r_{ij}$. Then the assertion follows for $r_0 = \max_{ij} r_{ij}$.

1.5 LEMMA *Let M be an A-module such that* $\text{Ext}_A^i(M, M) = 0$ *for* $i > 0$. If $M\text{-codim}\,({}_AA) < \infty$, *then the functor* $\varphi: K^b(\text{add}M) \to D^b(A)$ *is dense.*

Proof. Since A has finite global dimension, $D^b(A)$ is triangle-equivalent to $K^b({}_A\mathcal{P})$ by Proposition II, 1.4 of [V]. As M-codim$\,({}_AA) < \infty$, also M-codim$\,(P) < \infty$ for a projective A-module P by 1.3 above. Since φ is exact we infer that φ is dense.

1.6 THEOREM. *Let M be an A-module such that* $\text{Ext}_A^i(M, M) = 0$ *for* $i > 0$ *and suppose that ${}_AA$ has finite M-codimension. Let* $B = \text{End}\, M$ *and suppose that* $\text{gl.dim}\, B < \infty$. *Then the functor* $F = \text{Hom}_A(M, -): \text{mod}\, A \to \text{mod}\, B$ *induces a triangle-equivalence* $\bar{F}: D^b(A) \to D^b(B)$.

Proof. Clearly F induces a triangle-equivalence $\bar{F}: K^b(\text{add}M) \to K^b({}_B\mathcal{P})$. By 1.1 and 1.5 the result follows since $K^b({}_B\mathcal{P})$ and $D^b(B)$ are triangle-equivalent again by Proposition II, 1.4 of [V].

1.7 The interest for us in studying these properties of A-modules comes from tilting theory [HR], see also [BB]. An A-module M is called a *tilting module* if the following conditions are satisfied: (i) $\text{proj.dim}\, M \leq 1$, (ii) $\text{Ext}_A^1(M, M) = 0$, (iii) M-codim$\,({}_AA) \leq 1$. We call the triple $(A, {}_AM_B, B)$ a *tilting triple* if ${}_AM$ is a tilting module and $B = \text{End}\, M$.

COROLLARY. *Let $(A, {}_AM_B, B)$ be a tilting triple. Then $D^b(A)$ and $D^b(B)$ are triangle-equivalent.*

Proof. This follows from 1.6 above using Corollary 1.7.1 of [Bo1].

Here we refrain from deriving the fundamental results in tilting theory (compare [HR] and [Bo1]). But we hope to come back to this in a subsequent publication.

We say that two-finite-dimensional k-algebras A and B are *tilting-equivalent* if there exists a sequence $(A_i, {}_{A_i}M^i_{A_{i+1}}, A_{i+1})$ of tilting triples for $0 \leq i < m$ such that $A = A_0$ and $B = A_m$.

2. Isometry of Grothendieck groups

2.1 The material discussed here is quite classical (compare [Gr]). We leave out the proofs, for they are straightforward from the definitions. Let $\mathcal{F}$ be the free abelian group generated by representatives of the isomorphism classes of objects in $D^b(A)$, where A is a basic finite-dimensional k-algebra. We denote by $[X^{\cdot}]$ such a representative. Let $\mathcal{F}_0$ be the subgroup generated by $[X^{\cdot}] - [Y^{\cdot}] + [Z^{\cdot}]$ for all triangles $X^{\cdot} \to Y^{\cdot} \to Z^{\cdot} \to TX^{\cdot}$ in $D^b(A)$. The *Grothendieck group* $K_0(D^b(A))$ is by definition the factor group $\mathcal{F}/\mathcal{F}_0$.

A $\mathbb{Z}$-valued function a defined on the objects of $D^b(A)$ is called additive if $a(X^{\cdot}) - a(Y^{\cdot}) + a(Z^{\cdot}) = 0$ for all triangles $X^{\cdot} \to Y^{\cdot} \to Z^{\cdot} \to TX^{\cdot}$ in $D^b(A)$. The condition implies that $a(X^{\cdot}) = -a(TX^{\cdot})$. It is shown in [Gr] that $K_0(A)$ and $K_0(D^b(A))$ are isomorphic, where $K_0(A)$ is the Grothendieck group of A (see [GR], [Ri6]). Indeed, the embedding of $\operatorname{mod} A$ into $D^b(A)$ (0.4) induces an isomorphism.

2.2 Let $P(1), \ldots, P(n)$ be a complete set of representatives of the isomorphism classes of indecomposable projective A-modules. For an A-module X the *dimension vector* is defined by $\underline{\dim}\, X = (\dim_k \operatorname{Hom}_A(P(i), X))$. The map $X \to \underline{\dim}\, X$ induces an isomorphism of $K_0(A)$ with $\mathbb{Z}^n$. Using 2.1 this can be extended to $D^b(A)$. If $X^{\cdot} = (X^i, d^i) \in D^b(A)$, we obtain $\underline{\dim}\, X^{\cdot} = \sum_{i \in \mathbb{Z}} (-1)^i \underline{\dim}\, X^i$. Since $X^{\cdot}$ is bounded, the sum is finite.

2.3 *Remark.* This shows that each component $\underline{\dim}^j$ of $\underline{\dim}$ is an additive function on the objects of $D^b(A)$.

2.4 For the rest of this section we assume that A has finite global dimension. The Grothendieck group $K_0(A)$ is endowed with a bilinear form. We recall the relevant definitions, referring to 2.4 of [Ri6] for a more thoroughful treatment.

Let $C = C_A$ be the *Cartan matrix* of A. This is an $n \times n$ integer-valued matrix with entries $C_{ij} = \dim_k \operatorname{Hom}_A(P(i), P(j))$ $(1 \leq i, j \leq n)$. Thus the jth column of C is $(\underline{\dim}\, P(j))^t$, where t denotes the transpose. By a classical result $C = C_A$ is invertible. (See [Ri6]).

The matrix $C^{-t} = (C^{-1})^t$ defines a bilinear form $\langle -, - \rangle_A$ on $K_0(A) = \mathbb{Z}^n$ by

$\langle x, y\rangle_A = xC^{-t}y^t$. The corresponding quadratic form $\chi_A(x) = \langle x, x\rangle_A$ is called the *Euler characteristic* of A.

The introduced bilinear form has the following homological interpretation (compare with 2.4 of [Ri6]). Let X, Y be A-modules, then

(*) $\langle \underline{\dim}\, X, \underline{\dim}\, Y\rangle_A = \sum_{i\geq 0} (-1)^i \dim_k \operatorname{Ext}_A^i (X, Y)$

2.5 Using 2.1 and (*) we obtain:

LEMMA. *Let $X^\cdot$, $Y^\cdot \in D^b(A)$. Then*

$$\langle \underline{\dim}\, X^\cdot, \underline{\dim}\, Y^\cdot\rangle_A = \sum_{i\in \mathbb{Z}} (-1)^i \dim_k \operatorname{Hom}_{D^b(A)}(X^\cdot, T^iY^\cdot).$$

2.6 Let A and B be basic finite-dimensional k-algebras. We say that $K_0(A)$ and $K_0(B)$ are *isometric* if there exists an *isometry* $f: K_0(A)\to K_0(B)$ i.e. a linear bijection such that $\langle x, y\rangle_A = \langle xf, yf\rangle_B$ for all $x, y \in K_0(A)$. The use of $\langle -, -\rangle$ instead of χ will prove to be essential in section 5.

PROPOSITION. *Let A and B be basic finite-dimensional k-algebras and assume that A has finite global dimension. If $F: D^b(A)\to D^b(B)$ is a triangle-equivalence, there exists an isometry $f: K_0(A)\to K_0(B)$ such that $\underline{\dim}\, F(X^\cdot) = (\underline{\dim}\, X^\cdot)f$ for $X^\cdot \in D^b(A)$. In particular, A and B have the same number of simple modules up to isomorphism.*

Note that for a traingle-equivalence induced by a tilting triple $(A, {}_AM_B, B)$ (1.7) this is 3.2 of [HR].

3. Auslander–Reiten triangles

3.1 Let $\mathscr{C}$ be a triangulated category such that $\operatorname{Hom}_{\mathscr{C}} (X, Y)$ is a finite-dimensional k-vector space for all $X, Y \in \mathscr{C}$ and assume that the endomorphism ring of an indecomposable object is local. This assumption ensures that $\mathscr{C}$ is a Krull–Schmidt category (compare 2.2 of [Ri6]).

A triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX$ in $\mathscr{C}$ is called an *Auslander–Reiten triangle* if the following conditions are satisfied:

(AR1) X, Z are indecomposable
(AR2) $w \neq 0$

(AR3) If $f: W \to Z$ is not a retraction, then there exists $f': W \to Y$ such that $f'v = f$.

We will say that $\mathscr{C}$ has Auslander–Reiten triangles if for all indecomposable objects $Z \in \mathscr{C}$ there is a triangle satisfying the conditions above. Our motivation comes from Auslander–Reiten sequences, which are by definition non-split exact sequences $0 \to X \to Y \to Z \to 0$ of finite-dimensional modules satisfying (AR1) and (AR3).

Here we present some of the properties which carry over to Auslander–Reiten triangles. We will provide full proofs but acknowledge the influence of [AR] and [G2].

3.2 REMARK. The following are equivalent for a triangle as above: (i) (AR2); (ii) u is not a section; (iii) v is not a retraction.

Proof. In fact, if $w = 0$ consider the following morphism of traingles. The existence of u' is guaranteed by (TR3) (compare 0.3):

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & TX \\ \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle u'} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 1} \\ X & \xrightarrow{1} & X & \xrightarrow{0} & 0 & \xrightarrow{0} & TX \end{array}$$

This shows that u is a section. The converse is also proved using the diagram above. In the same way one can show that (i) and (iii) are equivalent.

3.3 REMARK. The following are equivalent for a triangle as above: (i) (AR3); (ii) If $f: W \to Z$ is not a retraction, then $fw = 0$.

Proof. The result follows since $\mathrm{Hom}_{\mathscr{C}}(W, -)$ is a cohomological functor by I.1 of [Ha].

3.4 LEMMA (Selfduality for Auslander–Reiten triangles). *Let $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX$ be an Auslander–Reiten triangle. If $f: X \to W$ is not a section, then there exists $f': Y \to W$ with $uf' = f$.*

Proof: By (TR1) (compare 0.3) the morphism $f: X \to W$ can be embedded into a triangle $X \xrightarrow{f} W \xrightarrow{g} W' \xrightarrow{h} TX$. Using (TR2) we see that $T^{-}W' \xrightarrow{-T^{-}h} X \xrightarrow{f} W \xrightarrow{g} W'$ is again a triangle. We apply the octahedral axiom (TR4) to the composition $(-T^{-}h)u$ and obtain the following diagram of triangles.

$$\begin{array}{ccccccc}
T^{-}W' & \xrightarrow{1} & T^{-}W' & & & & \\
\downarrow{\scriptstyle -T^{-}h} & & \downarrow{\scriptstyle -T^{-}hu} & & & & \\
X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & TX \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle Tf} \\
W & \xrightarrow{t_1} & Y' & \xrightarrow{t_2} & Z & \xrightarrow{wTf} & TW \\
\downarrow{\scriptstyle g} & & \downarrow{\scriptstyle s} & & & & \\
W' & \xrightarrow{1} & W' & & & &
\end{array}$$

If t_2 is a retraction, then t_1 is a section by 3.2. So there exists t_1' with $t_1t_1' = 1_W$. Now define $f' = rt_1'$. Then $uf' = urt_1' = ft_1t_1' = f$.

So assume that t_2 is not a retraction. Then there exists $t_2' : Y' \to Y$ with $t_2'v = t_2$ by (AR3).

Consider the following morphisms of triangles ($\bar{f}$ exists by (TR3)):

$$\begin{array}{ccccccc}
X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & TX \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle Tf} \\
W & \xrightarrow{t_1} & Y' & \xrightarrow{t_2} & Z & \xrightarrow{wTf} & TW \\
\downarrow{\scriptstyle \bar{f}} & & \downarrow{\scriptstyle t_2'} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle T\bar{f}} \\
X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & TX
\end{array}$$

Since f is not a section and X is indecomposable, we infer that $f\bar{f}$ is nilpotent. Hence there exists $n \in \mathbb{N}$ such that $(f\bar{f})^n = 0$. Therefore

$$\begin{array}{ccccccc}
X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & TX \\
\downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle (rt_2')^n} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 0} \\
X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & TX
\end{array}$$

is a morphism of triangles. But then $w = 0$ gives the required contradiction.

3.5 A morphism h between Z_1 and Z_2 of an arbitrary additive category is called *irreducible* if h is neither a section, nor a retraction but for any factorization $h = h_1h_2$ either h_1 is a section or h_2 is a retraction. For definitions using the radical of the category we direct the reader to [G2] or [Ri6].

PROPOSITION. *Let $X \xrightarrow{x} Y \xrightarrow{v} Z \xrightarrow{w} TX$ be an Auslander–Reiten triangle.*

(i) *Given Z it is unique up to isomorphism of triangles.*

(ii) *u and v are irreducible morphisms.*
(iii) *If $f: Z_1 \to Z$ is irreducible, there is a section $g: Z_1 \to Y$ with $f = gv$.*
(iv) *If $f: X \to X_1$ is irreducible, there is a retraction $g: Y \to X_1$ with $f = ug$.*

Proof. (i) Let $X' \xrightarrow{u'} Y' \xrightarrow{v'} Z \xrightarrow{w'} TX'$ be an Auslander–Reiten triangle. Since v' is not a retraction there exists g with $v' = gv$. By (TR3) we obtain a morphism of triangles:

$$\begin{array}{ccccccc} X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z & \xrightarrow{w'} & TX' \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle Tf} \\ X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & TX \end{array}$$

If f is not an isomorphism we obtain a morphism f' with $u'f' = f$ by 3.4. But $w = w'Tf = w'Tu'Tf' = 0$ gives a contradiction. Thus f is an isomorphism and so is g by I.1 of [Ha].

(ii) We will show that u is irreducible. In fact, consider a factorization $u = h_1 h_2$. If h_1 is not a section, there exists h_1' with $uh_1' = h_1$. By (TR3) we obtain a morphism of triangles:

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & TX \\ \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle h_1'h_2} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle 1} \\ X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & TX \end{array}$$

If h is not an isomorphism, then $w = hw = 0$ by 3.3, a contradiction. Thus $h_1'h_2$ is an isomorphism. Therefore h_2 is a retraction.

(iii) Let $f: Z_1 \to Z$ be irreducible. Since f is not a retraction we obtain $g: Z_1 \to Y$ with $f = gv$. As v is not a retraction, g is a section.

(iv) This is dual to (iii).

3.6 Let A be a finite-dimensional k-algebra of finite global dimension.

THEOREM. *The derived category $D^b(A)$ has Auslander–Reiten triangles.*

Proof. It is well-known that ${}_A\mathcal{P}$ and ${}_A\mathcal{I}$ (compare 0.5) are equivalent under the Nakayama functor $\nu = D\,\mathrm{Hom}_A(-, {}_AA)$, where D denotes the duality on mod A with respect to the base field k. There is an invertible natural transformation $\alpha_P: D\,\mathrm{Hom}(P, -) \to \mathrm{Hom}(-, \nu P)$. Equivalently, for each $X \in \mathrm{mod}\,A$, there is a vectorspace duality $\mathrm{Hom}(P, X) \times \mathrm{Hom}(X, \nu P) \to k$, $(\xi, \eta) \to (\xi \mid \eta)$ such that $(\xi\mu \mid \eta) = (\xi \mid \mu\eta)$ and $(\pi\xi \mid \eta) = (\xi \mid \eta\nu(\pi))$ for all morphisms μ in mod A and all π in ${}_A\mathcal{P}$.

The Nakayama functor ν induces an equivalence of triangulated categories again denoted by ν between $K^b({}_A\mathscr{P})$ and $K^b({}_A\mathscr{I})$ and an invertible natural transformation $\alpha_{P^\cdot}: D\operatorname{Hom}(P^\cdot, -)\to \operatorname{Hom}(-, \nu P^\cdot)$. In fact, if $X^\cdot$ is an object of $K^b(\operatorname{mod} A)$, the associated duality $\operatorname{Hom}(P^\cdot, X^\cdot)\times \operatorname{Hom}(X^\cdot, \nu P^\cdot)\to k$, $(\xi^\cdot, \eta^\cdot)\to(\xi^\cdot \mid \eta^\cdot)$ is defined by $(\xi^\cdot \mid \eta^\cdot)=\sum_{n\in\mathbb{Z}}(-1)^n(\xi^n \mid \eta^n)$.

Since A has finite global dimension $D^b(A)$ is triangle-equivalent to $K^b({}_A\mathscr{P})$ and to $K^b({}_A\mathscr{I})$. Thus an object in $D^b(A)$ can be written in the form $P^\cdot$, where $P^\cdot$ is contained in $K^b({}_A\mathscr{P})$.

Now asume that $P^\cdot$ is indecomposable in $D^b(A)$. Let $\varphi\in D\operatorname{Hom}(P^\cdot, P^\cdot)$ be the linear form on $\operatorname{End}(P^\cdot)$ which vanishes on the radical $\operatorname{rad}\operatorname{End}(P^\cdot)$ and satisfies $\varphi(1_{P^\cdot})=1$. We consider the image $\alpha_{P^\cdot}(\varphi)$; it is a non-zero linear map from $P^\cdot$ to $\nu P^\cdot$ such that $f\alpha_{P^\cdot}(\varphi)=0$ whenever the morphism f of $D^b(A)$ is not a retraction. This implies that

$$T^-\nu P^\cdot \xrightarrow{i} C_{(T^-\alpha_{P^\cdot}(\varphi))} \xrightarrow{-p} P^\cdot \xrightarrow{\alpha_{P^\cdot}(\varphi)} \nu P^\cdot$$

satisfies the axiom (AR3) by 3.3. Therefore this triangle is an Auslander–Reiten triangle.

3.7 By definition, the vertices of the *quiver* $\Gamma=\Gamma(\mathfrak{a})$ of a Krull–Schmidt category $\mathfrak{a}$ are the isomorphism classes [X] of the indecomposable objects X of $\mathfrak{a}$. The quiver has an arrow $[X]\to[Y]$ if there is an irreducible morphism from X to Y in $\mathfrak{a}$.

COROLLARY. *Let A be a finite-dimensional k-algebra of finite global dimension. Then $\Gamma(D^b(A))$ has the structure of a stable translation quiver* (see [R]).

Proof. Observe that $D^b(A)\simeq K^b({}_A\mathscr{P})$ is a Krull–Schmidt category and let $P^\cdot$ be as in 3.6. We define $\tau P^\cdot := T^-\nu P^\cdot$. It follows from 3.5 that $(\Gamma(D^b(A), \tau)$ is a stable translation quiver (*stable* means the translation τ is defined for all vertices). Note that in our situation τ is induced by an equivalence on $\operatorname{ind} D^b(A)$.

3.8 PROPOSITION. *Let A be a finite-dimensional k-algebra of finite global dimension and $X^\cdot$, $Y^\cdot\in D^b(A)$. Then*

$$D\operatorname{Hom}_{D^b(A)}(T^{2i-1}X^\cdot, Y^\cdot)\simeq \operatorname{Hom}_{D^b(A)}(Y^\cdot, \tau T^{2i}X^\cdot) \quad \textit{for all} \quad i.$$

Proof. Let $P_1^\cdot\simeq X^\cdot$ and $P_2^\cdot\simeq Y^\cdot$ with $P_1^\cdot, P_2^\cdot\in K^b({}_A\mathscr{P})$. Clearly T commutes with τ. So $\tau T^{2i}X^{\cdot} = \tau T^{2i}P_1^{\cdot} = \nu T^{2i-1}P_1^{\cdot}$. And the isomorphism is induced by the invertible natural transformation $\alpha_{T^{2i-1}P_1^{\cdot}}$ (compare 3.6).

4. The quiver of $D^b(k\vec{\Delta})$)

4.1. Let A be a hereditary, basic finite dimensional k-algebra (i.e. the path algebra $k\vec{\Delta}$ of a finite quiver without oriented cycle). We determine $\Gamma(D^b(k\vec{\Delta}))$ which we know to be isomorphic to $\Gamma(D^b(B))$ whenever B is tilting-equivalent to $k\vec{\Delta}$ (compare 1.7). Our results will be applied to indecomposable B-modules in section 7.

LEMMA. *Let $X^{\cdot}$ be an indecomposable object in $D^b(k\vec{\Delta})$. Then $X^{\cdot}$ is isomorphic to a stalk complex with indecomposable stalk.*

Proof. Since $D^b(k\vec{\Delta})$ is equivalent to $K^b({}_{k\vec{\Delta}}\mathcal{I})$, it is enough to show that each indecomposable object of $K^b({}_{k\vec{\Delta}}\mathcal{I})$ is isomorphic to some $\cdots 0 \to I^j \xrightarrow{d^j} I^{j+1} \to 0 \cdots$ where d^j is surjective.

Let $I^{\cdot}$ be indecomposable in $K^b({}_{k\vec{\Delta}}\mathcal{I})$. Applying T if necessary, we may assume that $I^{\cdot}$ has the form:

$$\cdots 0 \to I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} I^2 \to \cdots \quad \text{with} \quad I^0 \neq 0.$$

Consider a factorization $I^0 \xrightarrow{g} X \xrightarrow{h} I^1$ of d^0 in $\operatorname{mod} k\vec{\Delta}$ with g surjective and h injective. Then X is an injective $k\vec{\Delta}$-module, h is a section and we have an isomorphism $X \oplus C \xrightarrow{(hu)'} I^1$ in $\operatorname{mod} k\vec{\Delta}$. Since $hd^1 = 0$ we obtain an isomorphism of complexes.

$$\begin{array}{ccccccccccc}
\cdots 0 & \longrightarrow & I^0 & \xrightarrow{\quad d^0 \quad} & I^1 & \xrightarrow{\quad d^1 \quad} & I^2 & \xrightarrow{\quad d^2 \quad} & I^3 & \longrightarrow & \cdots \\
\| & & \| & & \uparrow{\scriptstyle \binom{h}{u}} & & \| & & \| & & \\
\cdots 0 & \longrightarrow & I^0 \oplus 0 & \xrightarrow{\begin{pmatrix} g & 0 \\ 0 & 0 \end{pmatrix}} & X \oplus C & \xrightarrow{\begin{pmatrix} 0 & 0 \\ 0 & ud^1 \end{pmatrix}} & 0 \oplus I^2 & \xrightarrow{\begin{pmatrix} 0 & 0 \\ 0 & d^2 \end{pmatrix}} & 0 \oplus I^3 & \longrightarrow & \cdots
\end{array}$$

Since $I^{\cdot}$ is indecomposable we conclude that $\cdots 0 \to I^0 \xrightarrow{g} X \to 0 \cdots$ or $\cdots 0 \to C \to I^2 \to I^3 \to \cdots$ is zero in $K^b({}_{k\vec{\Delta}}\mathcal{I})$ (i.e. acyclic). In the second case $I^{\cdot}$ is isomorphic to $\cdots 0 \to I^0 \xrightarrow{g} X \to 0 \cdots$ in $K^b({}_{k\vec{\Delta}}\mathcal{I}$.) In the first case, we are reduced to a complex of smaller width.

4.2. COROLLARY. *Let $X_0^{\cdot} \xrightarrow{f_0^{\cdot}} X_1^{\cdot} \to \cdots \to X_{r-1}^{\cdot} \xrightarrow{f_{r-1}^{\cdot}} X_0^{\cdot}$ be a cycle in*

$D^b(k\vec{\Delta})$ (compare 0.1). *Then each $X_i^{\cdot}$ is isomorphic to T^nX_i for some $X_i \in \operatorname{mod} k\vec{\Delta}$ and some fixed $n \in \mathbb{Z}$.*

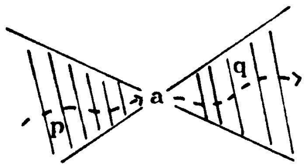
4.3 The computation of Auslander–Reiten triangles in $D^b(k\vec{\Delta})$ is divided into two steps.

First let $Z^{\cdot} = T^iZ$ for some $i \in \mathbb{Z}$ and some indecomposable non-projective $k\vec{\Delta}$-module Z. Then we have the Auslander–Reiten sequence $0 \to X \xrightarrow{u} Y \xrightarrow{v} Z \to 0$. Let $w \in \operatorname{Ext}^1_{k\vec{\Delta}}(Z, X) = \operatorname{Hom}_{D^b(k\vec{\Delta})}(Z, TX)$ be the corresponding element. Then we obtain a triangle

$$T^iX \xrightarrow{T^iu} T^iY \xrightarrow{T^iv} T^iZ \xrightarrow{T^iw} T^{i+1}X.$$

It is straightforward that the properties (AR1), (AR2) and (AR3) of 3.1 are satisfied.

Let us now turn to the case $Z^{\cdot} = T^iP(a)$, $i \in \mathbb{Z}$, where $P(a)$ is the indecomposable projective $k\vec{\Delta}$-module associated with the point a of $\vec{\Delta}$; for simplicity, we will assume that $i = 0$. Denote by E the following $k\vec{\Delta}$-module (considered as a contravariant representation of $\vec{\Delta}$): $E(x)$ is the vector space freely generated by the paths of the form $p: x \dashrightarrow a$ or $q: a \dashrightarrow x$ (so we have $E(x) = 0$ if x is not comparable with a in the order defined by the arrows of $\vec{\Delta}$); if $x \xrightarrow{\alpha} y$ is an arrow and $x < a$, $E(\alpha): E(y) \to E(x)$ maps p onto the composed path αp; if $x \geqslant a$, $E(\alpha)$ maps q onto q' or 0 according as q has the form $q'\alpha$ or not.



The paths (resp. the non-trivial paths) stopping at a generate a submodule of E which is identified with $P(a)$ (resp. with the radical $\underline{P}(a)$ of $P(a)$). The quotient $E/\underline{P}(a)$ (resp. $E/P(a)$) is identified with the indecomposable injective $I(a)$ attached to a (resp. with the quotient $\bar{I}(a)$ of $I(a)$ by its socle).

By w we denote the composition $P(a) \xrightarrow{i} E \xrightarrow{p} I(a)$, by $\eta \in \operatorname{Ext}^1_{k\vec{\Delta}}(\bar{I}(a), P(a)) \simeq \operatorname{Hom}_{D^b(k\vec{\Delta})}(\bar{I}(a), TP(a))$ and $\eta' \in \operatorname{Ext}^1_{k\vec{\Delta}}(I(a), \underline{P}(a))$ the extensions associated with the exact sequences $0 \to P(a) \xrightarrow{i} E \xrightarrow{p} \bar{I}(a) \to 0$ and $0 \to \underline{P}(a) \xrightarrow{i} E \xrightarrow{p} I(a) \to 0$ (i denotes an inclusion, p a projection).

LEMMA. *The sextuple associated with the sequence*

$$(*) \quad T^-I(a) \xrightarrow{[T^-p,\, -T^-\eta']} T^-\bar{I}(a) \oplus \underline{P}(a) \xrightarrow{[T^-\eta,\ ]} P(a) \xrightarrow{w} I(a)$$

is an Auslander–Reiten triangle.

Proof. Clearly, $I(a) = \nu P(a)$. By the proof of Theorem 3.6, the Auslander–Reiten triangle starting at $\tau P(a) = T^{-}I(a)$ has w as last morphism. So it suffices to verify that the sextuple of our lemma is a triangle. This directly follows from the diagram below, where E denotes an arbitrary module, $\underline{P} \subset P$ two submodules of E, I and $\bar{I}$ the quotients $E/\underline{P}$ and E/P respectively; by $[X \xrightarrow{d} Y]$ we denote a complex vanishing in degrees $\neq 0$, 1 which has X as 0-component, Y as 1-component. For the other notation, see the particular case above.

$$\begin{array}{ccccccc}
[0 \longrightarrow P] & \xrightarrow{[0\ w]} & [0 \longrightarrow I] & \xrightarrow{[0\ 1]} & [P \xrightarrow{w} I] & \xrightarrow{[1\ 0]} & [P \longrightarrow 0] \\
 & & \uparrow{\scriptstyle [0\ p]} & & & & \uparrow{\scriptstyle \left[\left[\begin{smallmatrix}\iota\\1\end{smallmatrix}\right] p\right]} \\
 & & [\underline{P} \xrightarrow{\iota} E] & \xrightarrow[{[[-1\ i]\,1]}]{} & & & [\underline{P} \oplus P \xrightarrow[{\left[\begin{smallmatrix}0\\i\end{smallmatrix}\right]}]{} E]
\end{array}$$

By construction, the first line is a triangle, and the vertical morphisms are quasi-isomorphims of $K^b(\operatorname{mod} k\vec{\Delta})$. Since $[\underline{P} \oplus P \rightarrow E]$ is quasi-isomorphic to $T\underline{P} \oplus \bar{I}$, the first line is isomorphic in $D^b(k\vec{\Delta})$ to the following sequence:

$$P \xrightarrow{w} I \xrightarrow{[p,\,-\eta']} \bar{I} \oplus T\underline{P} \xrightarrow{\left[\begin{smallmatrix}\eta\\ Ti\end{smallmatrix}\right]} TP.$$

The assertion now follows from (TR2).

The triangle (*) will be called a *connecting triangle*. We point out the analogy to connecting sequences in the theory of tilting modules [HR].

4.4 Using the results of 3.5 it is now easy to derive the structure of $\Gamma(D^b(k\vec{\Delta}))$. Let $\Gamma = \Gamma_{k\vec{\Delta}}$ be the Auslander–Reiten quiver of $k\vec{\Delta}$. Denote by Γ_i a copy of Γ for $i \in \mathbb{Z}$, by $\tilde{\Gamma}$ the quiver obtained from the disjoint union $\coprod_{i\in\mathbb{Z}} \Gamma_i$ by adding an arrow from the injective module $I(a)$ in Γ_i to the projective module $P(b)$ in Γ_{i+1} for each arrow from b to a in $\vec{\Delta}$.

PROPOSITION. *The quiver* $\Gamma(D^b(k\vec{\Delta}))$ *is* $\tilde{\Gamma}$.

4.5 From the structure of Γ ([G2], [Ri2], [Ri3]) it now follows:

COROLLARY.

(i) *If* $\vec{\Delta}$ *is a Dynkin diagram then* $\Gamma(D^b(k\vec{\Delta})) \xrightarrow{\sim} \mathbb{Z}\vec{\Delta}$.
(ii) *If* $\vec{\Delta}$ *is a tame quiver (i.e.* $k\vec{\Delta}$ *is representation-tame) then the components of* $\Gamma(D^b(k\vec{\Delta}))$ *are of the form* $\mathbb{Z}\vec{\Delta}$ *and* $\mathbb{Z}A_\infty/r$ *for some* $r \in \mathbb{N}$.
(iii) *If* $\vec{\Delta}$ *is a wild quiver (i.e.* $k\vec{\Delta}$ *is representation-wild) then the components of* $\Gamma(D^b(k\vec{\Delta}))$ *are of the form* $\mathbb{Z}\vec{\Delta}$ *and* $\mathbb{Z}A_\infty$.

4.6 Let $\vec{\Delta}$ be a Dynkin quiver and denote by $k(\mathbb{Z}\vec{\Delta})$ the *mesh category of* $\mathbb{Z}\vec{\Delta}$ (see [G2], [R]).

PROPOSITION. ind $D^b(k\vec{\Delta})$ *is equivalent to* $k(\mathbb{Z}\vec{\Delta})$.

Proof. By 4.5, both categories have the "same" quiver. Using 4.3 it is easy to see that we can represent the arrows of $\mathbb{Z}\vec{\Delta}$ by irreducible morphisms of ind $D^b(k\vec{\Delta})$ which satisfy the mesh relations and are globally stable under τ. This provides us with a full and dense functor $F: k(\mathbb{Z}\vec{\Delta}) \to \text{ind}\, D^b(k\vec{\Delta})$. Let $x, y \in k(\mathbb{Z}\vec{\Delta})$. Since F commutes with τ, we may assume that $F(x) = T^iP$ for some $i \in \mathbb{Z}$ and some indecomposable projective $k\vec{\Delta}$-module P. Under these assumptions, $\text{Hom}_{k(\mathbb{Z}\vec{\Delta})}(x, y) \neq 0$ implies $F(y) = T^iY$ for some indecomposable $k\vec{\Delta}$-module Y. But then [R] implies that

$$\text{Hom}_{k(\mathbb{Z}\vec{\Delta})}(x, y) \simeq \text{Hom}_{k\vec{\Delta}}(P, Y) \simeq \text{Hom}_{D^b(k\vec{\Delta})}(F(x), F(y)).$$

Thus F is faithful.

4.7 If $\vec{\Delta}$ is a Dynkin quiver, the Euler characteristic $\chi_{k\vec{\Delta}}$ is positive definite and the set of *roots* $\mathcal{R} = \{x \in \mathbb{Z}^n \mid \chi_{k\vec{\Delta}}(x) = 1\}$ is finite. A non-zero element $x = (x_1, \ldots, x_n) \in \mathbb{Z}^n$ is *positive* if $x_i \geqslant 0$ for all i. Then $\underline{\dim}$ induces a bijection between ind $k\vec{\Delta}$ and $\mathcal{R}^+ = \{x \in \mathcal{R} \mid x \text{ positive}\}$ [BGP], [G1].

COROLLARY. *Let* $\vec{\Delta}$ *be a Dynkin quiver. Then* $\underline{\dim}$ *induces a bijection between* ind $D^b(k\vec{\Delta})/T^2$ *and* $\mathcal{R}$.

Proof. By 4.1 and the previous remark $\chi_{k\vec{\Delta}}(\underline{\dim}\, X^{\cdot}) = 1$ for $X^{\cdot} \in \text{ind}\, D^b(k\vec{\Delta})$. Therefore $\underline{\dim}$ is a map from ind $D^b(k\vec{\Delta})$ to $\mathcal{R}$. As for $x \in \mathcal{R}$ either x or $-x$ is positive $\underline{\dim}$ is a surjective map. The definition of $\underline{\dim}$ shows that $\underline{\dim}^{-1}(x)$ is a T^2-orbit for $x \in \mathcal{R}$. Hence we obtain a bijective map from ind $D^b(k\vec{\Delta})/T^2$ to $\mathcal{R}$.

4.8 Let Δ be a finite graph and $\vec{\Delta}_1$, $\vec{\Delta}_2$ be quivers without oriented cycles and underlying graph equal to Δ. If $\vec{\Delta}_1$ can be obtained from $\vec{\Delta}_2$ by a sequence of "reflections" [BGP], [G2] and a quiver isomorphism we write $\vec{\Delta}_1 \sim \vec{\Delta}_2$.

The following lemma is straightforward.

LEMMA. $\mathbb{Z}\vec{\Delta}_1$ *and* $\mathbb{Z}\vec{\Delta}_2$ *are isomorphic as translation quivers if and only if* $\vec{\Delta}_1 \sim \vec{\Delta}_2$.

COROLLARY. *If* $D^b(k\vec{\Delta}_1)$ *is triangle-equivalent to* $D^b(k\vec{\Delta}_2)$, *then* $\vec{\Delta}_1 \sim \vec{\Delta}_2$.

Proof. By 4.5 the components of $\Gamma(D^b(k\vec{\Delta}))$ not isomorphic to $\mathbb{Z}\mathbb{A}_\infty$ or $\mathbb{Z}\mathbb{A}_\infty/r$ are isomorphic to $\mathbb{Z}\vec{\Delta}$. Thus $\mathbb{Z}\vec{\Delta}_1$ and $\mathbb{Z}\vec{\Delta}_2$ are isomorphic as translation quivers.

This corollary allows us to introduce the notion of type for the finite-dimensional k-algebras investigated in section 7.

5. Dynkin algebras

5.1 Let $\vec{\Delta}$ be a finite quiver without oriented cycle having n vertices. The square of the translation functor T is an automorphism on $D^b(k\vec{\Delta})$. The *root categrory* $\mathcal{R}(\vec{\Delta})$ is by definition the quotient category of $\operatorname{ind} D^b(k\vec{\Delta})$ by T^2. The canonical functor $\pi : \operatorname{ind} D^b(k\vec{\Delta}) \to \mathcal{R}(\vec{\Delta})$ is a Galois covering in the sense of Gabriel [G3].

If $\vec{\Delta}$ is a Dynkin quiver, the root category $\mathcal{R}(\vec{\Delta})$ coincides with the cylinder introduced in [H2].

In the following we will use the same notation for $\mathcal{R}(\vec{\Delta})$ and its quiver. Observe that $\mathcal{R}(\vec{\Delta})$ is not necessarily connected. We call a vertex $x \in \mathcal{R}(\vec{\Delta})$ *regular* if x is contained in a component of the form $\mathbb{Z}\mathbb{A}_\infty$ or $\mathbb{Z}\mathbb{A}_\infty/r$ (compare 4.5). All the other vertices are called *transjective vertices*. Note that this does not coincide with the definition of [R] since $\tau^n x$ is defined for each $n \in \mathbb{Z}$ if x is transjective in our sense.

5.2 In 2.2 we have defined the dimension vector $\underline{\dim}\, X^\cdot$ for $X^\cdot \in D^b(k\vec{\Delta})$. Let $x \in \mathcal{R}(\vec{\Delta})$ and $X^\cdot$, $Y^\cdot \in \pi^{-1}(x)$. Then clearly $\underline{\dim}\, X^\cdot = \underline{\dim}\, Y^\cdot$. Thus $\underline{\dim}\, x = \underline{\dim}\, X^\cdot$ does not depend on the choice of $X^\cdot$ in $\pi^{-1}(x)$. It will be called the *dimension vector* of x.

This definition allows us to consider two subcategories of $\mathcal{R}(\vec{\Delta})$. Let $\mathcal{R}^+(\vec{\Delta})$ and $\mathcal{R}^-(\vec{\Delta})$ be the full subcategories of $\mathcal{R}(\vec{\Delta})$ consisting of those $x \in \mathcal{R}(\vec{\Delta})$ such that $\underline{\dim}\, x$ is positive and negative respectively. Then $T\mathcal{R}^+(\vec{\Delta}) = \mathcal{R}^-(\vec{\Delta})$.

5.3 For all $x \in \mathcal{R}(\vec{\Delta})$ we define a function $f_x : \mathcal{R}(\vec{\Delta}) \to \mathbb{Z}$ by $f_x(y) = \dim_k \operatorname{Hom}_{\mathcal{R}(\vec{\Delta})}(x, y) - \dim_k \operatorname{Hom}_{\mathcal{R}(\vec{\Delta})}(y, \tau x)$.

LEMMA. $f_x(y) = \langle \underline{\dim}\, x, \underline{\dim}\, y \rangle$.

Proof. Let $X^\cdot$, $Y^\cdot \in \operatorname{ind} D^b(k\vec{\Delta})$ such that $\pi(X^\cdot) = x$ and $\pi(Y^\cdot) = y$. Since π is a covering functor we obtain:

$$\begin{aligned} f_x(y) &= \sum_{i\in\mathbb{Z}} \dim_k \operatorname{Hom}_{D^b(k\vec{\Delta})}(T^{2i}X^\cdot, Y^\cdot) - \sum_{i\in\mathbb{Z}} \dim_k \operatorname{Hom}_{D^b(k\vec{\Delta})}(Y^\cdot, \tau T^{2i}X^\cdot) \\ &= \sum_{i\in\mathbb{Z}} \dim_k \operatorname{Hom}_{D^b(k\vec{\Delta})}(T^{2i}X^\cdot, Y^\cdot) - \sum_{i\in\mathbb{Z}} \dim_k \operatorname{Hom}_{D^b(k\vec{\Delta})}(T^{2i-1}X^\cdot, Y^\cdot) \\ &\overset{(3.8)}{=} \langle \underline{\dim}\, X^\cdot, \underline{\dim}\, Y^\cdot \rangle = \langle \underline{\dim}\, x, \underline{\dim}\, y \rangle. \end{aligned}$$

5.4 A subset $\mathcal{T} = \{t_1, \ldots, t_n\}$ of vertices of $\mathcal{R}(\vec{\Delta})$ is called a *tilting set* if the following two conditions are satisfied:

(i) $\mathrm{Hom}_{\mathcal{R}(\vec{\Delta})}(t_i, \tau t_j) = 0$ for all i, j
(ii) $\underline{\dim}\, t_1, \ldots, \underline{\dim}\, t_n$ form a $\mathbb{Z}$-basis of $\mathbb{Z}^n$.

In [H2] we gave a different definition in the case of Dynkin quivers. But it is easy to see that both are equivalent.

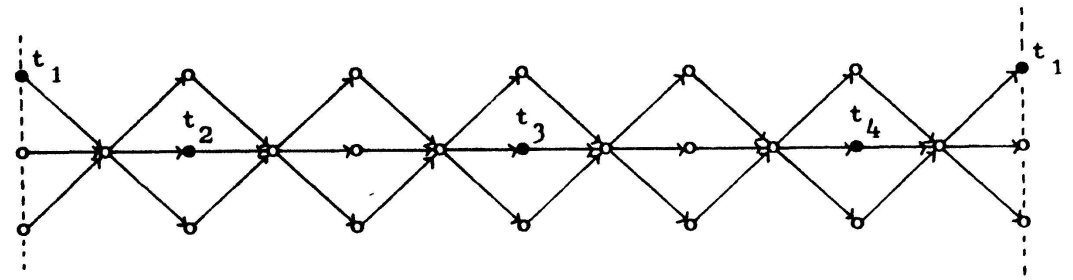
5.5 Let $\mathcal{T} = \{t_1, \ldots, t_n\}$ be a tilting set of $\mathcal{R}(\vec{\Delta})$. the coefficients of the *Cartan matrix* $C_{\mathcal{T}}$ of the k-algebra End $\mathcal{T}$ formed by all $n \times n$-matrices $f = (f_{ij})$ such that $f_{ij} \in \mathrm{Hom}_{\mathcal{R}(\vec{\Delta})}(t_i, t_j)$ are given by $(C_{\mathcal{T}})_{ij} = \dim_k \mathrm{Hom}_{\mathcal{R}(\vec{\Delta})}(t_i, t_j) = \langle \underline{\dim}\, t_i, \underline{\dim}\, t_j \rangle$.

A finite-dimensional k-algebra of the form End $\mathcal{T}$ is called a *$\vec{\Delta}$-root algebra*.

5.6 LEMMA. *Let $\vec{\Delta}$ be a tame quiver and $\mathcal{T} = \{t_1, \ldots, t_n\}$ be a tilting set of $\mathcal{R}(\vec{\Delta})$. Then $\mathcal{T}$ contains a transjective vertex.*

Proof. If $t_1, \ldots, t_n$ are regular vertices of $\mathcal{R}(\vec{\Delta})$, then $\underline{\dim}\, t_1, \ldots, \underline{\dim}\, t_n$ are linearly dependent, for they lie in the hyperplane of vectors of defect zero [DR].

5.7 We call a tilting set $\mathcal{T}$ of $\mathcal{R}(\vec{\Delta})$ *cycle-free* if the quiver of End $\mathcal{T}$ contains no oriented cycle. The tilting set formed by the marked vertices of the following picture is not cycle-free ($\Delta = \mathbb{D}_4$ and identification is along the dotted lines).



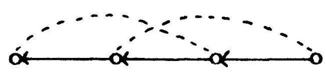
5.8 Let A be a finite-dimensional k-algebra, $(A, {}_AM_B, B)$ a tilting triple (compare 1.7), $\mathcal{X}$ and $\mathcal{Y}$ the full subcategories $\{Y \in \mathrm{mod}\, B \mid M \otimes_B Y = 0\}$ and $\{Y \in \mathrm{mod}\, B \mid \mathrm{Tor}_1^B(M, Y) = 0\}$ of mod B respectively. In [HR] it is shown that $(\mathcal{X}, \mathcal{Y})$ is a torsion theory on mod B. If every indecomposable B-modules lies either in $\mathcal{X}$ or in $\mathcal{Y}$, we say that the torsion theory splits. Following [AH], a finite-dimensional k-algebra A is called an *iterated tilted algebra* if there exists a finite quiver $\vec{\Delta}$ without oriented cycle and a sequence of tilting triples $A_i, {}_{A_i}M^i_{A_{i+1}}, A_{i+1})_{0 \le i < m}$ such that the associated torsion theories $(\mathcal{X}_{i+1}, \mathcal{Y}_{i+1})$ on mod A_{i+1} split and that $A_0 = k\vec{\Delta}$, $A_m = A$. This quiver $\vec{\Delta}$ is uniquely determined up to the relation $\sim$ introduced in 4.8 and will be called the *type* of A.

5.9 Of special interest to us are iterated tilted algebras of type $\vec{\Delta}$, where $\vec{\Delta}$ is a Dynkin quiver. This we assume for the rest of this section. It was shown in [AH] that these algebras are simply connected (for a definition see [BG] or [BLS]). By 2.6 we infer that $K_0(A)$ and $K_0(k\vec{\Delta})$ are isometric if A is an iterated tilted algebra of type $\vec{\Delta}$.

5.10 THEOREM. *Let A be a simply connected $\vec{\Delta}$-root algebra* (5.5). *Then A is an iterated tilted algebra of type $\vec{\Delta}$.*

For a proof of this theorem we refer to the appendix.

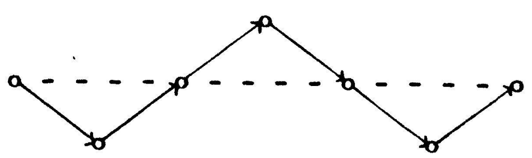
5.11 Let us give an example of the embedding of $\operatorname{ind} A$ into $\operatorname{ind} D^b(k\vec{\Delta})$ for an iterated tilted algebra of type $\vec{\Delta}$. We consider the algebra A defined by the bounded quiver



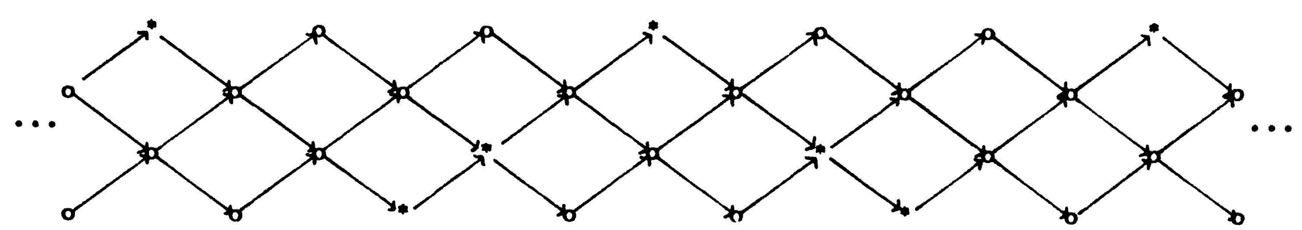
This is an iterated tilted algebra of type $\mathbb{A}_4$: With the notation of 5.8, we have $m = 2$; A_0 is the algebra of the quiver

and A_1 the algebra of the bounden quiver

M^1 is the direct sum of the A_0-modules with dimension vectors [1 0 0 0], [1 1 1 1], [0 0 1 1] and [0 0 0 1]; M^2 is the direct sum of the A_1-modules with dimension vectors [1 0 0 0], [1 1 0 0], [0 1 1 1] and [0 0 0 1]. The Auslander–Reiten quiver Γ_A has the following form. The dotted lines indicate the Auslander–Reiten translation.



Up to translation of T there are ten isomorphism classes of indecomposable objects in $D^b(A)$ (compare 5.13). The embedding of $\operatorname{ind} A$ into $\mathbb{Z}\mathbb{A}_4$ is illustrated in the following figure. The vertices marked by * correspond to indecomposable A-modules.



5.12 A finite-dimensional k-algebra A is called a *Dynkin algebra* of type $\vec{\Delta}$ if A is simply connected and there exists a Dynkin quiver $\vec{\Delta}$ such that $K_0(A)$ and $K_0(k\vec{\Delta})$ are isometric.

THEOREM. *Let A be a basic finite-dimensional k-algebra and $\vec{\Delta}$ a Dynkin quiver. Then the following are equivalent.*

(i) *$D^b(A)$ is triangle-equivalent to $D^b(k\vec{\Delta})$.*
(ii) *A is a Dynkin algebra of type $\vec{\Delta}$.*
(iii) *A is a simply connected $\vec{\Delta}$-root algebra.*
(iv) *A is an iterated tilted algebra of type $\vec{\Delta}$.*
(v) *A and $k\vec{\Delta}$ are tilting-equivalent.*

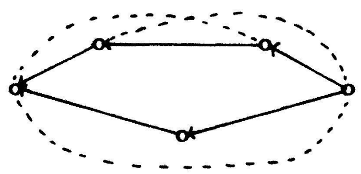
Proof. (i)$\Rightarrow$(iv) By 3.2a) of [H3].
(ii)$\Rightarrow$(iii) This follows from 8.8.
(iii)$\Rightarrow$(iv) By 5.10.
(iv)$\Rightarrow$(ii) By the remark in 5.9.
(iv)$\Rightarrow$(v) trivial.
(v)$\Rightarrow$(i) By 1.7.

COROLLARY. *Let A be a Dynkin algebra of type $\vec{\Delta}$ and set $\mathcal{R}_A = \{x \in \mathbb{Z}^n \mid \chi_A(x) = 1\}$. Then $\underline{\dim}$ induces a bijection between* $\operatorname{ind} D^b(A)/T^2$ *and $\mathcal{R}_A$.*

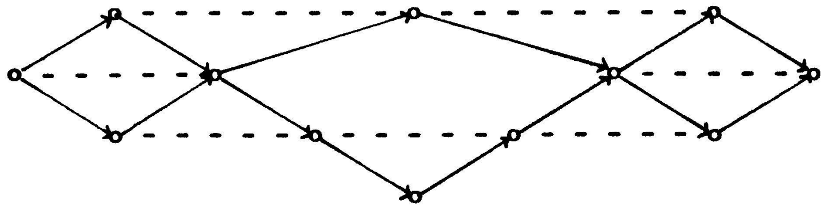
Proof. By 5.12 there is an equivalence F of triangulated categories from $D^b(A)$ to $D^b(k\vec{\Delta})$. By 2.6 we obtain an isometry $f: K_0(A) \to K_0(k\vec{\Delta})$ such that $\underline{\dim}\, F(X^\cdot) = (\underline{\dim}\, X^\cdot)f$ for $X^\cdot \in D^b(A)$. It follows that f induces a bijection between $\mathcal{R}_A$ and $\mathcal{R}_{k\vec{\Delta}}$. By 4.7 the assertion follows.

5.13 At this stage we want to point out why we used the bilinear form on the Grothendieck group to define isometries instead of using the quadratic form χ which might appear more natural.

Consider the algebra A given by the bounden quiver



Then $\operatorname{mod} A$ is directed as Γ_A shows:

Γ_A = 

Since Γ_A is not simply connected, we immediately see that $K_0(A)$ is not isometric to $K_0(k\vec{\Delta})$ for a Dynkin quiver $\vec{\Delta}$ (5.12).

But χ_A is congruent to $\chi_{\mathbb{D}_5}$ as the following calculation shows. The matrix representing χ_A is

$$\chi_A = \begin{pmatrix} 2 & -1 & 1 & -1 & 0 \\ -1 & 2 & -1 & 0 & 1 \\ 1 & -1 & 2 & 0 & -1 \\ -1 & 0 & 0 & 2 & -1 \\ 0 & 1 & -1 & -1 & 2 \end{pmatrix}$$

We choose

$$g = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & -1 \\ -1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Then $g \in \mathrm{GL}_5(\mathbb{Z})$ and

$$g\chi_A g^t = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & -1 \\ 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 2 \end{pmatrix}$$

This is the matrix representing $\chi_{\mathbb{D}_5}$.

The preceding calculation shows that χ_A is positive definite. We infer that A is not even an iterated tilted algebra of type $\vec{\Delta}$, where $\vec{\Delta}$, is an arbitrary quiver without oriented cycle.

6. Cycles in mod $k\vec{\Delta}$

Throughout this section let $A = k\vec{\Delta}$ for some finite quiver $\vec{\Delta}$ without oriented cycle.

6.1 LEMMA. *Let X_1, X_2, X_3 be A-modules. Suppose that $f: X_1 \to X_2$ is surjective and $g: X_2 \to X_3$ injective. Then there exists a module Y and linear maps $h_1: X_1 \to Y$ and $h_2: Y \to X_3$ such that*

$$0 \to X_1 \xrightarrow{(f h_1)} X_2 \oplus Y \xrightarrow{\binom{g}{-h_2}} X_3 \to 0 \quad \textit{is exact.}$$

Proof. Consider the following exact sequence

$$(*) \quad 0 \to X_2 \xrightarrow{g} X_3 \to X_3/X_2 \to 0.$$

Since A is hereditary, $\mathrm{Ext}^1_A\,(X_3/X_2, f)$ is surjective. Let $0 \to X_1 \xrightarrow{h_1} Y \to X_3/X_2 \to 0$ be a preimage of $(*)$ in $\mathrm{Ext}^1_A(X_3/X_2, X_1)$. Then we obtain the following commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X_1 & \xrightarrow{h_1} & Y & \longrightarrow & X_3/X_2 & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow h_2 & & \| & & \\ 0 & \longrightarrow & X_2 & \xrightarrow{g} & X_3 & \longrightarrow & X_3/X_2 & \longrightarrow & 0 \end{array}$$

with h_1 injective and h_2 surjective. By construction we have that

$$0 \to X_1 \xrightarrow{(f h_1)} X_2 \oplus Y \xrightarrow{\binom{g}{-h_2}} X_3 \to 0 \quad \text{is exact.}$$

6.2 THEOREM. *Let $\mathscr{C}$ be a full subcategory of* mod A *which is closed under extensions and direct summands. If $\mathscr{C}$ contains a cycle* (0.1), *$\mathscr{C}$ also contains an indecomposable Z such that* End $Z \neq k$.

The *proof* results from the following steps:
1) If X, $Y \in \mathscr{C}$ and $f \in \mathrm{Hom}_A\,(X, Y)$, then $\mathrm{im} f \in \mathscr{C}$ (apply 6.1).
2) If $\mathscr{C}$ contains a cycle, it contains an *even* cycle, i.e. a cycle of the form

$$\begin{array}{ccccccccc} & X_1 & & X_3 & & & & X_{2n-1} & \\ {}^{f_0}\nearrow & & \searrow^{f_1} \quad {}^{f_2}\nearrow & & \searrow^{f_3} & \cdots & & & \searrow^{f_{2n-1}} \\ X_0 & & X_2 & & X_4 & & & & X_{2n}=X_0 \end{array}$$

where all f_{2i} are injective and all f_{2i+1} surjective.

In the sequel, we suppose that among all even cycles the given one has minimal length $2n$. We also assume that $\operatorname{End} X_i = k$ for all i. This implies $n \geq 2$.

3) $\operatorname{Hom}_A(X_0, X_i) = 0$ if $2 \leq i < 2n$.

Suppose there exists $0 \neq f \in \operatorname{Hom}_A(X_0, X_i)$. By 1) all indecomposable summands of $\operatorname{im} f$ belong to $\mathscr{C}$. So there exists an indecomposable A-module Y in $\mathscr{C}$ and linear maps $X_0 \xrightarrow{f'} Y \xrightarrow{f''} X_i$ with f' surjective and f'' injective. This yields a cycle of length less than $2n$ in $\mathscr{C}$, contradicting the minimality of the given cycle.

In the sequel, we suppose that $\dim X_0 + \dim \operatorname{Hom}_A(X_1, X_3)$ is smaller or equal to the corresponding sum of any other even cycle of length $2n$.

4) Each non-zero $f \in \operatorname{Hom}_A(X_0, X_1)$ is injective.

In fact, suppose that there exists $0 \neq f \in \operatorname{Hom}_A(X_0, X_1)$ which is not injective. As $\operatorname{Hom}_A(X_0, X_2) = 0$ by 3) f is not surjective. By 1) there exists an indecomposable A-module $Y' \in \mathscr{C}$ and linear maps $X_0 \xrightarrow{f'} Y' \xrightarrow{f''} X_1$ with f' surjective and f'' injective. Consider the exact sequence $0 \to K \xrightarrow{i} X_0 \xrightarrow{f'} Y' \to 0$. As f_0 is injective its restriction to K is non-zero. Therefore $\operatorname{Hom}_A(i, X_1) \neq 0$. In particular, $\dim \operatorname{Hom}_A(Y', X_1) < \dim \operatorname{Hom}_A(X_0, X_1)$ and $\dim Y' \leq \dim X_0$, contradicting the minimality of $\dim X_0 + \dim \operatorname{Hom}_A(X_1, X_3)$.

By 6.1, there is a diagram

$$\begin{array}{ccccccc} & & & Y & & & \\ & & {}^{g}\nearrow & & \searrow^{h} & & \\ & X_1 & & & & X_3 & \\ {}^{f_0}\nearrow & & \searrow^{f_1} & & {}^{f_2}\nearrow & & \searrow^{f_3} \\ X_0 & & & X_2 & & & X_4 \end{array}$$

such that (+) $0 \to X_1 \xrightarrow{\binom{g}{f_1}} Y \oplus X_2 \xrightarrow{(h\ f_2)} X_3 \to 0$ is exact. Let $Y = \bigoplus_{i=1}^{r} Y_i$ be a decomposition of Y into indecomposables. Let g_i and h_i be the corresponding components of g and h.

5) $r \geq 2$. In particular $h_i \neq 0$ for all i.

The first follows from the minimality of the given cycle. Suppose $h_i = 0$ for one i. Then $Y_i \subset \ker (h\ f_2) = \operatorname{im} \binom{g}{f_1}$. Thus Y_i is a direct summand of $\operatorname{im} \binom{g}{f_1}$. But $X_1 \xrightarrow{\sim} \operatorname{im} \binom{g}{f_1}$ is indecomposable. Hence the sequence splits, contradicting again the minimality of the given cycle.

6) Each non-zero $f \in \operatorname{Hom}_A(X_0, Y_i)$ is injective.

In fact, apply $\operatorname{Hom}_A(X_0, -)$ to (+). By 3) it follows that $\operatorname{Hom}_A(X_0, X_1) \xrightarrow{\sim} \operatorname{Hom}_A(X_0, Y) = \bigoplus_{i=1}^{r} \operatorname{Hom}_A(X_0, Y_i)$. Let $u \in \operatorname{Hom}_A(X_0, X_1)$ be the preimage of f. Thus $u \neq 0$ and injective by 4). Therefore we have that $f = ug$ is injective.

Choose some i such that $f_0 g_i \neq 0$. Then

7) Since $2n$ is minimal, h_i is neither surjective nor injective.

8) $\operatorname{Ext}_A^1(X_3, X_3) \neq 0$

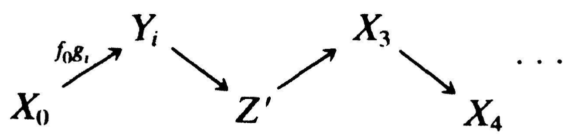
Otherwise, (+) induces an exact sequence

$$0 \to \operatorname{Hom}_A(X_3, X_3) \to \operatorname{Hom}(X_2 \oplus Y, X_3) \to \operatorname{Hom}_A(X_1, X_3) \to 0.$$

Denoting $\dim_k \operatorname{Hom}_A(M, N)$ by $\langle M, N \rangle$, we infer that

$$\begin{aligned}\langle X_1, X_3 \rangle &= \sum_{j=1}^{r} \langle Y_j, X_3 \rangle + \langle X_2, X_3 \rangle - 1 \\ &\geq \langle Y_i, X_3 \rangle + \sum_{j \neq i} \langle Y_j, X_3 \rangle \\ &> \langle Y_i, X_3 \rangle \quad \text{since} \quad \langle Y_j, X_3 \rangle \neq 0 \quad \text{for all} \quad j \text{ by 5).}\end{aligned}$$

It follows that the cycle



where Z' is an indecomposable direct summand of $\operatorname{im} h_i$ is a contradiction to the minimality of $\dim X_0 + \langle X_1, X_3 \rangle$.

9) If $0 \to X_3 \to E \to X_3 \to 0$ does not split, $\operatorname{End} E$ is isomorphic to the algebra of dual numbers.

This is a straightforward computation.

7. Piecewise hereditary algebras

7.1 We call a finite-dimensional k-algebra A *piecewise hereditary* if $D^b(A)$ is triangle-equivalent to $D^b(k\vec{\Delta})$ for some finite quiver $\vec{\Delta}$ (which is uniquely determined up to the relation $\sim$ introduced in 4.8). By 1.4 it follows that A has finite global dimension.

A finite-dimensional k-algebra is piecewise hereditary if it is tilting-equivalent to some $k\vec{\Delta}$ (compare 1.7). Note that in 5.12 we have shown the converse for Dynkin algebras.

7.2 LEMMA. *Let X, Y be indecomposable modules over a piecewise hereditary algebra A. Then*:

(a) $\mathrm{Ext}_A^i(X, X) = 0$ *for* $i > 1$.

(b) *If X, Y occur in a cycle* (0.1) *of* $\mathrm{mod}\, A$, $\mathrm{Ext}_A^i(X, Y) = 0$ *for* $i > 1$.

Proof. (a) Denote by F a triangle-equivalence from $D^b(A)$ to $D^b(k\vec{\Delta})$. By 4.1, $F(X) \simeq T^j X'$ for an indecomposable $k\vec{\Delta}$-module X' and some $j \in \mathbb{Z}$. Then

$$\begin{aligned}\mathrm{Ext}_A^i(X, X) &= \mathrm{Hom}_{D^b(A)}(X, T^i X) \xrightarrow{\sim} \mathrm{Hom}_{D^b(k\vec{\Delta})}(F(X), T^i F(X))\\ &\simeq \mathrm{Ext}_{k\vec{\Delta}}^i(X', X') = 0 \text{ for } i > 1.\end{aligned}$$

(b) If X, Y occur in a cycle of $\mathrm{mod}\, A$, it follows from 4.2 that $F(X)$ and $F(Y)$ are isomorphic to $T^j X'$ and $T^j Y'$ for some indecomposable $k\vec{\Delta}$-modules X' and Y' and some $j \in \mathbb{Z}$. Thus

$$\begin{aligned}\mathrm{Ext}_A^i(X, Y) &= \mathrm{Hom}_{D^b(A)}(X, T^i Y) \xrightarrow{\sim} \mathrm{Hom}_{D^b(k\vec{\Delta})}(X', T^i Y')\\ &= \mathrm{Ext}_{k\vec{\Delta}}^i(X', Y') = 0 \text{ for } i > 1.\end{aligned}$$

7.3 LEMMA. *A piecewise hereditary algebra A is a factor algebra of a finite-dimensional hereditary k-algebra.*

Proof. Assume $P_0 \xrightarrow{f_1} P_1 \xrightarrow{f_2} \cdots \to P_r = P_0$ is a cycle of indecomposable projective A-modules. Denote by F a triangle-equivalence from $D^b(A)$ to $D^b(k\vec{\Delta})$. Using 4.2 and F we obtain a cycle $X_0 \to X_1 \to \cdots \to X_r = X_0$ of indecomposable $k\vec{\Delta}$-modules, satisfying $\mathrm{Ext}_{k\vec{\Delta}}^1(X_i, X_j) = 0$ and $\mathrm{End}\, X_i \xrightarrow{\sim} k$. But this contradicts Corollary 4.2 of [HR] (or Theorem 6.2 above).

7.4 The following theorem is a generalization of a result due to Ringel [Ri5]. We closely follow his proof and recall that an A-module Z is called a *brick* if $\mathrm{End}\, Z = k$.

THEOREM. *Let A be a piecewise hereditary algebra and M be an indecomposable A-module which is not a brick. Then M contains a brick Z such that* $\mathrm{Ext}_A^1(Z, Z) \neq 0$.

Proof. It is enough to produce an indecomposable proper submodule X of M such that $\mathrm{Ext}_A^1(X, X) \neq 0$. Let $0 \neq f \in \mathrm{End}\, M$ be such that $\mathrm{im} f = S$ has minimal length. Then S is indecomposable. If $\mathrm{Ext}_A^1(S, S) \neq 0$, we set $X = S$. Otherwise, we choose an indecomposable $X \subset N = \ker f$ of minimal length such that $\mathrm{Hom}_A(S, X) \neq 0 \neq \mathrm{Ext}_A^1(S, X)$. Such an X exists by 1) below. We will show in 2) that $\mathrm{Ext}_A^1(X, X) \neq 0$.

1) Let $N = \bigoplus_{i=1}^s N_i$ with N_i indecomposable. Denote by p_i the canonical

projection from N to N_i. Consider the following diagram of exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & M & \xrightarrow{f} & S & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle p_i} & & \downarrow & & \| & & \\ 0 & \longrightarrow & N_i & \longrightarrow & E & \longrightarrow & S & \longrightarrow & 0 \end{array}$$

The lower sequence does not split. Otherwise, N_i would be a direct summand of M. Therefore $\operatorname{Ext}_A^1(S, N_i) \neq 0$ for all i and $S \subset \ker f$ implies that $\operatorname{Hom}_A(S, N_i) \neq 0$ for at least one i.

2) Let $0 \neq g \in \operatorname{Hom}_A(S, X)$. Since S has minimal length we infer that g is injective. $\operatorname{Ext}_A^1(S, S) = 0$ implies that g is not bijective. Consider the exact sequence $(*)$ $0 \to S \xrightarrow{g} X \xrightarrow{h} Q \to 0$. The exact sequence $\operatorname{Ext}_A^1(S, S) \to \operatorname{Ext}_A^1(S, X) \to \operatorname{Ext}_A^1(S, Q)$ yields $\operatorname{Ext}_A^1(S, Q) \neq 0$. So there is a non-split extension $0 \to Q \to E \to S \to 0$. This induces a sequence $Q \xrightarrow{u} E' \xrightarrow{v} S \xrightarrow{g} X \xrightarrow{h} Q$ for each indecomposable summand E' of E. Since Q is indecomposable by 3) below, u and v are non-zero and non-invertible. We infer that $\operatorname{Ext}_A^2(Q, X) = 0$ by 7.2, and the exact sequence $\operatorname{Ext}_A^1(X, X) \to \operatorname{Ext}_A^1(S, X) \to \operatorname{Ext}_A^2(Q, X)$ yields $\operatorname{Ext}_A^1(X, X) \neq 0$.

3) Suppose $Q = \bigoplus_{i=1}^r Q_i$ with Q_i indecomposable and $r > 1$. We may assume that $\operatorname{Ext}_A^1(S, Q_1) \neq 0$. Denote by i_1 the inclusion from Q_1 to Q. Consider the induced sequence:

$$\begin{array}{cccccccccc} (**) & 0 & \longrightarrow & S & \longrightarrow & Y & \longrightarrow & Q_1 & \longrightarrow & 0 \\ & & & \| & & \downarrow & & \downarrow{\scriptstyle i_1} & & \\ (*) & 0 & \longrightarrow & S & \longrightarrow & X & \longrightarrow & Q & \longrightarrow & 0 \end{array}$$

The upper sequence does not split, for X is indecomposable. Since $\operatorname{Ext}_A^2(S, S) = 0$ by 7.2 the exact sequence $\operatorname{Ext}_A^1(S, S) \to \operatorname{Ext}_A^1(S, Y) \to \operatorname{Ext}_A^1(S, Q_1) \to \operatorname{Ext}_A^2(S, S)$ yields $\operatorname{Ext}_A^1(S, Y) \neq 0$. Let $Y = \bigoplus_{i=1}^t Y_i$ with Y_i indecomposable. As $(**)$ does not split, $\operatorname{Hom}_A(S, Y_i) \neq 0$ for all i. So there exists j with $\operatorname{Ext}_A^1(S, Y_j) \neq 0$ and $\operatorname{Hom}_A(S, Y_j) \neq 0$. But this contradicts the choice of X. Hence Q is indecomposable.

7.5 COROLLARY. *Let A be a piecewise hereditary algebra. Then the following are equivalent.*

(i) *A is representation-finite.*

(ii) *For all bricks Z,* $\operatorname{Ext}_A^1(Z, Z) = 0$.

(iii) *Every indecomposable A-module is a brick.*

Proof. For the convenience of the reader we copy the proof from [Ri5].

(i)$\Rightarrow$(ii) Assume there exists a brick Z, with $\operatorname{Ext}_A^1(Z, Z) \neq 0$. By 7.2 the brick Z satisfies $\mathrm{Ext}_A^2(Z, Z) = 0$. Thus, we can construct indecomposable A-modules of arbitrary length, using the process of simplification (see [Ri1] or 3.1 of [Ri6]).

(ii) ⇒ (iii) This follows from 7.4.

(iii) ⇒ (i) This is true in general for finite-dimensional k-algebras. It follows directly from the representation theory of Schurian vector-space categories (for a survey, see [Ri3]). In our situation, where A is a factor algebra of a finite-dimensional hereditary k-algebra, one considers the category of A-modules as the category of representations of a bimodule of the form ${}_BM_k$, with B a proper factor algebra of A (see [Ri3] or 2.5 of [Ri6]), and uses induction.

7.6 COROLLARY. *Let A be a representation-finite piecewise hereditary algebra. Then* mod A *is directed.*

Proof. Let $X_0 \to X_1 \to \cdots \to X_n = X_0$ be a cycle of indecomposable A-modules. Denote by F a triangle-equivalence from $D^b(A)$ to $D^b(k\vec{\Delta})$. It follows from 4.2 that $F(X_0) \to F(X_1) \to \cdots \to F(X_n) = F(X_0)$ may be considered as a cycle of indecomposable $k\vec{\Delta}$-modules. Let $\mathscr{C}$ be the smallest full subcategory of mod $k\vec{\Delta}$ closed under extensions and direct summands containing $F(X_i)$ for $0 \le i \le n$. Then for $Y \in \mathscr{C}$ there exists an A-module Y' with $F(Y') \simeq Y$. In fact let $Y_1, Y_2 \in \mathscr{C}$ and (*) $0 \to Y_1 \xrightarrow{u} Y \xrightarrow{v} Y_2 \to 0$ be exact in mod $k\vec{\Delta}$. We may assume that $Y_1 = F(Z_1)$ and $Y_2 = F(Z_2)$ for some A-modules Z_1, Z_2. We have $\mathrm{Ext}^1_{k\vec{\Delta}}(Y_2, Y_1) \simeq \mathrm{Hom}_{D^b(k\vec{\Delta})}(Y_2, TY_1) \simeq \mathrm{Hom}_{D^b(A)}(Z_2, TZ_1) \simeq \mathrm{Ext}^1_A(Z_2, Z_1)$. Let $w \in \mathrm{Hom}_{D^b(k\vec{\Delta})}(Y_2, TY_1)$ be the element corresponding to (*). Then $w = F(w')$ for some $w' \in \mathrm{Hom}_{D^b(A)}(Z_2, TZ_1)$. Let $0 \to Z_1 \xrightarrow{f} Y' \xrightarrow{g} Z_2 \to 0$ be the corresponding element in $\mathrm{Ext}^1_A(Z_2, Z_1)$. So we obtain the triangle $Z_1 \xrightarrow{f} Y' \xrightarrow{g} Z_2 \xrightarrow{w'} TZ_1$ in $D^b(A)$. Thus also $F(Z_1) \xrightarrow{F(f)} F(Y') \xrightarrow{F(g)} F(Z_2) \xrightarrow{F(w')} F(TZ_1)$ is a triangle isomorphic to $Y_1 \xrightarrow{u} Y \xrightarrow{v} Y_2 \xrightarrow{w} TY_1$. In particular $F(Y') \simeq Y$. The assertion now follows from 6.2 and 7.5.

7.7 COROLLARY. *Let A be a representation-finite piecewise hereidtary algebra. Then the indecomposable A-modules are uniquely (up to isomorphism) determined by their composition factors.*

Proof. This follows from [H1] using 7.6.

7.8 THEOREM. *Let A be a piecewise hereditary algebra of type $\vec{\Delta}$ Then A is a cycle-free $\vec{\Delta}$-root algebra.*

Proof. Let $P(1), \ldots, P(n)$ be a complete list of representatives from the isomorphism classes of indecomposable projective A-modules. Let F be a triangle-equivalence from $D^b(A)$ to $D^b(k\vec{\Delta})$. Let $X_i^{\cdot} = F(P(i))$ for $1 \leq i \leq n$. Then $A \simeq \operatorname{End} \bigoplus_{i=1}^n X_i^{\cdot}$. In 5.1 we have introduced the covering functor $\pi : \operatorname{ind} D^b(k\vec{\Delta}) \to \mathscr{R}(\vec{\Delta})$. Let $t_i = \pi(X_i^{\cdot})$ for $1 \leq i \leq n$. We claim that $\mathscr{T} = \{t_1, \ldots, t_n\}$ is a tilting set of $\mathscr{R}(\vec{\Delta})$. Using 2.6 and 5.2 we see that $\underline{\dim}\, t_1, \ldots, \underline{\dim}\, t_n$ form a $\mathbb{Z}$-basis of $\mathbb{Z}^n$. Since π is a covering functor (see [BG] or [G3]) we obtain:

$$\begin{aligned}
\operatorname{Hom}_{\mathscr{R}(\vec{\Delta})}(t_i, \tau t_j) &= \operatorname{Hom}_{\mathscr{R}(\vec{\Delta})}(\pi(X_i^{\cdot}), \pi(\tau X_j^{\cdot}))\\
&= \operatorname{Hom}_{\mathscr{R}(\vec{\Delta})}(\pi(X_i^{\cdot}), \tau\pi(X_j^{\cdot}))\\
&= \coprod_{l \in \mathbb{Z}} \operatorname{Hom}_{D^b(k\vec{\Delta})}(X_i^{\cdot}, \tau T^{2l} X_j^{\cdot})\\
&= \coprod_{l \in \mathbb{Z}} \operatorname{Hom}_{D^b(k\vec{\Delta})}(T^{2l-1} X_j^{\cdot}, X_i^{\cdot}) \quad \text{(by 3.8)}\\
&= \coprod_{l \in \mathbb{Z}} \operatorname{Hom}_{D^b(A)}(T^{2l-1} P(j), P(i))\\
&= 0.
\end{aligned}$$

Using again that π is a covering functor we infer that $A \simeq \operatorname{End} \mathscr{T}$ and obviously is cycle-free.

8. Directed root algebras

8.1 In 2.4 we gave the definition of the Cartan matrix C_A for a basic finite-dimensional k-algebra A. For the formulation of our results we need some additional terminology. A matrix $C \in M_s(\mathbb{N})$ is called *schurian* if $C_{ij} \leq 1$ and $C_{ii} = 1$ for $1 \leq i, j \leq s$. We say that $C \in M_s(\mathbb{N})$ is *directed* if C is an upper triangular matrix up to conjugation by permutation matrices. A basic finite-dimensional k-algebra A is called *schurian* if C_A is schurian and *directed* if C_A is directed. Let A be a basic finite-dimensional k-algebra. Then A is given by a bounden quiver $(\vec{\Delta}, I)$ [G2]. This will be abbreviated by $A = k(\vec{\Delta}, I)$. We say that $(\vec{\Delta}, I)$ is *semi-commutative* if A is schurian, directed and for all vertices i, j of $\vec{\Delta}$ and paths w_1, w_2 from i to j in $\vec{\Delta}$ either both paths are contained in I or both paths are not contained in I.

Let $\Lambda(A)$ be the k-category associated with A in the following way [BG]. Let $e_1, \ldots, e_n$ be a complete set of primitive orthogonal idempotents of A. Then $e_1, \ldots, e_n$ are the objects of $\Lambda(A)$ and $\operatorname{Hom}_{\Lambda(A)}(e_i, e_j) = e_i A e_j$. The composition of morphisms is the multiplication of A. We say that $\Lambda(A)$ is $\tilde{\mathbb{A}}$-*free* [Bo2] if $\Lambda(A)$ does not contain a full subcategory isomorphic to $\Lambda(k\vec{E})$ with $E = \tilde{\mathbb{A}}_r$ for some $r \in \mathbb{N}$.

8.2 In this subsection we assume that $\vec{\Delta}$ is either a Dynkin or a tame quiver. We want to show how certain tilting sets arise quite naturally. Let A be a schurian, directed, basic finite-dimensional k-algebra such that $K_0(A)$ and $K_0(k\vec{\Delta})$ are isometric. Let f be an isometry and $P(1), \ldots, P(n)$ a complete set of representatives of the isomorphism classes of indecomposable projective A-modules. There exist uniquely determined vertices $t_1, \ldots, t_n$ of $\mathscr{R}(\vec{\Delta})$ (5.1) such that $\underline{\dim}\, t_i = f(\underline{\dim}\, P(i))$, for $\chi_{k\vec{\Delta}}(f(\underline{\dim}\, P(i))) = 1$.

LEMMA. *$\mathscr{T} = \{t_1, \ldots, t_n\}$ is a cycle-free tilting set of $\mathscr{R}(\vec{\Delta})$.*

Proof. By construction $\underline{\dim}\, t_1, \ldots, \underline{\dim}\, t_n$ form a $\mathbb{Z}$-basis. If $\mathscr{T}$ is a tilting set it is clearly cycle-free, for $C_{\mathscr{T}} = C_A$. So it remains to check the conditions $\mathrm{Hom}_{\mathscr{R}(\vec{\Delta})}(t_i, \tau t_j) = 0$.

Since A is schurian we have $0 \leqslant \langle \underline{\dim}\, t_i, \underline{\dim}\, t_j \rangle \leqslant 1$ for $1 \leqslant i, j \leqslant n$. Let $t_i, t_j \in \mathscr{T}$. We want to show that $\mathrm{Hom}_{\mathscr{R}(\vec{\Delta})}(t_j, \tau t_i) = 0$. If $\mathrm{Hom}_{\mathscr{R}(\vec{\Delta})}(t_i, t_j) = 0$ then $\mathrm{Hom}_{\mathscr{R}(\vec{\Delta})}(t_j, \tau t_i) = 0$, for $\langle \underline{\dim}\, t_i, \underline{\dim}\, t_j \rangle \geqslant 0$. So assume $\mathrm{Hom}_{\mathscr{R}(\vec{\Delta})}(t_i, t_j) \neq 0$.

We distinguish the following cases:

(1) t_i is a transjective vertex.

Applying T and τ if necessary, we may assume that t_i belongs to $\mathscr{R}^+(\vec{\Delta})$ and is projective as $k\vec{\Delta}$-module. Then $\mathrm{Hom}_{\mathscr{R}(\vec{\Delta})}(t_i, t_j) \neq 0$ implies $t_j \in \mathscr{R}^+(\vec{\Delta})$ and $\mathrm{Hom}_{\mathscr{R}(\vec{\Delta})}(t_j, \tau t_i) = 0$.

(2) t_j is a transjective vertex.

This is dual to (1).

This finishes the proof if $\vec{\Delta}$ is a Dynkin quiver.

(3) t_i, t_j are regular vertices.

(i) $t_i, t_j \in \mathscr{R}^+(\vec{\Delta})$.

By 5.6, $\mathscr{T}$ contains a transjective vertex t. Applying τ if necessary, we may assume that t is a projective $k\vec{\Delta}$-module. Since $\mathrm{Hom}_{k\vec{\Delta}}(t_i, t_j) \neq 0$, t_i and t_j belong to one component $\mathscr{C}$ of the Auslander–Reiten quiver of $k\vec{\Delta}$. Set $\mathscr{A} = \mathrm{add}\, \mathscr{C}$ and suppose that $\mathrm{Ext}^1_{k\vec{\Delta}}(t_i, t_j) = \mathrm{Ext}^1_{\mathscr{A}}(t_i, t_j) \neq 0$. Since $\mathscr{A}$ is a serial abelian category, the conditions $\mathrm{Hom}_{\mathscr{A}}(t_i, t_j) \neq 0 \neq \mathrm{Ext}^1_{\mathscr{A}}(t_i, t_j)$ imply that each simple object of $\mathscr{A}$ occurs as a Jordan–Hölder factor of $t_i \oplus t_j$, or equivalently of $\tau t_i \oplus \tau t_j$. It follows that each vertex of $\vec{\Delta}$ belongs to the support of $\tau t_i \oplus \tau t_j$, hence that $0 \neq \mathrm{Hom}(t, \tau t_i \oplus \tau t_j)$: contradiction.

(ii) $t_i \in \mathscr{R}^+(\vec{\Delta})$, $t_j \in \mathscr{R}^-(\vec{\Delta})$.

Suppose that $\mathrm{Hom}_{\mathscr{R}(\vec{\Delta})}(t_j, \tau t_i) = \mathrm{Hom}_{\mathscr{R}(\vec{\Delta})}(t_i, T t_j) \neq 0$. Set $t_j' = T t_j \in \mathscr{R}^+(\vec{\Delta})$. As in (i) above, the assumptions $\mathrm{Hom}_{k\vec{\Delta}}(t_i, t_j') \neq 0 \neq \mathrm{Ext}^1_{k\vec{\Delta}}(t_i, t_j') = \mathrm{Hom}_{\mathscr{R}(\vec{\Delta})}(t_i, t_j)$ lead to a contradiction.

It would be interesting to know if the previous lemma still holds when $\vec{\Delta}$ is a wild quiver. The proof does not generalize, for 5.6 is false in this situation.

8.3 For the next auxiliary results let A be a *hereditary basic finite-dimensional k-algebra.*

LEMMA. *Let X, Y be indecomposable A-modules such that* $\operatorname{Ext}_A^1(Y, X) = 0$. *Then every non-zero map $f: X \to Y$ is either injective or surjective.*

For a proof we refer to [HR], 4.1.

8.4 LEMMA. *Let X_1, X_2 be indecomposable A-modules such that* $\operatorname{Ext}_A^1(X_i, X_i) = 0$ *for* $1 \leqslant i \leqslant 2$ *and* $\operatorname{Ext}_A^1(X_2, X_1) = 0$. *If* $\operatorname{Hom}_A(X_1, X_2) \neq 0$ *then* $\operatorname{Ext}_A^1(X_1, X_2) = 0$.

Proof. Since $\operatorname{Ext}_A^1(X_2, X_1) = 0$ a non-zero map $f \in \operatorname{Hom}_A(X_1, X_2)$ is either injective or surjective by 8.3. If f is surjective, f induces a surjection $\operatorname{Ext}_A^1(X_1, X_1) \to \operatorname{Ext}_A^1(X_1, X_2)$. So $\operatorname{Ext}_A^1(X_1, X_1) = 0$ implies $\operatorname{Ext}_A^1(X_1, X_2) = 0$. If f is injective, f induces a surjection $\operatorname{Ext}_A^1(X_2, X_2) \to \operatorname{Ext}_A^1(X_1, X_2)$.

8.5 LEMMA. *Let X_1, X_2 be A-modules such that* $\operatorname{Ext}_A^1(X_i, X_i) = 0$, $\operatorname{End} X_i = k$ *for* $1 \leqslant i \leqslant 2$. $\operatorname{Hom}_A(X_1, X_2) = \operatorname{Hom}_A(X_2, X_1) = 0$, $\operatorname{Ext}_A^1(X_2, X_1) = 0$ *and* $\dim_k \operatorname{Ext}_A^1(X_1, X_2) = 1$. *If* $(*)$ $0 \to X_2 \to E \to X_1 \to 0$ *is a non-split extension then* $\operatorname{End} E = k$ *and* $\operatorname{Ext}_A^1(E, E) = 0$. *In particular, E is indecomposable.*

Proof. It follows from [Ri4], 2.1 that $\operatorname{End} E = k$. The exact sequence $(*)$ yields $\operatorname{Ext}_A^1(X_1, E) = 0 = \operatorname{Ext}_A^1(X_2, E)$. Applying $\operatorname{Ext}_A^1(-, E)$ gives now the assertion.

8.6 Let $\vec{\Delta}$ be a finite quiver without oriented cycle.

PROPOSITION. *Let $\mathcal{T} \in \mathcal{R}(\vec{\Delta})$ be a cycle-free tilting set and suppose that $C_{\mathcal{T}}$ is schurian. Let $t_1, t_2, t_3 \in \mathcal{T}$. If there exist $0 \neq f \in \operatorname{Hom}_{\mathcal{R}(\vec{\Delta})}(t_1, t_2)$ and $0 \neq g \in \operatorname{Hom}_{\mathcal{R}(\vec{\Delta})}(t_2, t_3)$ such that $fg = 0$ then $\operatorname{Hom}_{\mathcal{R}(\vec{\Delta})}(t_1, t_3) = 0$.*

Proof. Applying T if necessary, we may assume that $t_1 \in \mathcal{R}^+(\vec{\Delta})$. We distinguish the following cases:

(i) $t_2, t_3 \in \mathcal{R}^+(\vec{\Delta})$.

In this case we are dealing with *$k\vec{\Delta}$-modules. As $fg = 0$* and $\mathcal{T}$ is a tilting set f is injective and g is surjective. The exact sequence $(*)$ $0 \to K \to t_2 \xrightarrow{g} t_3 \to 0$ yields exact sequences

$$\operatorname{Hom}_{k(\vec{\Delta})}(t_1, t_2) \xrightarrow{\operatorname{Hom}(t_1, g)} \operatorname{Hom}_{k(\vec{\Delta}}(t_1, t_3) \to \operatorname{Ext}_{k\vec{\Delta}}^1(t_1, K) \to 0$$

and $\mathrm{Hom}_{k\vec{\Delta}}(t_2, t_2) \twoheadrightarrow \mathrm{Hom}_{k\vec{\Delta}}(t_2, t_3) \to \mathrm{Ext}^1_{k\vec{\Delta}}(t_2, K) \to 0$ using that $\mathcal{T}$ is a tilting set. By assumption $\mathrm{Hom}_{k\vec{\Delta}}(t_1, g) = 0$. So we have $\mathrm{Hom}(t_1, t_3) \twoheadrightarrow \mathrm{Ext}^1(t_1, K)$ and $\mathrm{Ext}^1(t_2, K) = 0$. Since f is injective, the induced map $\mathrm{Ext}^1(t_2, K) \to \mathrm{Ext}^1(t_1, K)$ is surjective. Hence $0 = \mathrm{Ext}^1(t_1, K) = \mathrm{Hom}(t_1, t_3)$.

(ii) $t_2 \in \mathcal{R}^-(\vec{\Delta})$, $t_3 \in \mathcal{R}^+(\vec{\Delta})$.

Let $f: t_1 \to t_3$ be non-zero. If f is injective, the induced map $\mathrm{Ext}^1(t_3, Tt_2) \to \mathrm{Ext}^1(t_1, Tt_2)$ is surjective; this contradicts the assumption that $\mathrm{Hom}(t_3, t_2) = 0$. If f is surjective, so is $\mathrm{Ext}^1(Tt_2, t_1) \to \mathrm{Ext}^1(Tt_2, t_3)$; this contradicts the assumption that $\mathrm{Hom}(t_2, t_1) = 0$.

(iii) $t_2, t_3 \in \mathcal{R}^-(\vec{\Delta})$.

Since $0 \neq \mathrm{Hom}_{\mathcal{R}(\vec{\Delta})}(t_1, t_2) = \mathrm{Ext}^1_{k\vec{\Delta}}(t_1, Tt_2)$ we obtain a non-split extension of $k\vec{\Delta}$-modules $(*)$ $0 \to Tt_2 \to E \to t_1 \to 0$. E is indecomposable and $\mathrm{Ext}^1_{k\vec{\Delta}}(E, E) = 0$ by 8.5. Applying $\mathrm{Hom}_{k\vec{\Delta}}(-, Tt_3)$ yields:

$(**)$ $0 \to \mathrm{Hom}(t_1, Tt_3) \to \mathrm{Hom}(E, Tt_3) \to \mathrm{Hom}(Tt_2, Tt_3) \xrightarrow{\partial} \mathrm{Ext}^1(t_1, Tt_3) \to \mathrm{Ext}^1(E, Tt_3) \to 0$.

By assumption $\mathrm{Hom}(Tt_2, Tt_3) \neq 0$ and $\partial = 0$. So $\mathrm{Hom}(E, Tt_3) \neq 0$. As $\mathcal{T}$ is a cycle-free tilting set $(*)$ yields that $\mathrm{Ext}^1(Tt_3, E) = 0$. By 8.4, applied to Tt_3, E, we conclude $\mathrm{Ext}^1(E, Tt_3) = 0$. Therefore $0 = \mathrm{Ext}^1(t_1, Tt_3) = \mathrm{Hom}_{\mathcal{R}(\vec{\Delta})}(t_1, t_3)$ from $(**)$.

(iv) $t_2 \in \mathcal{R}^+(\vec{\Delta})$, $t_3 \in \mathcal{R}^-(\vec{\Delta})$.

This is dual to (ii).

8.7 THEOREM. *Let $A = k(\vec{E}, I)$, where $(\vec{E}, I)$ is semi-commutative and $\Lambda(A)$ is $\mathbb{A}$-free. If $K_0(A)$ is isometric to $K_0(k\vec{\Delta})$ for some quiver $\vec{\Delta}$ which is either Dynkin or tame, then A is a cycle-free $\vec{\Delta}$-root algebra.*

Proof. Since $\Lambda(A)$ is $\mathbb{A}$-free, $H^2(A, k^*) = \{1\}$ by [Bo2], 2.3. So every directed schurian algebra B which has the same simplicial frame as A [BrG] is isomorphic to A.

Now let $\mathcal{T} \in \mathcal{R}(\vec{\Delta})$ be the tilting set constructed in 8.2 and $B = \mathrm{End}\, \mathcal{T}$. Then $C_B = C_{\mathcal{T}} = C_A =: C$, and B is schurian and directed. So it remains to show that $S_* B = S_* A$ (= simplicial frame of A). For this, we identify $\mathcal{T}$ with the set of vertices of the quivers of B and A. Let $t = (t_{i_0}, \ldots, t_{i_n}) \subset \mathcal{T}^{n+1}$ be a strictly increasing sequence ($C_{i_p i_{p+1}} = 1$ for all p). It follows from 8.6 that $t \in S_n B$ if and only if $C_{i_0 i_n} = 1$ (use induction on n, Proposition 8.6 being the case $n = 2$). Similarly, $t \in S_n A$ if and only if $C_{i_0 i_n} = 1$ (because $(\vec{E}, I)$ is semi-commutative). So $S_* A = S_* B$.

8.8 COROLLARY. *Let $A = k(\vec{E}, I)$ be a finite-dimensional k-algebra,* *where* mod A *is directed. If* $K_0(A)$ is isometric to $K_0(k\vec{\Delta})$ *for some quiver* $\vec{\Delta}$ *which is either Dynkin or tame, then* A *is a cycle-free* $\vec{\Delta}$*-root algebra.*

Proof. It follows from [Ri6], 2.4(9′) that A is representation-finite, whence $\Lambda(A)$ is $\tilde{\mathbb{A}}$-free. Moreover [BrG] 3.1 and 3.3b and [Bo2], 2.3 imply that $(\vec{E}, I)$ is semi-commutative. The assertion now follows from 8.7.

9. Frobenius categories

9.1 Let $\mathscr{A}$ be an abelian category and let $\mathscr{B}$ be a full subcategory of $\mathscr{A}$ which is closed under extensions. Let $\mathscr{S}$ be the set of exact sequences in $\mathscr{A}$ with terms in the subcategory $\mathscr{B}$. Following [Q] we call the pair $(\mathscr{B}, \mathscr{S})$ an *exact category*. An object X of $\mathscr{B}$ is called *$\mathscr{S}$-injective* if all exact sequences $0 \to X \to Y \to Z \to 0$ in $\mathscr{S}$ are split. Dually an object Z of $\mathscr{B}$ is called *$\mathscr{S}$-projective* if all exact sequences $0 \to X \to Y \to Z \to 0$ in $\mathscr{S}$ are split.

We say that the exact category $(\mathscr{B}, \mathscr{S})$ has *sufficiently many $\mathscr{S}$-injectives* if for all $X \in \mathscr{B}$ there is $0 \to X \to Y \to Z \to 0$ in $\mathscr{S}$ such that Y is $\mathscr{S}$-injective. Dually we say that $(\mathscr{B}, \mathscr{S})$ has *sufficiently many $\mathscr{S}$-projectives* if for all $Z \in \mathscr{B}$ there is $0 \to X \to Y \to Z \to 0$ in $\mathscr{S}$ such that Y is $\mathscr{S}$-projective.

An exact category $(\mathscr{B}, \mathscr{S})$ with sufficiently many $\mathscr{S}$-injectives and sufficiently many $\mathscr{S}$-projectives such that the $\mathscr{S}$-projectives and the $\mathscr{S}$-injectives coincide is called a *Frobenius category* [He].

We are mainly interested in the associated *stable category*. This is a category $\underline{\mathscr{B}}$ with the same objects as $\mathscr{B}$. For a pair $X, Y \in \mathscr{B}$ denote by $I(X, Y)$ the set of morphisms from X to Y which factor over an $\mathscr{S}$-injective. Then the morphisms in $\underline{\mathscr{B}}$ from X to Y are given by $\underline{\mathrm{Hom}}(X, Y) = \mathrm{Hom}_{\mathscr{B}}(X, Y)/I(X, Y)$ (compare [AB], [He]). The residue class of a morphism $u: X \to Y$ is denoted by $\underline{u}$.

Following [He], the suspension functor Ω^{-1} is a self-equivalence on $\underline{\mathscr{B}}$, where $(\mathscr{B}, \mathscr{S})$ is a Frobenius category. We assume that $T = \Omega^{-1}$ is an automorphism on $\underline{\mathscr{B}}$. This is possible if for all $X \in \underline{\mathscr{B}}$ the isomorphism classes of X and $\Omega^{-1}X$ have the same cardinality (compare [He]).

9.2 We include some examples to which these concepts may be applied. Let $\mathscr{B}'$ be an additive category with splitting idempotents and $\mathscr{B}$ the category of bounded complexes over $\mathscr{B}'$. The set $\mathscr{S}$ of exact sequences is given by pointwise split exact sequences over $\mathscr{B}'$. Then it is easily seen that $(\mathscr{B}, \mathscr{S})$ is an exact category. It is even a Frobenius category where the $\mathscr{S}$-projective complexes are built from complexes $\cdots 0 \to X \xrightarrow{1} X \to 0 \cdots$ with $X \in \mathscr{B}'$ by forming direct sums. The stable category $\underline{\mathscr{B}}$ is the homotopy category of $\mathscr{B}$. And the automorphism T is just the shift functor on $\underline{\mathscr{B}}$.

The category of finite-dimensional modules over a finite-dimensional selfinjective k-algebra is a Frobenius category. This example includes the case of a group algebra of a finite group over a field. A third kind of example is the category of graded modules over the exterior algebra (see [BGG]).

9.3 Let $(\mathcal{B}, \mathcal{S})$ be a Frobenius category and $\underline{\mathcal{B}}$ the stable category. We define a set $\mathcal{T}$ of sextuples in $\underline{\mathcal{B}}$.

Let $X, Y \in \mathcal{B}$ and $u \in \operatorname{Hom}_{\mathcal{B}}(X, Y)$. Consider the following diagram in $\mathcal{B}$:

$$\begin{array}{ccc} X & \xrightarrow{u} & Y \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle v} \\ I(X) & \xrightarrow{\bar{u}} & C_u \\ {\scriptstyle \bar{x}}\downarrow & & \downarrow{\scriptstyle w} \\ TX & = & TX \end{array}$$

where $0 \to X \xrightarrow{x} I(X) \xrightarrow{\bar{x}} TX \to 0$ is in $\mathcal{S}$ and $I(X)$ is $\mathcal{S}$-injective. C_u is the pushout of u and x.

Since $\mathcal{B}$ is closed under extensions in some abelian category $\mathcal{A}$ the pushout C_u in $\mathcal{B}$ coincides with the pushout in $\mathcal{A}$. A sextuple of the form $X \xrightarrow{u} Y \xrightarrow{v} C_u \xrightarrow{w} TX$ and its image in $\underline{\mathcal{B}}$ will be called *standard*. A sextuple $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} YX$ of objects and morphisms in $\underline{\mathcal{B}}$ lies in $\mathcal{T}$ if it is isomorphic in $\underline{\mathcal{B}}$ to a standard sextuple.

9.4 THEOREM. *The set $\mathcal{T}$ is a triangulation of $\underline{\mathcal{B}}$*

Proof. We check the axioms from 0.3.

(TR1). By definition $\mathcal{T}$ is closed under isomorphisms and every morphism can be embedded into a triangle. Clearly the sextuple $X \xrightarrow{1} X \xrightarrow{0} 0 \xrightarrow{0} TX$ lies in $\mathcal{T}$.

(TR3) It is easily seen that it suffices to consider the case of standard triangles.

Consider the following two standard sextuples.

$$\begin{array}{ccc} X & \longrightarrow & Y \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle v} \\ I(X) & \xrightarrow{\bar{u}} & C_u \\ {\scriptstyle \bar{x}}\downarrow & & \downarrow{\scriptstyle w} \\ TX & = & TX \end{array} \quad \text{and} \quad \begin{array}{ccc} X' & \xrightarrow{u'} & Y' \\ {\scriptstyle x'}\downarrow & & \downarrow{\scriptstyle v'} \\ I(X') & \longrightarrow & C_{u'} \\ {\scriptstyle \bar{x}'}\downarrow & & \downarrow{\scriptstyle w'} \\ TX' & \longrightarrow & TX' \end{array}$$

and two morphisms f and g such that $\underline{fu'} = \underline{ug}$ in $\underline{\mathcal{B}}$.

There exists a morphism $a: I(X) \to Y'$ such that $ug = fu' + xa$. We have morphisms $I_f: I(X) \to I(X')$ such that $fx' = xI_f$ and $Tf: TX \to TX'$ such that $I_f\bar{x}' = \bar{x}Tf$. We obtain morphisms $gv': Y \to C_{u'}$ and $I_f\bar{u}' + av': I(X) \to C_{u'}$. This yields a morphism $h: C_u \to C_{u'}$ such that $vh = gv'$ and $\bar{u}h = I_f\bar{u}' + av'$, for C_u is a pushout.

We claim that $hw' = wTf$. For this it is enough to show that $vhw' = vwTf$ and $\bar{u}hw' = \bar{u}wTf$. For the first observe that $vwTf = 0$ and $vhw' = gv'w' = 0$. For the second we have $\bar{u}wTf = \bar{x}Tf = I_f\bar{x}' = I_f\bar{u}'w' = I_f\bar{u}'w' + av'w' = (I_f\bar{u}' + av')w' = \bar{u}hw'$.

Thus $(\underline{f}, \underline{g}, \underline{h})$ is a morphism of triangles.

Before proving (TR2) let us state the following two remarks.

1) If $0 \to X \xrightarrow{u} Y \xrightarrow{v} Z \to 0$ is a short exact sequence in $\mathcal{S}$, we will say that u is a *proper monomorphism* and v a *proper epimorphism.* We claim that every morphism of $\mathcal{B}$ is isomorphic to the residue class $\underline{u}$ of a proper monomorphism u.

Indeed, given a morphism $X \xrightarrow{f} V$ of $\mathcal{B}$ and a proper monomorphism $X \xrightarrow{x} I$ of X into an $\mathcal{S}$-injective I, $\underline{f}$ is clearly isomorphic to the residue class of (f, x). On the other hand, $\mathcal{S}$ contains the short exact sequence

$$0 \to X \xrightarrow{(f\,x)} V \oplus I \xrightarrow{\binom{-g}{y}} C \to 0$$

where C is the pushout occuring in the following commutative diagram with rows in $\mathcal{S}$.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{x} & I & \xrightarrow{x'} & TX & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle y} & & \| & & \\ 0 & \text{——} & V & \xrightarrow{g} & C & \longrightarrow & TX & \text{——} & 0 \end{array}$$

2) Consider two exact sequences $0 \to X \xrightarrow{u} Y \xrightarrow{v} Z \to 0$ and $0 \to Y \xrightarrow{i} I \xrightarrow{i'} TY \to 0$ of $\mathcal{S}$, where I is $\mathcal{S}$-injective. They induce the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle w} & & \\ 0 & \longrightarrow & X & \xrightarrow{ui} & I & \xrightarrow{p} & TX & \longrightarrow & 0 \end{array}$$

We claim that $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{-w} TX$ belongs to $\mathcal{T}$. Indeed, this follows from the diagram:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \longrightarrow & 0 \\
 & & {\scriptstyle ui}\downarrow & & \downarrow{\scriptstyle (v\ i)} & & \| & & \\
0 & \longrightarrow & I & \xrightarrow{(0\ 1)} & Z \oplus I & \xrightarrow{\binom{1}{0}} & Z & \longrightarrow & 0 \\
 & & {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \binom{-w}{p}} & & & & \\
 & & TX & = & TX & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

It follows from (TR3) that it suffices to prove (TR2) and (TR4) for triangles constructed in 2) above.

Let us now turn to the proof of the sufficiency in (TR2). With the notation of 2), it suffices to prove that $Y \xrightarrow{v} Z \xrightarrow{-w} TX \xrightarrow{-Tu} TY$, or equivalently $Y \xrightarrow{v} Z \xrightarrow{w} TX \xrightarrow{Tu} TY$ belongs to $\mathcal{T}$. But this follows from the last two columns of the diagram

$$
\begin{array}{ccccc}
X & \xrightarrow{u} & Y & \xrightarrow{v} & Z \\
\| & & {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle w} \\
X & \xrightarrow{ui} & I & \xrightarrow{p} & TX \\
 & & {\scriptstyle i'}\downarrow & & \downarrow{\scriptstyle Tu} \\
 & & TY & = & TY
\end{array}
$$

The necessity in (TR2) is superfluous, since it follows from the other axioms:

Indeed, suppose that $B \xrightarrow{v} C \xrightarrow{w} TA \xrightarrow{-Tu} TB$ lies in $\mathcal{T}$. By the first part of (TR2), $TA \xrightarrow{-Tu} TB \xrightarrow{-Tv} TC \xrightarrow{-Tw} T^2A$ lies in $\mathcal{T}$. By (TR1), $\mathcal{T}$ contains a triangle $A \xrightarrow{u} B \xrightarrow{v'} C' \xrightarrow{w'} TA$, hence the induced triangle $TA \xrightarrow{-Tu} TB \xrightarrow{-Tv'} TC' \xrightarrow{-Tw'} T^2A$. By ((TR3) and consequences) there is an isomorphism

$$
\begin{array}{ccccccc}
TA & \xrightarrow{-Tu} & TB & \xrightarrow{-Tv} & TC & \xrightarrow{-Tw} & T^2A \\
\| & & \| & & {\scriptstyle \wr}\downarrow{\scriptstyle f} & & \| \\
TA & \xrightarrow{-Tu} & TB & \xrightarrow{-Tv'} & TC' & \xrightarrow{-Tw'} & T^2A
\end{array}
$$

which induces the wanted isomorphism

$$\begin{array}{ccccccc} A & \xrightarrow{u} & B & \xrightarrow{v} & C & \xrightarrow{w} & TA \\ \| & & \| & & \downarrow{\scriptstyle T^{-1}f} & & \| \\ A & \xrightarrow{u} & B & \xrightarrow{u'} & C' & \xrightarrow{w'} & TA \end{array}$$

TR4). Let $(X, Y, Z', \underline{u}, \underline{i}, \underline{i}')$, $(Y, Z, X', \underline{v}, \underline{j}, \underline{j}')$ and $(X, Z, Y', \underline{uv}, \underline{k}, \underline{k}')$ be triangles. By a previous remark, it is enough to consider the case where u and v are proper monomorphisms. In order to simplify our notations, we write B/A instead of coker m whenever m is a proper monomorphism and no confusion is possible. We also choose a proper monomorphism $Z \xrightarrow{m} I$, and a proper monomorphism $I/X \xrightarrow{n} J$, where I and J are $\mathcal{S}$-injective. This yields proper monomorphisms $X \xrightarrow{uvm} I$ and $Y \xrightarrow{vm} I$.

Axiom (TR4) now follows from the obvious commutativity of the following diagram, where u_1, u_2, v_1, v_2, f, f_1 denote the morphisms "naturally" induced by u, v, m and n.

$$\begin{array}{ccccccc} X & = & X & & & & \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle uv} & & & & \\ Y & \xrightarrow{v} & Z & \xrightarrow{v_1} & Z/Y & \xrightarrow{-v_2} & I/Y \\ \downarrow{\scriptstyle u_1} & & \downarrow{\scriptstyle w_1} & & \| & & \downarrow{\scriptstyle Tu_1} \\ Y/X & \xrightarrow{f} & Z/X & \xrightarrow{f_1} & Z/Y & \xrightarrow{-v_2(Tu_1)} & J/(Y/X) \\ \downarrow{\scriptstyle -u_2} & & \downarrow{\scriptstyle -w_2} & & \downarrow{\scriptstyle -v_2} & & \\ I/X & = & I/X & \xrightarrow{Tu} & I/Y & & \end{array}$$

9.5 Let $(\mathcal{B}, \mathcal{S})$ be an exact category. Let $\mathcal{F}$ be the free abelian group generated by the isomorphism classes of objects in $\mathcal{B}$ and $\mathcal{F}_0$ the subgroup generated by $[X]-[Y]+[Z]$ for all exact sequences $0 \to X \to Y \to Z \to 0$ in $\mathcal{S}$. The *Grothendieck group* $K_0(\mathcal{B}) = K_0((\mathcal{B}, \mathcal{S}))$ is by definition the factor group $\mathcal{F}/\mathcal{F}_0$.

Let $(\mathcal{B}, \mathcal{S})$ be a Frobenius category and denote by $\mathcal{P}$ the subgroup of $K_0(\mathcal{B})$ generated by [P] for all $\mathcal{S}$-projectives P in $\mathcal{B}$. The following corollary is an immediate consequence of the remarks 1) and 2) above.

COROLLARY. *The Grothendieck group $K_0(\underline{\mathcal{B}})$ of the triangulated category $\underline{\mathcal{B}}$ is the factor group $K_0(\mathcal{B})/\mathcal{P}$.*

10. Repetitive algebras

10.1 Let A be a finite-dimensional k-algebra and $Q = \mathrm{Hom}_k(A, k)$ the minimal injective cogenerator. Q carries a canonical A-bimodule structure. The *repetitive algebra* associated with A is by definition the doubly infinite matrix algebra, without identity,

$$\hat{A} = \begin{pmatrix} \ddots & & & & 0 \\ \ddots & A_{n-1} & & & \\ & Q_{n-1} & A_n & & \\ & & Q_n & A_{n+1} & \\ 0 & & & & \ddots \end{pmatrix}$$

in which matrices have only finitely many non-zero entries, $A_n = A$ is placed on the main diagonal, $Q_n = Q$ for all $n \in \mathbb{Z}$, all the remaining entries are zero, and the multiplication is induced from the canonical maps $A \otimes_A Q \to Q$, $Q \otimes_A A \to Q$ and the zero map $Q \otimes_A Q \to 0$.

This algebra was introduced in [HW] in connection with trivial extension algebras.

We define an $\hat{A}$-module X as a sequence $X = (X_n, f_n)$ of A-modules X_n and A-linear maps $f_n : X_n \to \mathrm{Hom}_A(Q, X_{n+1})$ satisfying $f_{n-1} \cdot \mathrm{Hom}_A(Q, f_n) = 0$ for all $n \in \mathbb{Z}$. Instead of (X_n, f_n) we also write

$$\cdots X_{-2} \overset{f_{-2}}{\sim} X_{-1} \overset{f_{-1}}{\sim} X_0 \overset{f_0}{\sim} X_1 \overset{f_1}{\sim} X_2 \cdots$$

or simply

$$\cdots X_{-2} \sim X_{-1} \sim X_0 \sim X_1 \sim X_2 \cdots$$

if we do not want to specify the maps f_i. A morphism $h : X = (X_n, f_n) \to Y = (Y_n, g_n)$ is a sequence $h = (h_n)$ of A-linear maps $h_n : X_n \to Y_n$ such that the following diagrams commute for all $n \in \mathbb{Z}$

$$\begin{array}{ccc} X_n & \xrightarrow{f_n} & \mathrm{Hom}_A(Q, X_{n+1}) \\ {\scriptstyle h_n}\downarrow & & \downarrow{\scriptstyle \mathrm{Hom}_A(Q, h_{n+1})} \\ Y_n & \xrightarrow{g_n} & \mathrm{Hom}_A(Q, Y_{n+1}) \end{array}$$

The category of $\hat{A}$-modules is equivalent to the category of modules over some locally bounded k-category [BG].

We denote by $\operatorname{Mod}\hat{A}$ the category of all $\hat{A}$-modules $X=(X_n, f_n)$ such that $\dim_k X_n < \infty$ for all $n \in \mathbb{Z}$, by $\operatorname{mod}\hat{A}$ the category of all $\hat{A}$-modules $X=(X_n, f_n)$ such that $\dim_k(\bigoplus_n X_n) < \infty$.

An alternative description of $\operatorname{mod}\hat{A}$ will be given at the end of this section.

We have a canonical embedding of $\operatorname{mod} A$ into $\operatorname{mod}\hat{A}$ which sends $X \in \operatorname{mod} A$ onto (X_n, f_n) where $X_0 = X$ and $X_n = 0$ for $n \neq 0$.

It is quite easy to see that $\operatorname{mod}\hat{A}$ is a Frobenius category and that the suspension functor can be chosen so as to be an automorphism (9.1, compare [HW]). The indecomposable projective-injective $\hat{A}$-modules are given by

$$\cdots 0 \sim X_i \overset{f_i}{\sim} X_{i+1} \sim 0 \cdots$$

where X_{i+1} is an indecomposable A-injective module, $X_i = \operatorname{Hom}_A(Q, X_{i+1})$ and $f_i = id_{X_i}$.

Using 9.4 we see that the stable category $\underline{\operatorname{mod}}\hat{A}$ is a triangulated category.

10.2 There is a rather useful notion in the theory of triangulated categories which was introduced in [BBD]. We recall the definition. A *t-category* is a triangulated category $\mathcal{D}$ endowed with two full sub-categories $\mathcal{D}^{\leq 0}$ and $\mathcal{D}^{\geq 0}$ which are closed under isomorphisms and such that for $\mathcal{D}^{\geq n} = T^{-n}(\mathcal{D}^{\geq 0})$ and $\mathcal{D}^{\leq n} = T^{-n}(\mathcal{D}^{\leq 0})$ the following three conditions are satisfied:

(1) For $X \in \mathcal{D}^{\leq 0}$ and $Y \in \mathcal{D}^{\geq 1}$ we have that $\operatorname{Hom}(X, Y) = 0$.

(2) $\mathcal{D}^{\leq 0} \subset \mathcal{D}^{\leq 1}$ and $\mathcal{D}^{\geq 1} \subset \mathcal{D}^{\geq 0}$.

(3) For $X \in \mathcal{D}$ there is a traingle $B' \to X \to B'' \to TB'$ such that $B' \in \mathcal{D}^{\leq 0}$ and $B'' \in \mathcal{D}^{\geq 1}$.

Under these conditions, we say that the pair $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is a *t-structure* on $\mathcal{D}$.

The derived category $D^b(\mathcal{A})$ of an abelian category $\mathcal{A}$ has a natural t-structure (see [BBD]).

Denote by $\mathcal{H}$ the full subcategory $\mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$ of $\mathcal{D}$. $\mathcal{H}$ is called the *heart* of the t-structure. It is shown in [BBD] that $\mathcal{H}$ is an abelian category.

PROPOSITION. *Let A be a finite-dimensional k-algebra. Then the triangulated category* $\underline{\operatorname{mod}}\hat{A}$ *has a natural t-structure* $(\mathcal{M}^{\leq 0}, \mathcal{M}^{\geq 0})$ *with heart equivalent to* $\operatorname{mod} A$.

Proof. Consider the full subcategory $\mathcal{M}^{\geq 0}$ (resp. $\mathcal{M}^{\leq 0}$) of $\underline{\operatorname{mod}}\hat{A}$ formed by the objects which admit a decomposition $Y \oplus Z$ in $\operatorname{mod}\hat{A}$ such that Z is projective-injective and $Y_n = 0$ for $n < 0$ (resp. for $n > 0$). We claim that $\mathcal{M}^{\geq 1} = T^{-1}(\mathcal{M}^{\geq 0})$ is the full subcategory of $\mathcal{M}^{\geq 0}$ formed by the objects $Y = (Y_n, f_n)$

such that the induced sequence

$$Q \otimes_A Y_{-1} \xrightarrow{f'_{-1}} Y_0 \xrightarrow{f_0} \operatorname{Hom}_A(Q, Y_1)$$

is exact in mod A. Indeed, this subcategory is obviously closed in $\underline{\operatorname{mod}} \hat{A}$ under isomorphisms. Therefore, in order to prove that $T^{-1}V$ belongs to it if $V \in \mathcal{M}^{\geq 0}$, it is enough to consider the case where $V_n = 0$ for $n < 0$. In this case V has a projective cover $P = (P_n, g_n)$ in mod $\hat{A}$ such that P_n is zero for $n < 0$ and $P_0 \xrightarrow{g_0} \operatorname{Hom}_A(Q, P_1)$ is injective. These two conditions are shared by all sub-objects of P, in particular by the kernel of $P \to V$ in mod $\hat{A}$, which is isomorphic to $T^{-1}V$ in $\underline{\operatorname{mod}} \hat{A}$. Conversely, suppose that $Y \in \mathcal{M}^{\geq 0}$ and that the sequence above is exact. In order to prove that $Y \in \mathcal{M}^{\geq 1}$, we may replace Y by an isomorphic object of $\underline{\operatorname{mod}} \hat{A}$, hence restrict to the case where $Y_n = 0$ for $n < 0$. We then choose an injective hull I_n of Y_n in mod A and set $P_n = \operatorname{Hom}_A(Q, I_n)$ for $n \geq 1$.

The injection $Y_0 \xrightarrow{f_0} \operatorname{Hom}_A(Q, Y_1) \to \operatorname{Hom}_A(Q, I_1) = P_1$ obviously extends to the monomorphism e below, so that Y is isomorphic to $T^{-1}(\operatorname{coker} e)$, where $\operatorname{coker} e \in \mathcal{M}^{\geq 0}$.

$$\begin{array}{ccccccccc}
\cdots 0 & \sim & Y_0 & \sim & Y_1 & \sim & Y_2 & \cdots \\
\downarrow{\scriptstyle e_{-1}} & & \downarrow{\scriptstyle e_0} & & \downarrow{\scriptstyle e_1} & & \downarrow{\scriptstyle e_2} & \\
\cdots 0 & \sim & P_1 \oplus 0 & \sim & I_1 \oplus P_2 & \sim & P_3 \oplus I_2 & \cdots
\end{array}$$

Now consider a morphism $h: X \to Y$, where $X \in \mathcal{M}^{\leq 0}$ and $Y \in \mathcal{M}^{\geq 1}$. In order to prove that $h = 0$ in $\underline{\operatorname{mod}} \hat{A}$, we may suppose that $X_n = 0$ for $n > 0$ and $Y_n = 0$ for $n < 0$. The diagram below then implies $h_0 = 0$, hence $h = 0$.

$$\begin{array}{ccccc}
 & & X_0 & \longrightarrow & \operatorname{Hom}_A(Q, 0) = 0 \\
 & & \downarrow{\scriptstyle h_0} & & \downarrow \\
0 & \longrightarrow & Y_0 & \longrightarrow & \operatorname{Hom}_A(Q, Y_1)
\end{array}$$

This proves condition (1). The inclusion $\mathcal{M}^{\geq 1} \subset \mathcal{M}^{\geq 0}$ is clear from the above. The inclusion $\mathcal{M}^{\leq -1} \subset \mathcal{M}^{\leq 0}$ follows from dual arguments. It implies $\mathcal{M}^{\leq 0} \subset \mathcal{M}^{\leq 1}$. Finally, if $X = (X_n, f_n) \in \underline{\operatorname{mod}} \hat{A}$, we construct a triangle $B' \to X \to B'' \to TB'$ such that $B' \in \mathcal{M}^{\leq 0}$ and $B'' \in \mathcal{M}^{\geq 1}$ by setting $B'_n = X_n$ and $B''_n = 0$ for $n < 0$, $B'_0 = \ker f_0$ and $B''_0 = \operatorname{im} f_0$, $B'_n = 0$ and $B''_n = X_n$ for $n > 0$.

10.3 PROPOSITION. *Let A have finite global dimension and $\mathscr{C}$ be a full triangulated subcategory of $\underline{\mathrm{mod}}\,\hat{A}$ which contains $\underline{\mathrm{mod}}\,A$ and is closed under isomorphisms. Then $\mathscr{C} = \underline{\mathrm{mod}}\,\hat{A}$.*

Proof. By assumption, $X \in \mathscr{C}$ implies $TX \in \mathscr{C}$ and $T^{-1}X \in \mathscr{C}$; moreover, a triangle (X, Y, Z, u, v, w) of $\underline{\mathrm{mod}}\,\hat{A}$ belongs to $\mathscr{C}$ if two of its objects do.

First we show that $X \in \mathscr{C}$ if $X \in \mathscr{M}^{\geq 0}$. We proceed by induction on $e(X) = \min\{e \in \mathbb{N} : X_n = 0 \text{ for } n > e\}$. If $e(X) = 0$, X is isomorphic to an object of $\mathrm{mod}\,A$ and belongs to $\mathscr{C}$. So we may suppose that $e(X) = e \geq 1$ and that $Y \in \mathscr{C}$ if $Y \in \mathscr{M}^{\geq 0}$ and $e(Y) < e$. We then proceed by induction on the injective dimension $\mathrm{id}\,(X_e)$ of X_e in $\mathrm{mod}\,A$: A minimal injection $i: X_e \to I$ into an injective A-module I extends to a morphism $j: X \to J$ of $\mathrm{mod}\,\hat{A}$, where J denotes the projective-injective such that $J_e = I$, $J_{e-1} = \mathrm{Hom}_A(Q, I)$ and $J_n = 0$ if $n \neq e, e-1$. Clearly, J belongs to $\mathscr{C}$, and so does $C = \mathrm{coker}\, j$ ($e(C) < e$ if $\mathrm{id}\,(X_e) = 0$ and $\mathrm{id}\,(C_e) < \mathrm{id}\,(X_e)$ if $\mathrm{id}\,(X_e) > 0$). It follows that $\mathrm{im}\, j \in \mathscr{C}$. Since $e(\ker j) < e$, we finally have that $\ker j \in \mathscr{C}$, hence $X \in \mathscr{C}$. By duality, we obtain that $\mathscr{M}^{\leq 0} \subset \mathscr{C}$. Our proposition now follows from axiom (3) of 10.2.

COROLLARY. *Let A have finite global dimension and $\mathscr{C}$ be a full triangulated subcategory of $\underline{\mathrm{mod}}\,\hat{A}$ which is closed under isomorphisms and contains the full subcategory ${}_A\mathscr{I}$ of* $\mathrm{mod}\,A$ *formed by the injectives. Then $\mathscr{C} = \underline{\mathrm{mod}}\,\hat{A}$.*

Proof. It is enough to show that each $X \in \mathrm{mod}\,A$ belongs to $\mathscr{C}$. For this we use induction on $r = \mathrm{id}\,(X)$ for $X \in \mathrm{mod}\,A$. For $r = 0$ there is nothing to show. So let $X \in \mathrm{mod}\,A$ with $\mathrm{id}\,(X) = r \geq 1$. Let $0 \to X \to I \to Y \to 0$ be exact in $\mathrm{mod}\,A$ with $I \in {}_A\mathscr{I}$. Then $\mathrm{id}\,(Y) < r$. The above exact sequence yields a triangle $X \to I \to Y \to TX$ in $\underline{\mathrm{mod}}\,\hat{A}$. As I and Y belong to $\mathscr{C}$, we infer that $X \in \mathscr{C}$.

Note that we have a corresponding result for the derived category $D^b(A)$. If A has finite global dimension, the smallest full triangulated subcategory of $D^b(A)$ which contains ${}_A\mathscr{I}$ and is closed under isomorphisms coincides with $D^b(A)$.

10.4 LEMMA. *There exist an exact functor $I: \mathrm{mod}\,\hat{A} \to \mathrm{mod}\,\hat{A}$ and a monomorphism $\mu: \mathrm{id} \to I$ such that $I(X)$ is injective for each $X \in \mathrm{mod}\,\hat{A}$.*

Proof. For each $X = (X_n, f_n) \in \mathrm{mod}\,\hat{A}$, we define $I(X) = (I_n, d_n)$ by $I_n = \mathrm{Hom}_k(Q, X_{n+1}) \oplus \mathrm{Hom}_k(A, X_n)$ and $d_n = \begin{pmatrix} 0 & \delta_n \\ 0 & 0 \end{pmatrix}$, where the left A-module structure of I_n is induced by the right A-module structure of Q and A, and where $\delta_n: \mathrm{Hom}_k(Q, X_{n+1}) \to \mathrm{Hom}_A(Q, \mathrm{Hom}_k(A, X_{n+1}))$ is the canonical isomorphism mapping φ onto $q \to (a \to \varphi(aq))$. We define $\mu(X): X \to I(X)$ by $\mu(X)_n = [f_n \xi_n]$, where $\xi_n: X_n \to \mathrm{Hom}_k(A, X_n)$ maps x onto $a \to ax$.

With the notation above, we set $S(X) = \operatorname{coker} \mu(X)$ and denote by $\pi(X): I(X) \to S(X)$ the canonical projection. This defines an exact functor $S: \operatorname{mod} \hat{A} \to \operatorname{mod} \hat{A}$ mapping injectives onto injectives. The induced functor $\underline{\operatorname{mod}}\, \hat{A} \to \underline{\operatorname{mod}}\, \hat{A}$ will be denoted by $\underline{S}$.

10.5 Let $C^{\leq 0}(\operatorname{mod} A)$ be the full subcategory of the category $C^b(\operatorname{mod} A)$ of bounded complexes which is formed by the complexes vanishing in positive degrees. The translation functor T is defined on $C^{\leq 0}(\operatorname{mod} A)$, and the mapping cone $C_{f^\cdot}$ of a morphism $f^\cdot$ in $C^{\leq 0}(\operatorname{mod} A)$ is contained in $C^{\leq 0}(\operatorname{mod} A)$.

For $i \geq 0$, denote by $C[-i, 0]$ the full subcategory of $C^{\leq 0}(\operatorname{mod} A)$ with objects $X^\cdot = (X^n, d^n)$ such that $X^n = 0$ for $n < -i$. Identify $C[0, 0]$ with $\operatorname{mod} A$. By induction on i we will construct functors $F_i: C[-i, 0] \to \operatorname{mod} \hat{A}$ such that $F_{i|C[-i+1,0]} = F_{i-1}$.

Let $i = 0$. Using the identification of $C[0, 0]$ with $\operatorname{mod} A$, we define F_0 to be the canonical embedding of $\operatorname{mod} A$ into $\operatorname{mod} \hat{A}$ (10.1). Suppose that F_{i-1}: $C[-i+1, 0] \to \operatorname{mod} \hat{A}$ is already constructed. Let $X^\cdot = (X^n, d_X^n)$ be in $C[-i, 0]$. Denote by $X'^\cdot = (X'^n, d_{X'}^n)$ the complex such that $X'^n = 0$ for $n \geq 0$, $X'^n = X^n$ for $n < 0$ and $d_{X'}^n = d_X^n$ for $n < -1$. Then $T^- X'^\cdot$ is contained in $C[-i+1, 0]$ and d_X^{-1} induces a morphism e_X from $T^- X'^\cdot$ to X^0 whose mapping cone is $X^\cdot$. The functor F_{i-1} is defined on $T^- X'^\cdot$, X^0 and e_X. Consider the following pushout diagram in $\operatorname{mod} \hat{A}$

$$
\begin{array}{ccc}
F_{i-1}(T^- X'^\cdot) & \xrightarrow{\;F_{i-1}(e_X)\;} & F_{i-1}(X^0) \\
\Big\downarrow {\scriptstyle \mu(F_{i-1}(T^- X'^\cdot))} & & \Big\downarrow {\scriptstyle u_{X^\cdot}} \\
I(F_{i-1}(T^- X'^\cdot)) & \longrightarrow & C_{F_{i-1}(e_X)} \\
\Big\downarrow {\scriptstyle \pi(F_{i-1}(T^- X'^\cdot))} & & \Big\downarrow {\scriptstyle v_{X^\cdot}} \\
S(F_{i-1}(T^- X'^\cdot)) & = & S(F_{i-1}(T^- X'^\cdot))
\end{array}
$$

Then we set $F_i(X^\cdot) = C_{F_{i-1}(e_X)}$.

Next we show by induction on i that $F_{i|C[-i+1,0]} = F_{i-1}$. If $i = 1$ and $X^\cdot \in C[0, 0]$, then $T^- X'^\cdot$ vanishes, and we have to consider the following pushout diagram in $\operatorname{mod} \hat{A}$:

$$
\begin{array}{ccc}
0 & \longrightarrow & F_0(X^0) \\
\big\downarrow & & \big\| \\
0 & & F_1(X^0) \\
\big\downarrow & & \big\downarrow \\
0 & = & 0
\end{array}
$$

Thus $F_1(X^\cdot) = F_0(X^\cdot)$.

Suppose that the assertion is true for $j<i$ and let $X^{\cdot} \in C[-i+1, 0]$. Then $T^{-}X'^{\cdot} \in C[-i+2, 0]$. We compute $F_i(X^{\cdot})$ and $F_{i-1}(X^{\cdot})$ by means of the following two pushout diagrams in $\operatorname{mod} \hat{A}$:

$$\begin{array}{ccc}
F_{i-2}(T^{-}X'^{\cdot}) & \xrightarrow{F_{i-2}(e_X)} & F_{i-2}(X^0) \\
\Big\downarrow{\scriptstyle \mu(F_{i-2}(T^{-}X'^{\cdot}))} & & \Big\downarrow \\
I(F_{i-2}(T^{-}X'^{\cdot})) & \longrightarrow & F_{i-1}(X^{\cdot}) \\
\Big\downarrow & & \Big\downarrow \\
S(F_{i-2}(T^{-}X'^{\cdot})) & = & S(F_{i-2}(T^{-}X'^{\cdot}))
\end{array}$$

$$\begin{array}{ccc}
F_{i-1}(T^{-}X'^{\cdot}) & \xrightarrow{F_{i-1}(e_X)} & F_{i-1}(X^0) \\
\Big\downarrow{\scriptstyle \mu(F_{i-1}(T^{-}X'^{\cdot}))} & & \Big\downarrow \\
I(F_{i-1}(T^{-}X'^{\cdot})) & \longrightarrow & F_i(X^{\cdot}) \\
\Big\downarrow & & \Big\downarrow \\
S(F_{i-1}(T^{-}X'^{\cdot})) & \longrightarrow & S(F_{i-1}(T^{-}X'^{\cdot}))
\end{array}$$

By induction $F_{i-2}(T^{-}X'^{\cdot}) = F_{i-1}(T^{-}X'^{\cdot})$, $F_{i-2}(X^0) = F_{i-1}(X^0)$ and $F_{i-2}(e_X) = F_{i-1}(e_X)$. Therefore $F_{i-1}(X^{\cdot}) = F_i(X^{\cdot})$.

LEMMA. *The functor* $F = \varinjlim_i F_i : C^{\leq 0}(\operatorname{mod} A) \to \operatorname{mod} \hat{A}$ *satisfies* $F_{|C[-i,0]} = F_i$ *for* $i \geq 0$ *and is associated with a canonical isomorphism* $\eta : FT \xrightarrow{\sim} SF$.

Proof. The first assertion is clear. So it remains to construct $\eta : FT \to SF$. Let $X^{\cdot} \in C^{\leq 0}(\operatorname{mod} A)$; then there exists i such that $X^{\cdot} \in C[-i, 0]$, but $X^{\cdot} \notin C[-i+1, 0]$. Clearly $X^{\cdot} = T^{-}(TX^{\cdot})'$ and $e_{TX} = 0$. Thus we have to consider the following pushout diagram in $\operatorname{mod} \hat{A}$:

$$\begin{array}{ccc}
F_i(X^{\cdot}) & \xrightarrow{0} & 0 \\
\Big\downarrow{\scriptstyle \mu(F_i(X^{\cdot}))} & & \Big\downarrow \\
I(F_i(X^{\cdot})) & \xrightarrow{\pi(F_i(X^{\cdot}))} & F_{i+1}(TX^{\cdot}) \\
\Big\downarrow{\scriptstyle \pi(F_i(X^{\cdot}))} & & \Big\downarrow{\scriptstyle v} \\
S(F_i(X^{\cdot})) & = & S(F_i(X^{\cdot}))
\end{array}$$

Note that $F_{i+1}(TX^{\cdot}) = F(TX^{\cdot})$ and that $S(F_i(X^{\cdot})) = S(F(X^{\cdot}))$. We set $\eta(X^{\cdot}) = v$.

10.6 The proof of the following two lemmas is clear.

LEMMA. *If* $X^{\cdot} \in C^{\leq 0}(\operatorname{mod} A)$, *the associated sequence* $F(T^{-}X^{\cdot\cdot}) \xrightarrow{F(e_X)} F(X^0) \xrightarrow{F(u)} F(X^{\cdot}) \xrightarrow{F(v)} F(X^{\cdot\cdot})$ *is a standard sextuple in* $\operatorname{mod} \hat{A}$. *In particular* $0 \to F(X^0) \xrightarrow{F(u)} F(X^{\cdot}) \xrightarrow{F(v)} F(X^{\cdot\cdot}) \to 0$ *is a short exact sequence in* $\operatorname{mod} \hat{A}$.

Consider $C^{\leq 0}(\operatorname{mod} A)$ as a full subcategory of the Frobenius category $C^b(\operatorname{mod} A)$ (9.2), and denote by $\mathcal{S}^{\leq 0}$ the set of exact sequences with terms in $C^{\leq 0}(\operatorname{mod} A)$.

LEMMA. *Let* $0 \to X^{\cdot} \xrightarrow{u^{\cdot}} Y^{\cdot} \xrightarrow{v^{\cdot}} Z^{\cdot} \to 0$ *be in* $\mathcal{S}^{\leq 0}$. *Then*

$$0 \to F(X^{\cdot}) \xrightarrow{F(u^{\cdot})} F(Y^{\cdot}) \xrightarrow{F(v^{\cdot})} F(Z^{\cdot}) \to 0$$

is a short sequence in $\operatorname{mod} \hat{A}$.

10.7 Let $C^{\leq 0}({}_A\mathcal{I})$ be the full subcategory of $C^{\leq 0}(\operatorname{mod} A)$ formed by complexes with components in ${}_A\mathcal{I}$. Denote by G' the restriction of F to $C^{\leq 0}({}_A\mathcal{I})$, by $K^{\leq 0}({}_A\mathcal{I})$ the residue-category of $C^{\leq 0}({}_A\mathcal{I})$ modulo homotopy.

LEMMA. G' *induces a functor* $G: K^{\leq 0}({}_A\mathcal{I}) \to \underline{\operatorname{mod}}\, \hat{A}$ *associated with a canonical isomorphism* $\xi: GT \xrightarrow{\sim} \underset{\sim}{S}G$.

Proof. It is enough to show that a projective-injective object in $C^{\leq 0}({}_A\mathcal{I})$ is transformed under G' into a projective-injective module in $\operatorname{mod} \hat{A}$. Let $I^{\cdot} \in C^{\leq 0}({}_A\mathcal{I})$ be projective-injective. We may assume that $I^{\cdot}$ is indecomposable. Applying T^{-} if necessary, we may assume that $I^{\cdot}$ is of the form $\cdots 0 \to I^{-1} \xrightarrow{1} I^0 \to 0 \cdots$. But then $G'(I^{\cdot}) = I(I^{-1})$ (with the notations of Lemma 10.4) which is a projective-injective module in $\operatorname{mod} \hat{A}$. Thus G' factors over $K^{\leq 0}({}_A\mathcal{I})$. Clearly η induces a natural transformation ξ.

10.8 As noted before S induces a functor on the stable category $\underline{\operatorname{mod}}\, \hat{A}$ denoted by $\underset{\sim}{S}$. It turns out that $\underset{\sim}{S}$ is a selfequivalence. Denote by S' a quasi-inverse of $\underset{\sim}{S}$ on $\underline{\operatorname{mod}}\, \hat{A}$ and by $\alpha: S'\underset{\sim}{S} \to \operatorname{id}$ an invertible natural transformation. We also choose an invertible natural transformation $\beta: \operatorname{id} \to \underset{\sim}{S}S'$.

We inductively construct an invertible natural transformation $\alpha_r: S'^r \underset{\sim}{S}^r \to \operatorname{id}$ for $r \geq 1$. Let $\alpha_1 = \alpha$ and suppose that α_i is constructed for $i < r$. Then we define $\alpha_r(X) = S'(\alpha_{r-1}(\underset{\sim}{S}(X))) \cdot \alpha_1(X)$. Clearly α_r is an invertible natural transformation.

Let $\alpha_0 = \operatorname{id}$. It follows that for $r, r' \geq 0$ we have

(1) $\alpha_{r+r'}(X) = S'^r(\alpha_{r'}(\underset{\sim}{S}^r(X))) \cdot \alpha_r(X)$.

Later we will need two consequences of this formula:

(2) $S'^r(\alpha_{s-r}(S^r(Y))) = \alpha_s(Y)\cdot\alpha_r^{-1}(Y)$ for $s-r\geq 0$.

(3) $S'^s(\alpha_{r-s}^{-1}(S^s(Y))) = \alpha_s(Y)\cdot\alpha_r^{-1}(Y)$ for $r-s\geq 0$.

LEMMA. *There exist a k-linear functor $\tilde{G}: K^b({}_A\mathcal{I})\to \underline{\mathrm{mod}}\,\hat{A}$ and an invertible natural transformation $\tilde{\xi}: \tilde{G}T \xrightarrow{\sim} S\tilde{G}$ such that $\tilde{G}|_{K^{\leq 0}({}_A\mathcal{I})} = G$ and $\tilde{\xi}|_{K^{\leq 0}({}_A\mathcal{I})} = \xi$.*

Proof. Let $X^\cdot\in K^b({}_A\mathcal{I})$. Then there exists $t(X)\geq 0$ such that $T^{t(X)}X^\cdot\in K^{\leq 0}({}_A\mathcal{I})$. Let $t(X)$ be minimal with this property. Then we define $\tilde{G}(X^\cdot) = S'^{t(X)}(G(T^{t(X)}X^\cdot)$.

Note that for $r\geq 0$ we have isomorphisms

$$S'^{t(X)}(\alpha_r(G(T^{t(X)}X^\cdot))): S'^{t(X)+r}G(T^{t(X)+r}X^\cdot)\to \tilde{G}(X^\cdot).$$

We define $\tilde{G}$ on morphisms as follows:

Let $f^\cdot: X^\cdot\to Y^\cdot$ be a morphism of $K^b({}_A\mathcal{I})$. If $t(Y)\geq t(X)$ we define $\tilde{G}(f^\cdot) = S'^{t(X)}(\alpha_r^{-1}(G(T^{t(X)}X^\cdot)))\cdot G(T^{t(Y)}f^\cdot)$ with $r = t(Y)-t(X)$. If $t(Y)\leq t(X)$ then we define

$$\tilde{G}(f^\cdot) = S'^{t(X)}G(T^{t(X)}f^\cdot)\cdot S'^{t(Y)}(\alpha_s(G(T^{t(Y)}Y^\cdot))) \quad \text{with} \quad s = t(X)-t(Y).$$

Observe that for $t(X) = t(Y)$ both definitions coincide. Clearly $\tilde{G}$ is k-linear and $\tilde{G}(\mathrm{id}_{X^\cdot}) = \mathrm{id}_{\tilde{G}(X^\cdot)}$.

Let us show that $\tilde{G}$ preserves the composition of morphisms. For this let $X^\cdot, Y^\cdot, Z^\cdot\in K^b({}_A\mathcal{I})$ and $f^\cdot: X^\cdot\to Y^\cdot$, $g^\cdot: Y^\cdot\to Z^\cdot$ be two morphisms. We consider the case $t(Y)\leq t(Z)\leq t(X)$:

Let $s = t(X)-t(Y)$, $r = t(Z)-t(Y)$. Thus $s-r = t(X)-t(Z)$.

By definition

$$\begin{aligned}\tilde{G}(f^\cdot\cdot g^\cdot) &= S'^{t(X)}G(T^{t(X)}f^\cdot g^\cdot)\cdot S'^{t(Z)}(\alpha_{s-r}(G(T^{t(Z)}Z^\cdot)))\\ &= S'^{t(X)}G(T^{t(X)}f^\cdot)\cdot S'^{t(X)}G(T^{t(X)}g^\cdot)\cdot S'^{t(Z)}(\alpha_{s-r}(GT^{t(Z)}Z^\cdot))\end{aligned}$$

by naturality of α_{s-r}

$$= S'^{t(X)}G(T^{t(X)}f^\cdot)\cdot S'^{t(Z)}(\alpha_{s-r}(GT^{t(Z)}Y^\cdot))\cdot S'^{t(Z)}G(T^{t(Z)}g^\cdot)$$

by (2)

$$= S'^{t(X)}G(T^{t(X)}f^\cdot)\cdot S'^{t(Y)}(\alpha_s(GT^{t(Y)}Y^\cdot))\cdot S'^{t(Y)}(\alpha_r^{-1}(GT^{t(Y)}Y^\cdot)\cdot S'^{t(Z)}G(T^{t(Z)}g^\cdot)$$

by definition of $\tilde{G}(f^\cdot)$ and $\tilde{G}(g^\cdot)$

$$= \tilde{G}(f^\cdot)\cdot\tilde{G}(g^\cdot).$$

The remaining five cases are similar.

Next we will define ξ.

For this let $X^{\cdot} \in K^b({}_A\mathcal{I})$. If $t(X)=0$, let $\xi(X^{\cdot})=\xi(X^{\cdot})$. Otherwise $t(TX^{\cdot}) = t(X)-1 \geq 0$. Then $\tilde{G}TX^{\cdot} = S'^{t(X)-1}G(T^{t(X)}X^{\cdot})$ and $\underset{\sim}{S}\tilde{G}X^{\cdot} = \underset{\sim}{S}S'^{t(X)}G(T^{t(X)}X^{\cdot}) = \underset{\sim}{S}S'S'^{t(X)-1}G(T^{t(X)}X^{\cdot})$.

Then we define $\xi(X^{\cdot}) = \beta(S'^{t(X)-1}G(T^{t(X)}X^{\cdot}))$, where $\beta:\mathrm{id} \xrightarrow{\sim} \underset{\sim}{S}S'$ is the chosen invertible natural transformation.

Clearly $\xi(X^{\cdot})$ is an isomorphism.

Let $f^{\cdot}: X^{\cdot} \to Y^{\cdot}$ be a morphism. We have to show that

$$\xi(X^{\cdot}) \cdot \underset{\sim}{S}\tilde{G}(f^{\cdot}) = \tilde{G}(Tf^{\cdot}) \cdot \xi(Y^{\cdot}).$$

We present the case $t(X) \geq t(Y)$. The other case is similar.

Let $s = t(X) - t(Y)$. Then

$$\begin{aligned}\tilde{G}(Tf^{\cdot}) \cdot \xi(Y^{\cdot}) &= S'^{t(X)-1}G(T^{t(X)-1}Tf^{\cdot}) \cdot S'^{t(Y)-1}(\alpha_s(G(T^{t(Y)}Y^{\cdot}))) \cdot \beta(S'^{t(Y)-1}GT^{t(Y)}Y^{\cdot})\\ &= S'^{t(X)-1}G(T^{t(X)-1}Tf^{\cdot}) \cdot \beta(S'^{t(X)-1}G(T^{t(Y)}Y^{\cdot})) \cdot \underset{\sim}{S}S'^{t(Y)}(\alpha_s(G(T^{t(Y)}Y^{\cdot})))\\ &= \beta(S'^{t(X)-1}G(T^{t(X)}X^{\cdot})) \cdot \underset{\sim}{S}S'^{t(X)}G(T^{t(X)}f^{\cdot}) \cdot \underset{\sim}{S}S'^{t(Y)}(\alpha_s(G(T^{t(Y)}Y^{\cdot})))\\ &= \xi(X^{\cdot}) \cdot \underset{\sim}{S}\tilde{G}(f^{\cdot}).\end{aligned}$$

Therefore ξ is an invertible natural transformation such that $\xi|_{K^{\leq 0}({}_A\mathcal{I})} = \xi$.

10.9 We have defined an automorphism T on $\underline{\mathrm{mod}}\,\hat{A}$ (9.1) which serves us as a translation functor for the triangulated category $\underline{\mathrm{mod}}\,\hat{A}$. There exists an invertible natural transformation $\gamma: \underset{\sim}{S} \xrightarrow{\sim} T$ [He]. In particular we obtain an invertible natural transformation $\xi: \tilde{G}T \xrightarrow{\sim} T\tilde{G}$ with $\xi = \xi(\gamma\tilde{G})$.

PROPOSITION. *$\tilde{G}$ is an exact functor of triangulated categories.*

Proof. We have noticed before that $\tilde{G}$ commutes with T up to isomorphism. Clearly we may restrict to the triangles constructed in Remark 2 of 9.4 and contained in $K^{\leq 0}({}_A\mathcal{I})$. But then the assertion follows from 10.6.

10.10 THEOREM. *Let A be a finite-dimensional k-algebra of finite global dimension. Then $\tilde{G}: K^b({}_A\mathcal{I}) \to \underline{\mathrm{mod}}\,\hat{A}$ is a triangle-equivalence.*

Proof. We have to show that $\tilde{G}$ is dense, full and faithful (compare [B1], [B2]). By 10.3 and 10.9 it follows that $\tilde{G}$ is dense. We show that $\tilde{G}$ is full and faithful by induction on the width $w(X^{\cdot})$ of the considered complexes $X^{\cdot}$ ($w(X^{\cdot})=0$ if $X^{\cdot}=0$, and $w(X^{\cdot}) = j-i+1$ if $X^i \neq 0 \neq X^j$ and $X^n = 0$ for $n<i$ or $n>j$).

For this let $I^{\cdot}, J^{\cdot} \in K^b({}_A\mathcal{I})$ with $w(I^{\cdot}) = w(J^{\cdot}) = 1$. Then $I^{\cdot} = T^iI$, $J^{\cdot} = T^jJ$ for some $i, j \in \mathbb{Z}$ and $I, J \in {}_A\mathcal{I}$. Applying T if necessary, we may assume that $i=0$. If

$j=0$ we use that $\tilde{G}$ restricted to ${}_A\mathcal{I}$ equals the identity. If $j<0$, note that $\tilde{G}(I^\cdot)\in\mathcal{M}^{\leq 0}$ and that $\tilde{G}(J^\cdot)\in\mathcal{M}^{\geq 1}$. Thus the assertion follows from the first property of the t-structure. So it remains to consider the case $j>0$, where $\tilde{G}(J^\cdot)\in\mathcal{M}^{\leq 0}$. Let $0\to T^{j-1}J\to I(T^{j-1}J)\to T^jJ\to 0$ be exact in $\operatorname{mod}\hat{A}$. Then we obtain in $\operatorname{mod}\hat{A}$:

$$
\begin{array}{ccccccc}
0 & & & 0 & & & \\
\downarrow & & & \downarrow & & & \\
T^{j-1}J = \cdots \sim & J_{-1} & \sim & & J_0 & \sim 0\cdots \\
\downarrow & & & \downarrow & & & \\
I(T^{j-1}J) = \cdots \sim & \begin{array}{c}\operatorname{Hom}_k(Q,J_0)\\ \oplus \\ \operatorname{Hom}_k(A,J_{-1})\end{array} & \sim & & \operatorname{Hom}_k(A,J_0) & \sim 0\cdots \\
\downarrow & & & \downarrow & & & \\
T^jJ = \cdots \sim & K_{-1} & \sim & & K_0 & \sim 0\cdots \\
\downarrow & & & \downarrow & & & \\
0 & & & 0 & & &
\end{array}
$$

K_0 is a direct summand of $\operatorname{Hom}_k(A, J_0)$, since $J_0\in{}_A\mathcal{I}$ (by induction on j). Thus $\underline{\operatorname{Hom}}(I, T^jJ)=0$.

Assume that the assertion is true for $I^\cdot, J^\cdot\in K^b({}_A\mathcal{I})$ with $w(I^\cdot)=1$ and $w(J^\cdot)<r$. Let $J^\cdot=(J^i, d^i)\in K^b({}_A\mathcal{I})$ with $w(J^\cdot)=r$. Then there exists $s\in\mathbb{Z}$ such that $J^\cdot$ is the mapping cone of $T^{s-1}J^s\to J'^\cdot$, where $w(J'^\cdot)=r-1$. This gives rise to triangles $T^{s-1}J^s\to J'^\cdot\to J^\cdot\to T^sJ^s$ in $K^b({}_A\mathcal{I})$ and $\tilde{G}T^{s-1}J^s\to\tilde{G}J'^\cdot\to\tilde{G}J^\cdot\to T\tilde{G}T^{s-1}J^s$ in $\underline{\operatorname{mod}}\,\hat{A}$. Applying the cohomological functors $\underline{\operatorname{Hom}}(I^\cdot, -)$ and $\underline{\operatorname{Hom}}(\tilde{G}I^\cdot, -)$ yields the following commutative diagram with exact rows:

$$
\begin{array}{ccccc}
\operatorname{Hom}(I^\cdot, T^{s-1}J^s) & \longrightarrow & \operatorname{Hom}(I^\cdot, J'^\cdot) & \longrightarrow & \operatorname{Hom}(I^\cdot, J^\cdot) \\
\downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} \\
\operatorname{Hom}(\tilde{G}I^\cdot, \tilde{G}T^{s-1}J^s) & \longrightarrow & \operatorname{Hom}(\tilde{G}I^\cdot, \tilde{G}J'^\cdot) & \longrightarrow & \operatorname{Hom}(\tilde{G}I^\cdot, \tilde{G}J^\cdot)
\end{array}
$$

$$
\begin{array}{ccccc}
\longrightarrow & \operatorname{Hom}(I^\cdot, T^sJ^s) & \longrightarrow & \operatorname{Hom}(I^\cdot, TJ'^\cdot) \\
& \downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle f_5} \\
\longrightarrow & \operatorname{Hom}(\tilde{G}I^\cdot, \tilde{G}T^sJ^s) & \longrightarrow & \operatorname{Hom}(\tilde{G}I^\cdot, T\tilde{G}J'^\cdot)
\end{array}
$$

By induction it follows that f_1, f_2, f_4, f_5 are isomorphisms, hence f_3. The remaining part of the proof is dual.

COROLLARY. *Let A be a finite-dimensional k-algebra of finite global dimension. Then $D^b(A)$ and $\underline{\mathrm{mod}}\,\hat{A}$ are triangle-equivalent.*

COROLLARY. *Let $(A, {}_AM_B, B)$ be a tilting triple with* $\mathrm{gl.\,dim}\,A < \infty$. *Then $\underline{\mathrm{mod}}\,\hat{A}$ and $\underline{\mathrm{mod}}\,\hat{B}$ are triangle-equivalent.*

For a result related to this we refer to [TW].

COROLLARY. *Let A be a piecewise hereditary algebra of type $\vec{\Delta}$. Then $\underline{\mathrm{mod}}\,\hat{A}$ and $\underline{\mathrm{mod}}\,(\widehat{k\vec{\Delta}})$ are triangle-equivalent.*

10.11 For the alternative description of $\mathrm{mod}\,\hat{A}$ we have to recall the definition of the *trivial extension algebra* $T(A)$ of A. The underlying vectorspace of $T(A) = A \oplus Q$, and the multiplication is defined by

$$(a, q) \cdot (a', q') = (aa', aq' + qa')$$

for $a, a' \in A$ and $q, q' \in Q$.

$T(A)$ is a $\mathbb{Z}$-graded algebra, where $A \oplus 0$ are the elements of degree zero, and $0 \oplus Q$ those of degree one. We denote by $\mathrm{gr\,mod}\,T(A)$ the category of finitely generated $\mathbb{Z}$-graded $T(A)$-modules with morphisms of degree zero.

It is straightforward that $\mathrm{gr\,mod}\,T(A)$ and $\mathrm{mod}\,\hat{A}$ are equivalent. Moreover, the forgetful functor from $\mathrm{gr\,mod}\,T(A)$ to $\mathrm{mod}\,T(A)$ is a Galois covering in the sense of Gabriel [G3].

Appendix: Proof of theorem 5.10

A1. The following demonstration replaces the proof of theorem 1 of [H2] for which P. Gabriel communicated us a counterexample.

Let $\mathcal{T} = \{t_1, \ldots, t_n\}$ be a tilting set (5.4) whose associated algebra $E = \mathrm{End}\,\mathcal{T}$ (5.5) is simply connected. Consider the canonical functor $\pi : \mathrm{ind}\,D^b(k\vec{\Delta}) \to \mathcal{R}(\vec{\Delta})$ (5.1) and the full subcategories $\mathcal{T}$ of $\mathcal{R}(\Delta)$ and $\pi^{-1}(\mathcal{T})$ of $\mathrm{ind}\,D^b(k\vec{\Delta})$ which are supported by $\mathcal{T}$ and $\pi^{-1}(\mathcal{T})$. Then $\pi^{-1}(\mathcal{T})$ is a Galois covering of $\mathcal{T}$ with Galois group $T^{2\mathbb{Z}}$ [G3]. Since $\mathrm{End}\,\mathcal{T}$ is simply connected, the connected components of $\pi^{-1}(\mathcal{T})$ are mapped isomorphically onto $\mathcal{T}$ by π. In the sense of the following definition, the points of such a component form a tilting set $\tilde{\mathcal{T}} = \{\tilde{t}_1, \ldots, \tilde{t}_n\}$ of the quiver $\tilde{\Gamma}$ of $D^b(k\vec{\Delta})$ (4.4), and we have $\mathrm{End}\,\tilde{\mathcal{T}} = \bigoplus_{i,j} \mathrm{Hom}_{D^b(k\vec{\Delta})}(t_i, t_j) \xrightarrow{\sim} \mathrm{End}\,\mathcal{T}$. Theorem 5.10 therefore follows from the theorem below.

DEFINITION. *A set of vertices* $\tilde{\mathcal{T}} = \{\tilde{t}_1, \ldots, \tilde{t}_n\}$ of $\tilde{\Gamma}$ *is called a* tilting set of

$\tilde{\Gamma}$ *if the following two conditions are satisfied*:
(i) $\mathrm{Hom}_{D^b(k\vec{\Delta})}(\tilde{t}_i, T^r\tilde{t}_j) = 0$ *for all* $r \neq 0$ *and all* i, j.
(ii) *dim* $\tilde{t}_1, \ldots,$ *dim* $\tilde{t}_n$ *form a* $\mathbb{Z}$*-basis of* $\mathbb{Z}^n$.

THEOREM. *If* $\vec{\Delta}$ *is a Dynkin quiver and* $\tilde{\mathcal{T}} = \{\tilde{t}_1, \ldots, \tilde{t}_n\}$ *a tilting set of* $\tilde{\Gamma}$, *then* End $\tilde{\mathcal{T}}$ *is an iterated tilted algebra of type* $\vec{\Delta}$.

A2. *Proof of theorem* A1.

Consider an element of $\tilde{E} = \mathrm{End}\,\tilde{\mathcal{T}}$ as a square matrix with entries in $\mathrm{Hom}_{D^b(k\vec{\Delta})}(\tilde{t}_i, \tilde{t}_j)$. Then each object $X \in D^b(k\vec{\Delta})$ gives rise to an $\tilde{E}$-module CX which consists of all columns with entries in $\mathrm{Hom}_{D^b(k\vec{\Delta})}(\tilde{t}_j, X)$, where $j = 1, \ldots, n$. In particular, the objects $\tilde{t}_i \in D^b(k\vec{\Delta})$ yield representatives $C\tilde{t}_i$ of the indecomposable projectives, and we have $\mathrm{Hom}_{D^b(k\vec{\Delta})}(\tilde{t}_i, \tilde{t}_j) \xrightarrow{\sim} \mathrm{Hom}_{\tilde{E}}(C\tilde{t}_i, C\tilde{t}_j)$ for all i, j.

Let $\tilde{t}_{i_0}$ be minimal in $\tilde{\mathcal{T}}$ for the order of $\tilde{\Gamma}$ defined by the arrows. Then $C\tilde{t}_{i_0}$ is simple projective; it is not injective (otherwise $K_0(k\vec{\Delta})$ would be the orthogonal sum of subgroups of rank 1 and $n-1$). So we have an almost split sequence of the form

$$0 \longrightarrow C\tilde{t}_{i_0} \xrightarrow{[Cu_p]} \bigoplus_{p=1}^{r} Ct_{j_p} \longrightarrow V \longrightarrow 0.$$

By [APR], the $\tilde{E}$-module

$$K = C\tilde{t}_1 \oplus \cdots \oplus C\tilde{t}_{i_0-1} \oplus V \oplus C\tilde{t}_{i_0+1} \oplus \cdots \oplus C\tilde{t}_n$$

is tilting. In order to show that End K is associated with a tilting set of $\tilde{\Gamma}$, we consider a triangle of $D^b(k\vec{\Delta})$ of the form

$$\tilde{t}_{i_0} \xrightarrow{[u_p]} \bigoplus_{p=1}^{r} \tilde{t}_{j_p} \xrightarrow{v} t' \xrightarrow{w} T\tilde{t}_{i_0}.$$

By A3 below, t' is indecomposable, say $t' \in \tilde{\Gamma}$; the set $\tilde{\mathcal{T}}' = (\tilde{\mathcal{T}} \backslash \{\tilde{t}_{i_0}\}) \cup \{t'\}$ is tilting in $\tilde{\Gamma}$, and $\tilde{E}' = \mathrm{End}\,\tilde{\mathcal{T}}'$ is identified with End K. So it remains to show that, for some choice of i_0, $\tilde{\mathcal{T}}'$ is "better" than $\tilde{\mathcal{T}}$ if $\tilde{\mathcal{T}}$ is not a *slice* (= the set of vertices of a connected full subquiver of $\tilde{\Gamma}$ which contains one representative of each τ-orbit). In fact, by A4 we can proceed by induction on the cardinality of the *convex hull* $\langle \tilde{\mathcal{T}} \rangle$ of $\tilde{\Gamma}$ (= the set of vertices occurring in the paths of $\tilde{\Gamma}$ which start and stop in $\tilde{\mathcal{T}}$).

A3. The long exact Hom-sequence of $D^b(k\vec{\Delta})$ provides us with an exact sequence

$$C\tilde{t}_{i_0} \xrightarrow{C[u_p]} \bigoplus_{p=1}^{r} C\tilde{t}_{j_p} \xrightarrow{Cv} Ct' \xrightarrow{Cw} CT\tilde{t}_{i_0}.$$

By A1(i) we have $CT\tilde{t}_{i_0}=0$, and Ct' is identified with V.

By construction, we have $\mathrm{Hom}(\tilde{t}_i, X) \xrightarrow{\sim} \mathrm{Hom}(C\tilde{t}_i, CX)$ for all i and all $X \in D^b(k\vec{\Delta})$ (Yoneda-lemma!). The diagram below shows that $\mathrm{Hom}(t', X) \xrightarrow{\sim} \mathrm{Hom}(Ct', CX))$ whenever $\mathrm{Hom}(T\tilde{t}_{i_0}, X)=0$, so in particular if $X=\tilde{t}_{i_0}$ or $X=t'$ (use the injectivity of $\mathrm{Hom}(\tilde{t}_{i_0}, [u_p]) \xrightarrow{\sim} \mathrm{Hom}(C\tilde{t}_{i_0}, C[u_p])$ to prove $\mathrm{Hom}(\tilde{t}_{i_0}, T^{-1}t')=0$). We infer that $\mathrm{End}\, t' \xrightarrow{\sim} \mathrm{End}\, V$, that t' is indecomposable and that $\mathrm{End}\, K \xrightarrow{\sim} \mathrm{End}(\tilde{t}_1 \oplus \cdots \oplus t' \oplus \cdots \oplus \tilde{t}_n)$.

$$\begin{array}{ccccccc}
\mathrm{Hom}(T\tilde{t}_{i_0}, X) & \to & \mathrm{Hom}(t', X) & \longrightarrow & \bigoplus_p \mathrm{Hom}(\tilde{t}_{j_p}, X) & \longrightarrow & \mathrm{Hom}(\tilde{t}_{i_0}, X) \\
\downarrow & & \downarrow & & \downarrow \wr & & \downarrow \wr \\
0 & \longrightarrow & \mathrm{Hom}(Ct', CX) & \to & \bigoplus_p \mathrm{Hom}(C\tilde{t}_{j_p}, CX) & \to & \mathrm{Hom}(C\tilde{t}_{i_0}, CX).
\end{array}$$

Because of A1(i) and of the minimality of $\tilde{t}_{i_0}$, we have $\mathrm{Hom}(T^{-r}\tilde{t}_i, \tilde{t}_{j_p})=0=\mathrm{Hom}(T^{-r}\tilde{t}_i, T\tilde{t}_{i_0})$, hence $\mathrm{Hom}(T^{-r}\tilde{t}_i, t')=0$ for $r\neq 0$ and $i\neq i_0$. The exact sequence below and the surjectivity of $\mathrm{Hom}([Tu_p], T\tilde{t}_i) \xrightarrow{\sim} \mathrm{Hom}([Cu_p], C\tilde{t}_i)$ show that $\mathrm{Hom}(t', T^r\tilde{t}_i)=0$ if $r\neq 0$ and $i \neq i_0$:

$$\mathrm{Hom}\left(\bigoplus_p T\tilde{t}_{j_p}, T^r\tilde{t}_i\right) \to \mathrm{Hom}(T\tilde{t}_{i_0}, T^r\tilde{t}_i) \to \mathrm{Hom}(t', T^r\tilde{t}_i) \to \mathrm{Hom}\left(\bigoplus_p \tilde{t}_{j_p}, T^r\tilde{t}_i\right).$$

In case $i=i_0$, the same sequence shows that $\mathrm{Hom}(t', T^r\tilde{t}_{i_0})=0$ if $r\neq 1$ and $\dim \mathrm{Hom}(t', T\tilde{t}_{i_0})=1$. Finally the exact sequence

$$\bigoplus_p \mathrm{Hom}(t', T^r\tilde{t}_{j_p}) \to \mathrm{Hom}(t', T^r t') \to \mathrm{Hom}(t', T^{r+1}\tilde{t}_{i_0})$$

shows that $\mathrm{Hom}(t', T^r t')=0$ if $r\neq 0$. We conclude that $\tilde{\mathcal{T}}' = (\tilde{\mathcal{T}}\backslash\{t_{i_0}\}) \cup \{t'\}$ is a tilting set of $\tilde{\Gamma}$.

A4. Suppose that $\tilde{\mathcal{T}}$ is not a slice. Then we have $\tau^{-1}\tilde{t}_i \in \langle \tilde{\mathcal{T}} \rangle$ for some i, and we choose i_0 so that $\tilde{t}_{i_0}$ is minimal in $\tilde{\mathcal{T}}$ and satisfies $\tilde{t}_{i_0} \leq \tilde{t}_i$. The last assumption implies that $\tau^{-1}\tilde{t}_{i_0} \in \langle \tilde{\mathcal{T}} \rangle \backslash \{\tilde{t}_{i_0}\}$.

If there is an index $j \neq i_0$ such that $\mathrm{Hom}(t', \tilde{t}_j) \neq 0$, then $t' \in \langle \tilde{\mathcal{T}} \rangle \backslash \{\tilde{t}_i\}$ and $\langle \tilde{\mathcal{T}}' \rangle \subset \langle \tilde{\mathcal{T}} \rangle \backslash \{\tilde{t}_{i_0}\}$.

On the contrary, if $\operatorname{Hom}(t', \tilde{t}_j) = 0$ for all $j \neq i_0$, then $\langle \underline{\dim}\, t', \underline{\dim}\, t_j \rangle = 0 = \langle \underline{\dim}\, \tilde{t}_j, \underline{\dim}\, \tilde{t}_{i_0} \rangle = \langle \underline{\dim}\, \tilde{t}_{i_0}, \underline{\dim}\, \tau T \tilde{t}_j \rangle = -\langle \underline{\dim}\, \tau^{-1} \tilde{t}_{i_0}, \underline{\dim}\, \tilde{t}_j \rangle$ and $\langle \underline{\dim}\, t', \underline{\dim}\, \tilde{t}_{i_0} \rangle = -\dim \operatorname{Hom}(t', T\tilde{t}_{i_0}) = -1 = \langle \underline{\dim}\, \tau^{-1}\tilde{t}_{i_0}, \underline{\dim}\, \tilde{t}_{i_0} \rangle$. We infer that $\underline{\dim}\, t' = \underline{\dim}\, \tau^{-1}\tilde{t}_{i_0}$ and $t' = T^{2r}\tau^{-1}\tilde{t}_{i_0}$ for some r. Now $\tilde{t}_{j_1}$ belongs to the convex hull $\langle \tilde{t}_{i_0}, \tau T \tilde{t}_{i_0} \rangle$, because $\operatorname{Hom}(\tilde{t}_{i_0}, \tilde{t}_{j_1}) \neq 0$; so we have $t' \in \langle \tilde{t}_{i_0}, (\tau T)^2 \tilde{t}_{i_0} \rangle$ because $\operatorname{Hom}(\tilde{t}_{j_1}, t') \neq 0$. Since $\tau^{-1}\tilde{t}_{i_0}$ is the only vertex of the form $T^{2r}\tau^{-1}\tilde{t}_{i_0}$ within $\langle \tilde{t}_{i_0}, (\tau T)^2 \tilde{t}_{i_0} \rangle$, we obtain $t' = \tau^{-1}\tilde{t}_{i_0} \in \langle \tilde{\mathcal{T}} \rangle \setminus \{\tilde{t}_{i_0}\}$ and again $\langle \tilde{\mathcal{T}}' \rangle \subset \langle \tilde{\mathcal{T}} \rangle \setminus \{\tilde{t}_{i_0}\}$.

The induction announced in A2 works!

BIBLIOGRAPHY

[AH] ASSEM, I. and HAPPEL, D., *Generalized tilted algebras of type* $\mathbb{A}_n$, Comm. Algebra, *9* (1981), 2101–2125.

[AB] AUSLANDER, M. and BRIDGER, M., *Stable module theory,* Mem. Amer. Math. Soc., No. 94, Providence (1969).

[AR] AUSLANDER, M. and REITEN, I., *Representation theory of Artin Algebras III, VI,* Comm. Algebra *3* (1975), 239–294, *5* (1977) 443–518.

[APR] AUSLANDER, M., PLATZECK, M. I. and REITON, I., *Coxetor functors without diagrams,* Trans. Amer. Math. Soc. 250 (1979), 1–46.

[BLS] BAUTISTA, R., LARRIÓN, F. and SALMERÓN, L., *On simply connected algebras,* J. London Math. Soc., Vol. 27, Part 2 (1983), 212–220.

[B1] BEILINSON, A. A., *Coherent sheaves on* $\mathbb{P}^n$ *and problems of linear algebra,* Func. Anal. and Appl., Vol 12, No. 3, 214–216.

[B2] BEILINSON, A. A., *The derived category of Coherent Sheaves on* $\mathbb{P}^n$, Sel. Math. Sov., Vol. 3, No. 3, 1983/84, 233–237.

[BBD] BEILINSON, A. A., BERNSTEIN, J. and DELIGNE, P., *Faisceaux pervers,* Astérique (100), 1982.

[BGG] BERNSTEIN, I. N., GEL'FAND, I. M. and GEL'FAND, S. I., *Algebraic bundles over* $\mathbb{P}^n$ *and problems of linear algebra,* Func. Anal. and Appl., Vol 12, No. 3, 212–214.

[BGP] BERNSTEIN, I. N., GEL'FAND, I. M. and PONOMAREV, V. A., *Coxeter functors and Gabriel's theorem,* Uspecki Mat. Nauk. *28* (1973), translated in Russian Math. Surveys *28* (1973), 17–32.

[Bo1] BONGARTZ, K., *Tilted algebras,* Springer Lecture Notes 903 (1982), 26–38.

[Bo2] BONGARTZ, K., *A criterion for finite representation type,* Math. Ann. *269* (1984), 1–12.

[BG] BONGARTZ, K. and GABRIEL, P., *Covering spaces in representation theory,* Invent. Math. *65* (1981), 331–378.

[BB] BRENNER, S. and BUTLER, M. C. R., *Generalization of the Bernstein–Gel'fand–Ponomarev reflection functors,* Springer Lecture Notes *832* (1980), 103–169.

[BrG] BRETSCHER, O. and GABRIEL, P., *The standard form of a representation-finite algebra,* Bull. Soc. Math. France, 111 (1983), 21–40.

[DR] DLAB, V. and RINGEL, C. M., *Indecomposable representations of graphs and algebras,* Mem. Amer. Math. Soc., No. 173, Providence (1973).

[G1] GABRIEL. P., *Unzerlegbare Darstellungen I,* Manuscripta Math. *6* (1972), 71–102.

[G2] GABRIEL, P., *Auslander–Reiten sequences and representation-finite algebras,* Springer Lecture Notes *831* (1980), 1–71.

[G3] GABRIEL, P., *The universal cover of a representation-finite algebra,* Springer Lecture Notes *903* (1981), 68–105.

[Gr] GROTHENDIECK, A., *Groupes des classes des catégories abeliennes et triangulées. Complexes parfait, in SGA* 5, *Exposé VIII,* Springer Lecture Notes *589* (1977).

[H1] HAPPEL, D., *Composition factors for indecomposable modules,* Proc. Amer. Math. Soc. *86* (1982), 29–31.

[H2] HAPPEL, D., *Tilting sets on cylinders,* Proc. London Math. Soc. (3), *51* (1985), 21–55.

[H3] HAPPEL, D., *Iterated tilted algebras of affine type,* to appear Comm of Alg.

[HR] HAPPEL, D. and RINGEL, C. M., *Tilted algebras,* Trans. Amer. Math. Soc., *274* (2), (1982), 399–443.

[Ha] HARTSHORNE, R., *Residues and Duality,* Springer Lecture Notes 20 (1966).

[He] HELLER, A., *The loop-space functor in homological algebra.* Trans. Amer. Math. Soc. *96* (1960), 382–394.

[HW] HUGHES, D. and WASCHBÜSCH, J., *Trivial extensions of tilted algebras,* Proc. London Math. Soc. (3), *46* (1983), 347–364.

[ML] MAC LANE, S., *Categories for the working mathematician,* Springer Verlag 1971.

[Q] QUILLEN, D., *Higher algebraic K-theory I,* Springer Lecture Notes *341* (1973), 85–147.

[R] RIEDTMANN, C., *Algebren, Darstellungsköcher, Überlagerungen und zurück,* Comm. Math. Helv. *55* (1980), 199–224.

[Ri] RINGEL, C. M., *Representations of K-species and bimodules,* J. Algebra *41* (1976), 269–302.

[Ri2] RINGEL, C. M., *Finite dimensional hereditary algebras of wild representation type,* Math. Z. *161* (1978), 235–255.

[Ri3] RINGEL, C. M., *Report on the Brauer–Thrall Conjectures,* Springer Lecture Notes *831* (1980), 104–136, and Tame algebras, same volume, 137–287.

[Ri4] RINGEL, C. M., *The rational invariants of tame quivers,* Invent. Math. *58* (1980), 217–239.

[Ri5] RINGEL, C. M., *Bricks in hereditary length categories,* Resultate der Mathematik *6*(1), (1983), 64–70.

[Ri6] RINGEL, C. M., *Tame algebras and quadratic forms,* Springer Lecture Notes *1099* (1984).

[TW] TACHIKAWA, H. and WAKAMATSU, T., *Applications of reflection functors for selfinjective algebras, Proc. ICRA IV,* Carleton Lecture Notes, 1984.

[V] VERDIER, J. L., *Catégories dérivés,* état 0, Springer Lecture Notes *569* (1977), 262–311.

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Received November 15, 1985/February 2, 1987