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Minimal surfaces in foliated manifolds

JOEL HASS

Introduction

In this paper the interaction between minimal surfaces and foliations of manifolds is studied. Techniques from the theory of foliations are used to give information about minimal surfaces, and minimal surface techniques to examine foliations.

The main idea is to introduce a special metric in which the leaves of a foliation are either minimal surfaces or totally geodesic. Finding such metrics is often possible, via results of Sullivan [Su]. In any case, such metrics exist in manifolds such as R^n and H^n, which have totally geodesic foliations. The minimal surfaces in such minimally foliated manifolds must then intersect the foliation in a restricted manner.

Section one contains some preliminary lemmas about the nature of minimal surfaces and how they intersect, and also about the nature of their singularities, branch points.

Section two examines Euler characteristic information obtained by examining the induced foliation on a minimal surface in a minimally foliated 3-manifold. Exploiting this leads to a simple proof of Novikov's theorems on foliation [N] via Theorem 2.7, which finds obstructions to the existence of a metric making each leaf of a foliation minimal. A key technical point arising is to show that singular points of the induced foliation on a minimal surface in a minimally foliated manifold are isolated. This is done in a series of lemmas that deal with both immersed and branched minimal surfaces, and with interior and boundary singularities. The section concludes with some constructions of minimal foliations of hyperbolic space.

In Section 3 the Bernstein problem is examined. In R^3 the Bernstein theorem states that a complete minimal graph is a plane. This has been generalized in [S–Y II] and [F-S] to show that a stable, complete, orientable minimal surface in

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a 3-manifold of non-negative Ricci curvature is totally geodesic. We consider the corresponding question in the eight 3-dimensional geometries. Counterexamples are known for H^3 [A], [H I], [W–W]. We show it to be false, except in the geometries of non-negative Ricci curvature, by constructing counter-examples in Nil, Sol, $SL(2, \mathrm{R})$ and $H^2 \times R$. The technique is to find foliations by minimal surfaces which are not totally geodesic.

In Section 4 we examine branch points of minimal surfaces. In the presence of a minimal foliation, a branch point induces certain types of singularities in the induced foliation. This enables us to count how many branch points can occur in certain situations. In Euclidean space, the number of critical points for a height function on a curve implies bound on the number of branch points for a spanning minimal surface. Via Milnor's results on the relationship of curvature and height functions, we obtain bounds on the number of branch points for a spanning minimal surface in terms of the total curvature of a boundary curve, generalizing results of [Ni] and [H–H]. The above results also apply in a more general foliated manifold than R^3.

In Section 5 we generalize the results to higher dimensions. Here the appropriate category is a totally geodesic foliation of codimension one. Using these, we show that certain least area surfaces in certain 4-manifolds have no branch points, analogous to the 3-manifold case proved in [O] and [G]. We also obtain a topological theorem analogous to the sphere theorem of 3-manifold theory [P], [St]. This result gives the existence of a non-trivial embedded 2-sphere in a 4-manifold with non-trivial second homotopy group which admits a totally geodesic codimension one foliation.

Finally, in the Appendix, we prove a simplified version of Sullivan's theorem and extend it to the case of a manifold with boundary. We then show how minimal surface techniques imply the existence of Reeb components in certain foliations.

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§1. Preliminaries

We will assume that all maps and manifolds are smooth unless otherwise specified. A surface is *minimal* if it is critical for the area functional under normal variation. A surface is *least area* if it is compact and has less area than any homotopic surface (rel boundary). A surface is *area minimizing* if it is least area on compact sets. A surface is *homology area minimizing* if any compact subsurface has less area than any competing homologous surface (rel boundary). The foliations we work with will always be assumed orientable. We say that a foliation of a Riemannian manifold is *minimal* if each leaf is a minimal surface. Let f be a map from a surface F with boundary ∂F to a manifold M with boundary ∂M. We say f is *proper* if

1) $f^{-1}(\partial M) = \partial F$
2) For any compact set $K \subset M$, $f^{-1}(K)$ is compact.

The following lemma generalizes the "Maximal Principle for minimal surfaces".

LEMMA 1.1. *Let M be a Riemannian 3-manifold and let F_1, F_2 be surfaces. Let $p_i \in F_i$, f_i: $F_i \to M$, $i = 1, 2$ be minimal immersions with $f_1(p_1) = f_2(p_2)$, and such that the tangent planes of the two immersions agree at $f_1(p_1)$. Let (x_1, x_2, x_3) be any smooth coordinates in a neighborhood V of $f_1(p_1)$ such that $f_1(p_1) = (0, 0, 0)$ and such that $f_i(F_i)$ restrict to graphs over the (x_1, x_2) plane in V, of functions ϕ_1, ϕ_2 respectively. Then either $\phi_1 = \phi_2$ in a neighborhood of $(0, 0, 0)$ or in V there is a C^1 change of coordinates in the (x_1, x_2) plane such that in the new coordinates the function $\phi_1 - \phi_2$ is given by $C \cdot \mathrm{Re}\,(z^n)$ where $z = x_1 + ix_2$ and n is an integer, $n \geq 2$, C is some non-zero constant.*

Proof. This is proved in Section 1 of [F–H–S].

Remark. It follows that pairs of minimal surfaces either coincide on open sets or have isolated tangencies. However more must be said to understand tangencies of families of minimal surfaces. A key point used later about Lemma 1.1 is that we need to change the x_1, x_2 coordinates but not the x_3 coordinate.

LEMMA 1.2. *Let $(M, \partial M)$ be a compact Riemannian 3-manifold whose boundary has zero mean curvature. Let $(F, \partial F)$ be a properly immersed minimal surface in a neighborhood of a point $x \in \partial M$. Then F is transverse to ∂M at x.*

Proof. F does not lie completely in ∂M as we assume it's proper. Suppose F is tangent to ∂M at $x \in \partial F \cap \partial M$. We pick (x_1, x_2, x_3) coordinates near x so that $x = (0, 0, 0)$, ∂F is the x_1-axis, $\partial M =$ the $x_1 - x_2$ plane. Then since F is smooth and tangent to ∂M at x, we can find a small neighborhood where it is a non-negative graph ϕ over the 1/2 plane $\{x_3 = 0, x_2 > 0\}$. ϕ satisfies a homogeneous linear elliptic P.D.E. $(a^{ij}(x_1, x_2)\phi_{x_i})_{x_j} = 0$, $a^{ij}(0, 0) = \delta^{ij}$ [G]. Also, we have $\nabla\phi(x_1, x_2) \to 0$ as $(x_1, x_2) \to (0, 0)$. It follows from the Hopf boundary point lemma [P–W, p. 67] that ϕ changes signs in every neighborhood of $(0, 0)$ in

$\{x_2 > 0, x_3 = 0\}$. But ϕ is non-negative in such neighborhoods by construction, so we have a contradiction.

LEMMA 1.3. *Let M^n be a Riemannian manifold, let $(F, \partial F)$ be a compact surface and let $f: F^2 \to M$ be a branched minimal immersion. Then the branch points of f are isolated. If $x \in F$ is an interior branch point, then there are coordinates $(x_1, x_2 \cdots x_n)$ about $f(x)$ with $f(x) = (0, \ldots, 0)$ such that if $\pi(x_1, x_2, x_3, \ldots, x_n) = (x_1, x_2)$ is the projection to the first two coordinates, then $\pi \circ f$ is a finite sheeted branched cover in some neighborhood of x, and there are at least two sheets. Similarly, if $x \in \partial F$ is a boundary branch point, we can find coordinates $(x_1, \ldots, x_n)$ with $f(\partial F) =$ the x_1-axis, and $\pi \circ f$ a branched cover which is k-sheeted on $\{x_2 > 0\}$ and $k - 1$ sheeted on $\{x_2 < 0\}$, in some neighborhood of x. Moreover, the tangent plane to $f(F)$, $T_{f(y)}f(F)$, smoothly approaches the (x_1, x_2) plane as $y \to x$.*

Proof. For interior branch points this is a consequence of the normal form of f near a branch point [G]. For boundary branch points there are similar expansions [Ni I].

We will say that a boundary branch point has *degree k* if $\pi \circ f$ is a k-sheeted cover on $(x_2 > 0)$ and $(k - 1)$-sheeted on $\{x_2 < 0\}$.

§2. Two dimensional foliations

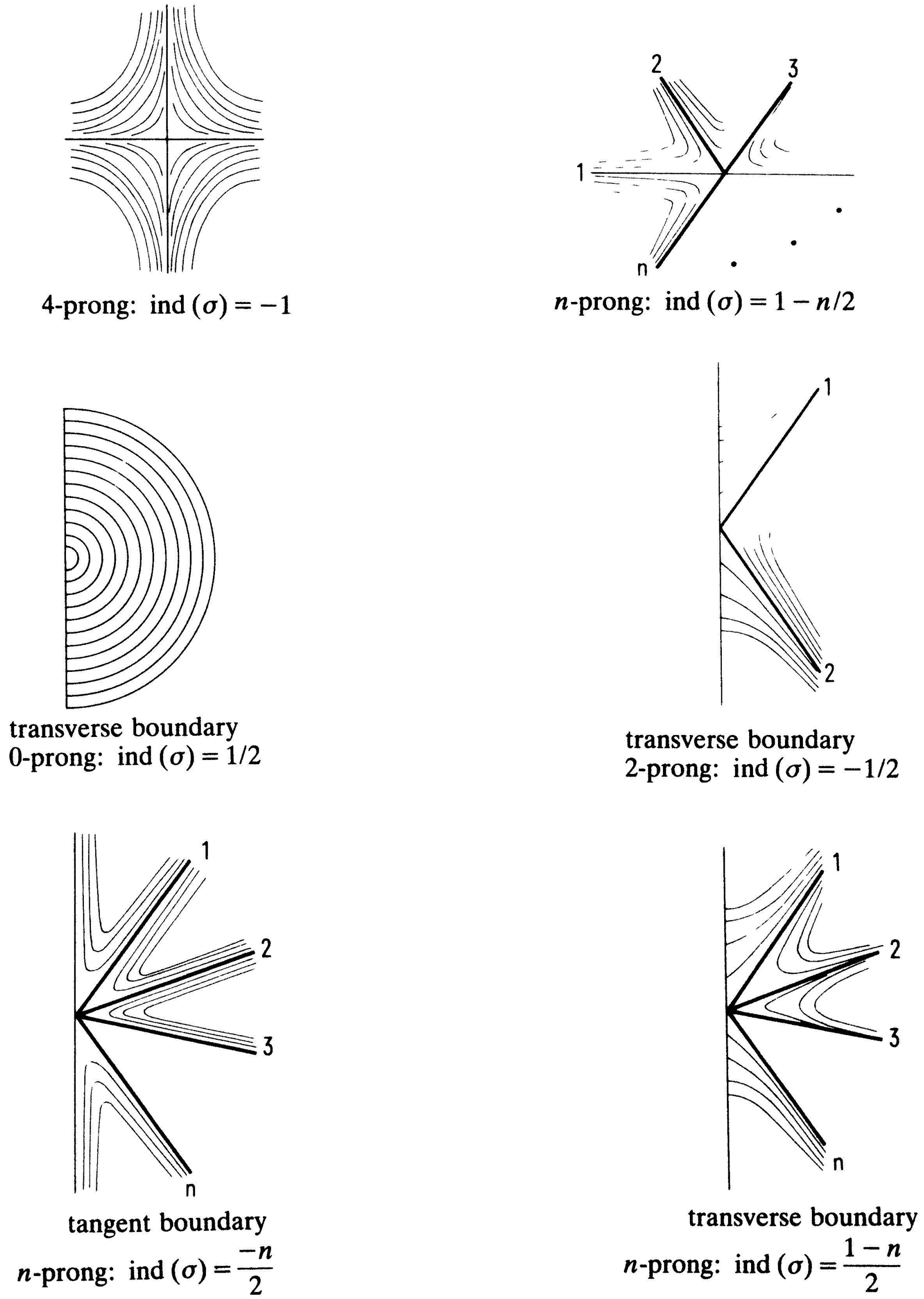
In this section we will use the tools of minimal surface theory to examine codimension one foliations. The philosophy behind this is that the minimal surface often picks out a canonically positioned surface, and one does not need to move such a surface by an isotopy or homotopy before applying a topological argument or technique.

The basic technique we will use is to examine the induced foliation on a minimal surface inside a manifold with a codimensions one foliation. For such an induced foliation to be advantageous, we need a geometric structure on the foliation. For a 2-dimensional foliation of a 3-manifold, we need the leaves to each be minimal surfaces. In higher dimensions we need the leaves to be totally geodesic. These conditions are not as restrictive as one might think.

We will show that the induced foliation on the surface has isolated singularities. These cause contributions to the Euler characteristic of the surface as indicated in the next lemma.

LEMMA 2.1. *The Euler characteristic χ of a compact surface with a codimension one foliation containing k isolated singularities σ_i satisfies $\chi = \sum_{i=1}^{k} \operatorname{ind}(\sigma_i)$ where* $\operatorname{ind}(\sigma_i)$ *is computed as in Figure* 1.

Proof. This result is well-known and follows easily from the Hopf Index Theorem.



4-prong: $\operatorname{ind}(\sigma) = -1$

n-prong: $\operatorname{ind}(\sigma) = 1 - n/2$

transverse boundary
0-prong: $\operatorname{ind}(\sigma) = 1/2$

transverse boundary
2-prong: $\operatorname{ind}(\sigma) = -1/2$

tangent boundary
n-prong: $\operatorname{ind}(\sigma) = \dfrac{-n}{2}$

transverse boundary
n-prong: $\operatorname{ind}(\sigma) = \dfrac{1-n}{2}$

Figure 1

Note. Other types of singularities can occur in general.

A foliation of a Riemannian 3-manifold is called a *minimal foliation* if every leaf is a minimal surface. Such foliations are sometimes called *geometrically taut.* The following lemma indicates when one can find such foliations. It is due to Sullivan [Su]. A proof is given in the Appendix.

THEOREM 2.2. *Let M be a closed orientable 3-manifold and let $\mathscr{F}$ be a codimension one foliation of M. M admits a metric such that every leaf of $\mathscr{F}$ is a minimal surface if and only if every compact leaf of $\mathscr{F}$ intersects some closed curve which is transverse to $\mathscr{F}$.*

Note. It follows that if $\mathscr{F}$ has no compact leaves then M admits a metric making $\mathscr{F}$ minimal.

The type of foliation induced on a minimal surface which is in a minimally foliated 3-manifold is of very restricted type. A key technical point which must be settled before analyzing the induced foliation is that singularities of the induced foliations are isolated.

LEMMA 2.3. *Let $\mathscr{F}$ be a minimal foliation in a 3-manifold M. Let F be an immersed minimal surface in M. The induced foliation on F is either*

1. *Trivial, and F lies in a leaf of $\mathscr{F}$ or*
2. *Non-trivial, and non-singular in interior(F) except at isolated points where singularities of $2n$-prong type occur, $n \geqslant 2$.*

Proof. Assume that F does not lie in a leaf. If F meets $\mathscr{F}$ transversely at x, then so does some neighborhood of x in F. If F meets $\mathscr{F}$ non-transversely at x, let L_0 be the leaf of $\mathscr{F}$ meeting x. Then the picture for the intersection of F and L_0 is given by Lemma 1.1, so that the intersection of F and L_0 has isolated tangencies. However, we must show that every leaf of $\mathscr{F}$ is transverse to F in a deleted neighborhood of x in F.

Pick coordinates (x^1, x^2, x^3) in a neighborhood of x in M so that the tangent space to F at x is tangent to $\{x^3 = 0\}$, and so that $\{x^3 = c\}$ is part of a leaf of $\mathscr{F}$. By appropriate choice of such coordinates, it was shown in Lemma 1.1 that up to a C^1-diffeomorphism, F is given by $x^3 = \text{Re}\,(x^1 + ix^2)^n$ in a small neighborhood of x. Since this graph is transverse to $\{x^3 = c\}$ except at the origin, the result follows.

We now state a similar lemma analyzing what happens for bounded minimal surfaces near their boundary.

LEMMA 2.4. *Let $\mathcal{F}$ be a minimal foliation in a 3-manifold M and let $(F, \partial F)$ be a minimal immersed surface in M. If a component of ∂F is transverse to $\mathcal{F}$ then there are no singularities in the induced foliation near this component of ∂F. If a component of ∂F lies in a leaf of $\mathcal{F}$, then singularities on this component are isolated, and are of tangent boundary n-prong type, $n \geqslant 1$, or F lies entirely within a leaf of $\mathcal{F}$.*

Proof. The first statement follows from the observation that transversality is an open condition. So assume that a component C of ∂F lies in a leaf of $\mathcal{F}$. Suppose that there is a point $x \in C$ such that each deleted neighborhood of x in F contains a point of tangency of F with a leaf of the foliation. Such points consist of either a sequence of interior singularities of the induced foliation of F, approaching x, or a sequence of boundary tangencies approaching x.

To study the latter case we consider $L_0 \cap F$, where L_0 is the leaf containing the boundary component C of F containing x and TF_x is tangent to L_0. Pick a coordinate neighborhood U of x with coordinates (x^1, x^2, x^3) so that the disks $\{x^3 = c\}$ give the leaves of $\mathcal{F}$, L_0 is given by $\{x^3 = 0\}$ and $F \cap U$ is a disk which is a graph $x^3 = \phi(x^1, x^2)$. If $F \cap L_0$ contains a closed curve in U, then there would be a leaf L_i in U which F meets at an interior point and which lies on one side of F locally. This contradicts the maximal principle described in Lemma 1.1. Similarly no other leaf intersects F in a closed curve in U. It follows that the intersection of F and L_0 in U consists of a number of graphs which are trees, with vertices consisting of n-prongs of the induced foliation. If there is a non-finite number of such graphs, then they accumulate in $F \cap U$, either to $C \subset \partial F$ or to a part of some tree in interior(F), as the set $F \cap L_0$ is closed. Note that if the graphs accumulate to a point $y \in C$ then they also accumulate to the entire arc between x and y on C, as otherwise there would be a violation of the maximal principle as above. In this case, there is an open arc of C where F agrees with L_0 and where the two tangent planes agree. This implies that F lies in L_0 by Lemma 5 of [M–Y 1] and Lemma 1.1. Consider now the case of an interior accumulation point. An interior point cannot be an accumulation point of distinct components of $F \cap L_0$ by Lemma 1.1. Thus it follows that there are only a finite number of trees of $L_0 \cap F$ in $F \cap U$. It also follows that these trees have only a finite number of vertices, as otherwise there would again be an infinite number of arcs leaving U, forcing an accumulation point. Thus x is isolated on C as a tangency point, and by picking U smaller, we can assume that $(L_0 \cap F) \cap U$ consists of a finite number of arcs A_j with one boundary point at x, together with C. Moreover, the number of arcs is at least one, as if no such arcs exist near x, then a small neighborhood of x in F lies on one side of L_0. It then follows from Lemma 1.2 that F is transverse to L_0 at x, contradicting our assertion that x is a point of tangency. To conclude the proof we need to show that there are no interior tangencies of F with $\mathscr{F}$ accumulating to x. Suppose a sequence of tangency points $\{x_i\}$ of leaves L_i with F accumulated to x. As $i \to \infty$, $L_i \to L_0$. We restrict attention to a small 1/2 disk D in F about x. $\partial D \cap L_0$ consists of $(\partial D \cap C)$ together with a finite number of points y_j. We can assume by taking D small, that each point of ∂D meets L_0 transversely, except for x. Thus we know that the local picture for $\mathscr{F} \cap D$ is as depicted in Figure 2, away from a neighborhood of x.

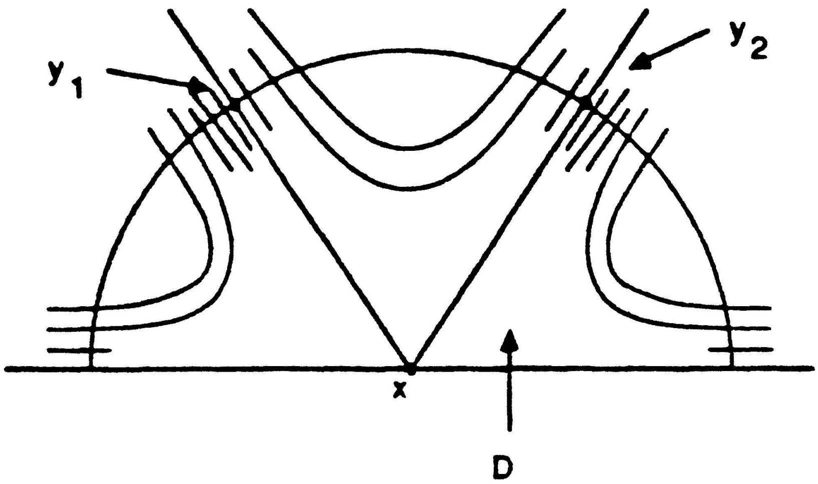


Figure 2

Since $L_i \to L_0$, for i sufficiently large $L_i \cap \partial D$ is arbitrarily close to the points $\{y_j\}$. Consider an L_i with $L_i \cap \partial D$ contained in a transverse neighborhood of $\{y_j\}$, as depicted. Since L_i is tangent to D to x_i, there are at least 4 arcs, B_1, B_2, B_3, B_4 leaving x_i and running out to $\partial D - C$ from x_i as in Figure 3, or giving a closed loop in D, or accumulating in D. Since we are working in a small neighborhood of x, each leaf intersects this neighborhood in a disk and thus its intersection with D is compact and can not have an accumulation point. It can not have a closed

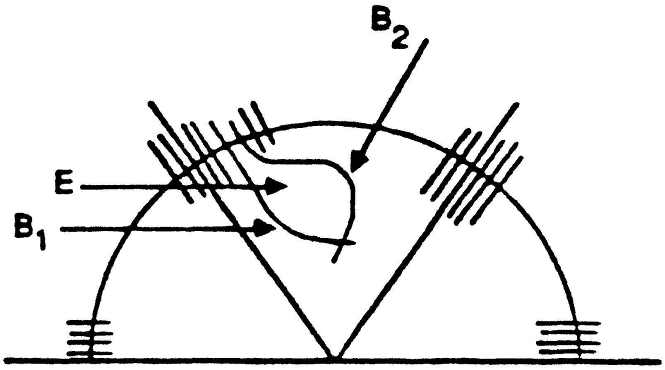


Figure 3

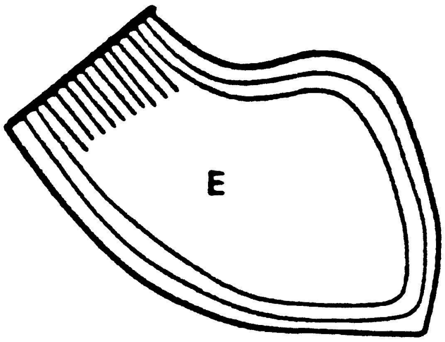


Figure 4

loop by the maximal principle. Thus each of B_1, B_2, B_3, B_4 run out to $\partial D - C$. At least two of these arcs, say B_1 and B_2, run to a transverse neighborhood of the same point y_k. These cut off a subdisk E of D. The induced foliation on E has a finite number of singularities as it is a bounded distance from the boundary of D. This induced foliation is as in Figure 4 near the boundary.

By doubling along $\partial E \cap \partial D$ we obtain a foliation of the 2-disk with no boundary singularities and only n-prong interior singularities. This contradicts $\chi(D^2) = +1$, and so we have a contradiction to the assumption of an accumulation point at x. The lemma now follows.

The above lemma does not apply if the boundary point is a branch point of the minimal surface F. Interior branch points can not accumulate to x unless x is a boundary branch point and thus did not need to be considered in Lemma 2.4. We deal with this possibility in the following lemma.

LEMMA 2.5. *Let $\mathcal{F}$ be a minimal foliation in a 3-manifold M and let $(F, \partial F)$ be a minimal surface in M (possibily having branch points on the boundary). If a component C of ∂F is transverse to $\mathcal{F}$ then the induced foliation is non-singular on ∂F except at isolated boundary branch points of F where there are transverse boundary n-prong singularities, $n \geq 3$.*

Proof. Suppose that C is transverse to $\mathcal{F}$. We must consider the induced foliation in a neighborhood of a boundary branch point x. The behavior of a least area surface with smooth boundary near a boundary branch point is described in Lemma 1.3. There is a well-defined tangent plane at the branch point. As $\mathcal{F}$ is transverse to this plane at x, and branch points are isolated, the induced foliation has an isolated singularity at x. Let (x^1, x^2, x^3) be coordinate about x such that TF_x is tangent to $\{x^3 = 0\}$. The projection to the (x^1, x^2) plane gives a map of degree k on $\{x^2 > 0\}$ and $k-1$ on $\{x^2 < 0\}$ of F to a neighborhood of $(0, 0)$, as F is locally a graph over the (x^1, x^2) plane away from $(0, 0)$. The intersection with F in a neighborhood of x of a surface transverse to the (x^1, x^2) plane and transverse to ∂F consists of $(2k-1)$ arcs converging to the origin. Thus there is a transverse boundary $(2k-1)$-prong.

LEMMA 2.6. *Let M be a Riemannian 3-manifold with a minimal foliation $\mathcal{F}$, let $(F, \partial F)$ be a minimal surface in M with ∂F contained in a leaf L_0 of $\mathcal{F}$ and suppose that F is tangent to L_0 at a boundary branch point x. Then there is a neighborhood U of x such that x is the only point of tangency of a neighborhoood of x in F and $\mathcal{F}$ in U, or F coincides with L_0 in U. Moreover, in the former case the induced foliation at x has a tangent boundary n-prong for some $n \geqslant 1$.*

Proof. We can pick a small neighborhood U with coordinates (x_1, x_2, x_3) so that $x = (0, 0, 0)$, $L_0 = \{x_3 = 0\}$, the leaves of $\mathcal{F}$ are given by $\{x_3 = \text{constant}\}$, $\partial F = \{x_3 = x_2 = 0\}$ and the projection of F to the (x_1, x_2) plane is a cover with degree k on $\{x_2 > 0\}$ and $k-1$ on $\{x_2 < 0\}$.

Consider now the induced foliation on F obtained by intersecting with the leaves of $\mathcal{F}$ in U. This foliation contains no closed curves in some neighborhood of x in F. To see this, it is convenient to pass to k-fold branched cover $\tilde{U}$ of U, branched along the x_3-axis. This has a smooth metric away from the pre-image of the x_3-axis, and is foliated by the pre-images of the leaves of $\mathcal{F}$. $F \cap U$ lifts to an embedded disk $\tilde{F}$ in $\tilde{U}$ and the induced foliation on $\tilde{F}$ in $\tilde{U}$ is the same as that on F in U. If there is a closed curve in this foliation, then a curve on a leaf $\tilde{L}_1$ in $\tilde{U}$ bounds two distinct disks, each embedded and minimal, one on $\tilde{L}_1$ and one on F. The maximal principle gives a contradiction as the disk on F must meet a leaf in a manner contradicting Lemma 1.1, namely the last leaf it meets. Note that it is irrelevant whether $L_1 = L_0$ or whether the branch locus in on this closed curve. Thus the induced foliation has no closed curves. The techniques of Lemma 2.4 now show that the branch point is an isolated tangency point, and the induced foliation has a tangent boundary n-prong for some $n \geqslant 1$. Note that F does not lie on one side of L_0, by the Hopf boundary maximum principle, in any neighborhood of x, as in Lemma 1.2.

Remark. The previous lemmas have concentrated on the intersection of a minimal surface with the particular leaf on which a branch point lies. They imply that the nearby leaves intersect transversely and thus in smooth curves. It is then straightforward to construct a homeomorphism of a neighborhood of the singularity so the local picture corresponds to some standard model for a prong singularity as given in Lemma 2.1.

We will apply these lemmas now to find conditions on a foliation which are obstructions to finding some metric in which it is a minimal foliation.

THEOREM 2.7. *Let M be a compact, closed 3-manifold with a 2-dimensional foliation $\mathcal{F}$. Any of the following conditions are obstructions to the existence of a metric on M in which each leaf of $\mathcal{F}$ is minimal.*

i) *M contains a null-homotopic curve transverse to the foliation.*

ii) *$\pi_1(M)$ is finite.*

(iii) *$\pi_2(M)$ is non-trivial, and $\mathcal{F}$ contains no leaves which are 2-spheres or projective planes.*

iv) *M is reducible, and $\mathcal{F}$ contains no leaves which are 2-spheres or projective planes.*

v) *There is a compressible leaf in M, i.e. a leaf L such that $\pi_1(L)$ does not inject into $\pi_1(M)$.*

Remark. If $\mathcal{F}$ contains a leaf with finite π_1 then all the leaves of $\mathcal{F}$ have finite π_1 [Re]. It follows that M is one of $S^2 \times S^1$, $S^2 \tilde{\times} S^1$, $P^2 \times S^1$ or $P^3 \# P^3$, as M is a bundle over a 1-dimensional orbifold with the leaves as fibers.

Proof. Suppose there exists a foliation on M making each leaf minimal. We will show that assuming any of i–v will lead to a contradiction.

i) In this case, let D be a least area disk bounding the null homotopic transversal. Such a disk exists [Mo] and is immersed in its interior [O]. The induced foliation on D has isolated singularities, by Lemmas 2.3 and 2.4. Using Lemma 2.1, and $\chi(D) = +1$, we get a contradiction, as the induced foliation on a minimal disk has only singularities of n-prong or transverse boundary n-prong type, $n \geqslant 2$.

ii) $\pi_1(M)$ is finite. Put a point in M and consider its orbit under the normal flow to the leaves of $\mathcal{F}$. Thus orbit has an accumulation point as M is compact. A small perturbation gives a closed transverse curve to the foliation. Taking a finite multiple of such a curve gives a null homotopic closed transversal, which after a small perturbation can be taken to be embedded. We now apply case i) to get a contradiction.

iii) If $\pi_2(M) \neq 0$, then a result of Sacks and Uhlenbeck [S–U] shows that there is a minimal 2-sphere immersed in M. Again, the induced foliation has isolated singularities, each of which contributes a negative number to $\chi(S^2) = +2$, by Lemma 2.1. This is a contradiction.

iv) If M is reducible, the existence of an embedded minimal 2-sphere has been established by Meeks, Simon and Yau [M–S–Y]. The result now follows as in case iii.

v) If M has a compressible leaf L, let C be a simple closed curve on L bounding a compressing disk. Let D be a least area disk bounding L. D can not lie in L as ∂D is non-trivial in L, so the induced foliation on D is non-trivial. We now get a contradiction as before by applying Lemmas 2.1, 2.4, 2.5 and 2.6.

Note. We can extend some parts of this theorem to non-compact 3-manifolds. All conditions except (ii) are obstructions as long as M is required to possess a homogeneously regular metric in the sense of [M–S–Y]. There exist foliations of R^3 with all leaves minimal so (ii) is not an obstruction in the non-compact case. For conditions (iii) and (iv) we must also require that M covers a compact manifold to establish the existence of minimal 2-spheres, and rule out $S^2 \times R$ and $P^2 \times R$.

COROLLARY 2.8 (Nokikov). *Let $\mathcal{F}$ be a foliation of a compact 3-manifold M. If one of the conditions i)–v) of Theorem 2.5 holds. then $\mathcal{F}$ has a compact leaf.*

Proof. If not, then every leaf of $\mathcal{F}$ is non-compact. It follows from Lemma 2.2 that $\mathcal{F}$ is minimal in some metric, contradicting Theorem 2.5.

Note. Novikov stated a somewhat different result, but his techniques, which were purely topological, can be extended to prove the above result, c.f. [Ro]. In particular, Novikov did not consider case (iv). See the appendix for an extension of Corollary 2.8 to show the existence of Reeb components.

We can use the techniques developed here to further analyze surfaces in foliated 3-manifolds. For example we prove the following.

THEOREM 2.9. *Let F be a closed minimal surface in a minimally foliated 3-manifold. If F is a 2-sphere or projective plane then F lies in a leaf of $\mathcal{F}$. If F is a Klein Bottle or torus then either F is transverse to $\mathcal{F}$ or F lies in a leaf of $\mathcal{F}$. If F* is a surface of higher genus and F does not lie in a leaf then

$$\#(\text{tangencies of } F \text{ to } \mathcal{F}) \leq |\chi(F)|$$

Proof. We apply the above technique to count the singularities of the induced foliation on F. Since each singularity contributes a negative integer to $\chi(F)$ the result follows.

We can also apply these techniques to obtain results on foliations of non-compact manifolds, as in the following theorem.

EXAMPLE. Let $\mathcal{F}$ be a foliation of a compact manifold whose leaves satisfy the hypothesis of Theorem 2.2 Then if $f: F \to M$ induces an injection of the fundamental group, there exists a map f' homotopic to f with $\#$(tangencies of $f'(F)$ to $\mathcal{F}) \leq |\chi(F)|$. This follows as the result of Schoen–Yau [S–Y] show that f can be homotoped to a minimal surface in the metric given by Theorem 2.2. Thus if F is a torus, or Klein bottle, there exists a homotopic map with no tangencies. See [Ga] for a topological argument.

COROLLARY 2.10. *Let M^3 be a Riemannian 3-manifold with a complete, homogeneously regular metric of non-negative Ricci curvature. Then every foliation of M by minimal surfaces has each leaf a totally geodesic surface.*

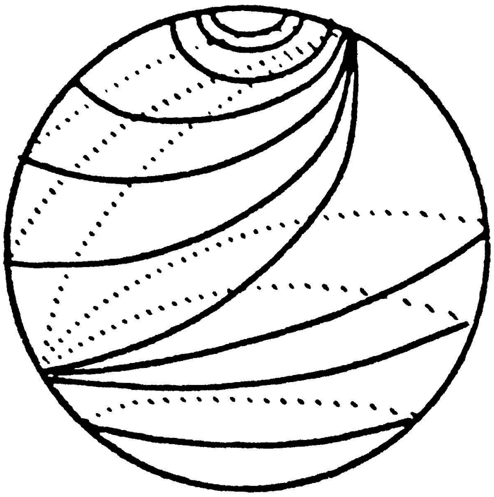
Proof. It suffices to prove such a result in the universal cover of M so we can assume M is simply connected. Then by the remark following Theorem 2.5 it follows that each leaf is also simply connected, else there is a compressible leaf. We will show that every leaf is stable. A theorem of Schoen and Yau [S–Y II] will then imply that each leaf is totally geodesic.

If L is not stable, for some leaf L, then there is a compact disk D in L and a variation L_t of $L = L_0$, supported on D, such that if $A(t)$ is the area of D_t then $A''(t) < 0$. Thus for some small ϵ, Area $(D_\epsilon) <$ Area (D), D_ϵ and D agree on ∂D. Let D' be a least area disk spanning ∂D. D' does not lie in L. Applying Lemmas 2.1, 2.4 and 2.6 we obtain a contradiction as before.

Note. The above argument implies that each leaf is area minimizing, and in fact homology area minimizing. An alternative approach is given in the Appendix.

COROLLARY 2.11. *Every foliation of E^3 by minimal surfaces is the standard planar foliation* $\{x^3 = \text{constant}\}$ *up to rotation.*

The situation is somewhat different in the case of a manifold with some negative curvature. In hyperbolic 3-space there are at least as many ways to construct a foliation by totally geodesic hyper-surfaces as there are ways of foliating the 2-sphere at infinity minus 2 points by circles as below.



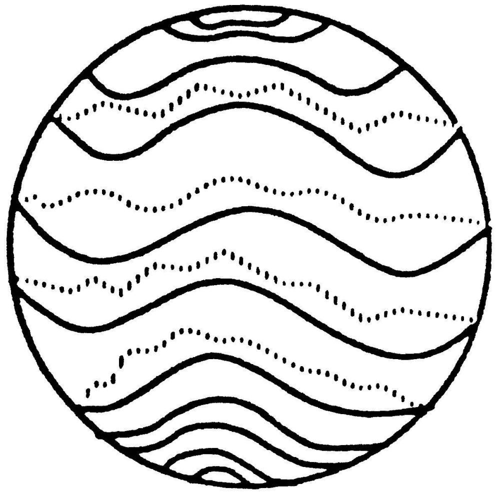
A foliation of S^2_∞ minus 2 points by round circles

Figure 5

However, we can construct a minimal foliation of H^3 in which the leaves are not totally geodesic.

EXAMPLE. There is a foliation of H^3 by minimal surfaces, no one of which is totally geodesic.

We construct this foliation by using a result of Anderson [A]. Pick a Jordan curve Γ on the sphere at $\Leftrightarrow$ of H^3 which intersects each longitude at precisely one point.



A foliation of S^2_∞ minus 2 points by circles which are not round, and which meet each longitude once.

Figure 6

Let γ_t be a one-dimensional family of hyperbolic isometries fixing the north pole and south pole of the sphere at ∞. Then $\{\gamma \cdot \Gamma)_{t \in R}$ is a foliation of the sphere at infinity—{north pole, south pole}. Anderson's result states that there is a stable minimal surface F asymptotic to Γ. $\gamma_t \cdot F$ will be the foliation we seek.

To see that this is a foliation it is necessary to check that $\gamma_t \cdot F \cap \gamma_{t'} \cdot F = \phi$ if $t \neq t'$, or equivalently, that $F \cap \gamma_t \cdot F = \phi$ for $t \neq 0$. If not, then note that $\gamma_t \cdot F$ is contained in the convex hull of $\gamma_t \cdot \Gamma$. For t large, this lies in a small neighborhood of the north pole. Moreover, the convex hull of Γ misses a neighborhood of the north pole. Thus $\gamma_t \cdot F$ doesn't intersect F for sufficiently large t. If $\gamma_t \cdot F \cap F \neq \phi$ for some t, let $t_M = \sup \{t \mid \gamma_t \cdot F \cap F \neq \phi\}$. Then $\gamma_{t_M} \cdot F \cap F \neq \phi$ and $\gamma_{t_M} \cdot F$ lies on one side of F. This contradicts the maximal principle, and so $\{\gamma_t \cdot F\}_{t \in R}$ gives a foliation of H^3 by minimal surfaces.

Each leaf on this foliation is stable and area minimizing by the methods of Corollary 2.10. As H^3 is homogeneously regular, the note following Theorem 2.10 implies that each leaf is incompressible and thus is a plane topologically. Moreover, each leaf is the unique surface in H^3 having the corresponding curve on the sphere at infinity as its asymptotic limit. If there were two such, the maximal principle again gives a contradiction. This answers a question of Anderson [A] in these cases. One can generalize the construction to a larger set of curves, those which bound asymptotically a minimal surface in H^3 which is part of a family of curves giving a non-singular foliation of $S^2_\infty - \{2 \text{ points}\}$, and with the property that any curve is carried to another by some hyperbolic isometry.

§3. The Bernstein Problem in the 3-geometries

In this section we consider the following question, which can be thought of as a generalization of the Bernstein theorem: When does there exist a stable minimal surface, in one of the eight 3-dimensional geometries on 3-manifolds, which is not totally geodesic? This question was settled for orientable surfaces in R^3 with the Euclidean metric by [D–P] and [F–S]. In [F–S] is it also shown that if M^3 has non-negative Ricci curvature then a stable surface is totally geodesic. Thus the stable minimal surfaces in S^3 and $S^2 \times \mathbb{R}$ are totally geodesic. S^3 in fact does not have such surfaces as great 2-spheres are unstable.

This leaves H^3, $H^2 \times R$, Sol and $SL(2, R)$. Examples of not totally geodesic, area minimizing surfaces in H^3 were constructed in [A], [W–W] and [H I]. The previous section gives a method to construct many such surfaces which are homology area minimizing planes. In [H I] such surfaces are also constructed in $H^2 \times R$. There are in fact lifts of least area surfaces in $F \times S^1$ where F is a compact hyperbolic surface. It follows from [F–H–S] that a least area surface in $F \times S^1$ is equivariant under the S^1 action, and this yields a foliation in $H^2 \times R$ by homology area minimizing planes.

Finally, we find explicitly foliations on each of Sol, Nil and $SL(2, R)$ consisting of minimal planes that are not totally geodesic. Since these are minimizing, and in fact homology area minimizing, the analog of the Bernstein theorem fails for these geometries.

For an exposition of the eight geometries, see [S].

Sol has a metric $ds^2 = e^{2z}\, dx^2 + e^{-2z}\, dy^2 + dz^2$ where (x, y, z) are coordinates on R^3. Orthonormal 1-forms are $w_1 = e^z\, dx$, $w_2 = e^{-z}\, dy$, $w_3 = dz$. Calculation gives $dw_1 = -w_1 \wedge w_3$, $dw_2 = w_2 \wedge w_3$, $dw_3 = 0$, and using the structure equations, $w_{12} = 0$, $w_{23} = w_2$, $w_{13} = -w_1$. We can then compute sectional curvatures $K_{1212} = +1$, $K_{2323} = -1$, $K_{1313} = -1$. In particular, we will use the fact that the x, y plane can not be totally geodesic, as its intrinsic curvature is given by the flat metric $dx^2 + dy^2$ from which it follows by the Gauss equation that its normal curvature is -1. Note that there is an isometry I of Sol given by $I\colon (x, y, z) \to (y, x, -z)$, and note also that $J\colon (x, y, z) \to (x + x', y + y', z)$ is an isometry. It follows that the (x, y) plane is minimal, as I reverses its orientation and leaves $(0, 0, 0)$ fixed, and

J takes the origin to any other point on the (x, y) plane, leaving the plane invariant, so that its mean curvature is zero everywhere.

Now consider the isometry $K\colon (x, y, z) \to (e^{-t}x, e^{t}y, z + t)$. This takes the plane $\{z = 0\}$ to the plane $\{z = t\}$. Thus all the horizontal planes are minimal and we have a foliation by surfaces not totally geodesic. Thus the Bernstein theorem does not hold in Sol.

Nil has metric $ds^2 = dx^2 + dy^2 + (dz - x\,dy)^2$ where (x, y, z) are coordinates on R^3. Orthonormal 1-forms are $w_1 = dx$, $w_2 = dy$, $w_3 = dz - x\,dy$. Calculation gives $dw_1 = 0$, $dw_2 = 0$, $dw_3 = -w_1 \wedge w_2$, and the structure equations give $w_{12} = -1/2w_3$, $w_{13} = -1/2w_2$, $w_{23} = +1/2w_1$ from which it follows that $K_{1212} = -3/4$, $K_{1313} = +1/4$, $K_{2323} = +1/4$. In particular, the $y - z$ plane is not totally geodesic, by the Gauss equation as before, as setting $x = 0$ we see that its intrinsic metric is flat. There is an isometry I: $(x, y, z) \to (-x, -y, z)$ showing that the mean curvature of $\{x = 0\}$ is zero at the origin. Moreover the isometry $J\colon (x, y, z) \to (x, y + a, z + b)$ for any constants a, b shows the mean curvature is everywhere zero. Finally, the isometry K, $K(x, y, z) = (a + x, y, 2 + ay)$ carries the plane $\{x = 0\}$ to the plane $\{x = a\}$, where a is any constant. Thus we get a foliation of Nil by minimal surfaces which are not totally geodesic.

$\widetilde{SL(2, R)}$ is the universal cover of the unit tangent bundle of hyperbolic 2-space. We do not give its metric explicitly, but note instead the isometries of SL$(2, R)$, which is the unit tangent bundle of the hyperbolic plane, include induced mappings from isometries of H^2 and rotations of the fibers by fixed angle. We consider the cylinder consisting of all the points above a hyperbolic geodesic, where we use the Poincaré disk model for H^2. This surface is carried to itself by a transitive group of isometries of $SL(2, R)$, and so has constant mean curvature. Reflection through the geodesic lifts to an isometry of $SL(2, R)$ which interchanges its two sides, and so its mean curvature is zero. Clearly if we can can foliate $SL(2, R)$ by non-totally geodesic minimal surfaces, lifting to the universal cover will give a foliation of $\widetilde{SL(2, R)}$.

The cylinders consisting of all unit vectors above a given geodesic have constant Gauss curvature, which must be zero, as they have parallel disjoint geodesics representing a generator of π_1. Flat cylinders have the property that any two geodesics on them which run from one end to the other intersect either zero or an infinite number of times. If these cylinders are totally geodesic in $SL(2, R)$ then geodesics on the cylinder are also geodesics in $SL(2, R)$. Assume this is the case, as else they serve as our desired example and we're done, by letting $\mathscr{F}$ be a foliation consisting of all the cylinders over a foliation of H^2 by geodesics.

We then consider a different foliation of $SL(2, R)$. Let q be a point on the circle at ∞ and let L_q consist of the unit tangent vectors to all geodesics emanating from q. Varying q on the circle at ∞ gives a foliation of $SL(2, R)$, $\mathcal{F}'$. Letting x be a point on L_q, reflection through the geodesic in H^2 to which x is tangent interchanges the sides of L_q, leaves x fixed and L_q invariant, and thus shows that L_q has zero mean curvature at x. x is arbitrary so L_q is minimal.

Consider the cylinder C over the geodesic from -1 to 1 and intersect it with L_q where $q = -i$. The intersection consists of a line l going from one end to the other end of C. If both L_q and C are totally geodesic then so is l as two totally geodesic surfaces intersect in a geodesic. But l intersects a geodesic γ on C which runs from one end to another in one point, therefore l is not a geodesic. The geodesic γ consists of unit vectors orthogonal to the geodesic running from -1 to 1 in H^2. Thus at least one of L_q and C is not totally geodesic.

Note. Explicit calculation can also show that these foliations are not totally geodesic. However the metric of $SL(2, R)$ is not as easy to write explicitly as that of the other geometries.

§4. Branch points

We have seen that the presence of branch points on a minimal surface which lies in a minimally foliated manifold leads to singularities in the induced foliation. We exploit this to give information on the number of possible branch points, both interior and boundary. Branch points can be either *true* or *false*, the latter occurring because of a parametrization rather than an actual point where the image of a surface is not immersed [G]. Our result apply to both types. Differences will occur only if the branched surface lies in a leaf, which will not happen in this section.

We have dealt in various cases with the effect on the induced foliation at a branch point. We summarize this in the following lemma.

LEMMA 4.1. *Let M be a Riemannian manifold, $\mathcal{F}$ a codimension one minimal foliation and let $f: (F, \partial F) \to M$ be a branched minimal immersion, with ∂F transverse to $\mathcal{F}$ except at isolated points. Then the induced foliation has singularities of negative Euler characteristic at each branch point and at each interior tangency point. If ∂F meets a leaf L_0 at a point x, and lies on one side of L_0, there is either a transverse boundary 0-prong or a transverse boundary k-prong, $k \geq 2$, and $k > 2$ only if the point x is also a branch point. At an interior branch point x of degree k, there is a $2k$-prong singularity induced if TF_x is transverse to $\mathcal{F}$ and $k = 2$ degree(x). If TF_x is tangent to $\mathcal{F}$ at x, then the induced foliation has a k-prong where $k \geq 2$ degree(x). At a boundary branch point x at* *which TF_x is transverse to $\mathcal{F}$, there is a transverse boundary $(2l-1)$-prong, where $l = \text{degree}(x)$. At a boundary point x at which TF_x is tangent to $\mathcal{F}$ there is a transverse boundary k-prong, $k \geqslant 2$. At an isolated tangency of ∂F to $\mathcal{F}$ the induced foliation has either a transverse boundary 0-prong or a transverse boundary k-prong, $k \geqslant 1$.*

Proof. Most of the results follow from Section 2. If an interior branch point x is tangent to $\mathcal{F}$ then the techniques of [G–L] show that F intersects the foliation in a k-prong with $k \geqslant 2\,\text{degree}(x)$. If a boundary branch point is tangent to $\mathcal{F}$, the argument of Lemma 2.6 yields a transverse boundary k-prong, $k \geqslant 2$. Finally, if ∂F is immersed at x and has a maximum or minimum relative to the foliation, then the induced foliation has a transverse boundary 0-prong or transverse boundary 2-prong. A saddle point tangency of ∂F yields no singularity.

Using Lemma 4.1 to measure the contribution of branch points to the Euler characteristic now gives the following result.

THEOREM 4.2. *Let Γ be a smooth curve in $\mathbb{R}^3$ bounding a minimal surface F. Let $h\colon R^3 \to R^3$ be a Morse height function such that $h \mid \Gamma$ has k critical points on Γ, and which has T critical points on interior(F). Let $\{x_b\}$ be the branch points of F, both interior and boundary. Then*

$$\chi(F) \leqslant \frac{k}{2} + \sum_{\{x_b\}} [1 - \text{degree}\,(x_b)] - T$$

Note. It follows from Lemma 1.3 that there are only finitely many branch points, so the sum makes sense.

Proof. By a small perturbation of h we can assume that the level sets of h are transverse to the tangent plane of F at critical points of ∂F and to the tangent planes of any branch points. These level sets are minimal surfaces, in fact totally geodesic. Contributions to the Euler characteristic of F as calculated from the induced foliation now yield $\sum_{\{x_b\}} [-\text{degree}(x_b) + 1]$ from interior and $\sum_{\{y_b\}} [-\text{degree}(y_b) + 1]$ from boundary branch points. Other contributions arise from tangencies, which always give negative contributions if interior, and can yield no more than $+1/2$ if a tangency of Γ to the level sets of h. The result follows, as there are k tangencies of the last type, and interior tangencies yield n-prongs, $n \geqslant 4$.

COROLLARY 4.3. *If Γ bounds a minimal disk D and Γ has a Morse height function with only one maximum then D has no branch points.*

Proof. $\chi(D)=1$ and $k=2$ in the formula of Theorem 4.2. Since degree$(x_b)\geqslant 2$ we conclude there are no branch points.

Note. A similar result was obtained by Rado, cf. [La].

COROLLARY 4.4. *If $k(\Gamma)=\int_\Gamma|\ddot{\Gamma}|$ satisfies $k(\Gamma)<2\pi(n+1)$, Γ a closed curve in R^3, and Γ bounds a branched minimal surface F, then $\sum_{\{x_b\}}[\text{degree}(x_b)-1]\leqslant n-\chi(F)$. Furthermore, if Γ is a simple closed curve then we can replace the hypothesis by $k(\Gamma)\leqslant 2\pi(n+1)$.*

Proof. It is shown in [Mi] that if $k(\Gamma)$ satisfies the above hypothesis, there is a height function h on Γ which is Morse with no more than $2n$ critical points. Perturbing to make h transverse to the branch points, the result follows from Theorem 4.2.

Note. Γ need only be C^3 for this to work.

For the disk, this yields the following, similar to results of Heinz–Hildebrandt obtained via a Gauss–Bonnet technique. Their results generalize to surfaces of bounded mean curvature, with extra terms.

COROLLARY 4.5. *With the previous situation, and F a disk*:

$$1+\sum_{\{x_b\}}[\text{degree}\,(x_b)-1]\leqslant\frac{k(\Gamma)}{2\pi}$$

Proof. If

$$r\leqslant\frac{k(\Gamma)}{2\pi}<r+1$$

then we can find a height function as in Corollary 4.4 with only r maxima. We then replace r with $k(\Gamma)/2\pi$ in Corollary 4.4.

We can generalize Theorem 4.2 in several directions. We first consider a more general theorem in $\mathbb{R}^3$.

THEOREM 4.6. *Let Q be a region in $\mathbb{R}^3$ foliated by a minimal foliation $\mathcal{F}$. Let Γ be a curve in $\mathbb{R}^3$ whose convex hull is contained in* interior(Q) *and which is* *tangent to* $\mathscr{F}$ *at k points. Let F be a branched minimal surface spanning* Γ*. Then*

$$\chi(F) \leqslant \frac{k}{2} + \sum_{\{x_b\}} [1 - \text{degree}\,(x_b)]$$

where $\{x_b\}$ *is the set of branch points.*

Proof. By perturbing F slightly, we can assume it intersects $\mathscr{F}$ transversely at any branch point or boundary tangency. The result now follows as in Theorem 4.2.

EXAMPLE. Let L_0 be a piece of the catenoid.

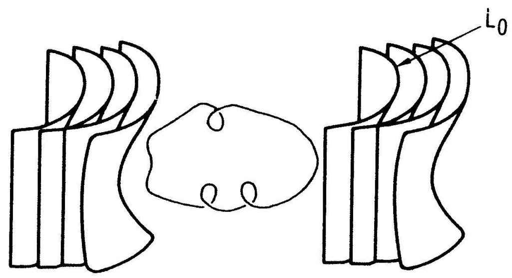


Figure 7

Let Γ be a curve with convex hull in a small ball. Translates of L_0 foliate this ball. Γ can be picked so each height function on Γ has $\geqslant 4$ critical points but Γ has only two critical points relative to $\mathscr{F}$. A minimal disk spanning Γ has no branch points by Theorem 4.6.

EXAMPLE. Let L_0 be the 1/2 plane $\{x_3 = 0, x_1 \geqslant 0\}$ in $\mathbb{R}^3$, and rotate about the x_2 axis to get a foliation $\mathscr{F}$ of $R^3 - (\{x_1 = x_3 = 0\}$ by geodesic planes. The curve below has no Morse function with two critical points, large total curvature, and is only tangent to $\mathscr{F}$ at two points. A minimal disk spanning Γ has no branch point by Theorem 4.6. Another generalization of Theorem 4.2 is to a general 3-manifold M. Here we can not naturally perturb Γ or $\mathscr{F}$, as M may not have a large isometry group like $\mathbb{R}^3$.

THEOREM 4.7. *Let M a Riemannian 3-manifold,* $\mathscr{F}$ *a minimal foliation of M,* $(F, \partial F)$ *a branched minimal surface with* ∂F *transverse to* $\mathscr{F}$ *except at k critical points. Then*

$$\chi(F) \leqslant \frac{k}{2} + \sum_{\{x_b\}} [-\text{degree}\,(x_b) + 1] + \sum_{\{y_b\}} [-1/2] - T$$

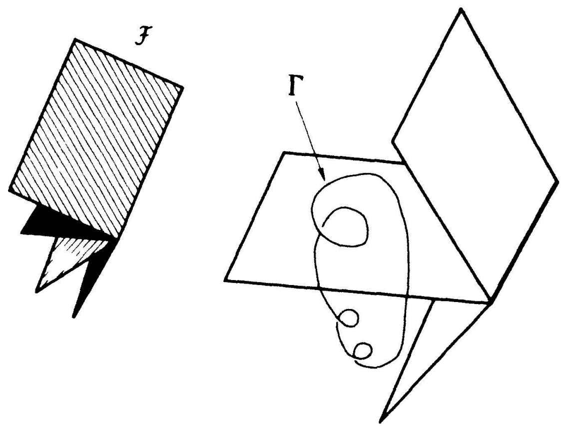


Figure 8

where $\{x_b\}$ *are the interior branch points,* $\{y_b\}$ *the boundary branch points and there are* T *points of tangency of* interior (F) *to* $\mathcal{F}$.

Proof. We can not now assume that TF_x is transverse to $\mathcal{F}$ at boundary tangencies or branch points. We still have the previous Euler characteristic contribution at interior branch points, even if they're tangent, but ∂-branch points may only have transverse boundary 2-prongs if tangent to $\mathcal{F}$, so we may only get a $-1/2$ contribution from them. Otherwise the proof is as in Theorem 4.2.

Further generalizations, to higher dimensions, are given in Section 5. Note that we do not need to weaken the results of 4.2 if we can perturb the foliation isometrically. Thus for curves in S^3, H^3 and so on, the stronger results of 4.2 hold.

§5. Higher dimensions, branch points and a sphere theorem

In the previous section we restricted attention to the case where M was a foliated 3-dimensional manifold. We now examine the situation in dimensions 4 and above. The appropriate analogue of a minimal foliation in these dimensions is one in which each leaf is totally geodesic. We call this a *totally geodesic foliation.*

Using these, we can extend all the results of the previous section on branch points, as well as prove a topological result on the existence of embedded 2-spheres in certain 4-manifolds.

The following lemma is the analogue of Lemma 4.1 in this context.

LEMMA 5.1. *Let* M *be a Riemannian manifold with a totally geodesic foliation* $\mathcal{F}$ *of codimension one, and let* $f: (F, \partial F) \to M$ *be a branched minimal* *immersion with ∂F transverse to $\mathcal{F}$ except at isolated points. Then either $f(F)$ lies entirely in a leaf of $\mathcal{F}$ or the induced foliation has isolated singularities of negative Euler characteristic at each tangency of F to $\mathcal{F}$ and at each branch point of F. At isolated points where ∂F is tangent to $\mathcal{F}$ there is either a transverse boundary 0-prong or a transverse boundary k-prong, $k \geqslant 1$. At interior branch points x of F there is a $2k$-prong singularity induced where $k =$ degree (x) if TF_x is transverse to $\mathcal{F}$, and a k-prong singularity, $k \geqslant 4$, if TF_x is tangent to $\mathcal{F}$. At boundary branch points x transverse to $\mathcal{F}$ there is a transverse boundary k-prong singularity where $k = 2 \cdot$ degree $(x) - 1$. If x is a boundary branch point tangent to $\mathcal{F}$ there is a transverse boundary k-prong at x, $k \geqslant 2$. If x is an interior tangency of F to $\mathcal{F}$ there is a k-prong singularity at x, $k \geqslant 4$.*

Proof. The existence of a tangent space of F at a branch point implies the statement where TF_x is transverse to $\mathcal{F}$ exactly as in dimension 3.

Suppose TF_x is tangent to $\mathcal{F}$. We pick coordinates $\{y_1, \ldots, y_n\}$ about $x = \{0, \ldots, 0\}$ such that $\mathcal{F}$ is given by $y_n =$ constant. Then the minimality of $f(F)$ implies that we can pick coordinates on F so that f is a harmonic, almost conformal map. Call these coordinates $\{x_1, x_2\}$. Then the harmonicity of f is given by the equation

$$\Delta(y_\alpha \circ f) + \Gamma^\alpha_{\beta,\gamma} \frac{\partial(y_\beta \circ f)}{\partial x_i} \frac{\partial(y_\gamma \circ f)}{\partial x_j} g^{ij} = 0 \qquad \begin{matrix} 1 \leqslant i, & j \leqslant 2 \\ 1 \leqslant \alpha & \beta, \gamma \leqslant n \end{matrix}$$

for each α, where Δ denotes the Laplacian on F and where g_{ij} is the metric on F, $(g^{ij}) = (g_{ij})^{-1}$, and $\Gamma^\alpha_{\beta,\gamma}$ are the Christoffel symbols of M. Since $y_n =$ constant gives a totally geodesic hypersurface of M, we have for $\alpha = n$ in the above, that

$$\begin{aligned} &\Delta(y_n \circ f) + \Gamma^n_{\beta,\gamma} \frac{\partial(y_\beta \circ f)}{\partial x_i} \frac{\partial(y_\gamma \circ f)}{\partial x_j} g^{ij} \\ &\quad = \Delta(y_n \circ f) + \left[\Gamma^n_{\beta,n} \frac{\partial(y_\beta \circ f)}{\partial x_i} g^{ij}\right] \frac{\partial(y_n \circ f)}{\partial x_j} = 0 \end{aligned}$$

as $\Gamma^n_{\beta,\gamma} = 0$ if $1 \leqslant \beta$, $\gamma \leqslant n-1$ by total geodesity, and $\Gamma^n_{n,\gamma} = 0$ by skew symmetry. Let $h^j = \left[\Gamma^n_{\beta,n} \dfrac{\partial(y_\beta \circ f)}{\partial x_i} g^{ij}\right]$. Then h^j is a smooth function on F and $u = (y_n \circ f)$ satisfies the equation $Lu = 0$ where $L = \Delta + h^j \dfrac{\partial}{\partial x_j}$. We now apply a theorem of Bers, as in [F–H–S, Section one] to show that after a C^1 change of coordinates in F, U agrees with the real part of the function $c \cdot z^m$, where c is a constant, $m \geqslant 2$, or else $u \equiv 0$. Thus the zeros of u consists of $2m$ lines crossing at $(0,0)$, $m \geqslant 2$, and this is the intersection of F with the leaf L_0 to which it is tangent. Moreover F is transverse to all leaves nearby to x, except at the one tangency point, as $\nabla(\operatorname{Re}(z^m))$ is not zero in a deleted neighborhood of $(0,0)$.

If F is tangent to $\mathcal{F}$ at an interior branch point, the above argument still applies, to show an induced k-prong singularity, $k \geqslant 4$. If F is tangent at a boundary branch point, the argument in Section two applies to show an induced transverse boundary k-prong, $k \geqslant 2$.

We first note that this extends the results of the previous section.

THEOREM 5.2. *The results of Theorem* 4.2, *Corollaries* 4.3, 4.4, 4.5 *and* 4.6, *and of Theorem* 4.7, *hold for dimension larger than* 3 *if* $\mathcal{F}$ *is a totally geodesic foliation of codimension one, except that the formula from Theorem* 4.7 *becomes the weaker*

$$\chi(F) \leqslant \frac{k}{2} + \sum_{\{x_b\}} [-1] + \sum_{\{y_b\}} [-1/2] - T$$

where $\{x_b\}$ *is the set of interior branch points,* $\{y_b\}$ *the set of boundary branch points.*

Proof. The proofs of all results are the same except that in the analog of Theorem 4.7 we do not have as much control over the induced foliation if a tangency of F to $\mathcal{F}$ occurs at a branch point. In this case we still know that the induced foliation has a $2k$-prong singularity where $k \geqslant 1$.

We next apply these techniques to get some new results in understanding the branch points of closed minimal surfaces in n-manifolds where $n \geqslant 4$. Results of Osserman and Gulliver [O] [G] have shown that least area surfaces in dimension three admit no true branch points. False branch points can occur, as for example when a least area torus is the two-fold branched cover of a 2-sphere in the 3-manifold $S^2 \times S^1$. However if the surface injects in π_1 or is 1–1 on its boundary, then false branch points can't exist for topological reasons. Branch points can occur for surfaces in 4-manifolds, as evidenced by complex submanifolds with singularities. Such submanifolds are always area minimizing. We next state a theorem giving information on the existence of branch points in this context.

THEOREM 5.3. *Let* M^4 *be a Riemannian* 4-*manifold with a totally geodesic foliation* $\mathcal{F}$. *Then if* $f: F^2 \to M^4$ *is a least area map of a surface F homeomorphic to a torus of Klein Bottle, then F has no true branch points. If f has false branch points its image lies in a leaf.*

Proof. We look at the induced foliation. All singularities contribute negative Euler characteristic, so none can exist. Thus f must be transverse to $\mathcal{F}$ or contained in a leaf. If contained in a leaf, then the results of Osserman and Gulliver yield the conclusion, as the leaf is a 3-manifold. If transverse, then any branch point, true or false, leads to a $2k$-prong singularity, $k \geq 2$, and thus none can exist.

COROLLARY 4.5. *Least area tori in $M^3 \times S^1$ have immersed image. Injective least area tori in $M^3 \times S^1$ have no branch points of any type, true or false.*

We state one more result relating the number of tangencies T, branch points B and the Euler characteristic of a surface.

THEOREM 5.5. *Let $f: F \rightarrow M^4$ be a branched minimal immersion into a Riemannian manifold M with a totally geodesic foliation $\mathcal{F}$. Then either $f(F)$ lies entirely in a leaf or*

$$\chi(F) \leq -T - B.$$

Proof. This follows as the previous results.

EXAMPLE. This implies for example that a minimal 2-sphere always lies in a leaf, as does a minimal RP^2. A surface of genus two can have at most two branch points or tangencies, or else it lies in a leaf.

The topological sphere theorem [Pa], [St], states that if $\pi_2(M^3) \neq 0$ then there is an embedded 2-sphere in M^3 which is non-trivial in π_2. We can extend this to certain 4-manifolds.

THEOREM 5.6. *If $\pi_2(M^4) \neq 0$ and M^4 admits a totally geodesic foliation of codimension one then there is a embedded 2-sphere in M which is non-trivial in $\pi_2(M)$.*

Proof. Equip M with the metric in which it has a totally geodesic foliation. Results of Sacks–Uhlenbeck give the existence of a least area 2-sphere S non-trivial in $\pi_2(M^4)$. Such a map is a branched minimal immersion. Considering its induced foliation, we get a contradiction unless it lies inside a leaf L. Now $\pi_2(L) \neq 0$, else the 2-sphere would be trivial. But the sphere theorem [St] gives that there is a collection of embedded 2-spheres generating $\pi_2(L)$ as a π_1-module. One of these at least is non-trivial in $\pi_2(M)$, as S is a linear combination of these, and so this serves as our desired sphere.

Appendix

For completeness and because we need to extend it slightly, we present here a proof of Theorem 2.2. The proof is a simplification of Sullivan's original argument which applies in codimension one, and is based on ideas related to me by D. Epstein and W. Thurston. The proof applies also in higher dimensions for codimension one foliations.

THEOREM A1. *Let M be an oriented closed 3-manifold and let $\mathscr{F}$ be a codimension one foliation of M. M admits a metric in which each leaf of $\mathscr{F}$ is a minimal surface if and only if every compact leaf of $\mathscr{F}$ intersects some closed curve which is transverse to $\mathscr{F}$. In such a metric, each leaf is homology area minimizing.*

Remark. Non-compact leaves always intersect closed curves transverse to $\mathscr{F}$.

Proof. We assume first that the foliation is transversely oriented. Note that this gives an orientation to the leaves of $\mathscr{F}$. Suppose first that each compact leaf intersects a closed transverse curve. Since M is compact, each non-compact leaf also intersects a closed transverse curve. Thus we can find a finite number of maps $f_i: S^1 \times D^2 \to M$, $i = 1, \ldots, k$, such that

1) Each f_i is an orientation preserving embedding and $f_i(pt \times D^2)$ is contained in a leaf.
2) $\bigcup_{i=1}^k f_i(S^1 \times \frac{1}{2}D^2) = M$.
3) $f_i(S^1 \times pt)$ are curves transverse to $\mathscr{F}$, and oriented according to the transverse orientation of $\mathscr{F}$.

On $S^1 \times D^2$ we can pick coordinates (θ, x, y) with $\theta \in S^1$ and $(x, y) \in D^2$. Let $\varphi(x, y)$ be a positive function on D^2 such that $\varphi \equiv 1$ on the disk of radius 1/2 and $\varphi \equiv 0$ outside the disk of radius 3/4. Let w_0 be the 2-form on $S^1 \times D^2$ given by $w_0 = \varphi(x, y)\, dx \wedge dy$. Then $dw_0 = 0$ as $dw_0 = \dfrac{\partial \varphi}{\partial \theta} d\theta \wedge dx \wedge dy = 0$, since φ is independent of θ. Using the embedding f_i gives a corresponding closed 2-form w_i on M, $i = 1, \ldots, k$. Let $w = \sum_{i=1}^k w_i$ be a closed 2-form on M. Note that w is non-zero restricted to the leaves of $\mathscr{F}$.

Let p be a point in M and consider the following map ψ from the tangent space of M at p to $\mathbb{R}^2$, whose kernel gives a "normal direction" to $\mathscr{F}$. ψ is the composition of the map $\varphi\ TM_p \to T^*M_p$ defined by $\varphi(x)(y) = w(x, y)$ and the map κ: $T^*M_p \to \mathbb{R}^2$ defined by $\kappa(\alpha) = (\alpha(u), \alpha(v))$ where (u, v) is a basis for the tangent space to the leaves of $\mathscr{F}$ in a neighborhood of p. Note that since $w(u, v) \neq 0$ by construction of w, ψ has a one-dimensional kernel. Let N be a vector field in ker (ψ) in a neighborhood of p. Then $w(N, u) = w(N, v) = 0$.

Now pick any volume form V on M and any Riemannian metric h on the leaves whose volume form equals w restricted to the leaves. We then rescale N so that $i_N(V) = w$ and prescribe a metric g on M to be h on the leaves and to have N as a unit vector orthogonal to the leaves. In the metric g, $\|w\| \leqslant 1$ and $\|w\| = 1$ on the tangent space to the leaves. We now show that each leaf is minimal in g. In fact we show that each leaf is homology area minimizing.

Let K be a compact region of a leaf and let K' be a homologous surface with $\partial K = \partial K'$. Then

$$\text{Area}\,(K') = \int_{K'} 1 \geqslant \int_{K'} w|_{K'} = \int_K w|_K = \text{Area}\,(K)$$

where the inequality holds because $\|w\| \leqslant 1$ and the following equality holds by Stoke's theorem, since $dw = 0$ and K, K' are homologous. This concludes the proof that each leaf is minimal in the metric g. Note that the same proof applies for codimension one foliations in manifolds of any dimension. This concludes the first implication of the theorem. We now prove the converse.

Suppose M has a metric in which each leaf of $\mathcal{F}$ is a minimal surface. Define a 2-form w by $w = i_N V$ where N is the unit normal vector field to the foliation and V is the volume form of M. Then $\|w\| \leqslant 1$ and $\|w\| = 1$ on the leaves of $\mathcal{F}$, so the above argument shows that each leaf is homology area minimizing.

We claim that $dw = 0$. If not, let p be a point where $dw > 0$ and pick L to be the leaf containing p. Let D be a small disk about p on L, and let D_t be the image of D under the leaf preserving normal flow to $\mathcal{F}$ at time t. Then

$$\text{Area}\,(D) = \int_D w = \int_{D_t} w + \int_A w + \int_R dw$$

where A is the annulus between ∂D and ∂D_t, R the region bounded by D, A, D_t. Since A is normal to $\mathcal{F}$, we have that $\int_A w = 0$. Thus

$$\int_R dw = \int_D w - \int_{D_t} w > 0$$

as $dw \neq 0$. Now w is the area form of both D and D_t. But D_t is a normal variation and so D_t minimal $\Rightarrow$ Area (D_A) constant, a contradiction. So we have $dw = 0$.

Now let L_N denote the Lie derivative associated to the flow along N. Then

$$L_N V = di_N V + i_N\, dV = dw = 0.$$

So the volume form V is preserved by the normal flow.

Now suppose that there is some compact leaf L_0 which does not intersect a closed transverse curve. For a small real number ϵ, the normal flow along N of distance ϵ takes L_0 off itself, tracing out a 3-dimensional region R_1. Iterating this flow along N, R_1 gets carried to a region R_2 of equal volume. The surface L_0 gets carried to a disjoint surface L_t by flowing along N distance t else there would be a closed transversal through L_0. Thus the region between L_0 and L_t is an embedded submanifold of M for all t, and its volume grows without bound. This is a contradiction in a finite volume manifold M. Thus there is a closed transversal through L_0.

We next deal with the situation where the foliation is not transversely oriented. Our arguments relied on this orientation, so they need modification as follows.

Assume first that every compact leaf intersects a closed transversal in M. As before, pick fibers $f_i(S^1 \times D^2)$ covering M. Note that there is a double cover $\tilde{M}$ of M, such that a loop in M lifts to $\tilde{M}$ if and only if the normal line field to $\mathcal{F}$ along the loop is orientable. Thus transversal closed curves lift to $\tilde{M}$ and thus $f_i(S^1 \times D^2)$ lifts. In $\tilde{M}$ we construct a metric making each leaf minimal as before, but equivariantly. We do so by picking the metric equivariantly on the two disjoint lifts of each $f_i(S^1 \times D^2)$. We thus get a metric on M in which each leaf is minimal.

The converse is easier. Lifting the minimal foliation $\mathcal{F}$ to M gives a minimal foliation $\tilde{\mathcal{F}}$ in $\tilde{M}$. Thus each compact leaf in $\mathcal{F}$ intersects a transversal. The projection of the transversal to M gives the same result in M.

Theorem A2 extends Theorem A1 to the case where M has boundary.

THEOREM A2. *Let M be a compact manifold with boundary and let $\mathcal{F}$ be a codimension one foliation, transverse to ∂M. Then there is a metric on M in which each leaf of $\mathcal{F}$ is minimal and ∂M is totally geodesic if and only if every compact leaf of $\mathcal{F}$ intersects a closed transversal.*

Proof. Assume each compact leaf of $\mathcal{F}$ intersects a closed transversal. Double M to obtain M', $\mathcal{F}'$. Every leaf of $\mathcal{F}'$ intersects a closed transversal. We now can carry out the construction of Theorem A1 equivariantly under the involution of M' to get a metric in which each leaf is minimal. The boundary is now totally geodesic.

Conversely, the argument of Theorem A1 applies.

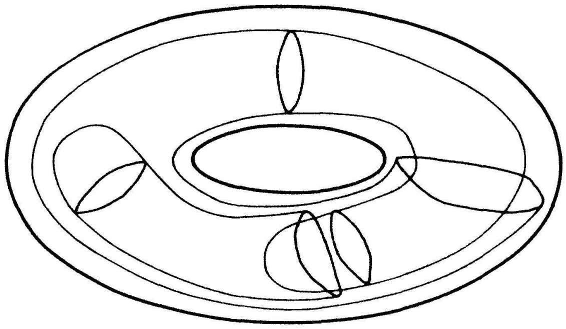


Figure 9

With additional argument one can go beyond the arguments of Theorem 2.7 and show the existence of a Reeb component in a foliation, rather than just the existence of a compact leaf. We give these arguments here.

A *Reeb component* is a solid torus in M^3 whose foliation is given up to diffeomorphism by the foliation on $\{\mathbb{R}^3_+ - \{0\}/\sim\}$ by the level planes $\{x^3 = \text{const}/\sim\}$ where the equivalence relation $\sim$ is given by $(x^1, x^2, x^3) \sim (2x^1, 2x^2, 2x^3)$. Here $\mathbb{R}^3_+$ denotes the set of points in $\mathbb{R}^3$ with $x^3 \geq 0$. A Reeb component is drawn in Figure 9.

We will prove the following result.

THEOREM A3 (Novikov). *Let M be a closed orientable 3-manifold and let $\mathcal{F}$ be a transversely oriented codiimension one foliation of M. If M has finite fundamental group or if M is reducible then $\mathcal{F}$ contains a Reeb component.*

Proof. Following Novikov [No], we put a partial ordering on the leaves of $\mathcal{F}$. We say that $L_1 > L_2$ if there is a transverse path, transversely oriented, from L_1 to L_2.

We can define an equivalence relation on those leaves L_i of $\mathcal{F}$ satisfying $L_i < L_i$ by $L_i \sim L_j$ if $L_i < L_j$ and $L_j < L_i$. The components of this equivalence relation are called Novikov components. Each non-compact leaf belongs to a Novikov component. We now put a partial ordering on the Novikov components of $\mathcal{F}$ by letting $N_1 > N_2$ if there is an oriented transverse curve leaving N_1 and entering N_2. Note that we can not have $N_1 > N_2 > N_1$, unless N_1, N_2 are the same Novikov component. By the Hausdorff Maximal Principle of set theory, there is a maximal linearly ordered subset of the set of Novikov components which we can assume contains N_1. As the set of Novikov components is countable, each being a disjoint open set in a manifold, we can pick a sequence $\{N_i\}$ such that $N_{j+1} > N_j$ and such that all Novikov components $>N_1$ which are in the maximal linearly ordered subset are in the sequence $\{N_i\}$. Pick a leaf $L_i \subset N_i$ for each i. The sequence L_i accumulates to a leaf L_∞. Either

1. There is a maximal Novikov component N_∞ with $L_\infty \subset N_\infty$, or
2. The leaf L_∞ is not in a Novikov component.

In the latter case, note that $L_\infty > L_i$ for all i. Now following backwards from L_∞ along a flow line of the normal flow to the foliation eventually gives rise to an accumulation point, which lies in a Novikov component. This Novikov component N' satisfies $N' > N_i$ for all i. Thus we get a larger linearly ordered subset, a contradiction. On the other hand, suppose there is a maximal Novikov component N_∞. ∂N_∞ consists of compact leaves as non-compact leaves lie in the interior of a Novikov component. The oriented normal flow to $\mathcal{F}$ is everywhere outward pointing on ∂N_∞, which implies that ∂N_∞ is a union of tori by Euler characteristic arguments.

Let K denote the open set of leaves which are not in a Novikov component, and not in the closure of the union of all the Novikov component. Then K is a manifold and no leaf in K intersects a closed transversal. ∂K consists of leaves which are limits of points of Novikov components.

CLAIM. *The boundary of K and the boundary of any Novikov component consists of tori leaves.*

Proof of Claim. We have seen that any maximal Novikov component has tori leaves. If there is a Novikov component with a non-torus ∂-component, let L_1 be such a ∂-component. The normal flow lines to L_1 each eventually enter a maximal Novikov component. Thus L_1 and a collection of tori bound a 3-manifold with flow transverse to the boundary. It follows that $\chi(L_1) = 0$. The boundary of K consists of leaves that are in the closure of Novikov components and thus are tori.

Suppose now that M has finite fundamental group and is irreducible. It follows from Theorems 2.2 and 2.7 that M has more than one Novikov component. Thus there is a Novikov component with tori on the boundary. A torus is compressible since it can not inject into the fundamental group of M. Either the compressing disk is inside the Novikov component or the compressing disk lies outside the Novikov component, and it then may or may not be disjoint from the other Novikov components.

By passing to an innermost Novikov component on the disk, one can reduce to two cases

Case 1. There is a compressing disk with interior in K and boundary on a torus leaf.

Case 2. There is a compressing disk in a Novikov component with boundary on a torus leaf T.

Case 1. By doubling K we get a closed manifold which is foliated so that each leaf intersects a closed transversal. Thus there is a metric on K in which every leaf is minimal, including the boundary. by Theorem A2. We can now solve the Plateau problem in K for a disk with boundary given by the compressing disk. This gives a contradiction as in Theorem 2.7.

Case 2. T is the boundary of a Novikov component N_0. so that there are non-compact leaves in N_0 whose accumulation points include T.

CLAIM *T can be isotoped into N_0 so that it becomes a torus T' transverse to the foliation in N_0, foliated by parallel circles or lines.*

Proof of Claim. Up to diffeomorphism, a neighborhood of T in N_0 is given by $[0, 1] \times [0, 1] \times [0, \epsilon]/\sim$ with $x^3 = \text{const}$ giving the foliation and the identification $\sim$ given by the holonomy of the foliation, yielding a foliation on a neighborhood of T in N_0.

Let α and β generate $\pi_1(T)$ with $[0, 1] \times 0 \times 0$ representing α and $0 \times [0, 1] \times 0$ representing β. Then since $\pi_1(T)$ is commutative, the holonomy $h(\alpha)$ of α and $h(\beta)$ of β can not both be the identity, else there would be tori parallel to T foliating a neighborhood of T in N_0, so we can find in $[0, 1] \times [0, 1] \times [0, \epsilon]$ a flat rectangle with boundary arcs a, b, $h(\alpha) \cdot a$, $h(\beta) \cdot b$, which will be transverse to the planes $\{x^3 = \text{constant}\}$. Note that the identification $\sim$ identifies $0 \times [0, 1] \times [0, \epsilon]$ to $1 \times [0, 1] \times [0, \epsilon]$ by $h(\alpha)$ and $[0, 1] \times 0 \times [0, \epsilon]$ to $[0, 1] \times 1 \times [0, \epsilon]$ by $h(\beta)$. This gives the desired torus T' in N_0, proving the claim.

Let R be the part of N_0 bounded by T'. R is foliated by leaves transverse to the boundary. Moreover, since N_0 was a single Novikov component, each leaf of the foliation $\mathscr{F}$ in R intersects a closed transversal. Thus there is a metric on R in which each leaf of $\mathscr{F}'$ is minimal and T' is totally geodesic. We can now find a minimal compressing disk D, whose boundary is either contained in a leaf or transverse to the leaves. The latter case gives a contradiction as before. In the former case we have that D is contained in a leaf, again by the argument of Theorem 2.7. So T' must be foliated by parallel circles and a similar result applies to each of these. Thus R is foliated by minimal disks. N_0 is constructed from R by adding on a collar with half cylindrical leaves converging to a torus. This gives a Reeb foliation on N_0.

We finally consider the case where M is reducible and $\mathscr{F}$ has no 2-sphere leaves. Let S be a 2-spheres that does not bound a ball. If S can be pushed off the Novikov components it lies in K. But K has a metric with each leaf minimal, giving a contradiction as in Theorem 2.7. Else S intersects a Novikov component in a compressing disk, giving a contradiction as in the previous case.

REFERENCES

[A] M. ANDERSON, *Complete minimal varieties in hyperbolic space,* Inventiones Math. *69* (1982), 477–494.

[D–P] M. DOCARMO and C. PENG, Stable complete minimal surfaces in R^3 are planes, *Bull. A.M.S.* (1979), 903–905.

[F–H–S] M. FREEDMAN, J. HASS and P. SCOTT, *Least area incompressible surfaces in 3-manifolds.* Inventiones Math. *71* (1983), 609–642.

[F–S] D. FISCHER COLBRIE and R. SCHOEN, *The structure of complete stable minimal surfaces in 3-manifolds of non-negative scalar curvature.* C.P.A.M. *33* (1980), 199–211.

[Ga] D. GABAI, *Foliations and the topology of 3-manifolds,* J. Diff. Geom., *18* (1983), 445–505.

[G] R. GULLIVER, *Regularity of minimizing surfaces of prescribed mean curvature,* Annals of Math. *97* (1973), 275–305.

[G–L] R. GULLIVER and F. LESLIE, *On boundary branch points of minimizing surfaces,* Arch. Rat. Mech. Anal., *52* (1973), 20–25.

[H–L] R. HARVEY and H. B. LAWSON, *Calibrated foliations,* Amer. J. Math., *103* (1981), 411–435.

[H I] J. HASS, *Complete area minimizing surfaces which are not totally geodesic,* Pacific Journal of Mathematics, *111* (1984), 35–38.

[H II] J. HASS, *Minimal surfaces in Seifert fiber spaces,* Topology & its Applications, *18* (1984), 145–151.

[H III] J. HASS, *Embedded minimal surfaces in three and four dimensional manifolds,* Ph.D. thesis, University of California, Berkeley, 1981.

[H–K] J. HASS and L. KARP, *Local properties of minimal submanifolds,* in preparation.

[H–H] E. HEINZ and S. HILDEBRANDT, *The number of branch points of surfaces of bounded mean curvature,* J. Diff. Geom. *4* (1970), 227–235.

[La] H. B. LAWSON Jr., *Lectures on Minimal submanifolds Vol. 1,* Publish or Perish 1980.

[M–Y I] W. H. MEEKS and S. T. YAU, *The classical Plateau problem and the topology of 3-dimensional manifolds,* Topology, *21* (1982), 409–442.

[M–Y II] W. H. MEEKS and S. T. YAU, *Topology of 3-dimensional manifolds and the embedding theorems in minimal surface theory,* Ann. of Math. *112* (1980), 441–484.

[M–Y III] W. H. MEEKS and S. T. YAU, *The existence of embedded minimal surfaces and the problem of uniqueness,* Math. Zeit., *179* (1982), 151–168.

[M–S–Y] W. H. MEEKS, L. SIMON and S. T. YAU, *Embedded minimal surface, exotic spheres and manifolds with positive Ricci curvature,* Annls of Math., *116* (1982), 621–659.

[Mi] J. MILNOR, *On the total curvature of knots,* Annals of Math. (2) *52*, (1950), 248–257.

[Mo] C. B. MORREY, *The problem of Plateau in a Riemannian manifold.* Annals of Math., *49* (1948), 807–851.

[Ni I] J. C. C. NITZE, *The boundary behavior of minimal surfaces, Kellog's theorem and branch points on the boundary,* Inventiones Math., *8* (1969), 313–333.

[Ni II] J. C. C. NITZE, *On new results in the theory of minimal surfaces.* Bull. A.M.S., *71* (1965), 195–270.

[No] S. P. NOVIKOV, *Topology of Foliations,* Trudy Mask. Math. Obshch. *14*, (1965) 248–278.

[O] R. OSSERMAN, *A proof of the regularity everywhere of the classical solution to Plateau's problem,* Annals of Math. *91* (1970), 550–569.

[P] C. D. PAPAKYRIAKOPOULOUS, *On Dehn's lemma and asphericity of knots,* Ann. of Math., *66* (1957), 1–26.

[P-W] M. PROTTER and H. WEINBERG, *Maximal Principles in Differential Equations,* Prentice-Hall.

[Re] G. REEB, *Sur certaines propietes topologiques des varites feulillitees,* Actual Sci. Ind. No. 1183, Hermann, Paris (1952).

[Ro] H. ROSENBERG, *Foliations by planes.* Topology, 7 (1968), 131–138.

[S-U I] J. SACKS and K. UHLENBECK, *Minimal immersions of closed Riemann surfaces,* preprint.

[S-U II] J. SACKS and K. UHLENBECK, *The existence of minimal immersions of 2-spheres,* Annals of Math., *113* (1981), 1–24.

[S-Y I] R. SCHOEN and S. T. YAU, *Existence of incompressible minimal surfaces and the topology of 3-dimensional manifolds with non-negative scalar curvature,* Annals of Math., *110* (1979), 127–142.

[S-Y II] R. SCHOEN and S. T. YAU, *Complete 3-dimensional manifolds with positive Ricci curvature and scalar curvatures, Seminar on differential geometry,* Annals of Math. Study 102, S. T. Yau, Ed., 209–228.

[S] P. SCOTT, *The geometries on 3-manifolds,* Bull. L.M.S., *15* (1983), 401–487.

[Su] D. SULLIVAN, *A homological characterization of foliations consisting of minimal surfaces,* Comm. Math. Helvetici, *54* (1979), 218–223.

[St] J. R. STALLINGS, *Group theory and 3-dimensional manifolds,* Yale University Press, New Haven, Conn., (1971).

[W-W] S. P. WANG and S. W. WEI, *Berstein conjecture in hyperbolic geometry, Seminar on minimal submanifolds,* Enrico Bombieri, Ed., Annals of Math., Studies 103.

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