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1. Introduction

It is well known that for any homeomorphism f of the unit disk $\mathbb{D}$ onto a domain Ω, where f is ACL and $\nabla f \in L^2(\mathbb{D})$, $f^{-1}(\Omega \cap L)$ has finite length for almost all rectifiable curves L. Suppose now that f is analytic, and let $\lambda(E)$ denote the Hausdorff linear measure of a set E. Hayman and Wu [8] proved that for any line L

$$\lambda(f^{-1}[\Omega \cap L]) < A, \tag{1}$$

for some absolute constant A. This was generalized by Garnett, Gehring and Jones [7] who gave conditions on a rectifiable Jordan curve in order that (1) holds for all Ω as above. It is necessary that L satisfy a regularity condition introduced by Ahlfors, i.e. there is a constant c_1:

$$\lambda[L \cap \{|\zeta - w| < r\}] \leq c_1 r \tag{2}$$

for all $\zeta \in L$ and $r > 0$. Garnett, Gehring and Jones conjectured that (1) could fail for a regular quasicircle, i.e. L satisfies (2) together with

$$|z_1 - z_2| > c_2 \min_i \operatorname{dia}(\gamma_i) \tag{3}$$

for any $z_1, z_2 \in L$ where γ_1, γ_2 are the two components of $L \backslash \{z_1, z_2\}$. In fact we show that the example suggested in [7] cannot work. A curve L is called quasismooth (or chord-arc) if there is a constant $M > 0$ such that for any $z_1, z_2 \in L$ we have

$$\min_{i=1,2} \lambda(\gamma_i) \leq M\, |z_1 - z_2|, \tag{4}$$

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sec Jerison and Kenig [9], Pommerenke [12]. Actually (2) and (3) are equivalent to (4). We prove:

THEOREM 1. *For any quasismooth curve L and any simply connected domain Ω with Riemann mapping f*

$$\lambda[f^{-1}(\Omega \cap L)] \leq A < \infty,$$

where A depends only on the chord arc constant M.

We conjecture that (2) is a necessary and sufficient condition on L in order that (1) hold for all conformal maps.

Next we consider the case of the universal covering mapping f of a multiply connected planar domain. Flinn [5] had obtained the following theorem: suppose that Ω is a hyperbolic planar domain and one component of $\mathbb{C}\backslash L$ is contained in Ω. Then if l is one component of $f^{-1}(\Omega \cap L)$ we have $\lambda(l) < \infty$. On the other hand if $\Omega = \mathbb{D}\backslash E$ where $E \subset (0, 1)$ is a closed set of zero logarithmic capacity then Belna, Cohn, Piranian and Stephenson [3] proved that there are circles L which do not satisfy (1).

Suppose that G is the Fuchsian group of Möbius transformations $T: \mathbb{D} \to \mathbb{D}$ which represents the cover group for Ω. The Dirichlet fundamental region $\mathcal{D}$ for G is

$$\mathcal{D} = \{z \in \mathbb{D} : |T'(z)| < 1, \forall T \in G\backslash\{I\}\}$$

We say that $\mathcal{D}$ is of finite length type if

$$\sum_G \lambda(\partial T\mathcal{D}) < \infty$$

THEOREM 2. *For any hyperbolic planar domain Ω of finite length type with universal covering map $f: \mathbb{D} \to \Omega$ and any quasismooth curve L:*

$$\lambda[f^{-1}(\Omega \cap L)] < \infty$$

From this we prove:

COROLLARY 1. *Suppose that Ω is a finitely connected hyperbolic planar domain with no point boundary components, and let f be the universal covering* *map. Then for any quasismooth curve L,*

$$\lambda[f^{-1}(\Omega \cap L)] < \infty.$$

The argument of the proof of Corollary 1 also shows how to construct infinitely connected domains for which the theorem holds.

However we do have:

COROLLARY 2. *Suppose that Ω is a Denjoy domain, i.e. $\partial\Omega \subset \mathbb{R}$. Then $\lambda[f^{-1}(\Omega \cap L)] < A$ for all quasismooth curves L if and only if Ω has finite length type.*

The fact that $\mathscr{D}$ has finite length type says something about the "size" of the group G. The usual way of measuring that size is through the exponent of convergence $\delta(G)$

$$\delta(G) = \inf\left\{\delta > 0: \sum_{T \in G} (1 - |T(0)|)^{\delta} < \infty\right\}$$

(see, e.g. [14]).

We have

COROLLARY 3. *Suppose that Ω is a planar domain and G the Fuchsian group uniformizing Ω then if $\delta(G) < \frac{1}{2}$ then for all quasismooth curves L*

$$\lambda(f^{-1}(\Omega \cap L)) < \infty.$$

The condition on $\delta(G)$ is sharp because for $\mathbb{D}\setminus\{0\}$ we have $\delta = \frac{1}{2}$ while $\lambda(f^{-1}(\mathbb{D}\setminus\{0\} \cap \mathbb{R}) = \infty$. But on the other hand the condition is not necessary because there are finitely connected domains with no point boundary components for which $\delta > \frac{1}{2}$, e.g. take $\Omega_{\varepsilon} = \{z \in \mathbb{C}: |z| < \varepsilon, |z-1| < \varepsilon, |z| > 1/\varepsilon\}$ with ε small enough (actually $\delta(\Omega_{\varepsilon}) \uparrow 1$ as $\varepsilon \to 0$).

2. Preliminary results

We shall be dealing with domains G which are regular for the Dirichlet problem. By dw_G^z we denote the unique probability measure such that if g is continuous on ∂G then the Perron solution u to the Dirichlet problem in G with boundary values g is given by

$$u(z)=\int_{\partial G} g\,dw_G^z.$$

The harmonic measure of a Borel subset E of ∂G at a point $z \in G$ with respect to G is then

$$\omega(z, E, G)=\int_E dw_G^z.$$

Also the disk $\{|z-a|<r\}$ is denoted by $\Delta(a, r)$.

We make frequent use of the following results which are simple consequences of the Carleman–Milloux inequality, see [1], and Hall's lemma respectively, see [6].

LEMMA 1. *There is a positive function* $c(\delta)$, $\delta>0$, *such that if the closure of a domain* Ω *contains continuum* E *which meets* $\partial\Omega$ *then for any* $z \in \Omega\backslash E$ *satisfying*

$$\operatorname{dist}(z, E) \leq c(\delta) \operatorname{dist}(z, \partial\Omega\backslash E)$$

we have

$$\omega(z, E, \Omega\backslash E) \geq 1-\delta.$$

Let us denote the upper half plane by H. Also if $0<a<b<1$ and $\theta \in (0, \pi/2)$ then

$$S(a, b, \theta)=\{z \in H: |z| \in (a, b), \arg z \in (\theta, \pi-\theta)\}.$$

LEMMA 2. *Given* $0<a<b$, *and* $\theta \in (0, \pi/2)$ *there exists* $R>0$ *and* $\eta>0$ *such that for any* $r \geq R$ *and for any continuum* $E \subset H$ *joining* $|z|=1$ *to* $|z|=r$

$$\omega(z, (\Delta(0, r) \cap H)\backslash E)>\eta$$

for each z *in the sector* $S(a, b, \theta)$.

Also we shall be using quasiconformal mappings. We need

LEMMA 3. *Given* $0<a<b<1$ *and* $\theta \in (0, \pi/2)$ *there exists* $R>0$ *and a positive function* $\eta(k)$ *such that for any* $r \geq R$:

If E is a continuum joining $|z|=1$ to $|z|=r$ in H, then for any k-quasiconformal mapping $\Phi:\mathbb{C}\to\mathbb{C}$ we have

$$\omega(\Phi(z), \Phi(E), \Phi(\Delta(0, r)\cap H\backslash E))\geq \eta(k)$$

for each $z\in S(a, b, \theta)$.

The lemma follows from Lemma 2 and the distortion theorem of Mori, see [2]. Let Ω be the component of $\Delta(0, r)\cap H\backslash E$ containing $z\in S(a, b, \theta)$ and suppose f and g are the Riemann mappings from the unit disk $\mathbb{D}$ onto (respectively) Ω and $\Phi(\Omega)$ with $f(0)=z$, $g(0)=\Phi(z)$. If $z\in S(a, b, \theta)$ then because of Lemma 2 $\omega(z, E, \Omega)>\eta$. On the other hand $2\pi\ \omega(z, E, \Omega)$ is the length of the subarc I_1 which is the closure of $\{e^{i\theta}:\lim_{r\to 1} f(re^{i\theta})\in E\}$. One should note that as E is a continuum in $\bar{H}$, by a theorem of Beurling (see Collingwood and Lohwater [4]), "$f^{-1}(E)$" is an arc of $\partial\mathbb{D}$ with a set of capacity zero removed. Similarly $2\pi\ \omega(\Phi(z), \Phi(\Omega))$ is the length of a subarc I_2. But $g^{-1}\circ\Phi\circ f=\psi$ is a quasiconformal mapping of $\mathbb{D}$ onto itself which fixes 0. Thus as $\psi(I_1)=I_2$ we see by Mori's theorem that:

$$\lambda(I_1)\geq c\{\lambda(I_2)\}^{\delta}$$

where c, $\delta>0$ depend only on k, which concludes the proof of the lemma.

The following is derived from estimates of Jerison and Kenig [9] and Kaufman and Wu [10, p. 269, 273].

LEMMA 4. *Suppose that U is a domain whose boundary is a quasismooth curve with constant M. If $z_0\in U$, $\zeta_0\in\partial U$ satisfy (for some $r>0$)*

$$\operatorname{dist}(z_0, \partial U)\geq ar$$

and

$$|z_0-\zeta_0|\leq br$$

for some $a, b>0$, then for any set $F\subset\partial U$ satisfying $F\subset\Delta(\zeta_0, r)$ and $\lambda(F)\geq r/2$ we have

$$\omega(z_0, F, U)\geq\eta$$

where $\eta>0$ depends only on a, b and M.

This is most easily proved by using Lemma 1 of [10] which provides us with a point $z_1 \in U$ satisfying

$$a_1^{-1}r \le \operatorname{dist}(z_1, \partial U) \le a_1 r$$
$$b_1^{-1}r \le |\xi_0 - z_1| \le b_1 r$$

and

$$\omega(z_1, F, U) \ge \eta_1 > 0$$

where a_1, b_1, $\eta_1 > 0$ depend only on a, b and M. Now U is an (ε, ∞) domain (see [11]) and so there exists a rectifiable arc $\gamma \subset U$ joining z_0 to z_1 and satisfying

$$\lambda(\gamma) \le a_2 r$$

and

$$\operatorname{dist}(\gamma, \partial U) \ge a_2^{-1} r$$

where a_2 depends only on a, b and K. Consequently Harnack's inequality is applied and we see that it is impossible that $\omega(z_0, F, U)$ may become arbitrarily small.

The connection between estimating harmonic measures and $\lambda(f^{-1}(\Omega \cap L))$ is derived from the notion of a Carleson measure (see [6]). Now a positive measure μ on the unit disk may be defined to be a Carleson measure if

$$\int_{\mathbb{D}} |T'(z)|\, d\mu < c \tag{5}$$

for any Möbius transformation $T: \mathbb{D} \to \mathbb{D}$. Clearly then, by considering $f \circ T$, any L satisfying (1) will have the property that arc length measure on $f^{-1}(\Omega \cap L)$ is a Carleson measure. This was observed in [7] and gives the extra conclusion that we have a Carleson measure.

LEMMA 5. *Suppose that L is a Jordan curve satisfying Ahlfors' regularity condition* (2). *Then to obtain $\lambda(f^{-1}[\Omega \cap L]) < c$ for all simply connected Ω the following are sufficient*:

There is a $\alpha > 0$, $\varepsilon > 0$ and $\beta < 1$ such that for any sequence $w_j \in L \cap \Omega$ with

$$|w_j - w_k| \ge \alpha \operatorname{dist}(w_j, \partial\Omega), \qquad j \ne k, \tag{6}$$

we have

$$\omega(w_j, K_j, \Omega \backslash K_j) \le \beta \tag{7}$$

where

$$K_j = \bigcup_{k \neq j} \bar{\Delta}(w_k, \alpha\varepsilon \operatorname{dist}(w_k, \partial\Omega)) \tag{8}$$

3. Proof of Theorem 1

We let M denote the chord arc constant of L, see (4). Now we fix $\alpha = \frac{1}{3}$ and determine ε and β so that (7) of Lemma 5 is verified for any sequence $\{w_k\} \subset L \cap \Omega$ satisfying (6).

Fix j and let $z = w_j$. Also we define $d = \operatorname{dist}(z, \partial\Omega)$ and $J = L \cap \Omega$. Denote by J_1 the component of J which contains z and by J_0 the component of $J_1 \cap \Delta(z, \alpha d)$ containing z. Consider now the closed (in Ω) set $K = \bigcup_{w \in J \backslash J_0} \bar{\Delta}(w, \alpha\varepsilon \operatorname{dist}(w, \partial\Omega))$. Clearly $K \supset K_j \cup (J - J_0)$ and in particular by the maximum principle

$$\omega(z, K, \Omega \backslash K) \ge \omega(z, K_j, \Omega \backslash K_j) \tag{9}$$

So we have only to show that if we choose ε appropriately (depending only on L) we obtain $\beta = \beta(\varepsilon, M) < 1$ so that

$$\omega(z, K, \Omega \backslash K) \le \beta. \tag{10}$$

Recall the function $c(\delta)$ of Lemma 1; then if $\varepsilon \le c(\delta)$ we have

$$\omega(z, J \backslash J_0, \Omega \backslash (J \backslash J_0)) \ge (1 - \delta)\omega(z, K, \Omega \backslash K). \tag{11}$$

To see this we write

$$\omega(z, J \backslash J_0, \Omega \backslash (J \backslash J_0)) = \int_{\partial K} \omega(\zeta, J \backslash J_0, \Omega \backslash (J \backslash J_0))\, dw^z_{\Omega \backslash K}(\zeta).$$

But by Lemma 1, if $\zeta \in K$ then $\omega(\zeta, J \backslash J_0, \Omega \backslash (J \backslash J_0) \ge (1 - \delta)$, and (11) follows. In fact, if $w \in J \backslash J_0$ and $|\zeta - w| \le \alpha\varepsilon \operatorname{dist}(w, \partial\Omega)$ then if E is the closure of the component of $J\backslash J_0$ containing w we have

$$\begin{aligned}\operatorname{dist}(\zeta, E) &\le \alpha\varepsilon \operatorname{dist}(w, \partial\Omega)\\ &\le \alpha\varepsilon(1-\alpha\varepsilon)^{-1}\operatorname{dist}(\zeta, \partial\Omega)\\ &\le c(\delta)\operatorname{dist}(\zeta, \partial\Omega).\end{aligned}$$

The next step is to estimate $\omega(z, J\backslash J_0, \Omega\backslash(J-J_0))$. Let U_1, U_2 be the complementary domains of L. Suppose that Ω_0 is the component of $\Omega\backslash(J\backslash J_0)$ containing z, and $\Omega_i = \Omega_0 \cap U_i$, $i=1, 2$. Also we define $J_{i,j}$ to be the components of $(J\backslash J_1)\cap \partial\Omega_i$. Note that $J_{i,j}$ belongs to only one of the boundaries $\partial\Omega_i$.

The disk $\Delta(z, d\alpha/2M)$ contains no point of $J\backslash J_0$. We set $r = d\alpha/2M$. Since L is chord arc we have subarcs I_j of $\partial\Delta(z, r)$ such that for some $\rho, \tau > 0$ (depending only on M)

$$I_i \subset \Omega_i \tag{12}$$

$$\lambda(I_i) = \frac{\rho}{2}\pi r \tag{13}$$

$$\operatorname{dist}(I_i, L) > \tau r. \tag{14}$$

Therefore

$$\begin{aligned}&\omega(z, J\backslash J_0, \Omega\backslash(J\backslash J_0))\\ &\quad= \int_{\partial\Delta(z,r)} \omega(\zeta, J\backslash J_0, \Omega\backslash(J\backslash J_0))\, dw^z_{\Delta(z,r)}(\zeta)\\ &\quad\le (1-\rho) + \frac{\rho}{2} + \frac{\rho}{2}\min_{i=1,2}\max_{I_i}\omega(\zeta, J\backslash J_0, \Omega\backslash(J\backslash J_0))\end{aligned} \tag{15}$$

But for $\zeta \in I_i$

$$\omega(\zeta, J\backslash J_0, \Omega\backslash(J\backslash J_0)) \le \omega(\zeta, J, \Omega_i) \tag{16}$$

and we see from (15) that it is enough to show that

$$\min_i \max_{I_i} \omega(\zeta, J, \Omega_i) \le \beta. \tag{17}$$

We consider now two cases. Let $B > 1$ be a constant to be determined later; B will depend only on M.

In this first case we suppose there exists $\zeta_0 \in (L\backslash J_1) \cap \partial\Omega_0$ so that

$$|\zeta_0 - z| \leq Bd. \tag{18}$$

In this case we let (following [10]) $S_i = L\backslash(\bigcup_j J_{i,j})$. Clearly $L\backslash J_1 = S_1 \cup S_2$. From the maximum principle we obtain for $\zeta \in I_i$

$$1 - \omega(\zeta, J, \Omega_i) = \omega(\zeta, \partial\Omega_i\backslash J, \Omega_i) \geq \omega(\zeta, S_i, U_i). \tag{19}$$

Now we use Lemma 4 for the chord arc domain U_i. Since $\zeta_0 \in L\backslash J_1$ we have that

$$\max_i \lambda(\Delta(\zeta_0, r) \cap S_i) \geq \frac{r}{2}. \tag{20}$$

Consequently by (18), (20) and Lemma 4 we obtain that

$$\max_{i=1,2} \min_{\zeta \in I_i} \omega(\zeta, S_i, U_i) \geq \eta > 0. \tag{21}$$

where η depends only on B and the chord arc constant M and so only on M. Then from (21), (19), (17), (15) and (11) we get

$$\omega(z, K, \Omega\backslash K) \leq \frac{\beta_1}{1-\delta} \tag{22}$$

where $\beta_1 < 1$ depends only on M.

This leaves the case where for each $\zeta \in (L\backslash J_1) \cap \partial\Omega_0$ we have $|\zeta - z| > Bd$. Let $w \in \partial\Omega \cap \partial\Omega_0$, $|w - z| = d$. Notice that $w \notin L$ for $w \notin L\backslash J_1$ by our assumption (and $w \notin J_1$ as $w \notin \Omega$). We can join w to a point on $\partial\Delta(z, Bd)$ with a continuum

$$F \subset \partial\Omega \cap \partial\Omega_0 \cap (\bar{\Delta}(z, Bd)\backslash\Delta(z, d)),$$

because if not there would exist $w_1 \in \partial\Omega \cap \partial\Omega_0$, $|w_1 - z| < Bd$ and $w_1 \in L\backslash J_1$ contradicting our assumption in the second case. Since $F \subset \partial\Omega_0$ we have $F \cap (L\backslash J_1) = \phi$ and $F \cap J_1 = \phi$ so $F \cap L = \phi$.

Use a quasiconformal mapping Φ from $\mathbb{C}$ to $\mathbb{C}$ mapping $\mathbb{R}$ onto L, $\Phi(0) = z$, $\Phi(\infty) = \infty$. Also we may assume $|\Phi(1) - z| = r$. Since F does not meet L, without loss of generality $F \subset U_1$ and Φ maps H onto U_1. Now from the uniform bounds for quasiconformal mapping, (13) and (14) we obtain constants a, b, θ depending only on the chord arc constant M so that

$$\Phi^{-1}(I_1) \subset S(a, b, \theta). \tag{23}$$

Now, if $\zeta \in I_1$

$$\omega(\zeta, \partial\Omega_1 \backslash J, \Omega_1) \geq \omega(\zeta, F, U_1). \tag{24}$$

But if $E = \Phi^{-1}(F)$ then E is a continuum running from $|z| = c_1$ to $|z| = c_2$, where c_1 depends only on k and c_2 depends on M and B, and $c_2 \to \infty$ as $B \to \infty$. Consequently from Lemma 3, (23) and (24) show that if $B \geq B_0(M)$

$$\omega(\zeta, J, \Omega_1) \leq \beta(M) < 1$$

for each $\zeta \in I_1$, and so, as in the first case, we obtain

$$\omega(z, K, \Omega \backslash K) \leq \frac{\beta_2}{1-\delta} \tag{25}$$

where $\beta_2 < 1$ depends only on M.

Therefore we choose $\beta = \max(\beta_1, \beta_2)$ and $\delta < 1 - \beta$ and with $\varepsilon = c(\delta)$ see that the proof of the theorem is complete.

4. Proof of Theorem 2

We need the following (see [7]):

LEMMA 6. *Let U be a simply connected domain with rectifiable boundary, and f a conformal mapping of U onto Ω. Then for any quasismooth curve L*

$$\lambda(f^{-1}(\Omega \cap L)) < c_1 \lambda(\partial U).$$

Let g be the Riemann mapping from $\mathbb{D}$ to U. Thus by Theorem 1 arc length $d\mu$ on $g^{-1} \circ f^{-1}(\Omega \cap L)$ is a Carleson measure and hence as $g' \in H^1$

$$\lambda(f^{-1}(\Omega \cap L)) = \int_{\mathbb{D}} |g'| \, d\mu \leq c_2 \int_{\partial\mathbb{D}} |g'| \, d\theta = c_2 \lambda(\partial U)$$

which proves the lemma.

Now let Ω be a hyperbolic planar domain and $f:\mathbb{D}\to\Omega$ be the universal covering map. Suppose that G is the Fuchsian group of Möbius transformations $T:\mathbb{D}\to\mathbb{D}$ which represents the cover group for Ω. The Dirichlet fundamental region $\mathscr{D}$ for G is

$$\{z\in\mathbb{D}:|T'(z)|<1,\ \forall T\in G\backslash\{I\}\}.$$

Now $\mathscr{D}$ is a convex set in the hyperbolic metric with rectifiable boundary. Thus by Lemma 6

$$\lambda\{f^{-1}(\Omega\cap L)\cap\overline{T(\mathscr{D})}\}\le c_3\lambda(\partial T\mathscr{D}) \tag{26}$$

for any quasismooth curve L and $T\in G$. This immediately proves theorem 2.

Corollary 1 follows from

LEMMA 7. *Suppose that G is the Fuchsian group of a finitely connected planar domain with no point boundary components. Let $\mathscr{D}$ be the Dirichlet region for G. Then*

$$\sum_G \lambda(T\,\partial\mathscr{D})<\infty.$$

The boundary $\partial\mathscr{D}$ consists of a finite number of disjoint nonconcentric circles orthogonal to the unit disk. Let us denote $\mathscr{D}$ as $\mathscr{D}_1$. The region $\mathscr{D}_2$ is obtained from $\mathscr{D}$ by reflecting $\mathscr{D}$ through each of the orthogonal circles, and adding $\mathscr{D}_1$. At the n^{th} stage we obtain $\mathscr{D}_n$ with boundaries exactly n reflections of the original circles. Thus $\sum_G\lambda(T\mathscr{D})=\sum_{n=1}^{\infty}\lambda(\partial\mathscr{D}_n)$. We need

LEMMA 8. *There is a constant $\beta<1$ such that*

$$\lambda(\partial\mathscr{D}_n\cap\mathbb{D})<\pi\beta^n$$

Let $E_n=\partial\mathscr{D}_n\cap\mathbb{D}$, F_n be a circle of E_{n-1} and G_n the part of E_n separated from the rest of E_n by F_n. By conformal invariance there is $\beta<1$ such that dia $(G_n)\le\beta$ dia (F_n). However as G_n consists of orthogonal semicircles $\lambda(G_n)\le\pi$ dia (G_n). Summing over the components of E_{n-1} gives

$$\lambda(E_n)\le\beta\lambda(E_{n-1})$$

Thus we prove Lemmas 7, 8 and complete the proof of Corollary 1.

The necessary part of Corollary 2 is derived from using the real line as our curve L. In this symmetric situation $f^{-1}(\mathbb{R})$ is $\bigcup_G T(\partial\mathcal{D})$.

Corollary 3 follows immediately from

LEMMA 9. *With the notations above if Ω is a hyperbolic planar domain then*

$$\sum_{T\in G} \lambda(\partial T\mathcal{D}) \leq 2\pi\Big(\inf_{T\neq I} |T(0)|\Big)^{-1} \sum_{T\in G} (1-|T(0)|^2)^{1/2}$$

where C is a universal constant.

Proof. To see this we will associate to each side of $\mathcal{D}$ and the $T(\mathcal{D})$'s an element $R \in G$ in a 1-1 fashion and in such a way that if $z \in s$ then $\rho(z, R(0)) \leq \rho(z, 0)$, where ρ denotes hyperbolic distance in $\mathbb{D}$. This is enough because then s is contained in a euclidean disk of radius $|R(0)|^{-1}\,(1-|R(0)|^2)^{1/2}$ and so

$$\lambda(s) \leq \pi\,|R(0)|^{-1}\cdot(1-|R(0)|^2)^{1/2}.$$

So consider the side s. It separates two contiguous images of $\mathcal{D}$, say $A(\mathcal{D})$, $B(\mathcal{D})$ with $A, B \in G$.

The transformations $\{T_i\}$ in G which pairwise identify the sides of $\mathcal{D}$ generate G and in fact since Ω is planar G is freely generated by the $\{T_i\}$.

Now $A = B \circ T_0$ for some generator T_0 so that if $B = T_n \circ \cdots \circ T_1$ is a reduced word then the word length of A is $n-1$ or $n+1$ according to $T_1 = T_0^{-1}$ or $T_1 \neq T_0^{-1}$. Changing the roles of A and B we may assume that the latter case occurs and to s we associate $A = T_n \circ \cdots \circ T_1 \circ T_0$. Notice that $A = T_n \circ \cdots \circ T_1 \circ T_0$ determines s by being the side separating $T_n \circ \cdots \circ T_1(\mathcal{D})$ from $A(\mathcal{D})$. Finally $s \subset \partial A(\mathcal{D})$ and so for each $z \in s$ $\rho(z, A(0)) \leq \rho(z, 0)$.

REFERENCES

[1] AHLFORS, L. V., *Conformal Invariants,* McGraw Hill, New York, (1973).
[2] ——, *Lectures on quasiconformal mapping,* Van Nostrand, Princeton, (1966).
[3] BELNA, C., COHN, W., PIRANIAN, G. and STEPHENSON, K., *Level sets of special Blaschke products,* Michigan Math. J. *29* (1982), 79–81.
[4] COLLINGWOOD, E. F. and LOHWATER, A. J., *Theory of Cluster Sets,* Cambridge University Press, (1966).
[5] FLINN, B. B., *Hyperbolic Convexity and Level Sets of Analytic Functions,* Indiana Math. J. *32*(6) (1983), 831–841.
[6] GARNETT, J. B., *Bounded Analytic Functions,* Academic Press, New York, (1981).
[7] GARNETT, J. F., GEHRING, F. W. and JONES, P. W., *Conformal invariant length sums,* Indiana Math. J. *32*(6) (1983), 809–829.

[8] HAYMAN, W. K. and WU, J. M., *Level sets of univalent functions,* Comment Mat. Helv. *56* (1981), 366–403.

[9] JERISON, D. and KENIG, C., *Boundary behaviour of harmonic functions in non-tangentially accessible domains,* Advances in Math. *46* (1982), 80–147.

[10] KAUFMAN, R. and WU, J. M., *Distortion of the boundary under conformal mapping,* Mich. Math. J. *29* (1982), 267–280.

[11] MARTIO, O. and SARVAS, J., *Injectivity theorems in the plane and space,* Ann. Acad. Sci. Fenn. Ser. AI. Math. *4* (1979), 383–401.

[12] POMMERENKE, C., *Boundary behaviour of conformal mapping,* in *Aspects of Contemporary Complex Analysis,* edited by Brannan, D. and Clunie, J., Academic Press, New York, 1980.

[13] SULLIVAN, D., *The density at infinity of a discrete group of hyperbolic motions,* Publ. IHES *50* (1979), 419–450.

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