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## Length of curves under conformal mappings

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### 1. Introduction

It is well known that for any homeomorphism  $f$  of the unit disk  $\mathbb{D}$  onto a domain  $\Omega$ , where  $f$  is ACL and  $\nabla f \in L^2(\mathbb{D})$ ,  $f^{-1}(\Omega \cap L)$  has finite length for almost all rectifiable curves  $L$ . Suppose now that  $f$  is analytic, and let  $\lambda(E)$  denote the Hausdorff linear measure of a set  $E$ . Hayman and Wu [8] proved that for any line  $L$

$$\lambda(f^{-1}[\Omega \cap L]) < A, \tag{1}$$

for some absolute constant  $A$ . This was generalized by Garnett, Gehring and Jones [7] who gave conditions on a rectifiable Jordan curve in order that (1) holds for all  $\Omega$  as above. It is necessary that  $L$  satisfy a regularity condition introduced by Ahlfors, i.e. there is a constant  $c_1$ :

$$\lambda[L \cap \{|\xi - w| < r\}] \leq c_1 r \tag{2}$$

for all  $\xi \in L$  and  $r > 0$ . Garnett, Gehring and Jones conjectured that (1) could fail for a regular quasicircle, i.e.  $L$  satisfies (2) together with

$$|z_1 - z_2| > c_2 \min_i \text{dia}(\gamma_i) \tag{3}$$

for any  $z_1, z_2 \in L$  where  $\gamma_1, \gamma_2$  are the two components of  $L \setminus \{z_1, z_2\}$ . In fact we show that the example suggested in [7] cannot work. A curve  $L$  is called quasismooth (or chord-arc) if there is a constant  $M > 0$  such that for any  $z_1, z_2 \in L$  we have

$$\min_{i=1,2} \lambda(\gamma_i) \leq M |z_1 - z_2|, \tag{4}$$

see Jerison and Kenig [9], Pommerenke [12]. Actually (2) and (3) are equivalent to (4). We prove:

**THEOREM 1.** *For any quasismooth curve  $L$  and any simply connected domain  $\Omega$  with Riemann mapping  $f$*

$$\lambda[f^{-1}(\Omega \cap L)] \leq A < \infty,$$

where  $A$  depends only on the chord arc constant  $M$ .

We conjecture that (2) is a necessary and sufficient condition on  $L$  in order that (1) hold for all conformal maps.

Next we consider the case of the universal covering mapping  $f$  of a multiply connected planar domain. Flinn [5] had obtained the following theorem: suppose that  $\Omega$  is a hyperbolic planar domain and one component of  $\mathbb{C} \setminus L$  is contained in  $\Omega$ . Then if  $l$  is one component of  $f^{-1}(\Omega \cap L)$  we have  $\lambda(l) < \infty$ . On the other hand if  $\Omega = \mathbb{D} \setminus E$  where  $E \subset (0, 1)$  is a closed set of zero logarithmic capacity then Belna, Cohn, Piranian and Stephenson [3] proved that there are circles  $L$  which do not satisfy (1).

Suppose that  $G$  is the Fuchsian group of Möbius transformations  $T: \mathbb{D} \rightarrow \mathbb{D}$  which represents the cover group for  $\Omega$ . The Dirichlet fundamental region  $\mathcal{D}$  for  $G$  is

$$\mathcal{D} = \{z \in \mathbb{D} : |T'(z)| < 1, \forall T \in G \setminus \{I\}\}$$

We say that  $\mathcal{D}$  is of finite length type if

$$\sum_G \lambda(\partial T\mathcal{D}) < \infty$$

**THEOREM 2.** *For any hyperbolic planar domain  $\Omega$  of finite length type with universal covering map  $f: \mathbb{D} \rightarrow \Omega$  and any quasismooth curve  $L$ :*

$$\lambda[f^{-1}(\Omega \cap L)] < \infty$$

From this we prove:

**COROLLARY 1.** *Suppose that  $\Omega$  is a finitely connected hyperbolic planar domain with no point boundary components, and let  $f$  be the universal covering*

map. Then for any quasismooth curve  $L$ ,

$$\lambda[f^{-1}(\Omega \cap L)] < \infty.$$

The argument of the proof of Corollary 1 also shows how to construct infinitely connected domains for which the theorem holds.

However we do have:

**COROLLARY 2.** *Suppose that  $\Omega$  is a Denjoy domain, i.e.  $\partial\Omega \subset \mathbb{R}$ . Then  $\lambda[f^{-1}(\Omega \cap L)] < A$  for all quasismooth curves  $L$  if and only if  $\Omega$  has finite length type.*

The fact that  $\mathcal{D}$  has finite length type says something about the “size” of the group  $G$ . The usual way of measuring that size is through the exponent of convergence  $\delta(G)$

$$\delta(G) = \inf \left\{ \delta > 0: \sum_{T \in G} (1 - |T(0)|)^\delta < \infty \right\}$$

(see, e.g. [14]).

We have

**COROLLARY 3.** *Suppose that  $\Omega$  is a planar domain and  $G$  the Fuchsian group uniformizing  $\Omega$  then if  $\delta(G) < \frac{1}{2}$  then for all quasismooth curves  $L$*

$$\lambda(f^{-1}(\Omega \cap L)) < \infty.$$

The condition on  $\delta(G)$  is sharp because for  $\mathbb{D} \setminus \{0\}$  we have  $\delta = \frac{1}{2}$  while  $\lambda(f^{-1}(\mathbb{D} \setminus \{0\}) \cap \mathbb{R}) = \infty$ . But on the other hand the condition is not necessary because there are finitely connected domains with no point boundary components for which  $\delta > \frac{1}{2}$ , e.g. take  $\Omega_\varepsilon = \{z \in \mathbb{C}: |z| < \varepsilon, |z - 1| < \varepsilon, |z| > 1/\varepsilon\}$  with  $\varepsilon$  small enough (actually  $\delta(\Omega_\varepsilon) \uparrow 1$  as  $\varepsilon \rightarrow 0$ ).

## 2. Preliminary results

We shall be dealing with domains  $G$  which are regular for the Dirichlet problem. By  $dw_G^z$  we denote the unique probability measure such that if  $g$  is continuous on  $\partial G$  then the Perron solution  $u$  to the Dirichlet problem in  $G$  with

boundary values  $g$  is given by

$$u(z) = \int_{\partial G} g dw_G^z.$$

The harmonic measure of a Borel subset  $E$  of  $\partial G$  at a point  $z \in G$  with respect to  $G$  is then

$$\omega(z, E, G) = \int_E dw_G^z.$$

Also the disk  $\{|z - a| < r\}$  is denoted by  $\Delta(a, r)$ .

We make frequent use of the following results which are simple consequences of the Carleman–Milloux inequality, see [1], and Hall's lemma respectively, see [6].

**LEMMA 1.** *There is a positive function  $c(\delta)$ ,  $\delta > 0$ , such that if the closure of a domain  $\Omega$  contains continuum  $E$  which meets  $\partial\Omega$  then for any  $z \in \Omega \setminus E$  satisfying*

$$\text{dist}(z, E) \leq c(\delta) \text{dist}(z, \partial\Omega \setminus E)$$

*we have*

$$\omega(z, E, \Omega \setminus E) \geq 1 - \delta.$$

Let us denote the upper half plane by  $H$ . Also if  $0 < a < b < 1$  and  $\theta \in (0, \pi/2)$  then

$$S(a, b, \theta) = \{z \in H : |z| \in (a, b), \arg z \in (\theta, \pi - \theta)\}.$$

**LEMMA 2.** *Given  $0 < a < b$ , and  $\theta \in (0, \pi/2)$  there exists  $R > 0$  and  $\eta > 0$  such that for any  $r \geq R$  and for any continuum  $E \subset H$  joining  $|z| = 1$  to  $|z| = r$*

$$\omega(z, (\Delta(0, r) \cap H) \setminus E) > \eta$$

*for each  $z$  in the sector  $S(a, b, \theta)$ .*

Also we shall be using quasiconformal mappings. We need

**LEMMA 3.** *Given  $0 < a < b < 1$  and  $\theta \in (0, \pi/2)$  there exists  $R > 0$  and a positive function  $\eta(k)$  such that for any  $r \geq R$ :*

If  $E$  is a continuum joining  $|z|=1$  to  $|z|=r$  in  $H$ , then for any  $k$ -quasiconformal mapping  $\Phi: \mathbb{C} \rightarrow \mathbb{C}$  we have

$$\omega(\Phi(z), \Phi(E), \Phi(\Delta(0, r) \cap H \setminus E)) \geq \eta(k)$$

for each  $z \in S(a, b, \theta)$ .

The lemma follows from Lemma 2 and the distortion theorem of Mori, see [2]. Let  $\Omega$  be the component of  $\Delta(0, r) \cap H \setminus E$  containing  $z \in S(a, b, \theta)$  and suppose  $f$  and  $g$  are the Riemann mappings from the unit disk  $\mathbb{D}$  onto (respectively)  $\Omega$  and  $\Phi(\Omega)$  with  $f(0) = z$ ,  $g(0) = \Phi(z)$ . If  $z \in S(a, b, \theta)$  then because of Lemma 2  $\omega(z, E, \Omega) > \eta$ . On the other hand  $2\pi \omega(z, E, \Omega)$  is the length of the subarc  $I_1$  which is the closure of  $\{e^{i\theta} : \lim_{r \rightarrow 1} f(re^{i\theta}) \in E\}$ . One should note that as  $E$  is a continuum in  $\bar{H}$ , by a theorem of Beurling (see Collingwood and Lohwater [4]), " $f^{-1}(E)$ " is an arc of  $\partial\mathbb{D}$  with a set of capacity zero removed. Similarly  $2\pi \omega(\Phi(z), \Phi(\Omega))$  is the length of a subarc  $I_2$ . But  $g^{-1} \circ \Phi \circ f = \psi$  is a quasiconformal mapping of  $\mathbb{D}$  onto itself which fixes 0. Thus as  $\psi(I_1) = I_2$  we see by Mori's theorem that:

$$\lambda(I_1) \geq c \{\lambda(I_2)\}^\delta$$

where  $c, \delta > 0$  depend only on  $k$ , which concludes the proof of the lemma.

The following is derived from estimates of Jerison and Kenig [9] and Kaufman and Wu [10, p. 269, 273].

**LEMMA 4.** *Suppose that  $U$  is a domain whose boundary is a quasismooth curve with constant  $M$ . If  $z_0 \in U$ ,  $\zeta_0 \in \partial U$  satisfy (for some  $r > 0$ )*

$$\text{dist}(z_0, \partial U) \geq ar$$

and

$$|z_0 - \zeta_0| \leq br$$

for some  $a, b > 0$ , then for any set  $F \subset \partial U$  satisfying  $F \subset \Delta(\zeta_0, r)$  and  $\lambda(F) \geq r/2$  we have

$$\omega(z_0, F, U) \geq \eta$$

where  $\eta > 0$  depends only on  $a, b$  and  $M$ .

This is most easily proved by using Lemma 1 of [10] which provides us with a point  $z_1 \in U$  satisfying

$$a_1^{-1}r \leq \text{dist}(z_1, \partial U) \leq a_1r$$

$$b_1^{-1}r \leq |\xi_0 - z_1| \leq b_1r$$

and

$$\omega(z_1, F, U) \geq \eta_1 > 0$$

where  $a_1, b_1, \eta_1 > 0$  depend only on  $a, b$  and  $M$ . Now  $U$  is an  $(\varepsilon, \infty)$  domain (see [11]) and so there exists a rectifiable arc  $\gamma \subset U$  joining  $z_0$  to  $z_1$  and satisfying

$$\lambda(\gamma) \leq a_2r$$

and

$$\text{dist}(\gamma, \partial U) \geq a_2^{-1}r$$

where  $a_2$  depends only on  $a, b$  and  $K$ . Consequently Harnack's inequality is applied and we see that it is impossible that  $\omega(z_0, F, U)$  may become arbitrarily small.

The connection between estimating harmonic measures and  $\lambda(f^{-1}(\Omega \cap L))$  is derived from the notion of a Carleson measure (see [6]). Now a positive measure  $\mu$  on the unit disk may be defined to be a Carleson measure if

$$\int_{\mathbb{D}} |T'(z)| d\mu < c \tag{5}$$

for any Möbius transformation  $T: \mathbb{D} \rightarrow \mathbb{D}$ . Clearly then, by considering  $f \circ T$ , any  $L$  satisfying (1) will have the property that arc length measure on  $f^{-1}(\Omega \cap L)$  is a Carleson measure. This was observed in [7] and gives the extra conclusion that we have a Carleson measure.

**LEMMA 5.** *Suppose that  $L$  is a Jordan curve satisfying Ahlfors' regularity condition (2). Then to obtain  $\lambda(f^{-1}[\Omega \cap L]) < c$  for all simply connected  $\Omega$  the following are sufficient:*

*There is a  $\alpha > 0, \varepsilon > 0$  and  $\beta < 1$  such that for any sequence  $w_j \in L \cap \Omega$  with*

$$|w_j - w_k| \geq \alpha \text{dist}(w_j, \partial \Omega), \quad j \neq k, \tag{6}$$

we have

$$\omega(w_j, K_j, \Omega \setminus K_j) \leq \beta \quad (7)$$

where

$$K_j = \bigcup_{k \neq j} \bar{\Delta}(w_k, \alpha \varepsilon \operatorname{dist}(w_k, \partial \Omega)) \quad (8)$$

### 3. Proof of Theorem 1

We let  $M$  denote the chord arc constant of  $L$ , see (4). Now we fix  $\alpha = \frac{1}{3}$  and determine  $\varepsilon$  and  $\beta$  so that (7) of Lemma 5 is verified for any sequence  $\{w_k\} \subset L \cap \Omega$  satisfying (6).

Fix  $j$  and let  $z = w_j$ . Also we define  $d = \operatorname{dist}(z, \partial \Omega)$  and  $J = L \cap \Omega$ . Denote by  $J_1$  the component of  $J$  which contains  $z$  and by  $J_0$  the component of  $J_1 \cap \Delta(z, \alpha d)$  containing  $z$ . Consider now the closed (in  $\Omega$ ) set  $K = \bigcup_{w \in J \setminus J_0} \bar{\Delta}(w, \alpha \varepsilon \operatorname{dist}(w, \partial \Omega))$ . Clearly  $K \supset K_j \cup (J - J_0)$  and in particular by the maximum principle

$$\omega(z, K, \Omega \setminus K) \geq \omega(z, K_j, \Omega \setminus K_j) \quad (9)$$

So we have only to show that if we choose  $\varepsilon$  appropriately (depending only on  $L$ ) we obtain  $\beta = \beta(\varepsilon, M) < 1$  so that

$$\omega(z, K, \Omega \setminus K) \leq \beta. \quad (10)$$

Recall the function  $c(\delta)$  of Lemma 1; then if  $\varepsilon \leq c(\delta)$  we have

$$\omega(z, J \setminus J_0, \Omega \setminus (J \setminus J_0)) \geq (1 - \delta) \omega(z, K, \Omega \setminus K). \quad (11)$$

To see this we write

$$\omega(z, J \setminus J_0, \Omega \setminus (J \setminus J_0)) = \int_{\partial K} \omega(\zeta, J \setminus J_0, \Omega \setminus (J \setminus J_0)) dw_{\Omega \setminus K}^z(\zeta).$$

But by Lemma 1, if  $\zeta \in K$  then  $\omega(\zeta, J \setminus J_0, \Omega \setminus (J \setminus J_0)) \geq (1 - \delta)$ , and (11) follows. In fact, if  $w \in J \setminus J_0$  and  $|\zeta - w| \leq \alpha \varepsilon \operatorname{dist}(w, \partial \Omega)$  then if  $E$  is the closure of the

component of  $J \setminus J_0$  containing  $w$  we have

$$\begin{aligned} \text{dist}(\zeta, E) &\leq \alpha \varepsilon \text{dist}(w, \partial \Omega) \\ &\leq \alpha \varepsilon (1 - \alpha \varepsilon)^{-1} \text{dist}(\zeta, \partial \Omega) \\ &\leq c(\delta) \text{dist}(\zeta, \partial \Omega). \end{aligned}$$

The next step is to estimate  $\omega(z, J \setminus J_0, \Omega \setminus (J - J_0))$ . Let  $U_1, U_2$  be the complementary domains of  $L$ . Suppose that  $\Omega_0$  is the component of  $\Omega \setminus (J \setminus J_0)$  containing  $z$ , and  $\Omega_i = \Omega_0 \cap U_i, i = 1, 2$ . Also we define  $J_{i,j}$  to be the components of  $(J \setminus J_1) \cap \partial \Omega_i$ . Note that  $J_{i,j}$  belongs to only one of the boundaries  $\partial \Omega_i$ .

The disk  $\Delta(z, d\alpha/2M)$  contains no point of  $J \setminus J_0$ . We set  $r = d\alpha/2M$ . Since  $L$  is chord arc we have subarcs  $I_j$  of  $\partial \Delta(z, r)$  such that for some  $\rho, \tau > 0$  (depending only on  $M$ )

$$I_i \subset \Omega_i \tag{12}$$

$$\lambda(I_i) = \frac{\rho}{2} \pi r \tag{13}$$

$$\text{dist}(I_i, L) > \tau r. \tag{14}$$

Therefore

$$\begin{aligned} \omega(z, J \setminus J_0, \Omega \setminus (J \setminus J_0)) &= \int_{\partial \Delta(z, r)} \omega(\zeta, J \setminus J_0, \Omega \setminus (J \setminus J_0)) d\omega_{\Delta(z, r)}^z(\zeta) \\ &\leq (1 - \rho) + \frac{\rho}{2} + \frac{\rho}{2} \min_{i=1, 2} \max_{I_i} \omega(\zeta, J \setminus J_0, \Omega \setminus (J \setminus J_0)) \end{aligned} \tag{15}$$

But for  $\zeta \in I_i$

$$\omega(\zeta, J \setminus J_0, \Omega \setminus (J \setminus J_0)) \leq \omega(\zeta, J, \Omega_i) \tag{16}$$

and we see from (15) that it is enough to show that

$$\min_i \max_{I_i} \omega(\zeta, J, \Omega_i) \leq \beta. \tag{17}$$

We consider now two cases. Let  $B > 1$  be a constant to be determined later;  $B$  will depend only on  $M$ .

In this first case we suppose there exists  $\zeta_0 \in (L \setminus J_1) \cap \partial\Omega_0$  so that

$$|\zeta_0 - z| \leq Bd. \quad (18)$$

In this case we let (following [10])  $S_i = L \setminus (\bigcup_j J_{i,j})$ . Clearly  $L \setminus J_1 = S_1 \cup S_2$ . From the maximum principle we obtain for  $\zeta \in I_i$

$$1 - \omega(\zeta, J, \Omega_i) = \omega(\zeta, \partial\Omega_i \setminus J, \Omega_i) \geq \omega(\zeta, S_i, U_i). \quad (19)$$

Now we use Lemma 4 for the chord arc domain  $U_i$ . Since  $\zeta_0 \in L \setminus J_1$  we have that

$$\max_i \lambda(\Delta(\zeta_0, r) \cap S_i) \geq \frac{r}{2}. \quad (20)$$

Consequently by (18), (20) and Lemma 4 we obtain that

$$\max_{i=1,2} \min_{\zeta \in I_i} \omega(\zeta, S_i, U_i) \geq \eta > 0. \quad (21)$$

where  $\eta$  depends only on  $B$  and the chord arc constant  $M$  and so only on  $M$ . Then from (21), (19), (17), (15) and (11) we get

$$\omega(z, K, \Omega \setminus K) \leq \frac{\beta_1}{1 - \delta} \quad (22)$$

where  $\beta_1 < 1$  depends only on  $M$ .

This leaves the case where for each  $\zeta \in (L \setminus J_1) \cap \partial\Omega_0$  we have  $|\zeta - z| > Bd$ . Let  $w \in \partial\Omega \cap \partial\Omega_0$ ,  $|w - z| = d$ . Notice that  $w \notin L$  for  $w \notin L \setminus J_1$  by our assumption (and  $w \notin J_1$  as  $w \notin \Omega$ ). We can join  $w$  to a point on  $\partial\Delta(z, Bd)$  with a continuum

$$F \subset \partial\Omega \cap \partial\Omega_0 \cap (\bar{\Delta}(z, Bd) \setminus \Delta(z, d)),$$

because if not there would exist  $w_1 \in \partial\Omega \cap \partial\Omega_0$ ,  $|w_1 - z| < Bd$  and  $w_1 \in L \setminus J_1$  contradicting our assumption in the second case. Since  $F \subset \partial\Omega_0$  we have  $F \cap (L \setminus J_1) = \emptyset$  and  $F \cap J_1 = \emptyset$  so  $F \cap L = \emptyset$ .

Use a quasiconformal mapping  $\Phi$  from  $\mathbb{C}$  to  $\mathbb{C}$  mapping  $\mathbb{R}$  onto  $L$ ,  $\Phi(0) = z$ ,  $\Phi(\infty) = \infty$ . Also we may assume  $|\Phi(1) - z| = r$ . Since  $F$  does not meet  $L$ , without loss of generality  $F \subset U_1$  and  $\Phi$  maps  $H$  onto  $U_1$ . Now from the uniform bounds for quasiconformal mapping, (13) and (14) we obtain constants  $a, b, \theta$  depending

only on the chord arc constant  $M$  so that

$$\Phi^{-1}(I_1) \subset S(a, b, \theta). \quad (23)$$

Now, if  $\zeta \in I_1$

$$\omega(\zeta, \partial\Omega_1 \setminus J, \Omega_1) \geq \omega(\zeta, F, U_1). \quad (24)$$

But if  $E = \Phi^{-1}(F)$  then  $E$  is a continuum running from  $|z| = c_1$  to  $|z| = c_2$ , where  $c_1$  depends only on  $k$  and  $c_2$  depends on  $M$  and  $B$ , and  $c_2 \rightarrow \infty$  as  $B \rightarrow \infty$ . Consequently from Lemma 3, (23) and (24) show that if  $B \geq B_0(M)$

$$\omega(\zeta, J, \Omega_1) \leq \beta(M) < 1$$

for each  $\zeta \in I_1$ , and so, as in the first case, we obtain

$$\omega(z, K, \Omega \setminus K) \leq \frac{\beta_2}{1 - \delta} \quad (25)$$

where  $\beta_2 < 1$  depends only on  $M$ .

Therefore we choose  $\beta = \max(\beta_1, \beta_2)$  and  $\delta < 1 - \beta$  and with  $\varepsilon = c(\delta)$  see that the proof of the theorem is complete.

#### 4. Proof of Theorem 2

We need the following (see [7]):

**LEMMA 6.** *Let  $U$  be a simply connected domain with rectifiable boundary, and  $f$  a conformal mapping of  $U$  onto  $\Omega$ . Then for any quasismooth curve  $L$*

$$\lambda(f^{-1}(\Omega \cap L)) < c_1 \lambda(\partial U).$$

Let  $g$  be the Riemann mapping from  $\mathbb{D}$  to  $U$ . Thus by Theorem 1 arc length  $d\mu$  on  $g^{-1} \circ f^{-1}(\Omega \cap L)$  is a Carleson measure and hence as  $g' \in H^1$

$$\lambda(f^{-1}(\Omega \cap L)) = \int_{\mathbb{D}} |g'| d\mu \leq c_2 \int_{\partial\mathbb{D}} |g'| d\theta = c_2 \lambda(\partial U)$$

which proves the lemma.

Now let  $\Omega$  be a hyperbolic planar domain and  $f: \mathbb{D} \rightarrow \Omega$  be the universal covering map. Suppose that  $G$  is the Fuchsian group of Möbius transformations  $T: \mathbb{D} \rightarrow \mathbb{D}$  which represents the cover group for  $\Omega$ . The Dirichlet fundamental region  $\mathcal{D}$  for  $G$  is

$$\{z \in \mathbb{D} : |T'(z)| < 1, \forall T \in G \setminus \{I\}\}.$$

Now  $\mathcal{D}$  is a convex set in the hyperbolic metric with rectifiable boundary. Thus by Lemma 6

$$\lambda\{f^{-1}(\Omega \cap L) \cap \overline{T(\mathcal{D})}\} \leq c_3 \lambda(\partial T\mathcal{D}) \quad (26)$$

for any quasismooth curve  $L$  and  $T \in G$ . This immediately proves theorem 2.

Corollary 1 follows from

**LEMMA 7.** *Suppose that  $G$  is the Fuchsian group of a finitely connected planar domain with no point boundary components. Let  $\mathcal{D}$  be the Dirichlet region for  $G$ . Then*

$$\sum_G \lambda(T \partial \mathcal{D}) < \infty.$$

The boundary  $\partial \mathcal{D}$  consists of a finite number of disjoint nonconcentric circles orthogonal to the unit disk. Let us denote  $\mathcal{D}$  as  $\mathcal{D}_1$ . The region  $\mathcal{D}_2$  is obtained from  $\mathcal{D}$  by reflecting  $\mathcal{D}$  through each of the orthogonal circles, and adding  $\mathcal{D}_1$ . At the  $n^{\text{th}}$  stage we obtain  $\mathcal{D}_n$  with boundaries exactly  $n$  reflections of the original circles. Thus  $\sum_G \lambda(T\mathcal{D}) = \sum_{n=1}^{\infty} \lambda(\partial \mathcal{D}_n)$ . We need

**LEMMA 8.** *There is a constant  $\beta < 1$  such that*

$$\lambda(\partial \mathcal{D}_n \cap \mathbb{D}) < \pi \beta^n$$

Let  $E_n = \partial \mathcal{D}_n \cap \mathbb{D}$ ,  $F_n$  be a circle of  $E_{n-1}$  and  $G_n$  the part of  $E_n$  separated from the rest of  $E_n$  by  $F_n$ . By conformal invariance there is  $\beta < 1$  such that  $\text{dia}(G_n) \leq \beta \text{dia}(F_n)$ . However as  $G_n$  consists of orthogonal semicircles  $\lambda(G_n) \leq \pi \text{dia}(G_n)$ . Summing over the components of  $E_{n-1}$  gives

$$\lambda(E_n) \leq \beta \lambda(E_{n-1})$$

Thus we prove Lemmas 7, 8 and complete the proof of Corollary 1.

The necessary part of Corollary 2 is derived from using the real line as our curve  $L$ . In this symmetric situation  $f^{-1}(\mathbb{R})$  is  $\bigcup_G T(\partial\mathcal{D})$ .

Corollary 3 follows immediately from

LEMMA 9. *With the notations above if  $\Omega$  is a hyperbolic planar domain then*

$$\sum_{T \in G} \lambda(\partial T\mathcal{D}) \leq 2\pi \left( \inf_{T \neq I} |T(0)| \right)^{-1} \sum_{T \in G} (1 - |T(0)|^2)^{1/2}$$

where  $C$  is a universal constant.

*Proof.* To see this we will associate to each side of  $\mathcal{D}$  and the  $T(\mathcal{D})$ 's an element  $R \in G$  in a 1-1 fashion and in such a way that if  $z \in s$  then  $\rho(z, R(0)) \leq \rho(z, 0)$ , where  $\rho$  denotes hyperbolic distance in  $\mathbb{D}$ . This is enough because then  $s$  is contained in a euclidean disk of radius  $|R(0)|^{-1} (1 - |R(0)|^2)^{1/2}$  and so

$$\lambda(s) \leq \pi |R(0)|^{-1} \cdot (1 - |R(0)|^2)^{1/2}.$$

So consider the side  $s$ . It separates two contiguous images of  $\mathcal{D}$ , say  $A(\mathcal{D})$ ,  $B(\mathcal{D})$  with  $A, B \in G$ .

The transformations  $\{T_i\}$  in  $G$  which pairwise identify the sides of  $\mathcal{D}$  generate  $G$  and in fact since  $\Omega$  is planar  $G$  is freely generated by the  $\{T_i\}$ .

Now  $A = B \circ T_0$  for some generator  $T_0$  so that if  $B = T_n \circ \cdots \circ T_1$  is a reduced word then the word length of  $A$  is  $n - 1$  or  $n + 1$  according to  $T_1 = T_0^{-1}$  or  $T_1 \neq T_0^{-1}$ . Changing the roles of  $A$  and  $B$  we may assume that the latter case occurs and to  $s$  we associate  $A = T_n \circ \cdots \circ T_1 \circ T_0$ . Notice that  $A = T_n \circ \cdots \circ T_1 \circ T_0$  determines  $s$  by being the side separating  $T_n \circ \cdots \circ T_1(\mathcal{D})$  from  $A(\mathcal{D})$ . Finally  $s \subset \partial A(\mathcal{D})$  and so for each  $z \in s$   $\rho(z, A(0)) \leq \rho(z, 0)$ .

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