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Links of surface singularities and CR space forms

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In memory of Peter Scherk

Abstract. We classify 3-dimensional compact locally homogeneous non-degenerate CR-manifolds ("CR space-forms"). Most of them are links of normal complex surface singularities, and we classify these singularities.

Introduction

In the last century, F. Klein studied the group Γ of orientation-preserving symmetries of the triangular tesselation of the 2-sphere with angles $\pi/2$, $\pi/3$, $\pi/5$, and of the hyperbolic plane with angles $\pi/2$, $\pi/3$, $\pi/7$. He showed that the algebra of Γ-automorphic forms has generators x, y, z satisfying $x^2+y^3+z^5=0$, respectively $x^2+y^3+z^7=0$. This connection was pursued in [D1], [M1], [N1], [N2], and [D2]. It can also be interpreted in terms of geometric structures on the link of the singularities. For example, the link of $x^2+y^3+z^5=0$ is Poincaré's dodecahedral space, which is a spherical space form.

Let (X, x) be a normal complex analytic surface singularity and let M be a link of (X, x). In [N3] it is shown that for (X, x) in one of the following four classes there is a one–one correspondence between geometric structures on M (i.e. locally homogeneous Riemannian metrics) and complex analytic structures on (X, x):

(i) quotient singularities,
(ii) simple elliptic singularities, or rational singularities which are quotients thereof,
(iii) quasihomogeneous hyperbolic singularities (i.e. quasihomogeneous singularities not in (i) or (ii)),
(iv) cusp singularities and quotients of them by involutions.

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The four geometries which occur are (notation will be explained below):

(i) S^3 (ii) N (iii) $(PSL)^{\sim}$ (iv) S

This relationship between the geometric structure on M and the complex analytic structure on (X, x), as described in [N3], is not intrinsic. On the other hand, any singularity link carries an intrinsic CR structure which determines the singularity analytically (cf. [S]). In terms of the CR structure we find the correspondence between the locally homogeneous structure on M and the analytic structure on X in a more natural form. In fact (Theorem 7.1) the singularities which have a CR space form (i.e. a locally homogeneous non-degenerate CR manifold) as a link are precisely the above together with an additional class:

(iii)′ non-quasihomogeneous hyperbolic singularities.

This new class is related to the Gorenstein singularities of type (iii) as explained below.

We now describe the above classes of singularities in more detail. The first three classes together comprise precisely the class of all normal quasihomogeneous surface singularities. According to Pinkham [P] (cf. [D1] and [D2] also) any such singularity can be constructed in the following way. Let D be $P_1\mathbb{C}$, $\mathbb{C}$ or the unit disc Δ in $\mathbb{C}$, and let Γ be a discrete, cocompact subgroup of $\mathrm{Aut}(D)$. Γ has a normal subgroup Γ' of finite index which acts freely on D. Let L be a Γ-line bundle on D such that the line bundle $\Gamma'\backslash L$ on $\Gamma'\backslash D$ is negative. The 0-section of $\Gamma'\backslash L$ can then be collapsed to a point to obtain a normal surface X' on which Γ/Γ' acts. Set $X(\Gamma, L) = (\Gamma/\Gamma')\backslash X'$. $X(\Gamma, L)$ has one singular point 0. The classes (i), (ii), and (iii) correspond to the cases $D = P_1\mathbb{C}$, $\mathbb{C}$, and Δ respectively.

The singularity $X(\Gamma, L)$ is Gorenstein if and only if $L^r \cong T_*\Delta$ as a Γ-bundle, for some $r > 0$ (cf. [D2], prop. 1). In this case there is a non-quasihomogeneous singularity, of the same topological type, associated with $(X(\Gamma, L), 0)$. We give a construction of it in 3.5 and 4.4. This is a "non-quasihomogeneous hyperbolic singularity" of the class (iii)′. In a future paper we identify it as the highest weight deformation of $X(\Gamma, L)$.

Finally, the singularities of class (iv) can be described as the singularities which have a resolution diagram of one of the two types in figure 1 (cf. [N3],

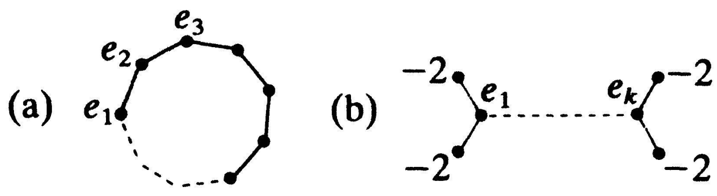

Figure 1

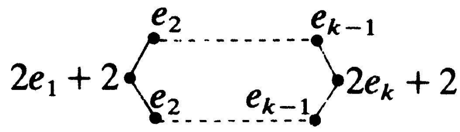

Figure 2

section 3). The weights are self-intersection numbers and all curves in the resolution are rational. To be a resolution diagram, the weights must satisfy $e_i \leq -2$ for all i and some $e_i < -2$. By Laufer [L] there is just one singularity with each of these resolution diagrams. (The cusp which double covers the singularity of figure 1b has the resolution diagram of figure 2.)

Our classification of 3-dimensional CR space forms is as follows ("$\tilde{}$" always means "universal cover"):

(i) $\Gamma \backslash S^3$, $\Gamma \subset U(2)$;

(ii) $\Gamma \backslash N$, where N carries the standard spherical CR structure, $\Gamma \subset \mathrm{Aut}_{\mathrm{CR}}(N) = \mathbb{C}^* \rtimes N$;

(iii) $\Gamma \backslash SU(1,1)\tilde{}$, where $SU(1,1)$ carries the standard spherical CR structure, $\Gamma \subset \mathrm{Aut}_{\mathrm{CR}}(SU(1,1)\tilde{})$, which is an extension of $PSU(1,1)$ by $\mathbb{R}$;

(iii)′ $\Gamma \backslash SU(1,1)\tilde{}$, where $SU(1,1)\tilde{}$ carries one of a 1-parameter family of aspherical CR structures, $\Gamma \subset SU(1,1)\tilde{}$;

(iv) $\Gamma \backslash S$,where S carries an aspherical CR structure, $\Gamma \subset (\mathbb{Z}/2) \rtimes S$;

(v) $\Gamma \backslash T_* P\tilde{}(s)$, where P is the elliptic, hyperbolic, or Euclidean plane, $T_*P(s)$ is the tangent circle bundle of radius s with its "natural" CR structure and $\Gamma \subset Is(P)\tilde{}$.

In each case, Γ is a discrete subgroup which acts with compact quotient. (i)–(iv) are links of the singularities (i)–(iv) respectively listed above. The "natural" CR structure on $T_*P(s)$ can be realized by looking at it as a G-orbit in $G_{\mathbb{C}}/K_{\mathbb{C}}$, where $G = Is(P)$ and K is the isotropy group of a point. We give a more explicit description in sections 5.1, 5.3, and 5.5. We also show (theorem 4.1) that $\Gamma \backslash T_*P(s)$ bounds an analytic variety if and only if Γ contains the centre of $Is^+(P)$. This gives examples of compact, 3-dimensional CR manifolds which do not bound analytic varieties and therefore, by [H–L], cannot be embedded in an affine space.

In section 1 we define a link of an isolated singularity as a CR manifold. In section 2 we discuss classifying spaces for non-degenerate CR structures on 3-dimensional real Lie groups; these are used in section 3 to classify simply-connected homogeneous 3-dimensional CR manifolds. CR space forms are discussed and classified in sections 4 and 5; the classification results are collected as a theorem in 5.6. In section 6 we show that the "non-quasihomogeneous hyperbolic singularities" are indeed not quasihomogeneous. In section 7 we use the results from section 5 to prove the classification of surface singularities having

CR space forms as links (Theorem 7.1) and in section 8 we discuss deformations of space forms.

We are grateful to E. Looijenga for making us aware of his description of the non-quasihomogeneous triangle singularities in [Lo], pp. 19, 20, which is essentially the same as the one we give in 3.5 and 5.3. We thank D. Burns and W. Goldman for useful conversations.

1. Links of isolated singular points

1.1. Let X be a complex analytic variety of dimension at least 2 with only one singular point x. Let $r: X \to [0, \infty)$ be a real analytic function on X such that $r^{-1}(0) = x$, and r is strictly plurisubharmonic on $X - \{x\}$. Then as in [Lo], 2.2, for ε sufficiently small, r has no critical points in $r^{-1}(0, \varepsilon]$. We define a link M of the germ (X, x) to be the boundary of such a neighbourhood $U = r^{-1}[0, \varepsilon)$ of x, i.e. $M = r^{-1}(\varepsilon)$. By extending the proof of 2.5 in [Lo] slightly, one sees that any two links of (X, x) are diffeomorphic.

Such a link M is a strongly pseudoconvex CR manifold. The CR structure on M determines the singularity (X, x) in the following sense.

THEOREM 1.1 (cf. [S]). *Let (X_1, x_1) and (X_2, x_2) be normal isolated singularities with links M_1 and M_2 respectively. Suppose $M_1 = \partial U_1$ and $M_2 = \partial U_2$ where U_1 and U_2 are neighbourhoods of x_1 and x_2 as above. Let $f: M \to M_2$ be a* CR *isomorphism. Then f extends to an analytic isomorphism $F: U_1 \to U_2$. In particular $(X_1, x_1) \cong (X_2, x_2)$.*

This result is a special case of the following:

THEOREM 1.2. *Let $X_1, X_2 \subset \mathbb{C}^N$ be normal varieties whose boundaries are C^∞, compact, connected, and pseudoconvex. Let $f: \partial X_1 \to \partial X_2$ be a* CR *covering map. Then f extends to a finite analytic map $F: X_1 \to X_2$.*

The proof of theorem 1.1 given in [S] extends to theorem 1.2 if one makes use of theorem 12.1′ of Harvey and Lawson ([H–L]).

2. Classifying spaces

2.1 DEFINITIONS. (i) A CR manifold M is homogeneous if its group of CR automorphisms, $\mathrm{Aut}_{\mathrm{CR}}(M)$, acts transitively on M.

(ii) M is locally homogeneous if $\tilde{M}$ is homogeneous.
(iii) A CR space form is a compact, non-degenerate, locally homogeneous CR manifold.

A weaker definition of local homogeneity would be that any two points in M have neighbourhoods which are isomorphic. This is not equivalent to (ii) (cf. section 8.1 below), unlike in Riemannian geometry.

A CR structure on a 3-dimensional manifold M may be given by a 2-dimensional subbundle $V \subset T_*M$ with a complex structure J on it. M is *strongly pseudoconvex*, or equivalently (in dimension 3), *non-degenerate* if

(*) for ξ, η linearly independent sections of V, $[\xi, \eta]$ is everywhere transverse to V.

In $T_*M \otimes \mathbb{C}$,

$$V \otimes \mathbb{C} = T_{1,0} \oplus T_{0,1}$$

where $T_{1,0}$ and $T_{0,1}$ are respectively the i and $-i$ eigenbundles of J. In particular $T_{1,0} = \bar{T}_{0,1}$. Conversely, a complex line bundle $L \subset T_*M \otimes \mathbb{C}$ such that $L \neq \bar{L}$ determines a bundle $V \subset T_*M$ with a complex structure on it. (*) is equivalent to

(**) if ζ is a nowhere-vanishing section of L, then $[\zeta, \bar{\zeta}]$ is everywhere transverse to $L \oplus \bar{L}$.

If M is a real hypersurface in a complex surface Y, then $T_{0,1}$ can also be described in the following way. Let J denote the complex structure on Y. Define a mapping of complex bundles

$$C: T_*M \otimes \mathbb{C} \to T_*Y$$

by

$$C(\xi + i\eta) = \xi + J\eta \qquad \xi, \eta \in T_pM, \qquad p \in M.$$

Then $T_{0,1} = \ker C$ and C is surjective.

It turns out that all strongly pseudoconvex, homogeneous, 3-dimensional CR manifolds are isomorphic to Lie groups with left invariant CR structures on them. For this reason we shall examine such CR manifolds first before going on to the actual classification of homogeneous manifolds in section 3.

Suppose G is a 3-dimensional real Lie group with Lie algebra $\mathfrak{g}$. A CR structure on G is then given by a line bundle $L \subset T_*G \otimes \mathbb{C} \cong G \times (\mathfrak{g} \otimes \mathbb{C}) = G \times \mathfrak{g}_{\mathbb{C}}$. So it corresponds to a mapping from G into the complex projective plane $P(\mathfrak{g}_{\mathbb{C}})$ and L is the pull-back of the canonical line bundle. In particular, left invariant CR structures on G correspond to points in $P(\mathfrak{g}_{\mathbb{C}})$. The CR structure corresponding to $[\zeta]$, $\zeta \in \mathfrak{g}_{\mathbb{C}}$ is non-degenerate if and only if $[\zeta, \bar{\zeta}]$ is not a linear combination of ζ and $\bar{\zeta}$. If $f \in \mathrm{Aut}(G)$, then f maps the CR structure corresponding to a point $z \in P(\mathfrak{g}_{\mathbb{C}})$ isomorphically to the CR structure corresponding to $f'(z)$, where $f' \in \mathrm{Aut}(\mathfrak{g})$ is the derivative of f. If G is the adjoint group of $\mathfrak{g}$, then the following theorem realizes G with these CR structures as hypersurfaces in $P(\mathfrak{g}_{\mathbb{C}})$.

THEOREM 2.1. *Let $\mathfrak{g}$ be a 3-dimensional real Lie algebra, and G its adjoint group. Suppose $z \in P(\mathfrak{g}_{\mathbb{C}})$ corresponds to a left invariant* CR *structure on G and $G_z = \{1\}$. Then G with this* CR *structure is isomorphic to the hypersurface $Gz \subset P(\mathfrak{g}_{\mathbb{C}})$.*

Proof. Let $L \subset \mathfrak{g}_{\mathbb{C}}$ be the line corresponding to z. For fixed $\zeta \in L$, $\zeta \neq 0$, we may identify $T_zP(\mathfrak{g}_{\mathbb{C}}) = T_\zeta \mathfrak{g}_{\mathbb{C}}/L = \mathfrak{g}_{\mathbb{C}}/L$. Then, since $T_\zeta(G\zeta) = ad\zeta(\mathfrak{g})$,

$$T_z(Gz) = (ad\zeta(\mathfrak{g}) + L)/L$$

and the diffeomorphism $\Pi : G \to Gz$ has derivative $\Pi' : \mathfrak{g} \to T_z(Gz)$ given by

$$\Pi'(\eta) = (ad\zeta(\eta) + L)/L \qquad \eta \in \mathfrak{g}.$$

Thus the mapping $C : \mathfrak{g}_{\mathbb{C}} \to T_zP(\mathfrak{g}_{\mathbb{C}})$ is simply $\Pi' \otimes \mathbb{C}$, i.e.

$$C(\eta) = (ad\zeta(\eta) + L)/L \qquad \eta \in \mathfrak{g}_{\mathbb{C}},$$

whose kernel is L. Therefore the orbit mapping Π is a CR isomorphism.

2.2. A left invariant non-degenerate CR structure on G induces a non-degenerate, locally homogeneous CR structure on any quotient $\Gamma \backslash G$ by a discrete subgroup. For any $g \in G$, right translation by g on $\Gamma \backslash G$ maps the CR structure corresponding to $z \in P(\mathfrak{g}_{\mathbb{C}})$ to the one corresponding to $Ad(g^{-1})z$. Thus CR structures on $\Gamma \backslash G$ corresponding to points in $P(\mathfrak{g}_{\mathbb{C}})$ lying in the same G-orbit are isomorphic.

2.3. There are many parallels between CR geometry and Riemannian geometry. The CR manifolds which correspond to flat Riemannian manifolds are those locally isomorphic to $S^3 \subset \mathbb{C}^2$.

DEFINITIONS. (i) A CR manifold locally isomorphic to S^3 is called spherical.

(ii) A locally homogeneous, non-degenerate CR manifold which is not spherical is called aspherical.

Spherical CR manifolds can also be characterized as those whose local automorphism groups have dimension greater than 3 (cf. [C], section 81). In [C] Cartan defines a scalar curvature R for any non-degenerate hypersurface M in $\mathbb{C}^2$. He proves that R vanishes identically if and only if M is spherical. (The definition of R and most of his results about it extend to any non-degenerate 3-dimensional CR manifold).

For a spherical manifold M, Burns and Shnider ([B–S], section 1) define a development map $\tilde{M} \to S^3$. They show that if M is locally homogeneous, then the image $\bar{M}$ in S^3 is a homogeneous domain and the map is a covering of $\bar{M}$. $\mathrm{Aut}_{\mathrm{CR}}(\bar{M})$ is a subgroup of $\mathrm{Aut}_{\mathrm{CR}}(S^3) \cong PSU(2, 1)$.

For aspherical manifolds the situation is quite different. Cartan proves the following ([C], section 77).

THEOREM 2.2. *Let M be a homogeneous, aspherical, 3-dimensional* CR *manifold. Then* $\dim \mathrm{Aut}_{\mathrm{CR}}(M) = 3$.

So if M is simply connected, then M is isomorphic to $G = \mathrm{Aut}_{\mathrm{CR}}(M)_0$ with a left invariant CR structure on it.

To classify homogeneous aspherical CR manifolds, we must determine when two such structures on a 3-dimensional Lie group G are isomorphic. Let $z_1, z_2 \in P(\mathfrak{g}_{\mathbb{C}})$ represent two such structures. Suppose f is a diffeomorphism of G mapping the first isomorphically to the second. We can assume that $f(1) = 1$. Then, as is shown in section 84 of [C], f must preserve the fundamental invariants of the CR structures, which include the structure constants of $\mathfrak{g}$. Therefore $f' \in \mathrm{Aut}(\mathfrak{g})$ and $z_2 = f'(z_1)$. So to classify aspherical CR structures on G, we need only determine the orbits of $\mathrm{Aut}(\mathfrak{g})$ in $P(\mathfrak{g}_{\mathbb{C}})$.

For aspherical CR structures, theorem 2.1 can therefore be strengthened.

THEOREM 2.1. *Let $\mathfrak{g}$ be a real 3-dimensional Lie algebra, and G its adjoint group. Suppose $z \in P(\mathfrak{g}_{\mathbb{C}})$ corresponds to a non-degenerate, left invariant, aspherical* CR *structure on G and $G_z = \{1\}$. Then G with this* CR *structure is isomorphic to the hypersurface $Gz \subset P(\mathfrak{g}_{\mathbb{C}})$. Two such hypersurfaces are isomorphic if and only if there is an $f \in$* Aut ($\mathfrak{g}$) *such that the mapping induced by f on $P(\mathfrak{g}_{\mathbb{C}})$ maps the one to the other.*

3. Classification of homogeneous CR manifolds

3.1. Cartan classified homogeneous real hypersurfaces in $\mathbb{C}^2$ ([C], 63 bis). His classification was extended to higher dimensions in the spherical case by Burns and Shnider ([B–S]).

As mentioned in 2.3 all simply-connected, homogeneous, spherical manifolds are covering spaces of homogeneous domains in S^3. Here is a list of the five possibilities, together with a description of their automorphism groups (cf. [B–S], p. 229).

(i) $D = S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\} \cong SU(2)$;
$\mathrm{Aut}_{\mathrm{CR}}(D) \cong PSU(2, 1)$

(ii) $D = S^3 - \{(1, 0)\} \cong N$, the Heisenberg group; (A)
$\mathrm{Aut}_{\mathrm{CR}}(D) \cong \mathbb{C}^* \rtimes N$ (cf. [B–S], p. 236)

(iii) $D = (S^3 - \{z_2 = 0\}) \cong T_*\Delta(1) \cong PSU(1, 1)$; $\tilde{D}$ is an infinite cyclic cover of D; (C)
$1 \to \mathbb{R} \to \mathrm{Aut}_{\mathrm{CR}}(\tilde{D}) \to PSU(1, 1) \to 1$ (cf. [B–S], p. 234)

(iv) $D = (S^3 - (S^3 \cap \mathbb{R}^3)) \cong PSU(1, 1)$;
$\tilde{D}$ is an infinite cyclic cover of D; (K′)
$\mathrm{Aut}_{\mathrm{CR}}(\tilde{D}) \cong SO(2, 1)^{\tilde{}} \cong \mathbb{Z}/2 \rtimes PSU(1, 1)^{\tilde{}}$ (cf. [C], 59)

(v) $D = \{(z_1, z_2) \in \mathbb{C}^2 \mid \operatorname{im} z_2 = |z_1|^2, \operatorname{im} z_1 > 0\}$; (E)
$\mathrm{Aut}_{\mathrm{CR}}(D) \cong \mathbb{R} \rtimes \mathbb{R}^2$ with $t \in \mathbb{R}$ acting by $t(x, y) = (e^t x, e^{2t} y)$.

(A), (C), (E), and (K′) are Cartan's notations. (v) does not admit any compact quotients ([B–S], 5.5). In 3.5 we shall make explicit the two spherical CR structures on $PSU(1, 1)$ in (iii) and (iv).

3.2. To classify simply-connected, homogeneous, aspherical CR manifolds we shall use section 2 rather than follow Cartan's approach. It is easy to obtain his equations from theorem 2.1′. A Lie group admits compact quotients only if its Lie algebra is unimodular (cf. e.g. [M2], p. 317 ff.). The 3-dimensional real unimodular Lie algebras are

(i) $\mathbb{R}^3$
(ii) $\mathfrak{su}(2)$; (L)
(iii) $\mathfrak{n}$, the Heisenberg algebra; (A)
(iv) $\mathfrak{su}(1, 1)$; (K)
(v) $\mathfrak{s}$, the algebra of infinitesimal isometries of $\mathbb{R}^2$ with a non-degenerate indefinite bilinear form; (E) with $m = -1$
(vi) $\mathfrak{s}'$, the algebra of infinitesimal isometries of $\mathbb{E}^2$; (H) with $m = 0$.

Again (A), (E), (H), (K), (L) are Cartan's notations. (i) clearly admits no non-degenerate left invariant CR structures. In 3.3–3.7 we shall determine up to isomorphism all left invariant CR structures, spherical and aspherical, on the simply-connected groups corresponding to (ii)–(vi).

3.3. $\mathfrak{su}(2)$

Left invariant CR structures on $SU(2)$ correspond to non-real points in $P(\mathfrak{su}(2)_{\mathbb{C}})$. Choosing a basis of $\mathfrak{su}(2)$ gives an isomorphism $P(\mathfrak{su}(2)_{\mathbb{C}}) \cong P_2(\mathbb{C})$.

We take a basis $\{\xi_1, \xi_2, \xi_3\} \subset \mathfrak{su}(2)$ such that

$$[\xi_1, \xi_2] = \xi_3, \quad [\xi_2, \xi_3] = \xi_1, \quad [\xi_3, \xi_1] = \xi_2.$$

With respect to this basis, the Killing form is the diagonal form $2(x_1y_1 + x_2y_2 + x_3y_3)$. So we can identify the adjoint action of $SU(2) = \mathrm{Aut}\,(\mathfrak{su}(2))$ on $\mathfrak{su}(2)$ or $P(\mathfrak{su}(2)_{\mathbb{C}})$, with the standard action of $SO(3)$ on $\mathbb{R}^3$ or $P_2(\mathbb{C})$.

To identify the orbits of $SO(3)$ in $P_2(\mathbb{C})$, we define a mapping $\Phi: T_*P_2(\mathbb{R}) \to P_2(\mathbb{C})$ by

$$\Phi(x, y) = [x_1 + iy_1, x_2 + iy_2, x_3 + iy_3] \in P_2(\mathbb{C}),$$
$$x = (x_1, x_2, x_3), \qquad y = (y_1, y_2, y_3), \qquad |x| = 1, \qquad x \cdot y = 0.$$

It is easy to check that Φ is onto. Moreover, $z = [x + iy]$ lies in the image of the zero-section if and only if $z = \bar{z}$, in which case the CR structure corresponding to z is degenerate. Φ is $SO(3)$-equivariant, and the orbits of $SO(3)$ in $T_*P_2(\mathbb{R})$ are the circle bundles. Therefore the orbits of $SO(3)$ in the image of Φ are the orbits of the points $z_s = [1, is, 0]$, $s \geq 0$. For $s > 0$ these points correspond to non-degenerate CR structures. Since z_s and $z_{1/s}$ lie in the same orbit, we can in fact take $0 < s \leq 1$. Using the formulae in [C], section 82, we find that the scalar curvature R of the CR structure corresponding to z_s is $\frac{3}{8}(s^2 - 1/s^2)$. This shows that for $0 < s < 1$, these *CR* structures are not isomorphic, and correspond to the possible left invariant aspherical CR structures on $SU(2)$. The orbit of z_1 is the conic $z \cdot z = 0$. It represents the isomorphism class of the standard spherical CR structure on S^3 (cf. also [D–G]). We denote by M_s, $SU(2)$ with the left invariant CR structure corresponding to z_s and by $\bar{M}_s$, the orbit of z_s. Notice that Φ is an embedding of the disc bundle of radius s onto a domain $X_s \subset P_2(\mathbb{C})$, for $s \leq 1$. This provides these disc bundles with a "natural" complex structure, and the circle bundles with a "natural" CR structure.

3.4. $\mathfrak{n}$

It turns out that all non-degenerate left invariant CR structures on N are isomorphic to the spherical one 3.1(ii). To see this, we take a basis $\{\xi_1, \xi_2, \xi_3\}$ of $\mathfrak{n}$ such that

$$[\xi_1, \xi_2] = \xi_3, \quad [\xi_3, \xi_1] = [\xi_3, \xi_2] = 0.$$

Then $z = [\zeta] \in P(\mathfrak{n}_{\mathbb{C}})$ with $\zeta = a_1\xi_1 + a_2\xi_2 + a_3\xi_3$ corresponds to a non-degenerate

CR structure if and only if

$$[\zeta, \bar{\zeta}] = (a_1\bar{a}_2 - \bar{a}_1 a_2)\xi_3 \neq 0.$$

This choice of basis gives

$$\operatorname{Aut}(\mathfrak{n}) = \left\{ \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & d \end{pmatrix} \in GL(3, \mathbb{R}) \mid d = a_{11}a_{22} - a_{12}a_{21} \neq 0 \right\}.$$

The spherical CR structure in 3.1 corresponds to the point $z = [1, i, 0] \in P_2(\mathbb{C})$. But the orbit of z is

$$\begin{aligned} X &= \{[a_1, a_2, a_3] \in P_2(\mathbb{C}) \mid \bar{a}_1 a_2 - a_1\bar{a}_2 \neq 0\} \\ &= \{[1, a_2, a_3] \mid \operatorname{im} a_2 \neq 0\}. \end{aligned}$$

Thus all non-degenerate CR structures are isomorphic to the standard one.

The following remark will be used in our discussion of compact quotients in section 5.2. The image of N in $\operatorname{Aut}(\mathfrak{n})$ is

$$\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & 1 \end{pmatrix} \mid a, b \in \mathbb{R} \right\}.$$

So the orbits of N in X are affine lines; in particular, the N-orbit of $[1, i, 0]$ is $\{[1, i, a + bi]\}$.

3.5. $\mathfrak{su}(1, 1)$

For $\xi, \eta \in \mathfrak{su}(1, 1)$ let $\xi \cdot \eta$ be the Killing form. For $\zeta \in \mathfrak{su}(1, 1)_{\mathbb{C}}$, ζ not real, we have that the Hermitian form $\zeta \cdot \bar{\zeta}$ is degenerate, indefinite and non-degenerate, or definite on the plane generated by ζ and $\bar{\zeta}$ according to whether its discriminant $D(\zeta) = (\zeta \cdot \bar{\zeta})^2 - |\zeta \cdot \zeta|^2$ is $=0$, <0, or >0.

These conditions have the following geometric significance for the point $z = [\zeta] \in P(\mathfrak{su}(1, 1)_{\mathbb{C}})$. Corresponding to the three cases, the line $z\bar{z}$ joining z and $\bar{z}$ is either tangent to the conic $C = \{z \mid z \cdot z = 0\}$, meets C in two distinct real points, or meets C in two distinct complex conjugate points. We set $X' = \{z = [\zeta] \mid D(\zeta) < 0\}$; $X'' = \{z = [\zeta] \mid D(\zeta) > 0\}$. X'' has two components which are conjugate to each other.

Suppose the Hermitian form is degenerate on the plane spanned by ζ and $\bar{\zeta}$. Since $[\zeta, \bar{\zeta}] \cdot \zeta = [\zeta, \bar{\zeta}] \cdot \bar{\zeta} = 0$, $[\zeta, \bar{\zeta}]$ must lie in this plane. Thus the CR structure given by the line $[\zeta]$ will be degenerate. Hence the non-degenerate CR structures correspond to points in X' and X''.

We want to describe the orbits of $SU(1, 1)$ in $P(\mathfrak{su}(1, 1)_{\mathbb{C}})$. Let $\{\xi_1, \xi_2, \xi_3\}$ be a basis of $\mathfrak{su}(1, 1)$ such that

$$[\xi_1, \xi_2] = -\xi_3, \qquad [\xi_2, \xi_3] = \xi_1, \qquad [\xi_3, \xi_1] = -\xi_2.$$

With respect to this basis, the Killing form is diagonal: for $\xi = x_1\xi_1 + x_2\xi_2 + x_3\xi_3$, $\eta = y_1\xi_1 + y_2\xi_2 + y_3\xi_3$, $\xi \cdot \eta = -2x_1y_1 + 2x_2y_2 + 2x_3y_3$. So the adjoint action of $SU(1, 1)$ can be identified with the standard action of $G = SO(2, 1)_0$ on $\mathbb{R}^3$, where $\mathbb{R}^3$ carries the bilinear form above. We first discuss the orbits of G in X'.

The tangent bundles of the hyperbolic plane $\mathbb{H}$ is given by

$$T_*\mathbb{H} = \{(x, y) \mid x, y \in \mathbb{R}^3, x \cdot x = -1, x_1 > 0, x \cdot y = 0\}.$$

Define $\Phi: T_*\mathbb{H} \to P_2(\mathbb{C})$ by $\Phi(x, y) = z = [x + iy]$. Φ is an open embedding. This gives $T_*\mathbb{H}$ a "natural" complex structure and the circle bundles in $T_*\mathbb{H}$ a "natural" CR structure. Φ is G-equivariant and maps $T_*\mathbb{H} - \{\text{zero-section}\}$ onto X'. Therefore the orbits of G in X' are the orbits $\bar{M}'_s$ of the points $z_s = [1, is, 0]$, $s \geq 0$. The CR structures corresponding to the points z_s for $s > 0$ are non-degenerate and therefore all the points in X' give non-degenerate CR structures. The curvature of the CR structure corresponding to z_s is $\frac{3}{8}(s^2 - 1/s^2)$. So for $s \neq 1$, the orbits of the z_s represent aspherical CR structures on $SU(1, 1)$. By theorem 2.1′ no two of them are isomorphic. (The automorphism group of $\mathfrak{su}(1, 1)$ is $O(2, 1)$, whose orbits on X' are the same as those of G). The orbit of z_1 lies in $\{z \mid z \cdot \bar{z} = 0\} \cong S^3$. It is clearly $S^3 - \{S^3 \cap P_2(\mathbb{R})\}$, which is just the spherical manifold (iv) in 3.1. $SU(1, 1)^{\sim}$ with the CR structure corresponding to z_s will be denoted by M'_s. Let X'_s be the image under Φ of the disc bundle of radius s.

To finish, we identify the orbits of G in X''. $z \in X''$ if and only if the line $z\bar{z}$ meets C in two distinct complex conjugate points. One such line is $z_1 = 0$, which meets C in the points $[0, 1, \pm i]$. Let X be the component of X'' containing $[0, 1, i]$. It is sufficient to find the orbits of G in X. The isotropy group of $z_1 = 0$ is isomorphic to $SO(2)$. Any non-real point on $z_1 = 0$ lies in the orbit of a point $z_s = [0, 1, is]$, $s > 0$, under this group. Therefore any point in X lies in the G-orbit $\bar{M}_s$ of some z_s. Since z_s and $z_{1/s}$ lie in the same orbit, we may take $0 < s \leq 1$. The curvature of the CR structure corresponding to z_s is $\frac{3}{8}(s^2 - 1/s^2)$. Thus for $0 \leqslant s \leq 1$, the points z_s correspond to non-isomorphic CR structures. The orbit $D = \bar{M}_1$ of z_1 is one of the half-planes in $C - C_{\mathbb{R}}$. It corresponds to the standard spherical CR structure on $SU(1, 1)$ ((iii) in 3.1). We denote $SU(1, 1)^{\sim}$ with the CR structure corresponding to z_s by M_s. We set $X_s = \bigcup_{s<t\leq 1} \bar{M}_t$, $0 \leq s < 1$. X_s is strongly pseudoconvex, and is G-diffeomorphic to a disc bundle over D.

3.6. $\mathfrak{s}$

Let $\{\xi_1, \xi_2, \xi_3\}$ be a basis of $\mathfrak{s}$ such that

$$[\xi_1, \xi_2] = 0, \qquad [\xi_3, \xi_1] = \xi_1, \qquad [\xi_3, \xi_2] = -\xi_2.$$

$z = [a_1\xi_1 + a_2\xi_2 + a_3\xi_3]$ corresponds to a degenerate CR structure on S if and only if $a_3 = 0$ or $z = [a_1\xi_1 + a_2\xi_2 + \xi_3]$ with a_1 or a_2 real. So the domain in $P_2(\mathbb{C})$ corresponding to non-degenerate structure is

$$\{[a_1, a_2, 1] \mid (\operatorname{im} a_1)(\operatorname{im} a_2) \neq 0\}.$$

On the other hand,

$$\operatorname{Aut}(\mathfrak{s}) \cong \left\{ \begin{pmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & a_{12} & a_{13} \\ a_{21} & 0 & a_{23} \\ 0 & 0 & -1 \end{pmatrix} \in GL(3, \mathbb{R}) \right\}.$$

so the domain above is just the orbit of $[i, i, 1]$ under $\operatorname{Aut}(\mathfrak{s})$, and all non-degenerate left invariant CR structures on S are isomorphic. In the chosen coordinates

$$S \cong \left\{ \begin{pmatrix} r & 0 & a \\ 0 & r^{-1} & b \\ 0 & 0 & 1 \end{pmatrix} \in GL(3, \mathbb{R}) \mid r > 0 \right\}.$$

Therefore the orbits of S are isomorphic to the hypersurfaces in $\mathbb{C}^2$ given by

$$(\operatorname{im} z_1)(\operatorname{im} z_2) = s, \qquad s \neq 0, \qquad \operatorname{im} z_1 > 0$$

or

$$(\operatorname{im} z_1)(\operatorname{im} z_2) = s, \qquad s \neq 0, \qquad \operatorname{im} z_1 < 0.$$

These are exchanged by an automorphism of $\mathfrak{s}$, so we need only consider the former, which we denote by M_s. For $s > 0$ we set

$$X_s = \{(z_1, z_2) \mid (\operatorname{im} z_1)(\operatorname{im} z_2) > s, \operatorname{im} z_1 > 0\}.$$

3.7. $\mathfrak{s}'$

Take a basis $\{\xi_1, \xi_2, \xi_3\}$ satisfying

$$[\xi_1, \xi_2] = 0, \qquad [\xi_3, \xi_1] = -\xi_2, \qquad [\xi_3, \xi_2] = \xi_1.$$

Then the domain of non-degenerate left invariant CR structures is

$$X = \{[a_1, a_2, 1] \in P_2(\mathbb{C}) \mid (\operatorname{im} a_1)^2 + (\operatorname{im} a_2)^2 \neq 0\}.$$

This is just the orbit of $(i, 0, 1)$ under

$$\operatorname{Aut}(\mathfrak{s}') \cong \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ -a_{12} & a_{11} & a_{23} \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & -a_{11} & a_{23} \\ 0 & 0 & 1 \end{pmatrix} \in GL(3, \mathbb{R}) \right\}.$$

Thus all non-degenerate, left invariant CR structures on $\tilde{S}'$ are isomorphic, where $S' = Is^+(\mathbb{E}^2)$,

$$S' \cong \left\{ \begin{pmatrix} \cos\vartheta & -\sin\vartheta & a \\ \sin\vartheta & \cos\vartheta & b \\ 0 & 0 & 1 \end{pmatrix} \in GL(3, \mathbb{R}) \right\}.$$

If $T_*\mathbb{E}^2$ is given its "natural" complex structure, then $\Phi: T_*\mathbb{E}^2 \to P_2(\mathbb{C})$ given by

$$\Phi(x, y) = [x_1 + iy_1, x_2 + iy_2, 1] \in P_2(\mathbb{C}), \qquad x = (x_1, x_2), \qquad y = (y_1, y_2)$$

is a holomorphic embedding. The non-degenerate orbits of S' are then the hypersurfaces

$$M_s = \{(a_1, a_2) \in \mathbb{C}^2 \mid (\operatorname{im} a_1)^2 + (\operatorname{im} a_2)^2 = s^2\}, \qquad s > 0.$$

4. Non-embeddable CR space forms

Theorem 10.4 of [H–L] shows that a compact, connected, pseudoconvex CR manifold which can be embedded in some $\mathbb{C}^N$ (as a CR manifold) bounds an irreducible variety with finitely many singularities. In this section we shall prove that the examples mentioned in the introduction do not bound analytic varieties.

Let P be the elliptic, hyperbolic, or Euclidean plane. Denote by $Is(P)$, respectively $Is^+(P)$, the group of isometries, respectively orientation preserving isometries, of P. Set $X = T_*P$ and for $s > 0$, let

$$X_s = \{\xi \in X \mid \|\xi\| < s\}, \qquad \bar{M}_s = \{\xi \in X \mid \|\xi\| = s\},$$

and let M_s be the universal cover of $\bar{M}_s$. In 3.3 (for $s \leq 1$), 3.5, and 3.7 respectively, we give embeddings of X_s in $P_2(\mathbb{C})$ which provide X_s with a natural complex structure, such that $\bar{M}_s$ is strongly pseudoconvex as the boundary of X_s and $P \subset X_s$ is totally real. $Is(P)$ acts analytically on X_s.

THEOREM 4.1. *Let $\Gamma \subset Is(P)^{\sim}$ be a discrete, cocompact subgroup which acts freely on M_s. Then $\Gamma \backslash M_s$ bounds an analytic variety if and only if Γ contains the centre of $Is^+(P)^{\sim}$.*

Proof. Let $\bar{\Gamma} \subset Is(P)$ be the projection of Γ. Then $\bar{\Gamma}$ acts freely on $X_s - P$ and $\bar{\Gamma} \backslash (X_s - P)$ is smooth. So the singular set of $\bar{\Gamma} \backslash X_s$ lies in $\bar{\Gamma} \backslash P$ which is totally real. Therefore $\bar{\Gamma} \backslash X_s$ has finitely many singularities. Since $\bar{\Gamma} \backslash X_s$ is relatively compact and its boundary $\bar{\Gamma} \backslash M_s$ is strongly pseudoconvex, $\bar{\Gamma} \backslash X_s$ is a Stein space. $\Gamma \supset Z(Is^+(P))$ if and only if the covering $f : \Gamma \backslash M_s \to \bar{\Gamma} \backslash \bar{M}_s$ is of degree 1. Assume that $\Gamma \backslash M_s$ does bound a variety Y. By theorem 1.2, f extends to a finite analytic mapping $F : Y \to \bar{\Gamma} \backslash X_s$. Since $\bar{\Gamma} \backslash X_s$ is Stein, the branch locus in $\bar{\Gamma} \backslash X_s$ is a finite set of points. Let $p \in \bar{\Gamma} \backslash X_s$ be a smooth point. Then p does not lie in the branch locus of F. For if $q \in F^{-1}(p)$, then the link of q is a covering space of the link of p, which is simply- connected, so q cannot be a branch point. It follows that

$$F : F^{-1}(\bar{\Gamma} \backslash (X_s - P)) \to \bar{\Gamma} \backslash (X_s - P)$$

is unbranched.

Let $D_t = \pi^{-1}(p)$, where $\pi : \bar{\Gamma} \backslash X_t \to \bar{\Gamma} \backslash P$, is the projection, $p \in \bar{\Gamma} \backslash P$, $0 < t \leq s$. Since $F^{-1}(\partial D_s)$ is a circle, $F^{-1}(\partial D_t)$ must be too, for all $t > 0$. On the other hand, if $p \in \bar{\Gamma} \backslash P$ is smooth, then for t small $F^{-1}(\partial D_t)$ will be a disjoint union of circles. So Y does not exist if $\deg f > 1$.

The manifolds $\Gamma \backslash M_s$ will be discussed further in 5.1, 5.3, and 5.5. It turns out that for $P = \mathbb{E}^2$, $\Gamma \backslash M_s$ is always isomorphic to $\bar{\Gamma} \backslash \bar{M}_s$.

5. Classification of CR space forms

5.1. $SU(2)$

In 3.3, non-degenerate, left invariant CR structures on $SU(2)$ were classified by the orbits of $SU(2)$ in $P(\mathfrak{su}(2)_{\mathbb{C}})$. The result was a 1-parameter family M_s, $0 < s \leq 1$, where M_1 is spherical and M_s, for $s < 1$, aspherical.

We first deal with the spherical structure $M_1 \cong S^3$. Suppose Γ is a discrete subgroup of $\mathrm{Aut}_{\mathrm{CR}}(S^3) \cong PSU(2, 1)$ which acts properly discontinuously on S^3.

Then Γ must be conjugate to a finite subgroup of $U(2)$ ([B–S], p. 233). The quotients $\Gamma\backslash S^3$ are precisely the links of the quotient singularities.

For $s<1$, $\mathrm{Aut}_{\mathrm{CR}}(M_s)\cong SU(2)$. Therefore a discrete cocompact subgroup $\Gamma\subset\mathrm{Aut}_{\mathrm{CR}}(M_s)$ must be a finite subgroup of $SU(2)$. If $\Gamma\supset\{\pm I\}$, then $\Gamma\backslash M_s$ is isomorphic to $\bar\Gamma\backslash\bar M_s$, where $\bar\Gamma=\Gamma/\{\pm I\}$. This bounds the Stein space $\bar\Gamma\backslash X_s$, which has cyclic quotient singularities in $\bar\Gamma\backslash P_2(\mathbb{R})$. If $\Gamma\not\supset\{\pm I\}$, i.e. if Γ is a cyclic group of odd order, then by theorem 4.1, $\Gamma\backslash M_s$ does not bound an analytic variety (cf. [R], p. 135 for the case $\Gamma=\{I\}$).

5.2. N

In 3.4 we showed that any non-degenerate left invariant CR structure on N can be mapped by an automorphism of N to the standard spherical structure corresponding to the point $[1,i,0]\in P_2(\mathbb{C})$. The automorphism group of this spherical structure on N is a semi-direct product: $\mathrm{Aut}_{\mathrm{CR}}(N)\cong\mathbb{C}^*\ltimes N$ (cf. [B–S], p. 236). $\mathbb{C}^*$ acts on N in the following way:

$$\text{for } re^{i\vartheta}\in\mathbb{C}^*,\ \begin{pmatrix}1&a&c\\0&1&b\\0&0&1\end{pmatrix}\in N,$$

$$r\begin{pmatrix}1&a&c\\0&1&b\\0&0&1\end{pmatrix}=\begin{pmatrix}1&ra&r^2c\\0&1&rb\\0&0&1\end{pmatrix},\qquad e^{i\vartheta}\begin{pmatrix}1&a&c\\0&1&b\\0&0&1\end{pmatrix}=\begin{pmatrix}1&a'&c\\0&1&b'\\0&0&1\end{pmatrix},$$

$$\text{where } \begin{pmatrix}a'\\b'\end{pmatrix}=\begin{pmatrix}\cos\vartheta&-\sin\vartheta\\\sin\vartheta&\cos\vartheta\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}.$$

If $\Gamma\subset\mathrm{Aut}_{\mathrm{CR}}(N)$ is a discrete cocompact subgroup, acting properly discontinuously on N, then (cf. [B–S], 5.6) it has the form $\Gamma=C\ltimes\Gamma'\subset U(1)\ltimes N$, where $\Gamma'=\Gamma\cap N$ is a normal subgroup of finite index in Γ and $C\cong\Gamma/\Gamma'$ is a cyclic group of rotations, of order 1, 2, 3, 4, or 6, leaving the lattice $\Gamma'/(Z(N)\cap\Gamma')$ in $N/Z(N)$ invariant.

Any discrete cocompact subgroup Γ' of N can be mapped by a CR automorphism to one of the form

$$\Gamma_{k,\tau}=\left\{\begin{pmatrix}1&a+bx&cy/k\\0&1&by\\0&0&1\end{pmatrix}\,\middle|\,a,b,c\in\mathbb{Z}\right\},\qquad \tau=x+iy\in\mathbb{C},\qquad k\in\mathbb{Z}^+.$$

therefore $\Gamma'\backslash N$ is CR isomorphic to $\Gamma_{k,\tau}\backslash N$ for some k, τ. $\Gamma_{k,\tau}\backslash N$ is isomorphic to the unit circle bundle of a line bundle with Chern class $-k$ over the elliptic curve $(\mathbb{Z}+\mathbb{Z}\tau)\backslash\mathbb{C}$. By definition, $\Gamma'\backslash N$ is a link of a simple elliptic singularity. For arbitrary $\Gamma\subset \mathrm{Aut}_{\mathrm{CR}}(N)$, we have that $\Gamma\backslash N\cong(\Gamma/\Gamma')\backslash(\Gamma'\backslash N)$. If $\Gamma/\Gamma'\neq\{1\}$, then this is the link of the rational singularity obtained from the appropriate simple elliptic singularity by taking its quotient by Γ/Γ'.

An alternative way of seeing this classification is the following. It is well known that a discrete cocompact subgroup Γ of N can be mapped by an automorphism of N to one of the form

$$\Gamma_k=\left\{\begin{pmatrix}1 & a & c/k\\ 0 & 1 & b\\ 0 & 0 & 1\end{pmatrix}\,\middle|\, a,b,c\in\mathbb{Z}\right\},\qquad k\in\mathbb{Z}^+.$$

In 3.4 we remarked that the orbits of N in $p(\mathfrak{n}_{\mathbb{C}})$ are lines parallel to $\{[1,i,a+bi]\}$, the orbit of $[1,i,0]$. As a slice transversal to this line at $[1,i,0]$, we can take $\Sigma=\{[1,-1/\tau,0]\mid \mathrm{im}\,\tau>0\}$. Any non-degenerate left invariant CR structure on N can be mapped to one corresponding to a point on Σ by an inner automorphism. Therefore, if N carries any such CR structure and $\Gamma'\subset N$ is discrete and cocompact, $\Gamma'\backslash N$ is CR isomorphic to $\Gamma_k\backslash N$ where N carries the CR structure corresponding to a point $[1,-1/\tau,0]$ in the slice Σ. Then $\Gamma_k\backslash N$ is again isomorphic to a circle bundle over the elliptic curve $(\mathbb{Z}+\mathbb{Z}\tau)\backslash\mathbb{C}$. (This curve can be described as $\mathbb{Z}^2\backslash\mathbb{R}^2$ with the complex structure given by $T_{0,1}=\mathbb{C}(1,-1/\tau)$). In particular, CR structures on $\Gamma'\backslash N$ corresponding to points in different N-orbits are in general not isomorphic.

5.3. $SU(1,1)$

In 3.5 it was shown that there are two 1-parameter families of left invariant non-degenerate CR structures on $SU(1,1)^{\sim}$, M_s, $0<s$, and M'_s, $0<s\leq 1$, corresponding to points in $P(\mathfrak{su}(1,1)_{\mathbb{C}})$ in the domains X and X' respectively. We first consider compact quotients of CR structures on $SU(1,1)^{\sim}$ coming from points in the domain X'. For any such CR structure, the automorphism group is $\mathbb{Z}/2\times SU(1,1)^{\sim}\cong SO(2,1)^{\sim}$. So let Γ be a discrete cocompact subgroup of $SO(2,1)^{\sim}$. Points in the same $SU(1,1)$ orbit $\bar{M}'_s$ give isomorphic CR structures on $\Gamma\backslash \mathrm{SU}(1,1)^{\sim}$.

The image $\bar{\Gamma}$ of Γ in $SO(2,1)$ is also discrete. $\bar{\Gamma}\backslash\bar{M}'_s$ bounds the Stein variety $\bar{\Gamma}\backslash X'_s$. It may have cyclic quotient singularities in $\bar{\Gamma}\backslash\mathbb{H}$. $\Gamma\backslash M'_s$ is a finite covering of $\bar{\Gamma}\backslash\bar{M}'_s$. For a description of the possible coverings see [N–R], theorem 3.1 (in that theorem $g+s$ should be $2g+s$). By theorem 4.1, if $\Gamma\backslash M'_s\not\cong\bar{\Gamma}\backslash\bar{M}'_s$, then $\Gamma\backslash M'_s$ does not bound an analytic variety.

Now suppose $SU(1, 1)^{\sim}$ carries the standard spherical CR structure M_1. M_1 is a principal $\mathbb{R}$-bundle over the unit disc Δ, and $\mathbb{R}$ acts by CR automorphisms. $\mathrm{Aut}_{\mathrm{CR}}(M_1)$ is a central extension of $PSU(1, 1)$ by $\mathbb{R}$ (cf. 3.1)

$$0 \to \mathbb{R} \to \mathrm{Aut}_{\mathrm{CR}}(M_1) \to PSU(1, 1) \to 1,$$

and $SU(1, 1)^{\sim} \cap \mathbb{R} = \mathbb{Z}$. Let $\Gamma \subset \mathrm{Aut}_{\mathrm{CR}}(M_1)$ be a discrete cocompact subgroup which acts properly discontinuously on $\bar{M}_1$. The image $\bar{\Gamma}$ of Γ in $PSU(1, 1)$ is also discrete (cf. [B–S], p. 235). Therefore $\Gamma \cap \mathbb{R} = r\mathbb{Z}$, for some $r \in \mathbb{Q}$. $Z(\Gamma)\backslash M_1$ is a circle bundle of a "negative" Γ-line bundle L on Δ, and thus $\Gamma\backslash M_1$ is a link of the quasihomogeneous hyperbolic singularity $(X(\Gamma, L), 0)$ (cf. Introduction).

$(X(\Gamma, L), 0)$ is Gorenstein if and only if $r \in \mathbb{Z}$ and $L^r \cong T_*\Delta$ (cf. [D2]), i.e. $\Gamma \subset SU(1, 1)^{\sim}$. For such Γ one can also construct a non-quasihomogeneous surface singularity. To do so, we consider compact quotients of CR structures on $SU(1, 1)^{\sim}$ coming from aspherical points in X. The automorphism group of such a CR structure is $SU(1, 1)^{\sim}$ itself, and the possible CR structures on a compact quotient $\Gamma\backslash SU(1, 1)^{\sim}$ are represented by the family $\Gamma\backslash M_s$, $0 < s < 1$. We want to show that the $\Gamma\backslash M_s$ are links of surface singularities.

Let $\bar{\Gamma} \subset PSU(1, 1)$ be the projection of Γ. Γ has a normal subgroup Γ' of finite index such that $\bar{\Gamma}' \subset \bar{\Gamma}$ is torsion free (cf. [M1], p. 184). Then $\bar{\Gamma}'\backslash X_s$ is a smooth surface with boundary $\bar{\Gamma}'\backslash \bar{M}_s$. We claim that there exists a smooth surface Y_s with boundary $\Gamma'\backslash M_s$ such that the covering map $f: \Gamma'\backslash M_s \to \bar{\Gamma}'\backslash \bar{M}_s$ extends to a finite analytic mapping $F: Y_s \to \bar{\Gamma}'\backslash X_s$. Indeed, f can be extended to a C^{∞} branched covering of $\bar{\Gamma}'\backslash X_s$, branched along $R = \bar{\Gamma}'\backslash D$. Since R is a complex analytic curve in $\bar{\Gamma}'\backslash X_s$, this cover can be given a complex analytic structure.

Since $\Gamma'\backslash M_s$ is strongly pseudoconvex, R can be blown down to a point, and Y_s/R is a Stein variety. The action of Γ/Γ' on $\Gamma'\backslash M_s$ extends to Y_s/R. So $Z_s = (\Gamma/\Gamma')\backslash(Y_s/R)$ is Stein variety with one singular point p. The singularities (Z_s, p) are still isomorphic for $s < 1$ and $\Gamma\backslash M_s$ is a link of this singularity. In section 6 we shall show that is is not quasihomogeneous. Singularities of this form we call *non-quasihomogeneous hyperbolic singularities.*

5.4. S

In 3.6 it was shown that all non-degenerate left invariant CR structures on S are isomorphic. The CR automorphism group is $\mathbb{Z}/2 \ltimes S$ (cf. [C], 47), where $\mathbb{Z}/2$ acts on M_s by exchanging z_1 and z_2. Let $\Gamma \subset \mathrm{Aut}_{\mathrm{CR}}(S)$ be a discrete, cocompact subgroup and let $\Gamma' = \Gamma \cap S$. It is well-known that Γ must be isomorphic to an extension of $\mathbb{Z}$ by $\mathbb{Z}^2$, where $1 \in \mathbb{Z}$ acts on $\mathbb{Z}^2$ by a hyperbolic element in $SL(2, \mathbb{Z})$.

Those points which lie in the same S-orbits give isomorphic CR structures on $\Gamma\backslash S$. So the possible non-degenerate locally homogeneous CR structures on $\Gamma\backslash S$ are represented by the hypersurfaces $\Gamma\backslash M_s \subset \Gamma\backslash \mathbb{H}^2$. In [H], section 2 (cf. also [K], p. 237) it is shown that $\Gamma\backslash X_s$ can be compactified by adding a point ∞. $\Gamma\backslash M_s$ is then a link of the singularity ∞. Since $\Gamma\backslash X_s$ does not admit a good $\mathbb{C}^*$ action, the links $\Gamma\backslash M_s$ are all distinct. The family of CR manifolds $\Gamma\backslash M_s$, $s>0$, can also be obtained with fixed s by deforming the subgroup $\Gamma \subset \mathrm{Aut}_{\mathrm{CR}}(S)$.

5.5. S'

All non-degenerate, left invariant CR structures on $\tilde{S}'$ are isomorphic to one another (cf. 3.7). The CR automorphism group is $Is(\mathbb{E}^2)^\sim \cong \mathbb{Z}/2 \ltimes \tilde{S}'$. Suppose $\Gamma \subset Is(\mathbb{E}^2)^\sim$ is a discrete cocompact subgroup which acts freely on $\tilde{S}'$. Then the projection $\bar{\Gamma}$ of Γ in S' is discrete as well and in fact $\Gamma\backslash\tilde{S}'$ is diffeomorphic to $\bar{\Gamma}\backslash S'$ ([R–V], 1.3). So we can restrict ourselves to discrete cocompact subgroups Γ of $Is(\mathbb{E}^2)$ which act freely on S'. These are the crystallographic groups which contain no reflections (up to similarity there are 7 types: three triangle groups plus four 1-parameter families). The different non-degenerate, locally homogeneous CR structures on $\Gamma\backslash S'$ are represented by the hypersurfaces $\Gamma\backslash M_s \subset \Gamma\backslash\mathbb{C}^2$, $s>0$. The Stein space $\Gamma\backslash\mathbb{C}^2$ has singularities of type A_2, A_3, A_4, or A_6 in $\Gamma\backslash\mathbb{R}^2$. Since $\Gamma\backslash\mathbb{C}^2$ does not admit a good $\mathbb{C}^*$ action, for fixed Γ the hypersurfaces $\Gamma\backslash M_s$ are in general distinct. This family of CR structures can also be obtained with fixed s by deforming the subgroup $\Gamma \subset Is(\mathbb{E}^2)$ by similarity transformations.

5.6. *Classification*

Collecting the results of 5.1–5.5 we get our classification of CR space forms.

THEOREM 5.1. *The 3-dimensional* CR *space forms are*

(i) $\Gamma\backslash S^3$, *with* $S^3 = M_1$, *the standard spherical structure*, $\Gamma \subset U(2)$ (5.1);
(ii) $\Gamma\backslash N$, *with* $\Gamma \subset \mathrm{Aut}_{\mathrm{CR}}(N)$ (5.2);
(iii) $\Gamma\backslash SU(1,1)^\sim$, *where* $SU(1,1)^\sim$ *carries the standard spherical* CR *structure* M_1, $\Gamma \subset \mathrm{Aut}_{\mathrm{CR}}(M_1)$ (5.3);
(iv) $\Gamma\backslash SU(1,1)^\sim$, *where* $SU(1,1)^\sim$ *carries the* CR *structure* M_s, $0<s<1$, $\Gamma \subset SU(1,1)^\sim$ (5.3);
(v) $\Gamma\backslash S$, *where* S *carries the* CR *structure* M_1, $\Gamma \subset \mathrm{Aut}_{\mathrm{CR}}(S)$ (5.4);
(vi) $\Gamma\backslash M_s$, $M_s = T_*P(s)^\sim$ *carrying the natural* CR *structure, where* $P = P_2(\mathbb{R})$, $\mathbb{H}$, *or* $\mathbb{E}^2$, *and* $0<s<1$, $0<s$, *or* $s=1$ *respectively*, $\Gamma \subset Is(P)^\sim$ (5.1, 5.3, 5.5).

If $M \to F$ is a Seifert fibration of a closed oriented 3-manifold M over a possibly non-orientable surface F with Seifert invariant $(g;b;$

$(\alpha_1, \beta_1), \ldots, (\alpha_k, \beta_k))$, we define

$$e(M \to F) = -b - \sum_{i=1}^{n} \beta_i/\alpha_i, \qquad x(M \to F) = x(F) - \sum_{i=1}^{n} (\alpha_i - 1)/\alpha_i.$$

Then the 3-manifolds M which admit CR space form structures in cases (i), (ii), and (iii) are precisely the manifolds admitting Seifert fibrations with F orientable and $e(M \to F) < 0$ and $x(M \to F) > 0$, $=0$, <0, respectively (cf. [N3]). The manifolds for cases (iv) and (vi) are also Seifert fibered, but are much more restricted (cf. [R–V] or [N–R], Theorem 3.1); case (vi) includes some manifolds Seifert fibered over non-orientable F. Case (v) gives all oriented manifolds which can be fibered over S^1 with fiber $S^1 \times S^1$ and monodromy $h \in SL(2, \mathbb{Z})$ satisfying $|\mathrm{trace}\,(h)| > 2$ and all manifolds double covered by these (twisted doubles of the orientation $[0, 1]$-bundle over the Klein bottle).

6. Non-quasihomogeneous hyperbolic singularities

In order to prove that the singularities (Z_s, p) with links $\Gamma \backslash M_s$, $s < 1$, of section 5.3 are not isomorphic to the quasihomogeneous hyperbolic singularities with links $\Gamma \backslash M_1$, we shall need the structure equations for the CR structures M_s. M_s corresponds to the point $[0, 1, is] \in P_2(\mathbb{C})$. As a representative in $\mathfrak{su}(1, 1)$, we take

$$\zeta_s = s^{-1/2}(\xi_2 + is\xi_3) \qquad 0 < s \leq 1$$

We have

$$[\zeta_s, \bar{\zeta}_s] = -2i\xi_1, \qquad [\xi_1, \zeta_s] = ia\zeta_s + ib\bar{\zeta}_s,$$
$$a = \frac{1}{2}\left(s + \frac{1}{s}\right), \qquad b = \frac{1}{2}\left(s - \frac{1}{s}\right).$$

Choose left invariant 1-forms α, ω_s, $\bar{\omega}_s$ on $SU(1, 1)^\sim$ dual to ξ_1, ζ_s, $\bar{\zeta}_s$. Then

$$d\alpha = 2i\omega_s \wedge \bar{\omega}_s, \qquad d\omega_s = (ia\omega_s - ib\bar{\omega}_s) \wedge \alpha.$$

Now suppose that (Z_s, p) is isomorphic to the corresponding quasihomogeneous singularity. Then for s close to 1, the $\mathbb{R}^+$ action gives the germ of an $\mathbb{R}^+$ action on Z_s. So for some $\varepsilon > 0$, there exists a complex structure J on

$Z=(\Gamma\backslash M_s)\times(-\varepsilon,\varepsilon)$ such that

(a) $\Gamma\backslash M_s\to\Gamma\backslash M_s\times\{0\}$ is a CR embedding, and

(b) ∂_t, $t\in(-\varepsilon,\varepsilon)$, is an infinitesimal biholomorphic mapping.

(a) and (b) imply that ζ_s is a $(1,0)$ vector field on Z. As a second $(1,0)$ vector field, we take one of the form

$$\eta=if_1\xi_1+J(f_1\xi_1)$$

where f_1 is real and nowhere 0. We choose f_1 so that

$$J(f_1\xi_1)=\partial_t+f_2\xi_1+g\zeta_s+\bar g\bar\zeta_s$$

with f_2 real. Thus

$$\eta=\partial_t+f\xi_1+g\zeta_s+\bar g\bar\zeta_s,\qquad f=f_2+if_1.$$

(b) implies that f and g do not depend on t. Now

$$\begin{aligned}[\zeta_s,\eta]&=(\zeta_sf)\xi_1+f[\zeta_s,\xi_1]+(\zeta_sg)\zeta+(\zeta_s\bar g)\bar\zeta_s+\bar g[\zeta_s,\bar\zeta_s]\\&=(\zeta_sf-2i\bar g)\xi_1+(\zeta_sg-iaf)\zeta_s+(\zeta_s\bar g-ibf)\bar\zeta_s.\end{aligned}$$

Therefore integrability implies that

$$\zeta_sf=2i\bar g,\qquad \zeta_s\bar g=ibf,$$

and thus

$$\zeta_s^2f=-2bf.$$

Now

$$\int_{\Gamma\backslash M_s}\zeta_s^2f\alpha\wedge\omega_s\wedge\bar\omega_s=-\int_{\Gamma\backslash M_s}d(\zeta_sf\alpha\wedge\bar\omega_s)=0$$

On the other hand,

$$\int_{\Gamma\backslash M_s}\zeta_s^2f\alpha\wedge\omega_s\wedge\bar\omega_s=-2b\int_{\Gamma\backslash M_s}f\alpha\wedge\omega\wedge\bar\omega_s\neq0$$

since $f_1=\operatorname{im}f$ vanishes nowhere and $b\neq0$ for $0<s<1$. Therefore the singularity (Z_s,p) is not quasihomogeneous.

7. Links of surface singularities

We now prove the first theorem in the introduction.

THEOREM 7.1. *A normal surface singularity* (X, x) *has a locally homogeneous link if and only if*

(i) (X, x) *is a quotient singularity, or*
(ii) (X, x) *is a simple elliptic singularity or a rational quotient thereof, or*
(iii) (X, x) *is a quasihomogeneous hyperbolic singularity, or*
(iv) (X, x) *is a non-quasihomogeneous hyperbolic singularity, or*
(v) (X, x) *is a cusp or a quotient of a cusp by an involution.*

The locally homogeneous links are the CR *space forms* (i)–(v) *respectively in theorem* 5.1.

Proof. Suppose (X, x) is a normal surface singularity with a locally homogeneous link. The classification of CR space forms in section 5 shows that (X, x) must be one of the singularities in the list above.

Conversely let (X, x) be one of the singularities in this list. If (X, x) is a cusp, then it has a link of type (v) in theorem 5.1 (cf. [H]). A non-quasi-homogeneous hyperbolic singularity has a link of type (iv) by definition. Suppose now that (X, x) is a quasihomogeneous hyperbolic singularity. We shall show that (X, x) has a canonical link which is a CR space form of type (iii).

Let R be a Riemann surface of genus $\geqslant 2$ with the standard metric. We have

$$T_*R(1) \cong \pi_1 R \backslash PSU(1, 1) \cong (\pi_1 R)_1 \backslash SU(1, 1)^\sim,$$

where $(\pi_1 R)_1 \subset SU(1, 1)^\sim$ is the extension of $\pi_1 R$ with $(\pi_1 R)_1 \cap Z(SU(1, 1)^\sim) = \mathbb{Z}$. More generally, the unit circle bundle of $(T_*R)^m$ for $m \in \mathbb{Z}$, $m > 0$ is isomorphic to $(\pi_1 R)_m \backslash SU(1, 1)^\sim$ where $(\pi_1 R)_m \subset \mathrm{Aut}_{\mathrm{CR}}(SU(1, 1)^\sim)$ is the extension of $\pi_1 R$ with $(\pi_1 R)_m \cap \mathbb{R} = (1/m)\mathbb{Z}$. Now suppose $L' \to R$ is a flat line bundle. L' is given by a character $x : \pi_1 R \to S^1$ which lifts to a character $x_m : (\pi_1 R)_m \to \mathbb{R}$. If we consider $\mathbb{R}$ as $Z(\mathrm{Aut}_{\mathrm{CR}}(SU(1, 1)^\sim))$, we can twist the embedding of $(\pi_1 R)_m$ with x_m in $\mathrm{Aut}_{\mathrm{CR}}(SU(1, 1)^\sim)$ to obtain a new embedding of $(\pi_1 R)_m$ in $\mathrm{Aut}_{\mathrm{CR}}(SU(1, 1)^\sim)$. Given an arbitrary negative line bundle $L \to R$, there exist a unique flat bundle L' and positive integers m and n such that

$$L^n \cong L' \otimes (T_*R)^m.$$

The n-fold covering $L \to L' \otimes (T_*R)^m$ determines a subgroup $(\pi_1 R)_{m/n} \subset (\pi_1 R)_m$

of index n such that the unit circle bundle $L(1)$ satisfies

$$L(1) \cong (\pi_1 R)_{m/n} \backslash SU(1, 1)^{\sim}.$$

Now a quasihomogeneous hyperbolic singularity is of the form $(\Gamma\backslash L/R, 0)$ where Γ is a finite group of automorphisms of L. The action of Γ lifts to T_*R. This gives an action of Γ on $L' \cong L^n \otimes (T_*R)^{-m}$. Because L' is uniquely determined, Γ must preserve the flat structure on L'. Therefore $L(1)$ is invariant under Γ and $\Gamma\backslash L(1)$, which is a link of $(\Gamma\backslash L/R, 0)$, is a CR space form of type (iii).

If (X, x) is a simple elliptic singularity, then $(X, x) \cong (L/R, 0)$ where R is an elliptic curve and L is a negative line bundle over R. L has a canonical metric such that $L(1)$ is a space form of type (ii) (cf. [B–S], p. 237). If Γ is a finite group of automorphisms of L, then Γ will preserve the metric, so that $(\Gamma\backslash L/R, 0)$ also has a link which is a space form of type (ii). The case (i) is straightforward.

8. Deformations

8.1. Local Homogeneity.

Our definition of a locally homogeneous CR manifold (2.1) raises the following question. Suppose any two points in M have neighbourhoods which are CR isomorphic. Is M then locally homogeneous? In this section we point out that if the curvature R of M is not 0, then the answer is yes, but that there exist spherical manifolds which are not locally homogeneous. On the other hand, we shall see in 8.2 that among spherical CR-manifolds, local homogeneity is preserved by deformation.

THEOREM 8.1. *Let M be a compact, connected, non-degenerate* CR *manifold. Suppose that any two points of M have neighbourhoods which are* CR *isomorphic, and that the curvature R of M is not* 0. *Then M is locally homogeneous, i.e. a space form.*

Proof. Since R is not 0, M has a canonical Riemannian metric (cf. [W]). With respect to this metric, the local CR automorphisms are isometries. Therefore M is a locally homogeneous Riemannian manifold, and $\tilde{M}$ is complete and homogeneous as a Riemannian manifold. Each local CR automorphism on $\tilde{M}$ extends to a global isometry. The subset of M on which this isometry is a CR mapping is analytic and contains an open set. Therefore the isometry is a CR automorphism. So $\mathrm{Aut}_{\mathrm{CR}}(\tilde{M}) \subset Is(\tilde{M})$ has dimension at least 3, and acts transitively on $\tilde{M}$.

Here are examples due to D. Burns of compact spherical M which are not locally homogeneous. Let $N^* = N - \{0\}$. N^* is simply connected, and

$\mathrm{Aut}_{\mathrm{CR}}(N^*) \cong \mathbb{C}^*$. So N^* is not homogeneous. Choose $z \in \mathbb{C}^*$ with $|z| \neq 1$ and let $\Gamma = \{z^n \mid n \in \mathbb{Z}\} \subset \mathbb{C}^*$. Then $M = \Gamma \backslash N^*$ is compact, in fact diffeomorphic to $S^1 \times S^2$. It is a hypersurface in a Hopf surface. Many other non-locally-homogeneous examples can be constructed by the observation in [B–S], p. 245, that connected sums can be formed in the category of manifolds with spherical CR structure.

It is not hard to show that among spherical CR manifolds, small deformations of "Hopf hypersurfaces" are still Hopf hypersurfaces and small deformations of connected sums are still connected sums. The next theorem gives the analogous result for space forms.

8.2. Deformation of space forms.

THEOREM 8.2. *Let M_0 be a space form and let M_t, $t \in T$ be a deformation of M_0 such that each* CR *manifold M_t is locally homogeneous in the weak sense (any two points have isomorphic neighbourhoods). Then all M_t for t near* 0 *are space forms.*

Proof. By the discussion in sections (5.1), (5.4), and (5.5), the theorem holds for space forms M_0 of type S^3, S, and S', so we will restrict ourselves to the other cases. If any M_t is aspherical, then it is a space form by Theorem 8.1; moreover, we may continuously deform the parameter s of theorem 5.1 to $s = 1$, thereby deforming the CR structure to a spherical space form structure. We may thus assume that all the M_t are spherical and M_0 is a space form of type N or $SU(1, 1)^{\sim}$.

For M_0 of type N, the result is proposition 5.10 of [B–S], who prove it by computing the infinitesimal deformations. It can also be seen more simply by observing that if one deforms the defining homomorphism $f: \Gamma \to \mathrm{Aut}_{\mathrm{CR}}(N) \subset PSU(2, 1)$ to a homomorphism $g: \Gamma \to PSU(2, 1)$, then the Zariski closure of the image of g will still be solvable and contained in a conjugate of $\mathrm{Aut}_{\mathrm{CR}}(N)$.

For M_0 of type $SU(1, 1)^{\sim}$ the desired result is theorem 4 of W. Goldman [G], which was generalized to higher dimensions by W. Goldman and J. Millson [G–M]. In this case it is not just a theorem about infinitesimal deformations; the second order obstruction to a deformation is used.

Remark. If one takes a fixed simply connected homogeneous CR manifold as "model", then our results show that the space of deformations of a CR space form M with this universal cover is obtained simply by deforming the subgroup $\Gamma \cong \pi_1(M)$ in $\mathrm{Aut}_{\mathrm{CR}}(\tilde{M})$. In cases (1), (ii), and (iii) of theorem 5.1 it is thus the "universal Jacobian" of the base orbifold of the Seifert fibering of M (i.e. the total space of the fibration over the moduli space of the orbifold with fibers the Jacobian varieties; cf. remarks following Theorem 5.1). In cases (iv) and (vi) of theorem 5.1 it is just the moduli space of the base orbifold, and it is $\mathbb{R}_+$ in case (v).

In the cases where M bounds a singularity (case (i) to (v) of theorems 5.1 and 7.1), this deformation space is the parameter space of singularities of the given type with link homeomorphic to M (i.e. different CR space forms with isomorphic universal covers bound different singularities) except in case (v), where the singularity is rigid and the $\mathbb{R}_+$ family corresponds to expanding and shrinking the link about the singularity. (Expanding and shrinking the link has no effect on the CR structure in the quasihomogeneous case and corresponds to varying the parameter s of theorem 5.1 in case (iv).)

Note that these parameter spaces of singularities are not in general complex analytic deformation spaces for the singularities; though they give complex analytic families of the resolved singularities, the resolutions cannot in general be simultaneously blown down. For example, in case (iii) the Gorenstein singularities correspond to a discrete set in the Jacobian of the base orbifold with a given structure. Since a deformation of a Gorenstein singularity is Gorenstein, other points in the Jacobian do not correspond to deformations of such a singularity.

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