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# Galois coverings of representation-infinite algebras

Piotr Dowbor and Andrzej Skowroński

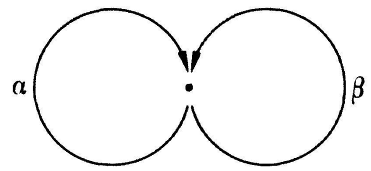
Coverings techniques in representation theory were introduced and developed for the research of representation-finite algebras and for computing their indecomposable representations. In this theory one of the important results is the following [20], [24]:

"Let $K$ be an algebraically closed field, $Q$ a locally finite quiver, $I$ an admissible ideal [10] in the path-category $KQ$ of $Q$, $R$ the (locally bounded) quotient category $KQ/I$, $\operatorname{mod} R$ the category of finite dimensional $R$-modules (representations of $R$) and $G$ a group of $K$-linear automorphisms of $R$ acting freely on the objects of $R$. Moreover, let $F: R \to R/G = \Lambda$ be the functor which assigns to each object $x$ of $R$ its $G$-orbit $G \cdot x$, and $F_\lambda: \operatorname{mod} R \to \operatorname{mod}(R/G)$ the push-down functor [10] associated with $F$ and such that $(F_\lambda M)(a) = \bigoplus_{F(x)=a} M(x)$ for any $M \in \operatorname{mod} R$ and $a \in R$. Then $R$ is locally representation-finite if and only if so is $\Lambda$. In this case, $F_\lambda$ induces a bijection between the $G$-orbits of isoclasses of indecomposable finite dimensional $R$-modules and the isoclasses of indecomposable finite dimensional $\Lambda$-modules".

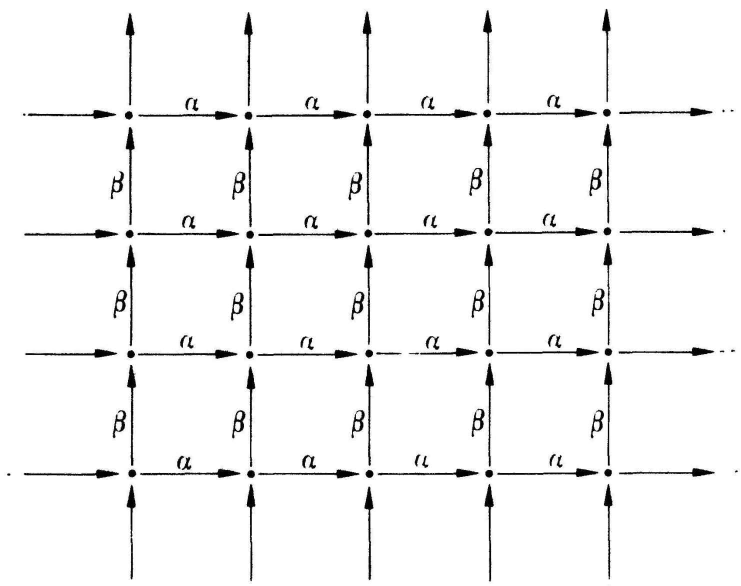
Therefore, if $R$ is locally representation-finite, $\operatorname{mod} \Lambda$ coincides with the full subcategory $\operatorname{mod}_1 \Lambda$ formed by all modules of the form $F_\lambda M$, $M \in \operatorname{mod} R$; in the general case we call these $F_\lambda M$ $\Lambda$-modules of the first kind. The authors showed in [15], [17] that $\operatorname{mod} \Lambda = \operatorname{mod}_1 \Lambda$ holds for a wider class of locally bounded categories consisting of all locally support-finite ones. The equality $\operatorname{mod} \Lambda = \operatorname{mod}_1 \Lambda$ is also discussed here (§2).

The main object investigated in this paper is the full subcategory $\operatorname{mod}_2 R/G$ of $\operatorname{mod} R/G$ formed by all modules having no direct summands of the first kind; we call them $R/G$-modules of the second kind (with respect to a fixed Galois covering $F: R \to R/G$). Our main theorem (3.1) asserts that for some class of Galois coverings $F: R \to R/G$ the investigation of $\operatorname{mod}_2 R/G$ can be reduced to the quotient categories associated with the supports of some periodic, indecomposable, locally finite dimensional $R$-modules. In particular, we obtain the following covering interpretation of the Gelfand–Ponomarev classification [21] of indecomposable finite dimensional modules over the algebras $\Lambda_n = \mathbb{C}[X, Y]/$

$(XY, X^n, Y^n) = \mathbb{C}Q_n/I_n$, $n \geq 2$. Here $Q_n$ is the quiver




and $I_n$ is the ideal generated by $\alpha\beta$, $\beta\alpha$, $\alpha^n$ and $\beta^n$. Let $R$ be the residue-category $\mathbb{C}Q/I$ where $Q$ is the following locally finite quiver




and $I$ is the ideal generated by all elements of the form $\alpha\beta$, $\beta\alpha$, $\alpha^n$ and $\beta^n$. Consider the action of the free abelian group $G = \mathbb{Z} \times \mathbb{Z}$ on $R$ given by the vertical and horizontal shifts of $Q$. Then $\Lambda_n$ is isomorphic to $R/G$ and we have a Galois covering $F: R \to R/G = \Lambda_n$. A line in $R$ is a full convex subcategory $L$ of $R$ which is isomorphic to the path category of a linear quiver (of type $\mathbb{A}_n$, $\mathbb{A}_\infty$ or $\mathbb{A}_\infty^\infty$). A line $L$ is $G$-periodic if its stabilizer $G_L = \{g \in G; gL = L\}$ is nontrivial. With each line $L$ in $R$ we associate a canonical indecomposable $R$-module $B_L$ by setting $B_L(x) = K$ for $x \in L$, $B_L(x) = 0$ for $x \notin L$ and $B_L(\gamma) = id_K$ for each path $\gamma$ in $L$. It is well-known that the modules $B_L$, where $L$ ranges over all finite lines in $R$, are representatives of the isoclasses of finite dimensional indecomposable $R$-modules. Therefore every indecomposable module in $\mathrm{mod}_1\,\Lambda_n$ is isomorphic to $F_\lambda(B_L)$ for some finite line $L$ in $R$. Let $\mathscr{L}$ be the set of all $G$-periodic lines in $R$ and $\mathscr{L}_0$ a fixed set of representatives of the $G$-orbits in $\mathscr{L}$. Then, according to our main theorem (3.1), there is an equivalence of categories

$$\coprod_{L \in \mathscr{L}_0} (\mathrm{mod}\, L/G_L)/[\mathrm{mod}_1\, L/G_L] \xrightarrow{\sim} (\mathrm{mod}\, R/G)/[\mathrm{mod}_1\, R/G]$$

where $[\mathrm{mod}_1\, R/G]$ (resp. $[\mathrm{mod}_1\, L/G_L]$) denotes the ideal of all morphisms factorized through an object of $\mathrm{mod}_1\, R/G$ (resp. $\mathrm{mod}_1\, L/G_L$). In our example each $(\mathrm{mod}\, L/G_L)/[\mathrm{mod}_1\, L/G_L]$ is equivalent to the category $\mathrm{mod}\, KG_L$ of finite dimensional modules over the algebra $KG_L \xrightarrow{\sim} K[T, T^{-1}]$ of Laurent polynomials. Moreover, for any $L \in \mathscr{L}_0$, the canonical action of $G_L$ on $L$ supplies a left $KG_L$-module structure on $F_\lambda B_L$. For each $a \in R/G$, the $K[T, T^{-1}]$-module $F_\lambda B_L(a)$ is free of finite rank. We will prove that the equivalence

$$\coprod_{L \in \mathscr{L}_0} \mathrm{mod}\, K[T, T^{-1}] \xrightarrow{\sim} (\mathrm{mod}\, R/G)/\mathrm{mod}\, R/G$$

described above is given by the functors

$$- \bigotimes_{K[T,T^{-1}]} F_\lambda B_L : \mathrm{mod}\, K[T, T^{-1}] \to \mathrm{mod}\, R/G.$$

In particular every indecomposable module in $\mathrm{mod}_2\, \Lambda_n$ is isomorphic to $V \otimes_{K[T,T^{-1}]} F_\lambda B_L$ for some $L \in \mathscr{L}_0$ and some indecomposable finite dimensional $K[T, T^{-1}]$-module $V$.

We see that in the research of $\mathrm{mod}_2\, R/G$ an important role is played by locally finite dimensional indecomposable $R$-modules with nontrivial stabilizers. In §4 we show that these modules are limits of sequences of finite dimensional indecomposable modules. In §5 we apply our main theorem to the classification of indecomposable modules over interesting classes of tame algebras.

The methods we use are rather simple. We assume only basic results on Galois coverings of locally bounded categories proved by Gabriel in [20], elementary properties of adjoint functors [23], Krull–Schmidt–Warfield decomposition theorem [33] and the description of indecomposable representations of Dynkin quivers of type $A_n$ [18].

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## §1. Basic definitions and notations

1.1. Throughout this paper we denote by $K$ an algebraically closed field and by $R$ a connected, locally bounded $k$-category [see 5, 10].

Let $M$ be an $R$-module [5, 10]. The *support* of $M$ is the full subcategory $\operatorname{supp} M$ of $R$ formed by all objects $x \in R$ such that $M(x) \neq 0$. The *dimension-vector* of $M$ is the family $\underline{dim}\, M = (M(x):K)_{x\in R}$; its *dimension* is the number $\dim M = \sum_{x\in R}(M(x):K)$. The $R$-module $M$ is called *locally-finite dimensional* if $(M(x):K)$ is finite for all $x \in R$. We denote by $\operatorname{MOD} R$ category of all $R$-modules, by $\operatorname{Mod} R$ (resp. $\operatorname{mod} R$) the full subcategory formed by all locally finite dimensional (resp. finite dimensional) $R$-modules, by $\operatorname{Ind} R$ (resp. $\operatorname{ind} R$) the full subcategory of $\operatorname{Mod} R$ (resp. $\operatorname{mod} R$) formed by all indecomposable objects, by $\operatorname{Ind} R/\simeq$ (resp. $\operatorname{ind} R/\simeq$) the set of isoclasses of objects in $\operatorname{Ind} R$ (resp. $\operatorname{ind} R$).

If $X$, $Y \in \operatorname{MOD} R$, we write $Y \subset_\oplus X$ whenever $Y$ is isomorphic to a direct summand of $X$. If $\mathscr{C}$ is a full subcategory of $R$ and $Z \in \operatorname{MOD} \mathscr{C}$, we write $Z \subset_\oplus | X$ if $Z \subset_\oplus X \mid \mathscr{C}$. We say that $\mathscr{C}$ is *convex* if each path of the ordinary quiver $Q_R$ of $R$ with origin and terminus in $\mathscr{C}$ has all its points in $\mathscr{C}$. By $\hat{\mathscr{C}}$ we denote the full subcategory of $R$ formed by all $x \in R$ such that $R(x, y) \neq 0$ or $R(y, x) \neq 0$ for some $y \in \mathscr{C}$.

If $\mathscr{V}$ is an additive category and $\mathscr{V}_0$ a full subcategory of $\mathscr{V}$, $\mathscr{V}/[\mathscr{V}_0]$ denote the factor category of $\mathscr{V}$ modulo the ideal $[\mathscr{V}_0]$ of all morphisms in $\mathscr{V}$ factorized through a direct sum of some objects of $\mathscr{V}_0$.

1.2. In the sequel, $G$ denotes a group of $K$-linear automorphisms of $R$. For each full subcategory $L$ of $R$, we denote by $G_L$ the stabilizer $\{g \in G, gL = L\}$ of $L$, by $GL$ the full subcategory of $R$ formed by the $G$-orbits of all objects of $L$. The group $G$ acts on $\operatorname{MOD} R$ by the translations ${}^g(-)$, which assign to each $M \in \operatorname{MOD} R$ the $R$-module ${}^gM = M \circ g^{-1}$. For each $M \in \operatorname{MOD} R$, we denote by $G_M$ the stabilizer $\{g \in G, {}^gM \simeq M\}$. *Through this paper we assume that $G$ acts freely on* $\operatorname{ind} R/\simeq$.

By $\operatorname{MOD}^G R$ we denote the category whose objects are the pairs $(M, \mu)$, where $\mu$ is an $R$-action on $M \in \operatorname{MOD} R$. The set of morphisms from $(M, \mu)$ to $(M', \mu')$, denoted by $\operatorname{Hom}_R^G(M, M')$, consists of all $R$-homomorphisms from $M$ to $M'$ compatible with the actions of $G$ (see [20]). $\operatorname{Mod}_f^G R$ is the full subcategory of $\operatorname{MOD}^G R$ formed by all $(M, \mu) \in \operatorname{MOD}^G R$ such that $M \in \operatorname{Mod} R$ and that $\operatorname{supp} M$ is contained in a finite number of $G$-orbits of $R$.

Let $F: R \to R/G$ be a Galois covering, $F.: \operatorname{MOD} R/G \to \operatorname{MOD} R$ the pull-up functor associated with $F$ and $F_\lambda: \operatorname{MOD} R \to \operatorname{MOD} R/G$ the push-down left adjoint to $F.$ (see [10; 3.2]). Then $F.$ induces an equivalence of categories [20; p. 94]

$$\operatorname{mod} R/G \xrightarrow{\sim} \operatorname{Mod}_f^G R.$$

Moreover, since $G$ acts freely on $\operatorname{ind} R/\simeq$, $F_\lambda$ induces an injection from the set $(\operatorname{ind} R/\simeq)/G$ of $G$-orbits of $\operatorname{ind} R/\simeq$ into $(\operatorname{ind} R/G)/\simeq$ (see [20; 3.5)].

1.3. Let $\operatorname{ind}_1 R/G$ be the full subcategory of $\operatorname{ind} R/G$ consisting of all objects isomorphic to $F_\lambda M$ for some $M \in \operatorname{mod} R$, and $\operatorname{ind}_2 R/G$ the full subcategory of $\operatorname{ind} R/G$ formed by remaining indecomposables.

## §2. $G$-exhaustive categories

2.1. The category $R$ is called *$G$-exhaustive* if $\operatorname{ind} R/G = \operatorname{ind}_1 R/G$. In order to characterize $G$-exhaustive categories we shall first give a characterization of modules of the first and second kind.

LEMMA. (i) *Each $M \in \operatorname{Mod} R$ is a direct sum of indecomposables.*

(ii) *For each $M \in \operatorname{Ind} R$, $\operatorname{End}_R(M)$ is a local ring.*

(iii) *Let $M = \bigoplus_{i \in I} M_i$ be a decomposition of $M \in \operatorname{Mod} R$ such that for $i \neq j$ no indecomposable summand of $M_i$ is isomorphic to one of $M_j$. Then, an endomorphism $f = (f_{ij})_{i,j \in I}$ of $M$ is invertible iff so is each $f_{ii}$.*

*Proof.* (i) See [4] or prove directly using transitive induction.

(ii) Use spectral decomposition.

(iii) $f$ is invertible iff so is each $f(x)$, $x \in R$. Therefore we can reduce the proof to the case when $M$ is a direct sum of finitely many indecomposable modules.

2.2. LEMMA [17]. *For $X \in \operatorname{ind} R/G$ the following conditions are equivalent*

(i) *$X \in \operatorname{ind}_1 R/G$*

(ii) *$F.X = \bigoplus_{i \in I} Z_i$, where $Z_i \in \operatorname{mod} R$ for all $i \in I$*

(iii) *$F.X$ has a finite dimensional direct summand.*

*Proof.* For a proof using Auslander–Reiten sequences see [17]. We give here an alternative elementary proof. The implication (ii)$\rightarrow$(iii) is obvious and (i)$\rightarrow$(ii) follows from [20, 3.2]. In order to prove (iii)$\rightarrow$(i), assume $Z \subset_\oplus F.X$ for some $Z \in \operatorname{ind} R$. Then there exists two morphisms $j \in \operatorname{Hom}_R(Z, F.X)$ and $p \in \operatorname{Hom}(F.X, Z)$ such that $p \cdot j = 1_Z$. The families of morphisms ${}^g j: {}^g Z \rightarrow {}^g F.X \xrightarrow{\sim} F.X$ and ${}^g p: {}^g F.X \rightarrow {}^g Z$, $g \in G$, produce a pair of morphisms $j': \bigoplus_{g \in G} {}^g Z \rightarrow F.X$ and $p': F.X \rightarrow \bigoplus_{g \in G} {}^g Z = \prod_{g \in G} {}^g Z$ in $\operatorname{Mod}_f^G R$. Since $G$ acts freely on $\operatorname{ind} R/\simeq$, $p' \circ j'$ is invertible, by Lemma 2.1. Consequently $F_\lambda Z \subset_\oplus X$ and by our assumption $X \simeq F_\lambda Z$.

2.3. In a characterization of modules of the second kind an important role is played by the following class of locally finite-dimensional modules. A module

$Y \in \operatorname{Ind} R$ is called *weakly-G-periodic* if $\operatorname{supp} Y$ is infinite and $(\operatorname{supp} Y)/G_Y$ is finite. This implies that $G_Y$ is infinite.

EXAMPLE. Let $R_{n,m}$; $n, m \in \mathbb{N}$, $n, m \geq 2$ be the locally bounded $K$-category defined by the locally finite quiver $Q$ as in introduction and ideal $I_{n,m}$ generated by all paths of the form $\alpha\beta$, $\beta\alpha$, $\alpha^n$ and $\beta^m$.

Recall that a full subcategory $L$ of $R$ is called a *line* if $L$ is convex and is isomorphic to the path category of a linear quiver (of type $\mathbb{A}_n$, $\mathbb{A}_\infty$ or $\mathbb{A}_\infty^\infty$). A line $L$ is called *G-periodic* if $G_L \neq \{1\}$.

With each $G$-periodic line $L$ in $R_{n,m}$ we associate a canonical weakly-$G$-periodic $R_{n,m}$-module $B_L$ by setting $B_L(x) = K$ if $x \in L$, $B_L(x) = 0$ if $x \notin L$ and $B_L(\alpha) = id_K$ for each morphism $\alpha$ in $L$. In fact the map $L \to B_L$ induces a bijection between the set $\mathscr{L}$ of all $G$-periodic lines in $R_{n,m}$ and the isoclasses of all weakly-$G$-periodic $R_{n,m}$-modules (see §4).

PROPOSITION. *Let $X \in \operatorname{mod} R/G$. Then $X \in \operatorname{mod}_2 R/G$ iff there exists a decomposition $F.X = \bigoplus_{i \in I} Y_i$ in* $\operatorname{Mod} R$ *where all $Y_i$ are weakly-G-periodic.*

In the proof of this Proposition and further we shall use the following lemma.

LEMMA. *Let the support of $Y \in \operatorname{Mod} R$ be stable under a subgroup $H$ of $G$, and denote by $U$ a set of representatives of the cosets of $G \bmod H$. Then $\bigoplus_{g \in U} {}^gY \in \operatorname{Mod} R$ iff for each G-orbit $\mathcal{O}$, $(\mathcal{O} \cap \operatorname{supp} Y)/H$ is finite.*

*Proof.* Obvious.

*Proof of Proposition* 2.3. The condition is sufficient by Lemma 2.2. Now take any $X \in \operatorname{mod}_2 R/G$. By Lemma 2.1 and 2.2, there exists a decomposition $F.X = \bigoplus_{i \in I} Y_i$, where $Y_i \in \operatorname{Ind} R$ and where $\operatorname{supp} Y_i$ is infinite for all $i \in I$. To prove the necessity it is therefore enough to show, that for any $Y \in \operatorname{Ind} R$ such that $Y \subset_\oplus F.X$, $\operatorname{supp} Y/G_Y$ is finite. For each $g \in G$, ${}^gY \subset_\oplus {}^g(F.X) \xrightarrow{\sim} F.X$ and hence $\bigoplus_{q \in U} {}^gY \subset_\oplus F.X$, where $U$ is a set of representatives of cosets of $G \bmod G_Y$ (use that $\operatorname{End}_R(Y)$ is local and that ${}^gY \not\simeq {}^{g'}Y$ if $g \neq g'$, $g, g' \in U$). On the other hand, $\operatorname{supp} Y \subset \operatorname{supp} F.X$ is contained in a finite number of $G$-orbits. That $(\operatorname{supp} Y)/G_Y$ is finite therefore follows from Lemma 2.3.

COROLLARY. *If a group G of K-linear automorphisms of a locally bounded K-category R acts freely on* $\operatorname{Ind} R/\simeq$, *then R is G-exhaustive.*

2.4. We will show that under some extra assumption the condition stated in Corollary 2.3 is also necessary.

PROPOSITION. *Let $G$ be a free (noncommutative) group of $K$-linear automorphisms of a locally bounded $K$-category $R$. Then $R$ is $G$-exhaustive iff there is no weakly-$G$-periodic $R$-module in* $\operatorname{Mod} R$. *If moreover $R/G$ is finite and $G = \mathbb{Z}$, then $R$ is $G$-exhaustive iff $G$ acts freely on* $\operatorname{Ind} R/\simeq$.

For the proof of this proposition we need some preparation.

Let $H$ be a subgroup of $G$, $U$ be a fixed set of representatives of the cosets of $G \bmod H$, $(N, v) \in \operatorname{Mod}^H R$ and $M = \bigoplus_{g \in U} {}^g N$. Then the isomorphisms $\mu(g, x): M(x) = \bigoplus_{g_1 \in U} N(g_1^{-1}x) \to M(gx) = \bigoplus_{g_2 \in U} N(g_2^{-1}gx)$ induced by $v(g_2^{-1} \cdot g \cdot g_1, g_1^{-1} \cdot x): Y(g_1^{-1} \cdot x) \to Y(g_2^{-1} \cdot gx)$, where $g \cdot g_1 H = g_2 H$, produce an $R$-action $\mu_v$ of $G$ on $M$.

LEMMA. *With the notation above, suppose that $N$ is weakly-$G$-periodic and that $H = G_N$. Then $M = \bigoplus_{g \in U} {}^g N$ is an indecomposable object of* $\operatorname{Mod}_f^G R$, *and the associated $R/G$-module is of second kind. In particular $R$ is not $G$-exhaustive.*

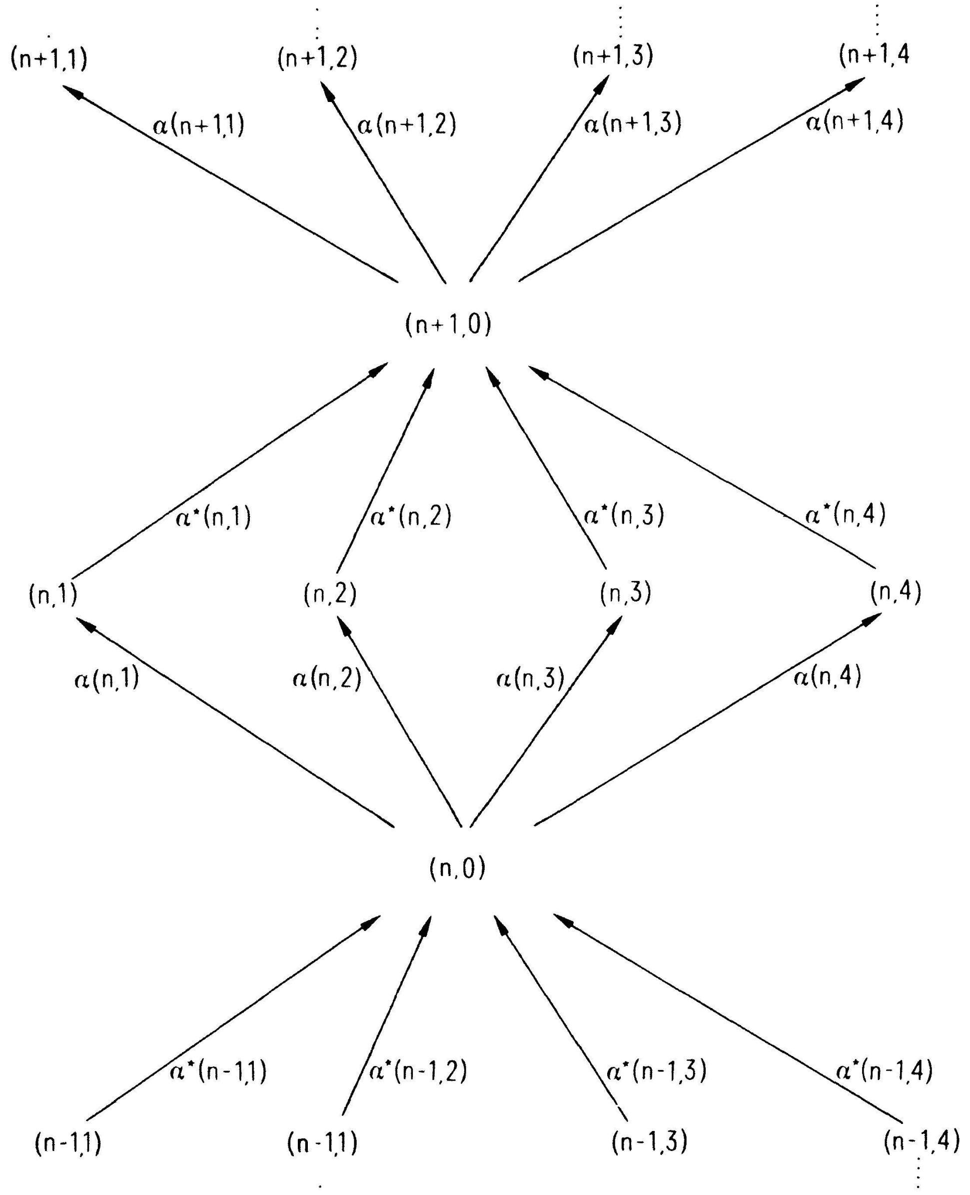
*Proof.* By Lemma 2.3 $M \in \operatorname{Mod}_f^G R$. Now the lemma follows by arguments similar to those given in [20; 3.5] and Proposition 2.3.

*Proof of the Proposition.* The condition is sufficient by Proposition 2.3. In order to prove the necessity, assume that there exists a weakly-$G$-periodic $R$-module $Y$. Since $G_Y$, as a subgroup of $G$, is free one can construct an $R$-action of $G_Y$ on $Y$ applying arguments from [20, pp. 94–95]. Consequently, by Lemma 2.4, $R$ is not $G$-exhaustive. Now in order to prove the second part of Proposition it is enough by Corollary 2.3 to show that, for each $Y \in \operatorname{Ind} R$ with $G_Y \neq \{1\}$, $\operatorname{supp} Y/G_Y$ is finite. But this is an immediate consequence of the fact that $R/G$ is finite and that $G_Y$ has finite index in $G = \mathbb{Z}$.

2.5. Now we formulate a more handy sufficient condition for $R$ to be $G$-exhaustive, which is a natural generalization of the definition of a locally representation-finite category [10]. For each $x \in R$, denote by $R_x$ the full subcategory of $R$ consisting of the points of all $\operatorname{supp} M$, where $M \in \operatorname{ind} R$ is such that $M(x) \neq 0$. Following [15], $R$ is called *locally support-finite* if $R_x$ is finite for all $x \in R$.

EXAMPLE 1. A locally representation-finite category $R$ is locally support-finite.

EXAMPLE 2. Let $R$ be the locally bounded $K$-category defined by the quiver




and the relations $\alpha_{(n+1,j)}\alpha^*_{(n,i)}=0$, $\alpha^*_{(n,i)}\alpha_{(n,i)}=\alpha^*_{(n,j)}\alpha_{(n,j)}$, $i\neq j$, $i, j=1, 2, 3, 4$, $n\in\mathbb{N}$.

Observe that $R$ contains the full subcategory $\mathscr{C}_0$ formed by the objects $(0,0)$, $(0,1)$, $(0,2)$, $(0,3)$, $(0,4)$; this is the path category of an extended Dynkin quiver $Q'$ of type $\tilde{\mathbb{D}}_4$; hence $R$ is not locally representation-finite. In fact the modules $M\in \operatorname{ind} R$ which are not projective-injective are annihilated by the radical square. Their support is therefore contained in a set of the form $\{(n, 0), (n, 1), (n, 2), (n, 3), (n, 4)\}$ or $\{(n, 1), (n, 2), (n, 3), (n, 4), (n+1, 0)\}$. Let $G$ be the cyclic group of automorphsms of $R$ generated by $(n, i)\to(n+1, i)$. Then $\operatorname{mod} R/G\simeq\operatorname{mod} T(A)$, where $T(A)$ is the trivial extension $A\ltimes D(A)$ of $A=kQ'$ by the injective cogenerator $D(A)$ of mod $A$. For other examples of locally support-finite categories we refer to [29], [30].

PROPOSITION. *Let $R$ be a locally support-finite $K$-category. Then* $\operatorname{Ind} R = \operatorname{ind} R$ *and each locally finite dimensional $R$-module is a direct sum of finite dimensional indecomposable $R$-modules. In particular, $R$ is $G$-exhaustive for any group $G$ of $K$-automorphisms of $R$ acting freely on* $\operatorname{ind} R/\simeq$.

For the proof of this proposition we should recall the following simple fact.

LEMMA [15]. *Let $\mathscr{C}$ be a full subcategory of $R$ and $M$ an $R$-module. Assume, that there exists $Z \in \operatorname{mod} \hat{C}$ such that $\operatorname{supp} Z \subset \mathscr{C}$ and $Z \subset_{\oplus} | M$. Then the $R$-module $\tilde{Z}$ such that $\tilde{Z} \mid \hat{\mathscr{C}} = Z$ and $\tilde{Z}(x) = 0$ for $x \notin \hat{\mathscr{C}}$ is a direct summand of $M$.*

*Proof of the Proposition.* Take any $Y \in \operatorname{Ind} R$, $x \in \operatorname{supp} Y$, and consider the $\hat{R}_x$-module $Y \mid \hat{R}_x$. Then there exists $Z \in \operatorname{ind} \hat{R}_x$ such that $Z(x) \neq 0$ and $Z \subset_{\oplus} | Y$. Observe that supp $Z$ is contained in $R_x$. Indeed, $\operatorname{supp} Z \subset \operatorname{supp} e_\lambda(Z) \subset R_x$, where $e_\lambda : \operatorname{MOD} \hat{R}_x \to \operatorname{MOD} R$ is the left adjoint to the restriction $e. : \operatorname{MOD} R \to \operatorname{MOD} \hat{R}_x$. Thus, by the above lemma $\tilde{Z} \subset_{\oplus} Y$ in MOD $R$ and hence $Y \simeq \tilde{Z}$, since $Y \in \operatorname{Ind} R$. Two remaining statements follow from Lemma 2.1 and Lemma 2.2.

## §3. Modules of the second kind

In this section $R$ is not supposed to be $G$-exhaustive. Our purpose is to describe $\operatorname{mod}_2 R/G$ under some assumptions which we are to make precise.

3.1. DEFINITION. A family $\mathscr{S}$ of full subcategories of $R$ is called *separating* (with respect to $G$) if $\mathscr{S}$ satisfies the following conditions:

(i) for each $L \in \mathscr{S}$ and $g \in G$, $gL \in \mathscr{S}$.

(ii) for each $L \in \mathscr{S}$ and each $G$-orbit $\mathscr{O}$ of $R$, $\mathscr{O} \cap L$ is contained in finitely many $G_L$-orbits.

(iii) for any two different $L, L' \in S$, $L \cap L'$ is locally support-finite.

(iv) for each weakly-$G$-periodic $R$-module $Y$ there exists an $L \in \mathscr{S}$ such that $\operatorname{supp} Y \subset L$.

*Remark.* If there exists a weakly-$G$-periodic $R$-module $Y$ with $\operatorname{supp} Y = R$, then $\mathscr{S} = \{R\}$ is the unique separating family of subcategories of $R$ with respect to $G$.

THEOREM. *Let $R$ be a locally bounded $K$-category and $G$ a group of automorphisms of $R$ which acts freely on* $(\operatorname{ind} R)/\simeq$. *Let $\mathcal{S}$ be a separating family of subcategories of $R$ with respect to $G$ and $\mathcal{S}_0$ a fixed set of representatives of $G$-orbits of $\mathcal{S}$. Then there is an equivalence of categories*

$$E: \coprod_{L\in\mathcal{S}_0} (\operatorname{mod} L/G_L)/[\operatorname{mod}_1 L/G_L] \to (\operatorname{mod} R/G)/[\operatorname{mod}_1 R/G].$$

*As a consequence, the Auslander–Reiten quiver $\Gamma_{R/G}$* [1, 2] *of $R/G$ is isomorphic to the disjoint union of translation-quivers $\Gamma_R/G \amalg (\coprod_{L\in\mathcal{S}_0} (\Gamma_{L/G_l})_2)$, where $(\Gamma_{L/G_L})_2$ is the union of connected components of $\Gamma_{L/G_l}$ whose points are $L/G_L$-modules of second kind.*

We may recall here that $\coprod_{L\in\mathcal{S}_0} (\operatorname{mod} L/G_L)/[\operatorname{mod}_1 L/G_L]$ denotes the full subcategory of the product $\prod_{L\in\mathcal{S}_0} (\operatorname{mod} L/G_L)/[\operatorname{mod}_1 L/G_L]$ whose object are the families $(M_L)_{L\in\mathcal{S}_0}$ such that $M_L \in \operatorname{mod}_1 L/G$ for almost all $L \in \mathcal{S}_0$ (i.e. $M_L$ is zero in the factor-category).

3.2. Let $H$ be a group of automorphisms of a locally bounded $K$-category $\mathscr{C}$, which acts freely on $\operatorname{ind} \mathscr{C}/\simeq$. Denote by $\operatorname{Mod}_{f1}^H \mathscr{C}$ (resp. $\operatorname{Mod}_{f2}^H \mathscr{C}$) the full subcategory of $\operatorname{Mod}_f^H \mathscr{C}$ consisting of all $M \in \operatorname{Mod}_f^H \mathscr{C}$ such that $M = \bigoplus_{i\in I} Z_i$, where $Z_i \in \operatorname{ind} \mathscr{C}$ (resp. $Z_i \in \operatorname{Ind} \mathscr{C}$ and $Z_i \subset_{\oplus} \operatorname{mod} C$) for each $i \in I$. Then the pull-up functor $F'_\cdot$, associated with the Galois covering $F': \mathscr{C} \to \mathscr{C}/H$ furnishes equivalences $\operatorname{mod}_1 \mathscr{C}/H \xrightarrow{\sim} \operatorname{Mod}_{f1}^H \mathscr{C}$ and $\operatorname{mod}_2 \mathscr{C}/H \xrightarrow{\sim} \operatorname{Mod}_{f2}^H \mathscr{C}$. Denote by $\underline{\operatorname{Mod}}_f^H \mathscr{C}$ the factor category $\operatorname{Mod}_f^H \mathscr{C}/[\operatorname{Mod}_{f1}^H \mathscr{C}]$. In order to prove the first part of the Theorem it is enough to produce an equivalence of categories

$$\coprod_{L\in\mathcal{S}_0} \underline{\operatorname{Mod}}_f^{G_L} L \xrightarrow{\sim} \underline{\operatorname{Mod}}_f^G R.$$

Let $L \in \mathcal{S}_0$ and $E_\lambda^L: \operatorname{MOD}^{G_L} L \to \operatorname{MOD}^G R$ be the left adjoint to the restriction functor $E_\cdot^L: \operatorname{MOD}^G R \to \operatorname{MOD}^{G_L} L$ defined by setting $E_\lambda^L(N) = \bigoplus_{g\in U_L} {}^g(e_\lambda^L N)$ (see 2.4). Here $U_L$ is a fixed set of representatives of the cosets of $G \bmod G_L$, $e_\lambda^L = -\otimes_L R: \operatorname{MOD} L \to \operatorname{MOD} R$ is a left adjoint to the restriction functor $e_\cdot^L: \operatorname{MOD} R \to \operatorname{MOD} L$. The $R$-module $e_\lambda^L(N)$ is endowed with an $R$-action of $G_L$ which is induced by the given $L$-action of $G_L$ on $N$.

The proof of the Theorem will be done in several steps.

LEMMA. *For each $L \in \mathcal{S}$, $E_\lambda^L$ and $E_\cdot^L$ induce functors $\mathbf{E}_\lambda^L: \underline{\operatorname{Mod}}_f^{G_L} L \to \underline{\operatorname{Mod}}_f^G R$ and $\mathbf{E}_\cdot^L: \underline{\operatorname{Mod}}_f^G R \to \underline{\operatorname{Mod}}_f^{G_L} L$.*

*Proof.* $E_\lambda^L(\mathrm{Mod}_f^{G_l} L) \subset \mathrm{Mod}_f^G R$ holds by Lemma 2.3; the inclusion $E_\lambda^L(\mathrm{Mod}_{f1}^{G_l} L) \subset \mathrm{Mod}_{f1}^G R$ is obvious; so $\mathbf{E}_\lambda^L$ is well defined. $E_\cdot^L(\mathrm{Mod}_f^G R) \subset \mathrm{Mod}_f^{G_l} L$ by 3.1(ii); again $E_\cdot^L(\mathrm{Mod}_{f1}^G R) \subset \mathrm{Mod}_f^{G_L} L$ is obvious; so $\mathbf{E}_\cdot^L$ is well defined.

3.3. LEMMA. *Let $Y$ be a weakly-$G_L$-periodic $L$-module, where $L \in S$. Then*
(a) $e_\lambda^L(Y) \mid L = Y$ *and* $e_\lambda^L(Y)(x) = 0$ *if* $x \notin L$.
(b) *for any* $L' \in \mathcal{S}$, $L' \neq L$, $(e_\lambda^L(Y)) \mid L'$ *is a direct sum of finite dimensional $L'$-modules.*

*Proof.* For the proof of (a) it suffices to show that $(\operatorname{supp} e_\lambda^L Y) \subset L$. Observe that $(\operatorname{supp} e_\lambda^L Y) \subset \widehat{\operatorname{supp} Y}$, $(G_L)_Y \subset G_{e_\lambda^L Y}$ and $(\widehat{\operatorname{supp} Y})/G_Y$ is finite. Now by 3.1(iv) there exists $L' \in \mathcal{S}$ such that $\operatorname{supp} e_\lambda^L Y \subset L'$. Hence 3.1(iii) and Proposition 2.5 imply $L = L'$. For the proof (b) take $L' \in \mathcal{S}$, $L \neq L'$. By (a) $\operatorname{supp}((e_\lambda^L Y) \mid L') \subset L \cap L'$, so by 3.1(iii) and Proposition 2.5, $e_\lambda^L(Y)$ satisfies the required condition.

3.4. LEMMA. *Let $L$, $L' \in \mathcal{S}_0$. Then*

$$\mathbf{E}_\cdot^{L'}\mathbf{E}_\lambda^L \cong \begin{cases} \mathbb{1}_{\underline{\mathrm{Mod}}_f^{G_L} L} & \text{if } L = L' \\ 0 & \text{if } L \neq L' \end{cases}$$

*Proof.* Let $\varphi^L : \mathbb{1}_{\underline{\mathrm{MOD}}^{G_L} L} \to E_\cdot^L E_\lambda^L$ be the unit of the adjoint pair $(E_\lambda^L, E_\cdot^L)$ and $N \in \mathrm{Mod}_f^{G_L} L$. Applying 3.3b to the indecomposable summands of $N$ (considered as an $L$-module), we infer that

$$\underline{\varphi^L(N)} : N \to \mathbf{E}_\cdot^L \mathbf{E}_\lambda^L(N)$$

is an isomorphism and that $E_\cdot^{L'} E._\lambda^L(N) = \bigoplus_{g \in U_L} {}^g(e_\lambda^L N) \mid L' \in \mathrm{Mod}_{f1}^{G_{L'}} L'$ if $L \neq L'$.

Let $\mathbf{E}. : \underline{\mathrm{Mod}}_f^G R \to \prod_{L \in \mathcal{S}_0} \underline{\mathrm{Mod}}_f^{G_L} L$ be the functor defined by the family of functors $(\mathbf{E}_\cdot^L)_{L \in \mathcal{S}_0}$, and let $I : \coprod_{L \in \mathcal{S}_0} \underline{\mathrm{Mod}}_f^{G_L} L \to \prod_{L \in \mathcal{S}_0} \underline{\mathrm{Mod}}_f^{G_L} L$ be canonical embedding. We denote by $\mathbf{E}_\lambda : \coprod_{L \in \mathcal{S}_0} \underline{\mathrm{Mod}}_f^{G_L} L \to \underline{\mathrm{Mod}}_f^G R$ the functor which maps the object $(M_L)_{L \in \mathcal{S}_0}$ onto $\coprod_{L \in T} \mathbf{E}_\lambda^L(M_L)$, where $T = \{L \in \mathcal{S}_0 : M_L \notin \mathrm{Mod}_{f1}^{G_L} L\}$.

COROLLARY. *The functors $I$ and $\mathbf{E}.\mathbf{E}_\lambda$ are isomorphic.*

3.5. PROPOSITION. $\mathbf{E}.$ *factors through* $\coprod_{L \in \mathcal{S}_0} \underline{\mathrm{Mod}}_f^{G_L} L$. *The induced functor* $\mathbf{E} : \underline{\mathrm{Mod}}_f^G R \to \coprod_{L \in \mathcal{S}_0} \underline{\mathrm{Mod}}_f^{G_L} L$ *is such that* $\mathbf{E}\mathbf{E}_\lambda$ *and* $\mathbf{E}_\lambda\mathbf{E}$ are isomorphic to the identical functors.

*Proof.* We will show that each indecomposable $M \in \mathrm{Mod}_{f2}^G R$ is isomorphic to some $E_\lambda^L N$, where $L \in \mathscr{S}_0$ and $N \in \mathrm{Mod}_{f2}^{G_L} L$. This fact and Lemma 3.4 clearly imply our proposition. So let $M \in \mathrm{Mod}_{f2}^G R$ be indecomposable. Then there exists a weakly-$G$-periodic $R$-module $Y$ such that $Y \subset_\oplus M$ in $\mathrm{Mod}\, R$. By 3.1(iv) there exists $L \in \mathscr{S}$ such that $\mathrm{supp}\, Y \subset L$. Without loss of generality one can assume that $L \in \mathscr{S}_0$, because for any $g \in G$, ${}^gY \subset_\oplus {}^gM \xleftarrow{\sim} M$ in $\mathrm{Mod}\, R$. It follows that $Y \mid L \subset_\oplus N \neq 0$ for any decomposition $E_\cdot^L M = N \oplus N'$ in $\mathrm{Mod}_f^{G_L} L$ such that $N \in \mathrm{Mod}_{f2}^{G_L} L$ and $N' \in \mathrm{Mod}_{f1}^{G_L} L$. Set $\tilde{N} = e_\lambda^L N$ ( = extension of $N$ to $R$ by 0). As in 3.3, we can show that $\tilde{N}$ is identified with $e_\rho^L N$, where $e_\rho^L$ is the right adjoint to the restriction $e_\cdot^L$. The inclusion $N \to E_\cdot^L M$ and the projection $E_\cdot^L M \to N$ are therefore associated with $G_L$-equivariant morphisms $i: \tilde{N} \to M$ and $p: M \to \tilde{N}$ such that $pi = 1_{\tilde{N}}$. The induced morphisms ${}^gi: {}^g\tilde{N} \to {}^gM \xrightarrow{\sim} M$ and ${}^gp: M \xrightarrow{\sim} {}^gM \to {}^g\tilde{N}$ define $G$-equivariant morphisms $j: \coprod_{g \in U_L} {}^g\tilde{N} \to M$ and $q: M \to \prod_{g \in U_L} {}^g\tilde{N} \xrightarrow{\sim} \coprod_{g \in U_l} {}^g\tilde{N}$. By Lemma 3.3(b) the morphism $qj$ satisfies the assumption of Lemma 2.1(iii); so is invertible and $M$ is isomorphic to $E_\lambda^L N = \coprod_{g \in U_l} {}^g\tilde{N}$.

In order to describe the structure of $\Gamma_{R/G}$ recall that by [20; 3.6] the modules of the first kind and second kind are contained in different components of $\Gamma_{R/G}$ and that the union $(\Gamma_{R/G})_1$ of the components containing all indecomposables of the first kind has the form $\Gamma_R/G$. Denote by $J_R$ (resp. $J_L$, $L \in \mathscr{S}_0$) the Jacobson radical of the category $\mathrm{Mod}_f^G R$ (resp. $\mathrm{Mod}_f^{G_l} L$). Since, for each $L \in \mathscr{S}_0$, $E_\lambda^L|: \mathrm{Mod}_{f2}^{G_L} L \to \mathrm{Mod}_f^G R$ is "exact" by Lemma 3.3(a), the structure of $(\Gamma_{R/G})_2 = \Gamma_{R/G} \setminus (\Gamma_{R/G})_1$ follows immediately from the formula

$$J_R/J_R^2(E_\lambda^L N, M) = \begin{cases} J_L/J_L^2(N, N') & \text{if } M = E_\lambda^L N' \quad \text{for some } N' \in \mathrm{Mod}_{f2}^{G_L} L \\ 0 & \text{if } M = E_\lambda^{L'} N' \quad \text{for some } N' \in \mathrm{Mod}_{f2}^{L'} L',\ L \neq L' \\ 0 & \text{if } M \in \mathrm{Mod}_{f1}^G R \end{cases}$$

where $M \in \mathrm{Mod}_f^G R$ and $N \in \mathrm{Mod}_{f2}^{G_l} L$ are indecomposable and $L$ is a fixed element of $\mathscr{S}_0$. This finishes the proof of Theorem 3.1.

3.6. Let $A$ be a $k$-algebra (not necessarily finite-dimensional). Then any contravariant functor $Q: R \to \mathrm{MOD}\, A^{op}$ will simply be called *$A$-$R$-bimodule.* Each $A$-$R$-bimodule $Q$ induces a functor $- \otimes_A Q: \mathrm{MOD}\, A \to \mathrm{MOD}\, R$, where $(V \otimes_A Q)(x) = V \otimes_A Q(x)$ for all $V \in \mathrm{MOD}\, A$ and $x \in R$.

Let $B$ be a weakly-$G$-periodic $R$-module together with an $R$-action $v$ of $G_B$ on $B$. Then $F_\lambda B$ carries the structure of a $KG_B$-$R/G$-bimodule, where $KG_B$ is the group algebra of $G_B$ over $K$. More precisely, for each $Gx \in R/G$, $(F_\lambda B)(Gx)$ is a free $KG_B$-module of rank $\sum_{y\in W_x}(B(y):K)$, where $W_x$ is a set of representatives of the $G_B$-orbits of $Gx$. In particular, if $B = B_L$ for some line $L$ in $R$ (see 2.3), then $F_\lambda B_L$ is a $K[T, T^{-1}]$-$R/G$-bimodule. In this case we will denote by $\Phi^L$ the functor $-\otimes_{K[T,T^{-1}]} F_\lambda B_L$: mod $K[T, T^{-1}] \to$ mod $R/G$.

Let $\mathscr{L}$ be the set of all subcategories supp $Y \subset R$, where $Y$ ranges over all weakly-$G$-periodic $R$-modules, and let $\mathscr{L}_0$ be a fixed set of representatives of the $G$-orbits of $\mathscr{L}$.

THEOREM. *Let $R$ be a locally bounded $K$-category and let a group $G$ of $K$-linear automorphisms of $R$ act freely on* ind $R/\simeq$. *Assume that $\mathscr{L}$ consists only of lines in $R$* (2.3). *The family of functors $\Phi^L$, $L \in \mathscr{L}_0$, induces an equivalence of categories*

$$\Phi: \coprod_{\mathscr{L}_0} \operatorname{mod} K[T, T^{-1}] \xrightarrow{\sim} (\operatorname{mod} R/G)/[\operatorname{mod}_1 R/G].$$

*In particular, $(\Gamma_{R/G})_2 = \coprod_{\mathscr{L}_0} \Gamma_{k[T,T^{-1}]}$, where $\Gamma_{k[T,T^{-1}]}$ is the translation-quiver of the category of finite dimensional $k[T, T^{-1}]$-modules.*

*Moreover, $R/G$ is tame iff so is $R$.*

*Proof.* First we show that $\mathscr{L}$ forms a separating family of subcategories of $R$. Properties (i), (ii), (iv) are trivially satisfied. Let $L, L' \in \mathscr{L}$, $L \neq L'$. Then $L \cap L'$ is a disjoint union of connected finite subcategories. Indeed, if $D$ is a half-line of $L$ and $L'$, it easily follows that the semigroup $\{g \in G, gD \subset D\}$ is infinite cyclic. Its generator $g$ is also a generator of the groups $G_L$ and $G_{L'}$ and satisfies $L = \bigcup_n g^{-n}D = L'$, a contradiction. Consequently, $L \cap L'$ is a disjoint union of finite connected subcategories, and (iii) is satisfied.

Let $L \in \mathscr{L}_0$. Then by [14] and [26] (see also [13]) the functor

$$\Psi^L = -\underset{k[T,T^{-1}]}{\otimes} F_\lambda^L(B_L \mid L) \colon \operatorname{mod} k[T, T^{-1}] \to \operatorname{mod} L/G_L,$$

where $F^L: L \to L/G_L$ is the canonical "projection", induces an equivalence of categories

$$\bar{\Psi}^L \colon \operatorname{mod} k[T, T^{-1}] \to (\operatorname{mod} L/G_L)/[\operatorname{mod}_1 L/G_L]$$

and an isomorphism of the translation-quivers $\Gamma_{k[T,T^{-1}]} \simeq (\Gamma_{L/G_L})_2$. (One can prove this statement in an elementary way using the fact that each weakly-$G_L$-periodic $L$-module is isomorphic to $B_L \mid L$ by Corollary 4.4 below.) The functors

$\bar{\Psi}^L$ give rise to the following diagram which is commutative up to isomorphism:

$$\begin{tikzcd}
\coprod_{\mathcal{L}_0} \mathrm{mod}\, k[T,T^{-1}] \arrow[dr, "(\varphi^L)"] \arrow[drr, "\Phi=(\Phi^L)"] \arrow[ddr, "\bar{\Psi}=(\bar{\Psi}^L)"'] & & \\
& \coprod_{L\in\mathcal{L}_0} \mathrm{mod}\, L/G_L \arrow[r] \arrow[d, "(\Pi^L)"] & \mathrm{mod}\, R/G \arrow[d, "\Pi"] \\
& \coprod_{L\in\mathcal{L}_0} (\mathrm{mod}\, L/G_L)/[\mathrm{mod}_1 L/G_L] \arrow[r, "E"] & (\mathrm{mod}\, R/G)/[\mathrm{mod}_1 R/G]
\end{tikzcd}$$

Since $E$ is an equivalence by Theorem 2.1 so is $\bar{\Phi} = \Pi \circ \Phi \simeq E \circ \bar{\Psi}$.

The required description of $(\Gamma_{R/G})_2$ follows from Theorem 2.1.

Before the proof of the last assertion, we recall that a locally bounded category $R$ is called *tame* if, for each finite dimension-vector $d$ of $R$ there exists a finite family of $k[T]$-$R$-bimodules $Q_i$ such that:

(a) For each $x \in R$, $Q_i(x)$ is a free $k[T]$-module of rank $d(x)$.

(b) Every indecomposable $R$-module $M$ with $\underline{dim}\, M = d$ is of the form $M \cong V \otimes_{k[T]} Q_i$ for some $i$ and some simple $k[T]$-module $V$.

In this definition, $k[T]$ can be replaced by $k[T, T^{-1}]$.

It is shown in [15, Proposition 2] that, if $R/G$ is tame, so is always $R$. Conversely, if $R$ is tame, the indecomposable $R/G$-modules of the first kind with fixed dimension-vector are parametrized by finite families of bimodules (see [15; Lemma 3]).

Let us now turn to the $R/G$-modules of second kind. By theorem 3.6 they are "parametrized" by $k[T, T^{-1}]$-$R/G$-bimodules $Q_{L,n} = k[S, T, T^{-1}]/(T-S)^n \otimes_{k[S]} F_\lambda B_L$, where $L \in \mathcal{L}_0$ and $n \geqslant 1$. The dimension of the $R/G$-module $V \otimes_{k[T]} Q_{L,n}$ attached to a simple $k[T, T^{-1}]$-module $V$ is equal to $n \cdot |L/G_L|$, where $|L/G|$ denotes the number of points of $L/G_L$. We infer that the number of bimodules $Q_{L,n}$ such that modules $V \otimes_{k[T]} Q_{L,n}$ have a fixed dimension-vector, is finite because, for each $y \in R$ and each $r \geqslant 1$, there are only finitely many lines passing through $y$ and such that $|L/G_L| \leqslant r$.

## §4. Fundamental sequences

In this section we shall show that each indecomposable locally finite dimensional $R$-module is a "limit" of a sequence of finite dimensional indecomposable modules over some finite full subcategories of $R$. In particular, weakly-$G$-periodic $R$-modules are "limits" of "$G$-periodic sequences". We will see in §5 that, for a class of locally bounded categories $R$, such sequences are related with a rather narrow class of modules in $\operatorname{ind} R$. This will enable us to describe weakly-$G$-periodic $R$-modules completely.

4.1. In the sequel $\mathscr{C}_n$, $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$, denotes a fixed family of finite full subcategories of $R$ such that

(1) For each $n \in \mathbb{N}$, $\mathscr{C}_{n+1} = \hat{\mathscr{C}}_n$ (1.1).

(2) $R = \bigcup_{n \in \mathbb{N}} \mathscr{C}_n$.

Since $R$ is connected, such a family always exists.

For each $n \in \mathbb{N}$, the restriction functors $e_\cdot^n : \operatorname{Mod} \mathscr{C}_{n+1} \to \operatorname{Mod} \mathscr{C}_n$ and $\varepsilon_\cdot^n : \operatorname{Mod} R \to \operatorname{Mod} \mathscr{C}_n$ admit left adjoint functors $e_\lambda^n : \operatorname{Mod} \mathscr{C}_n \to \operatorname{Mod} \mathscr{C}_{n+1}$ and $\varepsilon_\lambda^n : \operatorname{Mod} \mathscr{C}_n \to \operatorname{Mod} R$ such that $e_\cdot^n e_\lambda^n \xrightarrow{\sim} 1_{\operatorname{Mod} \mathscr{C}_n}$, $\varepsilon_\cdot^n, \varepsilon_\lambda^n \xrightarrow{\sim} 1_{\operatorname{Mod} \mathscr{C}_n}$ and $\varphi : \varepsilon_\lambda^{n+1} e_\lambda^n \xrightarrow{\sim} \varepsilon_\lambda^n$ ([23, chap X]).

DEFINITION. A *fundamental R-sequence* is a sequence $(Y_n, u_n)_{n \in \mathbb{N}}$ of modules $Y_n \in \operatorname{mod} \mathscr{C}_n$ and $\mathscr{C}_n$-homomorphisms $u_n : Y_n \to Y_{n+1} \mid \mathscr{C}_n$ satisfying the conditions (a), (b), (c), and (d) below:

(a) For each $n \in \mathbb{N}$, $Y_n = 0$ or $Y_n \in \operatorname{ind} \mathscr{C}_n$.

(b) $Y_n \neq 0$ for some $n \in \mathbb{N}$.

(c) For each $n \in \mathbb{N}$, $u_n$ is a splittable monomorphism in $\operatorname{mod} \mathscr{C}_n$.

(d) For each $x \in R$, the sequence $(\dim Y_n(x))_{n \in \mathbb{N}}$ is bounded. A fundamental $R$-sequence $(Y_n, u_n)_{n \in \mathbb{N}}$ is *bounded* if there is a common upper bound for $\dim_k Y_n(x)$, $x \in R$, $n \in \mathbb{N}$. Finally, a fundamental $R$-sequence $(Y_n, u_n)_{n \in \mathbb{N}}$ is *produced* by an $R$-module $X$ if $Y_n \subset_\oplus \mid X$ for all $n \in \mathbb{N}$.

4.2. *Remark*. Every locally finite dimensional $R$-module $X \neq 0$ produces a fundamental $R$-sequence $(Y_n, u_n)_{n \in \mathbb{N}}$. Indeed, take a point $a \in \operatorname{supp} X$ and an arbitrary indecomposable direct summand $Z$ of $X \mid \mathscr{C}_m$ with $Z(a) \neq 0$ for some $\mathscr{C}_m$ containing $a$. Put $Y_n = 0$ for $n < m$ and $Y_m = Z$. There exists $Y_{m+1} \in \operatorname{ind} \mathscr{C}_{m+1}$ and a splittable monomorphism $u_m : Y_m \to Y_{m+1} \mid \mathscr{C}_m$ such that $Y_{m+1} \subset_\oplus \mid X$. Repeating this procedure we can find, for all $n \geq m$, $Y_n \in \operatorname{ind} \mathscr{C}_n$ and splittable monomorphisms $u_n : Y_n \to Y_{n+1} \mid \mathscr{C}_n$ such that $Y_n \subset_\oplus \mid X$. Since $X \mid \mathscr{C}_n$ is finite dimensional the condition ($d$) is satisfied and $(Y_n, u_n)_{n \in \mathbb{N}}$ is a fundamental $R$-sequence produced by $X$.

4.3. Let $(Y_n, u_n)_{n \in \mathbb{N}}$ be a fundamental $R$-sequence. We shall define the limit $\lim (Y_n, u_n)$, shortly denoted by $\lim Y_n$. Since $e_\lambda^n$ is left adjoint to $e_\cdot^n$, $u_n : Y_n \to e_\cdot^n(Y_{n+1})$ induces canonical morphism $v_n : e_\lambda^n(Y_n) \to Y_{n+1}$ in $\operatorname{mod} \mathscr{C}_{n+1}$. Then we have the following homomorphisms in $\operatorname{Mod} R$

$$t_n = \varepsilon_\lambda^{n+1}(v_n) : \varepsilon_\lambda^{n+1} e_\lambda^n(Y_n) \to \varepsilon_\lambda^{n+1}(Y_{n+1})$$

and consequently

$$w_n = t_n\varphi_n^{-1} : \varepsilon_\lambda^n(Y_n) \to \varepsilon_\lambda^{n+1}(Y_{n+1}),\ n \in \mathbb{N}.$$

We set $\lim Y_n = \varinjlim (\varepsilon_\lambda^n(Y_n), w_n)$.

LEMMA. *Let $(Y_n, u_n)_{n\in\mathbb{N}}$ be a fundamental $R$-sequence. Then* $\lim Y_n$ *is an indecomposable locally finite dimensional $R$-module and, for each $m \in \mathbb{N}$, there exists $p \geq m$ such that $Y_p \mid \mathscr{C}_m \xrightarrow{\sim} (\lim Y_n) \mid \mathscr{C}_m$.*

*Proof.* The last property follows from (d) and the fact that $\varepsilon_\lambda^p(Y_p) \mid \mathscr{C}_m \cong (\varepsilon_\lambda^p(Y_p) \mid \mathscr{C}_p) \mid \mathscr{C}_m \cong Y_p \mid \mathscr{C}_m$ for $p \geq m$. Set $Y = \lim Y_n$ and suppose that $Y = X \oplus Z$ for some $X \neq 0$ and $Z \neq 0$ in $\operatorname{Mod} R$. Set $\mathbb{N}_X = \{n \in \mathbb{N};\ Y_n \subset_\oplus \mid X\}$ and $\mathbb{N}_Z = \{n \in \mathbb{N};\ Y_n \subset_\oplus \mid Z\}$. Then $\mathbb{N} = \mathbb{N}_X \cup \mathbb{N}_Z$ and one of the sets $\mathbb{N}_X$ or $\mathbb{N}_Z$, say $\mathbb{N}_X$, is infinite. Hence $\mathbb{N} = \mathbb{N}_X$ since $m \leq n$ and $n \in \mathbb{N}_X$ imply $m \in \mathbb{N}_X$. Therefore $\varepsilon_\cdot^n(Y) \subset_\oplus \mid \varepsilon_\cdot^n(X)$ for all $n \in \mathbb{N}$, $Z = 0$ and we have a contradiction.

4.4. PROPOSITION. *Let $X$ and $Z$ be modules in $\operatorname{Mod} R$. Then*

(i) *$X \subset_\oplus Z$ if and only if $X \mid \mathscr{C}_n \subset_\oplus Z \mid \mathscr{C}_n$ for all $n \in \mathbb{N}$.*

(ii) *$X \cong Z$ if and only if $X \mid \mathscr{C}_n \cong Z \mid \mathscr{C}_n$ for all $n \in \mathbb{N}$.*

We owe the *proof* to Gabriel: (ii) is a consequence of (i). In order to prove (i), we first consider arbitrary modules $V$, $W \in \operatorname{Mod} R$. We set $V_n = V \mid \mathscr{C}_n$ and denote the restriction map $\operatorname{Hom}_{\mathscr{C}_m}(V_m, W_m) \to \operatorname{Hom}_{\mathscr{C}_n}(V_n, W_n)$ by $\rho_n^m$ for $m \geq n$. We then put $\operatorname{Hom}'(V_n, W_n) = \bigcap_{m\geq n} \operatorname{Im} \rho_n^m$ and observe that $\operatorname{Hom}'(V_n, W_n) = \operatorname{Im} \rho_n^m$ for large $m$, that the maps $\Pi_n^m : \operatorname{Hom}'(V_m, W_m) \to \operatorname{Hom}'(V_n, W_n)$ induced by $\rho_n^m$ are surjective and that $\operatorname{Hom}_R(V, W) \xrightarrow{\sim} \varprojlim \operatorname{Hom}'(V_n, W_n)$. Now, for each $n \in \mathbb{N}$, our assumptions imply the existence of morphisms $a_n \in \operatorname{Hom}'(X_n, Z_n)$ and $b_n \in \operatorname{Hom}'(Z_n, X_n)$ such that $b_n a_n = 1_{X_n}$. The problem is to construct these $a_n$, $b_n$ in such a way that $\Pi_n^{n+1} a_{n+1} = a_n$ and $\Pi_n^{n+1} b_{n+1} = b_n$. We do this by induction on $n$, setting

$$A = \operatorname{Hom}'(X_{n+1} \oplus Z_{n+1}, X_{n+1} \oplus Z_{n+1}), \qquad A' = \operatorname{Hom}'(X_n \oplus Z_n, X_n \oplus Z_n)$$

and using the following simple lemma.

LEMMA. *Let $\rho : A \to A'$ be a surjective homomorphism of finite dimensional $k$-algebras, and $e$, $f$ two orthogonal idempotents of $A$. Suppose that there are elements $x \in fAe$ and $y \in eAf$ such that $yx = e$. Then, for all $a' \in \rho(f)A'\rho(e)$ and*

*$b' \in \rho(e)A'\rho(f)$ such that $b'a' = \rho(e)$, there are elements $a \in fAe$ and $b \in eAf$ such that $\rho(a) = a'$, $\rho(b) = b'$ and $ba = e$.*

*Proof.* Reduce to the semisimple case by factoring out the radicals of $A$ and $A'$.

COROLLARY. *Let $X$ be a module in $\operatorname{Mod} R$ and $(Y_n, u_n)_{n\in\mathbb{N}}$ a fundamental $R$-sequence produced by $X$. Then $\lim Y_n$ is a direct summand of $X$. In particular, $X = \lim Y_n$ if and only if $X$ is indecomposable.*

4.5. A fundamental $R$-sequence $(Y_n, u_n)_{n\in\mathbb{N}}$ with $Y = \lim Y_n$ is called $G$-*periodic* if $G_Y \neq \{1_R\}$. The following lemma gives a description of $G_Y$ in terms of the sequence $(Y_n)$.

LEMMA. *Let $(Y_n, u_n)_{n\in\mathbb{N}}$ be a fundamental $R$-sequence, $Y = \lim Y_n$ and $g \in G$. The following two conditions are equivalent*

(i) $Y \cong {}^gY$

(ii) *For any $n \in \mathbb{N}$ there is an $m \geqslant n$ such that $Y_n \subset_\oplus | {}^gY_m$.*

*Proof.* Assume (i) and take $n \in \mathbb{N}$. Then $\mathscr{C}_n \subset g\mathscr{C}_p$ for some $p \geqslant n$ and, by Lemma 4.3, there is $m \geqslant p$ such that ${}^gY | gC_p \cong {}^gY_m | gC_p$. Consequently $Y_n \subset_\oplus | {}^gY_m$ since $Y_n \subset_\oplus | Y \cong {}^gY$. Conversely, if (ii) holds, Lemma 4.3 implies that $Y | \mathscr{C}_n \subset_\oplus {}^gY | \mathscr{C}_n$ for all $n \in \mathbb{N}$; so, by Proposition 4.4, $Y \subset_\oplus {}^gY$ and finally $Y \cong {}^gY$, since ${}^gY$ is indecomposable.

## §5. Examples and applications

5.1. Assume that $R = kQ/I$ is a locally bounded $K$-category satisfying the following conditions: (a) $R$ is Schurian [12], (b) $Q$ is connected, directed and interval-finite [5], (c) $\Pi_1(Q, I) = 0$ [25], (d) the support of any indecomposable finite dimensional $R$-module is representation-finite or belongs to the Bongartz–Happel–Vossieck list ([5], [9], [22]) of critical algebras. Let $G$ be a group of $K$-linear automorphisms of $R$ acting freely on the objects of $R$. It follows from our assumption and [24] that $G$ acts also freely on $\operatorname{ind} R/\simeq$.

PROPOSITION. *Every weakly-$G$-periodic $R$-module is linear. As a consequence $R/G$ is tame.*

*Proof.* Let $Y$ be a weakly-$G$-periodic $R$-module and let $(Y_n, u_n)_{n\in\mathbb{N}}$ be a

($G$-periodic) fundamental $R$-sequence produced by $Y$. By Corollary 4.4, $Y = \lim Y_n = \varinjlim \varepsilon_\lambda^n(Y_n)$ (4.1). From our assumption and [23] we know that the support of any $\varepsilon_\lambda^n(Y_n)$ belongs to the Bongartz–Happel–Vossieck list or is a (representation-finite) simply connected algebra. Set $m = \min\{i \in \mathbb{N}, Y_i \neq 0\}$ and $n_0 = m + 336$. Since $Y$ is infinite dimensional and indecomposable, Lemma 2.5 implies $\operatorname{supp} Y_n \not\subset \mathscr{C}_{n-1}$ for $n \geqslant n_0$. Hence, for $n \geqslant n_0$, $\operatorname{supp} Y_n$ has at least $336 = 5 \cdot 67 + 1$ points and consequently $\operatorname{supp} \varepsilon_\lambda^n(Y_n)$ is either a Schurian algebra of type $\tilde{\mathbb{D}}_m$ (Bongartz–Happel–Vossieck list) or belongs to the 24 familes listed by Bongartz in [7, 2.4]. Moreover, for $n \geqslant n_0$, $1 \neq g \in G_Y$, we have $Y_n = \varepsilon_\lambda^n(Y_n) \mid \mathscr{C}_n$, $Y_n \subset_\oplus \mid Y_r$ and $Y_n \subset_\oplus \mid {}^gY_r$ for some $r \geqslant n$. Using the structure of indecomposable finite dimensional modules of the above families of algebras and the fact that $G$ acts freely on $\operatorname{ind} R/\cong$, we deduce that the support of any $Y_n$ is linear and consequently $Y = \lim Y_n$ is linear. The second part of the proposition follows from Theorem 3.6.

COROLLARY. *Assume that* $d = \dim R/G$ *is finite. Then the following statements are equivalent.*

(i) *$R$ is locally support-finite.*

(ii) $\operatorname{ind} R = \operatorname{Ind} R$.

(iii) *$G$ acts freely on* $\operatorname{Ind} R/\cong$.

(iv) *$R$ is $G$-exhaustive.*

(v) *$R$ does not contain convex subcategory $B \xleftarrow{\sim} kQ_B$, where $Q_B$ is a Dynkin quiver of type $A_{2d+1}$.*

*Proof.* The implications (i)$\Rightarrow$(ii)$\Rightarrow$(iii)$\Rightarrow$(iv) follow from Proposition 2.5 and Corollary 2.3. Assume that (iv) holds and suppose that $R$ contains a convex subcategory $B \xleftarrow{\sim} kQ_B$ for some Dynkin quiver $Q_B$ of type $A_{2d+1}$. We mark all sources and sinks in $Q_B$ and get, up to duality,

$$Q_B : q_1 \to t \to \cdots \to s_1 \leftarrow \cdots u \leftarrow q_t \to v \cdots x \leftarrow q_r \to y \to \cdots \to s$$

where the right part after $y$ can be missing. Then by our assumption, $Q_B$ contains three sources which have the same image under the Galois covering $F: R \to R/G$ (see [8, 3.2]). Changing notation, we have $F(q_1) = F(q_t) = F(q_r)$. Assume first that $F(u) \neq F(v)$. Then we have $F(t) \neq F(u)$ or $F(v) \neq F(x)$ or $F(x) \neq F(t)$ and $B$ contains a full subcategory $C$ of the form $x_0 \to x_1 \to \cdots \leftarrow x_n \leftarrow x_{n+1}$ such that $F(x_0) = F(x_{n+1})$ and $F(x_1) \neq F(x_n)$. In case $F(u) = F(v)$, since $F$ is a covering map and $u \neq v$, $R$ contains a convex subcategory $D$ of the form

$$\cdots z_{-2} \xleftarrow{\alpha_{-1}} y_{-1} \xrightarrow{\beta_{-1}} z_{-1} \xleftarrow{\alpha_0} y_0 \xrightarrow{\beta_0} z_1 \xleftarrow{\alpha_1} y_1 \xrightarrow{\beta_1} z_2 \cdots$$

where $F(\alpha_i) = F(\alpha_0)$, $F(\beta_j) = F(\beta_0)$, and $\alpha_i$ and $\beta_j$ are pairwise different for all integers $i$, $j$. In both cases $R$ contains a $G$-periodic line and we get a contradiction with the fact that $R$ is $G$-exhaustive (Lemma 2.4). Therefore (iv) implies (v). If (v) holds, our assumption implies the existence of an upper bound on the number of points of the supports of indecomposable finite dimensional $R$-modules. Consequently, $R$ is locally support-finite.

5.2. Biserial algebras. A locally bounded $k$-category $R$ is called *biserial* if the radical of each indecomposable projective left or right $R$-modules is a sum of two uniserial submodules whose intersection is simple or zero. A locally bounded category is called *special biserial* if it is isomorphic to a bounden quiver category (in the sense of [10]) $kQ/I$, where the bounden quiver satisfies the following conditions:

(i) the numbers of arrows starting and ending at any vertex of $Q$ are bounded by 2,

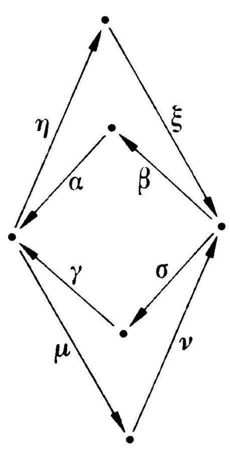
(ii) for any arrow $\alpha$ of $Q$ there is at most one arrow $\beta$ and at most one arrow $\gamma$ such that $\beta\alpha$ and $\alpha\gamma$ are not in $I$. By [31, Lemma 1], each special biserial category is biserial. By [5, 31], each locally representation-finite biserial category is special biserial. Well-known examples of representation-infinite special biserial algebras are group algebras of dihedral groups in characteristic 2 [6, 28] and algebras appearing in the Gelfand–Ponomarev classification of Harish–Chandra modules over the Lorentz group [21]. K. Erdmann has recently proved that in characteristic 2 each block with a dihedral defect group is a special biserial.

Recall that for an algebra $\Lambda$, we have two natural invariants $\alpha(\Lambda)$ and $\beta(\Lambda)$ introduced by Auslander and Reiten [2, 3]. The invariant $\alpha(\Lambda)$ is the largest possible number of indecomposable summands in the middle term of an almost split sequence and $\beta(\Lambda)$ is the largest possible number of such summands which are neither projective nor injective. In the research of biserial algebras we can assume that each indecomposable projective-injective is uniserial (see [3; 4.2]). Moreover, by [31, Corollary 1], each special biserial algebra having no nonuniserial projective-injective indecomposable modules is isomorphic to $kQ/I$, where $(Q, I)$ satisfies (i), (ii) and $I$ is generated by a set of paths. The universal cover $(\tilde{Q}, \tilde{I})$ of such a $(Q, I)$ is a bounden tree satisfying the conditions (i) and (ii). Finally, it is known [11, 27] that the support of any module in ind $k\tilde{Q}/\tilde{I}$ is a finite line. Therefore we obtain the following consequence of Theorem 3.6 (and Corollary 4.4).

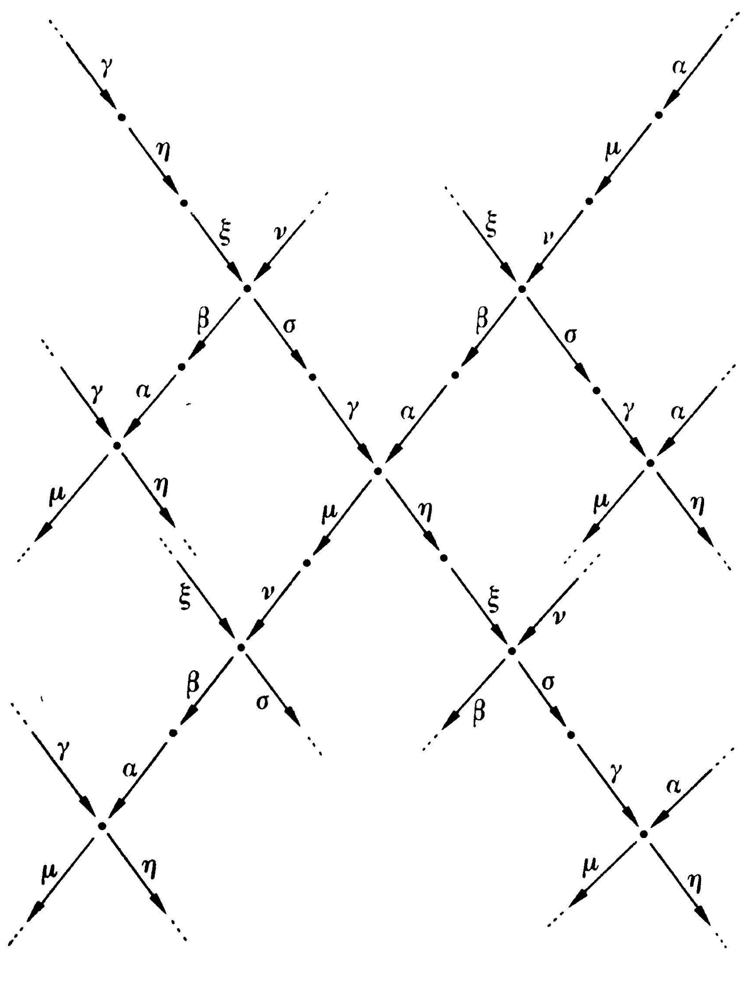
PROPOSITION. *Every special biserial algebra $\Lambda$ is tame and $\beta(\Lambda) \leq 2$.*

This result was proved in [32] using methods of Gelfand and Panomarev [21].

5.3. The following example shows that there are other locally representation-infinite $K$-categories satisfying the assumptions of (5.1). Let $\Lambda$ be the bounden quiver algebra $kQ/I$ where $Q$ is the quiver and $I$ is the ideal of $KQ$ generated by

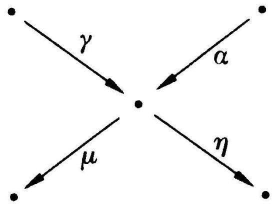



the elements $\beta\xi$, $\sigma\nu$, $\eta\alpha\beta$, $\mu\alpha\beta$, $\eta\gamma\sigma$, $\mu\gamma\sigma$, $\nu\mu\alpha$, $\nu\mu\gamma$, $\xi\eta\alpha$ and $\xi\eta\gamma$. Then the fundamental group $\Pi(Q, I)$ of $(Q, I)$ [25] is a (non-commutative) free group in three generators and there is a universal Galois covering $F: R \to R/G = \Lambda$ with group $G = \Pi(Q, I)$ where $R = K\tilde{Q}/\tilde{I}$ is given by the following quiver $\tilde{Q}$ the ideal $\tilde{I}$ being generated by all elements of the form $\beta\xi$, $\sigma\nu$, $\eta\alpha\beta$, $\mu\alpha\beta$, $\eta\gamma\sigma$, $\mu\gamma\sigma$, $\nu\mu\gamma$,

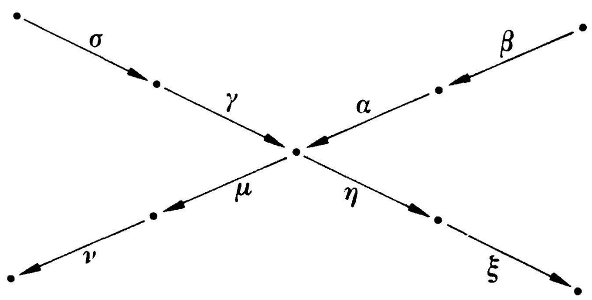


$\nu\mu\alpha$, $\xi\eta\alpha$ and $\xi\eta\gamma$. Observe that $R$ contains full subcategories given by the extended Dynkin quivers of type $\tilde{\mathbb{D}}_4$. Fix a quiver $D$ and consider the full

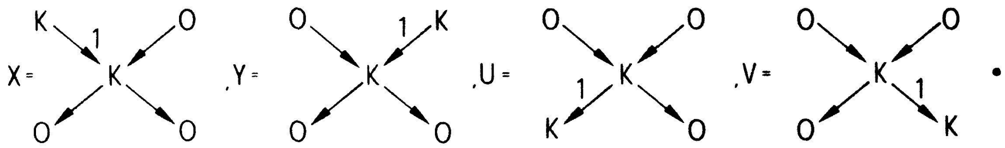
D:



bounden subquiver $E$ of $R$ of the form



(where all paths of length 3 equal zero). Then $E$ can be obtained from $D$ by two one-point extensions using the $D$-modules $X$ and $Y$ below which lie at the end of the preinjective component [13] of $\Gamma_D$, and two one-point coextensions using the $D$-modules $U$ and $V$ which lie at the beginning of the preprojective component of $\Gamma_D$:



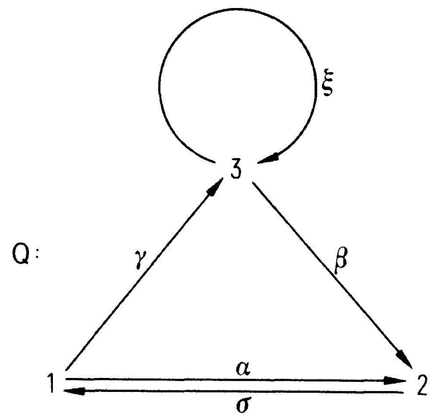
Hence the support of any $Z \in \operatorname{ind} E$ is contained in $D$ or in one of the following linear subquivers

$$L_1\colon . \xrightarrow{\sigma} . \xrightarrow{\gamma} . \xleftarrow{\alpha} . \xleftarrow{\beta} . \quad \text{or} \quad L_2\colon . \xleftarrow{\nu} . \xleftarrow{\mu} . \xrightarrow{\eta} . \xrightarrow{\xi} . .$$

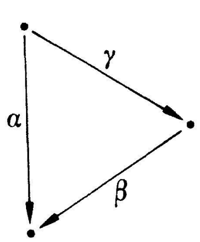
Further, $R$ can be obtained from $E$ by successive one-point extensions and coextensions using modules whose restriction to $E$ is either a representation of $L_1$, a representation of $L_2$ or zero. Thus, if the support of a module $M \in \operatorname{ind} R$ is not contained in some quiver $D$, $M$ is annihilated by the ideal $J$ of $R$ generated by all paths of the form $\mu\alpha$, $\mu\gamma$, $\eta\gamma$, $\eta\alpha$, that is, $M$ is a representation of the special biserial category $R/J$. Therefore the support of any indecomposable finite dimensional $R$-module is either a finite line or is contained in an extended Dynkin quiver of the form $D$; so $R$ satisfies the assumptions of 5.1.

5.4. We end the paper with an example showing that Theorem 3.1 can be also applied to locally bounded categories having nonlinear weakly periodic modules. In a forthcoming paper by the second author and Z. Pogorzały, Theorem 3.1 will be applied to the classification of indecomposable finite dimensional modules over arbitrary biserial algebras. In the example considered below we shall outline the covering part of this classification.

Let $\Lambda$ be the bounden quiver algebra $kQ/I$ defined by the quiver

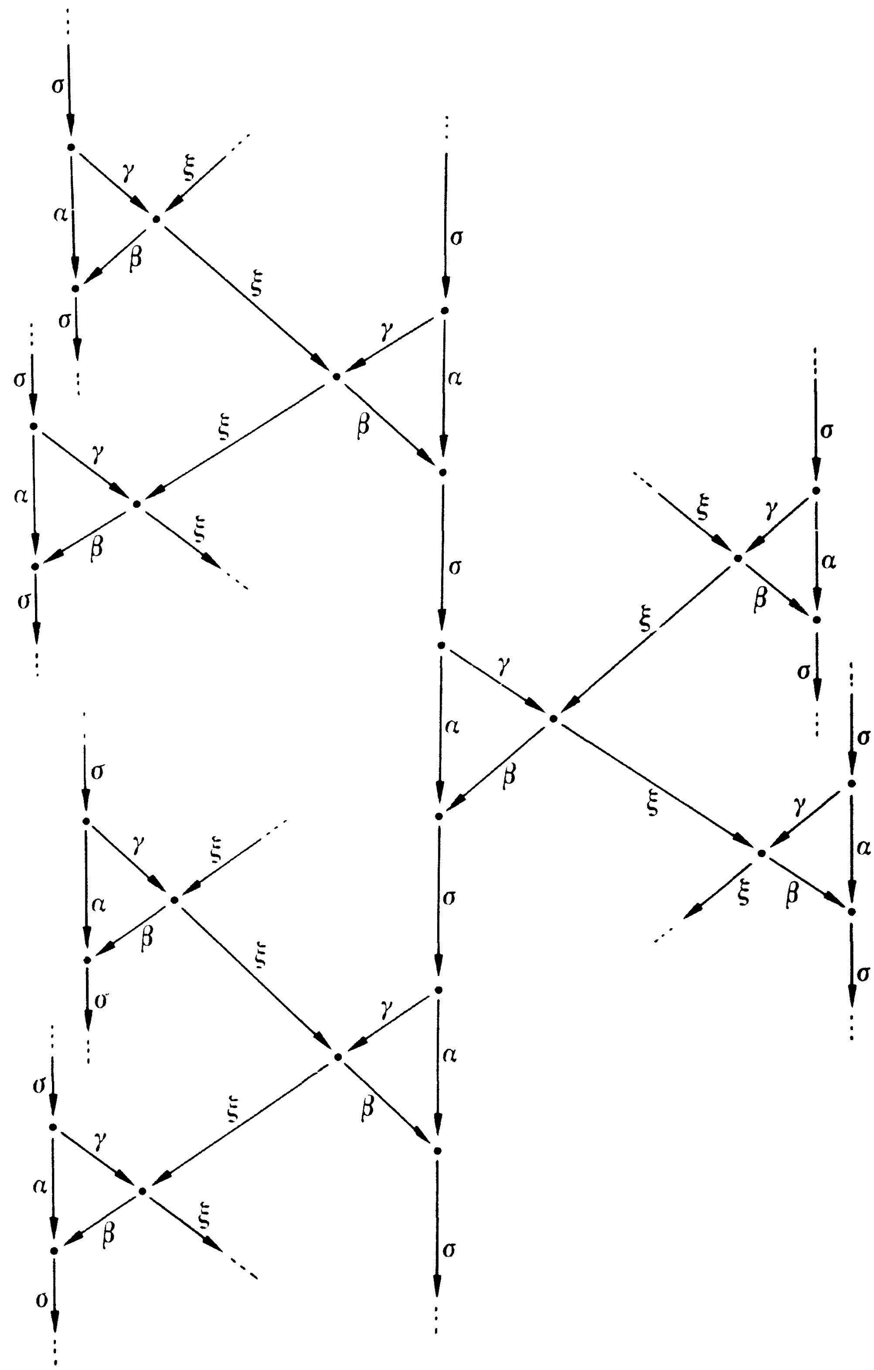


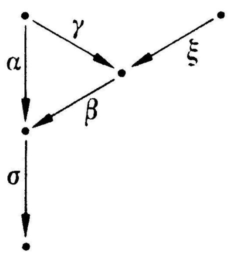

and the ideal $I$ generated by the elements $\alpha\sigma - \beta\gamma\sigma$, $\sigma\alpha$, $\xi^2$, $\xi\gamma$ and $\beta\xi$. It is easy to see that $\Lambda$ is a nonspecial biserial algebra and each indecomposable projective-injective $\Lambda$-module is uniserial. The fundamental group $\Pi(Q, I)$ of $(Q, I)$ is a free group in two generators and there is a universal Galois covering $F: R \to R/G$ with group $G = \Pi(Q, I)$, where $R = k\tilde{Q}/\tilde{I}$ is defined by the quiver $\tilde{Q}$ (page opposite) and the ideal $\tilde{I}$ generated by all elements of the form $\alpha\sigma - \beta\gamma\sigma$, $\sigma\alpha$, $\xi^2$, $\xi\gamma$ and $\beta\xi$. Observe that $k\tilde{Q}/\tilde{I}$ contains a finite full subcategory defined by the extended Dynkin quiver



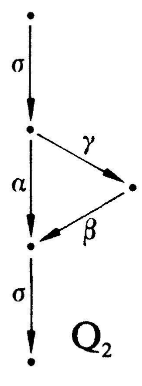

of type $\tilde{\mathbb{A}}_3$. If we identify the points of each triangle in $\tilde{Q}$ of the above form, we obtain an infinite tree. A convex subcategory $U$ of $R$ is called $t$-closed if it contains the whole of a triangle of the above form whenever it contains one of its vertices. A convex subcategory $V$ of $R$ is called admissible if its quiver does not contain a subquiver of the form $\tilde{Q}_i$ (page opposite). Observe that any admissible subcategory of $R$ is special biserial. Using one-point extensions and coextensions,

$\tilde{Q}$

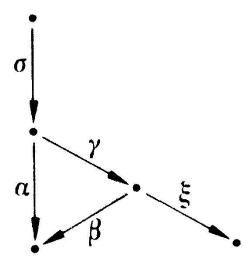


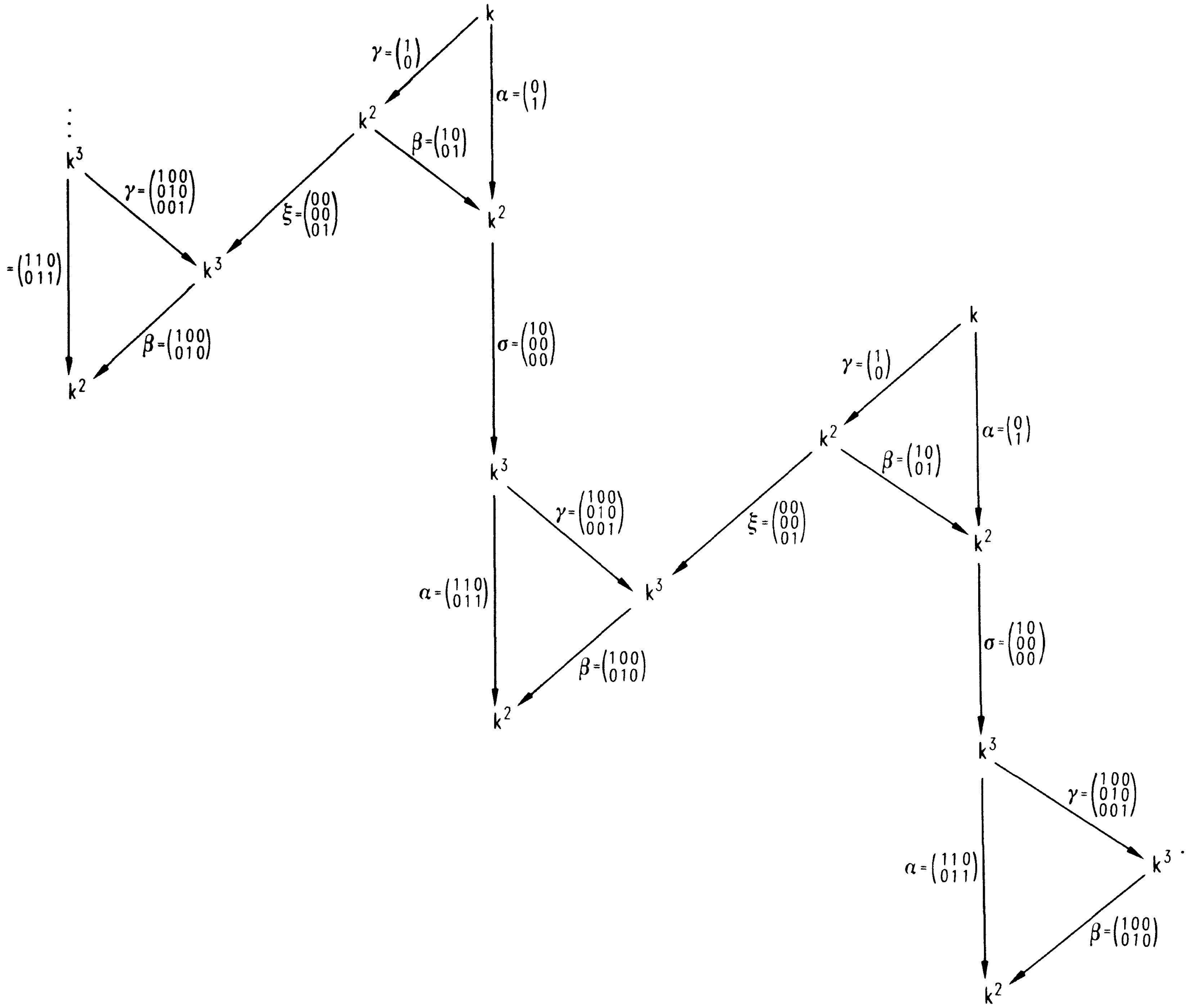

$Q_1$

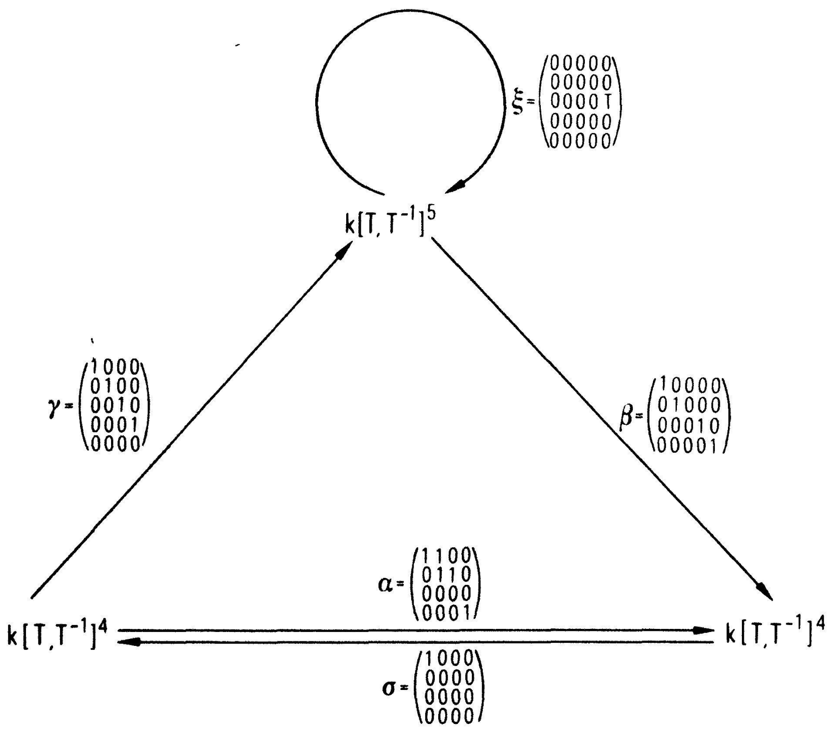



$Q_2$




$Q_3$





one can prove that the support of any indecomposable finite dimensional $R$-module is an admissible subcategory of $R$. Consequently, by Corollary 4.4 and Lemma 4.5, the support of any weakly-$G$-periodic $R$-module is an admissible subcategory $D$ with $G_D = \{g \in G; gD = D\}$ an infinite cyclic group. Denote by $\mathcal{S}$ the set of all $t$-closed admissible subcategories $D$ of $R$ with $G_D$ nontrivial and by $\mathcal{S}_0$ a set of representatives of the $G$-orbits in $\mathcal{S}$. It is easy to check that $\mathcal{S}$ is a separating family (3.1) in $R$ and then, according to Theorem 3.1, there is an equivalence of categories

$$\coprod_{D \in \mathcal{S}_0} (\operatorname{mod} D/G_D)/[\operatorname{mod}_1 D/G_D] \xrightarrow{\sim} (\operatorname{mod} R/G)/[\operatorname{mod}_1 R/G].$$

Moreover, since all $D/G_D$ with $D \in \mathcal{S}_0$ are special biserial, it is easy to deduce (c.f. 3.6) the following description of $(\operatorname{mod} R/G)/[\operatorname{mod}_1 R/G]$. Let $W$ be a set of representatives of the $G$-orbits in $\operatorname{Ind} R/\cong$ of weakly-$G$-periodic $R$-modules with supports contained in some $D \in \mathcal{S}_0$, and $W_0$ a set of representatives of isoclasses in $W$. Then for each $Y \in W_0$, $G_Y = \{g \in G; {}^gY \cong Y\}$ is an infinite cyclic group, $F_\lambda Y$ is a $K[T, T^{-1}]$-$R/G$-bimodule, and the functors

$$- \bigotimes_{K[T,T^{-1}]} F_\lambda Y\colon \operatorname{mod} K[T, T^{-1}] \to \operatorname{mod} R/G,\ Y \in W_0,$$

induce an equivalence of categories

$$\coprod_{Y \in W_0} \operatorname{mod} K[T, T^{-1}] \xrightarrow{\sim} (\operatorname{mod} R/G)/[\operatorname{mod}_1 R/G].$$

Here each $D \in \mathcal{S}_0$ is a support of infinitely nonisomorphic nonlinear weakly-$G$-periodic $R$-modules. The following $R$-module $Y$ is a typical example of such a module (see page opposite). Then the $k[T, T^{-1}]$-$\Lambda$-bimodule $F_\lambda Y$ is isomorphic to the bimodule (see page opposite).

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## Buchanzeigen

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Der vorliegende Band ist der dritte nunmehr erschienene der auf acht Bände veranschlagten IV. Serie der Gesamtausgabe (vorgängig der Registerband IV A 1; dann IV A 5, der die Korrespondenz Eulers mit Clairaut, d'Alembert und Lagrange enthält). Etwa 130 (3 lateinische, sonst französisch verfasste) Briefe von Euler an Maupertuis sind erhalten, leider nur 5 von M. an E. Eine flüssig verfasste Einleitung und ein hervorragender Kommentar von P. Costabel begleiten die Herausgabe. Wichtigster Inhalt ist die Diskussion um das Prinzip der kleinsten Wirkung. Der von Eduard Winter (†) eingeleitete Briefwechsel von Euler mit Friedrich II betrifft neben administrativen Fragen solche der Anwendung der Mathematik, u.a. auf Windmühlen, Pumpen, Glücksspiele, Finow-Kanal. Ein Sachverzeichnis verweist auf die Fülle der behandelten Probleme. Daher ist zu erwarten, dass der hervorragend edierte Band auf das Interesse breitester Kreise stossen wird.

J. J. Burckhardt

W. R. KNORR, **The Ancient Tradition of Geometric Problems**, Birkhäuser Verlag Boston–Basel–Stuttgart, 1986, 411 pp., SFr. 128.—

1. Sifting History from Legend 1 – 2. Beginnings and Early Efforts: The Duplication of the Cube; The Quadrature of the Circle; Problems and Methods – 3. The Geometers in Plato's Academy: Solutions of the Cube-Duplication; Geometric Methods in the Analysis of Problems; Efforts toward the Quadrature of the Circle; Geometry and Philosophy in the 4th Century – 4. The Generation of Euclid: A Locus-Problem in the Aristotelian Corpus; Euclid's Analytic Work; The Analysis of Conic Problems: Some Reconstructions; An Angle-Trisection via "Surface-Locus"; Euclid's Contribution to the Study of Problems – 5. Archimedes – The Perfect Eudoxean Geometer: Circle-Quadrature and Spirals; Problem-Solving via Conic Sections; Problem-Solving via Neuses, An Anonymous Cube-Duplication; The Impact of Archimedes' Work – 6. The Successors of Archimedes in the 3rd Century: Eratosthenes; Nicomedes; Diocles; On the Curve called "Cissoid"; Dionysodorus, Perseus and Zenodorus; In the Shadow of Archimedes – 7. Apollonius – Culmination of the Tradition: Apollonius; Archimedes and Heraclides; Apollonius and Nicomedes; Apollonius and Euclid; Apollonius and Aristaeus; Origins and Motives of the Apollonian Geometry – 8. Appraisal of the Analytic Field in Antiquity: The Ancient Classifications of Problems; Problems, Theorems and the Method of Analysis; "... and many and the greatest sought, but did not find"; Epilogue.

R. G. DOUGLAS, C. M. PEARCY, B. SZ-NAGY, F. -H. VASILESCU and D. VOICULESCU (Managing Editor Gr. ARSENE), **Advances in Invariant Subspaces and Other Results of Operator Theory** – 9th Int. Conf. on Operator Theory, Timişoara and Herculane, Romania, (Operator Theory: Advances a. Applications Vol. 17), Birkhäuser Verlag Basel–Boston–Stuttgart, 1986, 375 pp., SFr. 88.–.

Contributions by E. Albrecht, J. Eschmeier and M. N. Neumann, F. Altomare, C. Apostol, H. Bercovici, C. Foiaş and C. Pearcy, J. Barria and K. R. Davidson, Z. Ceauşescu and I. Suciu, B. Chevreau, T. Constantinescu, R. E. Curto, P. S. Muhly and T. Nakazi, J. Dazord, R. G. Douglas, H. Helson, P. Jonas, T. Kailath and A. M. Bruckstein, L. Kérchy, H. Langer, P. Legisa, B. Magajna, H. Neidhardt, M. Omladič, M. Putinar, C. J. Read, J. Rosenberg, K. Rudol, D. Timotin, D. Vuza and H. Widom.