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Growth of the coefficient of quasiconformality and the boundary correspondence of automorphisms of a ball

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Abstract. A homeomorphism $f: B^n \rightarrow B^n$ of the unit ball in R^n ($n \geq 2$) whose coefficient of quasiconformality in the ball of radius $r < 1$ has asymptotic rate of growth $K(r) = \sup_{|x| \leq r} k(x, f) = O(\log(1/(1-r)))$ can be continued to a homeomorphism $\tilde{f}: \bar{B}^n \rightarrow \bar{B}^n$ of the closed ball \bar{B}^n . For $n = 2$ this implies that the Caratheodory theory of prime ends for conformal mappings also holds for the class of homeomorphisms $f: B^2 \rightarrow D$ with $K(r) = O(\log(1/(1-r)))$.

The following theorem was recently given by Zorič [10]:

If $f: B^2 \rightarrow B^2$ is an automorphism of the unit disc B^2 such that

$$\int_0^1 \frac{dr}{(1-r)K(r)} = \infty, \quad \int_0^1 K(r) dr < \infty,$$

where $K(r)$ is the coefficient of quasiconformality of f in the disc $B^2(r)$, then f can be extended to a *continuous* mapping $\tilde{f}: \bar{B}^2 \rightarrow \bar{B}^2$ of the closed disc \bar{B}^2 into itself.

Zorič [10] also made the conjecture that the above theorem holds for $n \geq 3$ with $K^{n-1}(r)$ instead of $K(r)$.

In this paper we prove that every homeomorphism $f: B^n \rightarrow B^n$ of the unit ball B^n ($n \geq 2$) such that $K(r) = O(\log(1/(1-r)))$, i.e. $K(r)$ increases as the logarithm, can be continued to a *homeomorphism* $\tilde{f}: \bar{B}^n \rightarrow \bar{B}^n$ of the closed ball \bar{B}^n . We also give some consequences of this statement.

Turn to the precise formulations.

Let D and D' be regions in euclidean space R^n and $f: D \rightarrow D'$ a homeomorphism. The number

$$k(x, f) = \limsup_{t \rightarrow 0} \frac{\max_{|y-x|=t} |f(y) - f(x)|}{\min_{|y-x|=t} |f(y) - f(x)|}$$

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will be called the coefficient of quasiconformality of f at $x \in D$. If D is the unit ball B^n let

$$K_f(r) = K(r) = \sup_{|x| \leq r} k(x, f).$$

In connection with the sequel recall that the coefficient of quasiconformality of a homeomorphism is a Borel measurable function (cf. [8]).

The rest of the notation and terminology that we use here is generally the same as in [8].

LEMMA 1. *Let $f: B^n \rightarrow B^n$ be a homeomorphism with $\int_0^1 K^{n-1}(r) dr < \infty$. Then $g = f^{-1}$ has a continuous extension $\bar{g}: \bar{B}^n \rightarrow \bar{B}^n$ of the closed ball \bar{B}^n into itself.*

Proof. Since $k(x, f)$ is bounded in every ball $B^n(r)$ of radius $0 < r < 1$, it follows that $k(y, g)$ is locally bounded and f is in the Sobolev space $W_{n, \text{loc}}^1(B^n)$, i.e. ACL^n in the sense of [8], (cf. [8], 32.3). So, for coordinate functions g^i , $1 \leq i \leq n$, of g we have (cf. [5], [6]):

$$\begin{aligned} \int_{B^n} |\nabla g^i|^n dy &\leq \int_{B^n} k^{n-1}(y, g) J(y, g) dy \leq \int_{B^n} k^{n-1}(x, f) dx \\ &\leq \int_{S^{n-1}} d\omega_{n-1} \int_0^1 r^{n-1} K^{n-1}(r) dr \leq \omega_{n-1} \int_0^1 K^{n-1}(r) dr < \infty. \end{aligned}$$

By the standard argument (for example in the same way as in proof of theorem 10.1 in [3]), one concludes the proof of the lemma.

LEMMA 2 (fundamental lemma). *Let F be a compact subset of the unit ball B^n , $b \in S^{n-1} = \partial B^n$ and Γ the family of all curves γ in B^n such that γ has a common point with F and contains b in its closure. Let $f: B^n \rightarrow D$ be a homeomorphism such that*

$$\int_0^1 \frac{dr}{(1-r)K(r)} = \infty, \tag{a}$$

and for some $m > 1$

$$\int_{1-t^m}^1 K^{n-1}(r) dr = o(t) \quad \text{when } t \rightarrow 0, (t > 0). \tag{b}$$

Then $M(\Gamma') = 0$, where $\Gamma' = f(\Gamma)$.

Proof. Let (r_k) be an increasing sequence in $[0, 1)$ such that $r_k \rightarrow 1$ when $k \rightarrow \infty$ and $F \subset B^n(r_0)$. Let Γ_k be the family whose elements are subcurves of elements of Γ that connect through the spherical ring $R_k = \{x \in R^n : 1 - r_k < |x - b| < 1 - r_{k-1}\}$ its boundary spheres $S^{n-1}(b, 1 - r_k)$ and $S^{n-1}(b, 1 - r_{k-1})$. The condition (a) (as well as (b)) implies $K(r) < \infty$ for $0 \leq r < 1$ and by theorem 32.3 in [8] a homeomorphism f is in the class $W_{n, \text{loc}}^1(B^n)$. Consequently, families $\Gamma'_k = f(\Gamma_k)$ are separate and $\Gamma' > \Gamma'_k$ (cf. [8]). Therefore [2]

$$\frac{1}{M^{1-n}}(\Gamma') \geq \sum_{k=0}^{\infty} \frac{1}{M^{1-n}}(\Gamma'_k). \quad (2)$$

Standard arguments yield (cf. [4], Lemma 1)

$$M(\Gamma'_k) \leq \int_{R_k \cap B^n} \rho^n(x) k^{n-1}(x, f) dx,$$

for every ρ admissible for Γ_k . If for ρ we choose the extremal function of the ring R_k then we obtain

$$M(\Gamma'_k) \leq \frac{1}{\left(\log \frac{1 - r_{k-1}}{1 - r_k}\right)^n} \int_{R_k \cap B^n} \frac{k^{n-1}(x, f)}{|x - b|^n} dx. \quad (3)$$

Let $(t, \omega) \xrightarrow{P} x$, $\omega \in S^{n-1}(b, 1)$ be the spherical coordinate system with origin in b . Let τ_m be the hypersurface defined by $x \in \tau_m$ if and only if $|x| = 1 - t^m$, where $m > 1$ is such that the condition (b) is satisfied. Denote by A_t the central projection from b of the set $S^{n-1}(b, t) \cap \bar{B}^n$ onto the unit sphere $S^{n-1}(b, 1)$, by $A'_t \subset A_t$ the projection of that part of the set $S^{n-1}(b, t) \cap B^n$ which lies inside of the surface τ_m and by A''_t the difference $A_t - A'_t$. Then, taking into account that $k^{n-1}(P(t, \omega)) \leq K^{n-1}(|P(t, \omega)|)$, we get

$$\begin{aligned} \int_{R_k \cap B^n} \frac{k^{n-1}(x, f)}{|x - b|^n} dx &\leq \int_{1-r_k}^{1-r_{k-1}} \frac{dt}{t} \int_{A_t \subset S^{n-1}(b, 1)} k^{n-1}(P(t, \omega)) dS^{n-1} \\ &\leq \int_{1-r_k}^{1-r_{k-1}} \frac{dt}{t} \int_{A_t} K^{n-1}(|P(t, \omega)|) dS^{n-1}. \end{aligned} \quad (4)$$

Further, for $1 - r_k \leq t \leq 1 - r_{k-1}$

$$\int_{A'_t} K^{n-1}(|P(t, \omega)|) dS^{n-1} \leq \omega_{n-1} K^{n-1}(1 - t^m), \quad (t_k = 1 - r_k), \quad (5)$$

and for $0 < t \leq 1 - r_0$

$$\int_{A'_t} K^{n-1}(|P(t, \omega)|) dS^{n-1} = \int_{S^{n-2}} dS^{n-2} \int_{\theta_\tau}^{\theta_S} K^{n-1}(r(t, \theta)) d\theta,$$

where θ is the angle between the vectors $x - b$ and $-b$, $r(t, \theta) = |x|$ and θ_τ and θ_S correspond to these points of $S^{n-1}(b, t)$ that lie on τ_m and S^{n-1} respectively. It is easy to see that for $t \neq 0$

$$d\theta = \frac{1 - 2t \cos \theta + t^2}{rt \sin \theta} dr.$$

Consequently, there exist $0 < t' < 1$ and a constant $c > 0$ such that

$$d\theta \leq c \frac{dr}{t} \quad \text{for } 0 < t \leq t', \quad \theta_\tau < \theta < \theta_S (\theta_S < \pi).$$

So we have for $0 < t \leq t'$

$$\int_{A'_t} K^{n-1}(|P(t, \omega)|) dS^{n-1} \leq \frac{c\omega_{n-2}}{t} \int_{r(t, \theta_\tau)}^1 K^{n-1}(r) dr, \quad (6)$$

with $r(t, \theta_\tau) = 1 - t^m$. According to (b) there exist $0 < t'' < 1$ and $c_1 > 0$ such that

$$\int_{1-t^m}^1 K^{n-1}(r) dr \leq c_1 t \quad \text{for } 0 < t \leq t''. \quad (7)$$

Let $t_0 = \min \{t', t''\}$. Then from (6) and (7) it follows that

$$\int_{A''_t} K^{n-1}(|P(t, \omega)|) dS^{n-1} \leq \tilde{c} \omega_{n-2}, \quad (8)$$

for $0 < t \leq t_0$ and some $\tilde{c} > 0$. From (3), (4), (5) and (8) it follows that there exist $C > 0$ and $0 \leq R_0 < 1$ such that

$$M(\Gamma'_k) \leq C \frac{K^{n-1}(1 - (1 - r_k)^m)}{\left(\log \frac{1 - r_{k-1}}{1 - r_k}\right)^{n-1}} \quad (9)$$

whenever $r_{k-1} \geq R_0$. From (2) and (9) one gets

$$M^{1/1-n}(\Gamma') \geq C^{1/1-n} \sum_{r_{k-1} \geq R_0} \frac{\ln(1-r_{k-1}) - \ln(1-r_k)}{K(1-(1-r_k)^m)}$$

for every increasing sequence (r_k) , $r_k \rightarrow 1$. It follows that

$$M^{1/1-n}(\Gamma') \geq M \int_R^1 \frac{dr}{(1-r)K(1-(1-r)^m)},$$

for some $M > 0$ and $R \geq 0$. Changing variable by $1 - (1-r)^m = u$ we finally have

$$M^{1/1-n}(\Gamma') \geq \frac{M}{m} \int^1 \frac{dr}{(1-r)K(r)}. \quad (10)$$

If $M(\Gamma') > 0$ it follows from (10) that the integral in (a) converges. This yields a contradiction and the proof of the lemma is complete.

LEMMA 3. *Let $f: B^n \rightarrow D$ be a homeomorphism such that $\int^1 K^{n-1}(r) dr < \infty$. Then D is a proper subset of R^n .*

Proof. Suppose on the contrary that $D = R^n$. Let p, q be two different points of the unit sphere $S^{n-1} = \partial B^n$, let s be a fixed element of $(0, 1)$ and Γ the family of curves which through B^n join the segments $[sp, p)$ and $[sq, q)$. Let a be the distance between the points sp and sq . Then the function $x \mapsto \rho(x) = 1/a$ is admissible for Γ . Let $\Gamma' = f(\Gamma)$. Then we have

$$\begin{aligned} M(\Gamma') &\leq \int_{B^n} \rho^n(x) k^{n-1}(x, f) dx \leq \frac{1}{a^n} \int_{B^n} k^{n-1}(x, f) dx \\ &\leq \frac{\omega_{n-1}}{a^n} \int_0^1 K^{n-1}(r) dr < \infty. \end{aligned}$$

On the other side, since the modulus of curve family is a conformal invariant, we can suppose that Γ' is the family of curves which join two arcs that begin in the same point of R^n . This implies $M(\Gamma') \geq c \log(b/t)$ for a fixed b and each $0 < t \leq b$ (cf. [8], 10.12). This is a contradiction.

THEOREM 1. *If $f: B^n \rightarrow B^n$ is a homeomorphism with $K(r) = O(\log(1/(1-r)))$ than f can be extended to a homeomorphism $\bar{f}: \bar{B}^n \rightarrow \bar{B}^n$ of the closed ball \bar{B}^n .*

Proof. Let $b \in S^{n-1} = \partial B^n$. Let $C(f, b)$ be the cluster set of f at b . Let F, Γ, Γ' be as in Lemma 2, F being connected and having more than one point. Since the family $\Delta(f(F), C(f, b), B^n)$ of all curves that join $f(F)$ and $C(f, b)$ through B^n is a subfamily of Γ' , because of the monotonicity of the modulus and Lemma 2, we obtain $M(\Delta(f(F), C(f, b), B^n)) = 0$. (If $K(r) = O(\log(1/1-r))$ then the conditions (a) and (b) of Lemma 2 are satisfied). Since $C(f, b)$ is connected this means that $C(f, b)$ has exactly one point. It follows that f has a continuous extension $\tilde{f}: \bar{B}^n \rightarrow \bar{B}^n$. On the base of Lemma 1 we conclude that f is a homeomorphism.

Remarks. 1) Theorem 1 was in fact proved under the hypothesis (a) and (b) of Lemma 2. But the condition (b) is slightly stronger than the condition $\int_0^1 K^{n-1}(r) dr < \infty$ in [10]. 2) If $K(r)$ increases faster than $\log(1/1-r)$ then Theorem 1 does not hold. According to [10] for every nondecreasing function h such that

$$\int_0^1 \frac{dr}{(1-r)h(r)} < \infty \quad \text{or} \quad \int_0^1 \frac{dr}{(1-r)h(r)} = \infty \quad \text{and} \quad \int_0^1 h^{n-1}(r) dr = \infty$$

there exists a diffeomorphism $f: B^n \rightarrow B^n$ with $K(r) \leq h(r)$ having no continuous extension from \bar{B}^n into itself. 3) Theorem 1 also holds (under the conditions (a) and (b)) if we replace the ball B^n in the range by a region D which has property P_2 on the boundary (cf. [8], 17.5 and 17.15).

It was pointed out in [10] that the question about boundary behavior of different classes of homeomorphisms in the plane is reduced, from the metrical point of view, to the study of boundary behavior of automorphisms of a disc B^2 . (This is a consequence of the Riemann mapping theorem and the Caratheodory theory of prime ends).

THEOREM 2. *For the class of locally quasiconformal mappings $f: B^2 \rightarrow D$ which satisfy the condition $K(r) = O(\log(1/1-r))$ the Caratheodory theory of prime ends holds.*

Proof. It is enough to show that a region D is conformally equivalent to the unit disc B^2 . But that is a consequence of Lemma 3.

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