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# On a fundamental variational lemma for extremal quasiconformal mappings

RICHARD FEHLMANN

## 1. Introduction

In [R2] E. Reich considers the following extremal problem in qc (quasiconformal) mappings. Given are a closed set  $\sigma$  on the boundary  $\partial D$  of the unit disk  $D = \{w \mid |w| < 1\}$  which contains at least four points and a measurable set  $E$  in  $D$  where  $D \setminus E$  has positive area-measure and where, if  $\sigma$  is an infinite set,  $\bar{E}$  is assumed to be compact in  $\bar{D} \setminus \sigma$ . Furthermore a quasisymmetric boundary mapping  $h : \partial D \rightarrow \partial D$  is given and a measurable non-negative function  $b(w)$  on  $E$  with  $\text{ess sup}_{w \in E} b(w) < 1$  which is called the “dilatation bound function”.  $Q(h, \sigma, E, b)$  then denotes the class of all qc mappings  $F : D \rightarrow D$  which satisfy the side-condition

$$F|_{\sigma} = h|_{\sigma} \quad \text{and} \quad |\kappa_F(w)| \leq b(w) \quad \text{a.e. in } E,$$

where  $\kappa_F = F_{\bar{w}}/F_w$  is the complex dilatation of  $F$ . In this class a mapping  $F$  is called *extremal* if it minimizes the value

$$\text{ess sup}_{w \in D \setminus E} |\kappa_F(w)|$$

and is called *uniquely extremal* if it is the only such mapping.

In the case when  $E$  is the empty set a necessary and sufficient condition for extremality is the Hamilton-condition as has been shown in [H] and [RS]. In [R2] E. Reich has given a generalization of this condition which is necessary and sufficient for extremality in  $Q(h, \sigma, E, b)$  and by which extremal mappings can be characterized. But in his work an additional requirement had to be posed on  $b(w)$ , namely that it is bounded away from zero. Later F. Gardiner succeeded in proving the analogous condition in the case when  $\sigma$  is finite and  $b(w) \equiv 0$  in  $E$  [G2]. He used a result from Teichmüller theory which he had proved in [G1].

In this note we use Gardiner’s result to generalize a fundamental variational lemma which is needed in Reich’s treatment. In its generalized form it turns out

to be adequate for the general case. In section 3 we apply it to handle the case where  $\sigma$  is infinite and  $b(w) \equiv 0$  in  $E$ . The proof then follows exactly the same pattern as the one in Reich's paper. In a forthcoming paper of K. Sakan [Sa] it then will be applied to arbitrary dilatation bound functions  $b(w)$ .

In section 4 we give, based on Reich's treatment, alternative proofs of Gardiner's result in two special cases. Namely, if the area-measure of the boundary  $\partial E$  of the set  $E$  is zero, then this result follows immediately by approximation and if  $E$  is supposed to be a closed set, it can be proved similarly.

Finally, I want to add that the idea of setting variable dilatation bounds as a side-condition for extremal problems goes back to O. Teichmüller ([T], p. 15), and to my knowledge R. Kühnau has been the first one who attacked such problems successfully. In [K1] he solved a problem of this sort (Satz 1) which enables him in [K2] to give a complete solution of our extremal problem above in the case where  $\sigma$  consists of four points by an essentially different method. No requirements as  $b(w) \geq \varepsilon > 0$  had to be made except for some regularity assumptions on  $E$  and  $b(w)$ .

## 2. Notations and the variational lemma

For a qc mapping  $F$  we denote its complex dilatation by  $\kappa_F$ , the dilatation of  $F$  at the point  $w$  by  $D_F(w) = (1 + |\kappa_F(w)|)/(1 - |\kappa_F(w)|)$  and its maximal dilatation by  $K[F]$ . We put  $\sigma' = h(\sigma)$ ,  $E_0 = \{w \in E \mid b(w) = 0\}$  and for a fixed element  $F \in Q(h, \sigma, E, b)$  we introduce

$$f = F^{-1}, \quad \kappa = \kappa_f, \quad k_F = \operatorname{ess\,sup}_{w \in D \setminus E} |\kappa_F(w)|$$

and

$$\hat{\kappa}(z) = \begin{cases} \kappa(z) & z \in D \setminus F(E) \\ k_F \frac{\kappa(z)}{b(f(z))} & z \in F(E \setminus E_0) \\ 0 & z \in F(E_0) \end{cases} \quad (2.1)$$

We note that  $\|\hat{\kappa}\|_\infty := \operatorname{ess\,sup}_{z \in D} |\hat{\kappa}(z)| = k_F$ . Then the Banach-space  $B_{\sigma'} = \{\varphi \mid \varphi \text{ holomorphic in } D, \|\varphi\| < \infty, \varphi dz^2 \text{ real along } \partial D \setminus \sigma'\}$  over the field  $\mathbb{R}$  will be used, where

$$\|\varphi\| = \iint_D |\varphi(z)| dx dy, \quad (z = x + iy)$$

as well as the unit sphere in  $B_{\sigma'}$

$$B_{\sigma'1} = \{\varphi \in B_{\sigma'} \mid \|\varphi\| = 1\}.$$

For measurable sets  $A$  in  $D$  we will put

$$\|\varphi\|_A = \iint_A |\varphi(z)| \, dx \, dy.$$

If  $Q(h, \sigma, E, b)$  is not empty, then there exist extremal mappings in this class as follows by normality and the following result of Strebel [St]: If a sequence of qc mappings  $F_n$  converge locally uniformly in  $D$  to a qc mapping,  $F$ , then

$$|K_F(w)| \leq \overline{\lim_{n \rightarrow \infty}} |K_{F_n}(w)| \quad \text{a.e. in } D.$$

The result of Gardiner then is the

**THEOREM 2.1 [G2].** *If  $\sigma$  is finite and  $b(w) \equiv 0$  in  $E$ , then  $F \in Q(h, \sigma, E, b)$  is extremal iff*

$$\sup_{\substack{\varphi \in B_{\sigma'} \\ \|\varphi\|_{D \setminus F(E_0)} = 1}} \operatorname{Re} \iint_D \hat{k}(z) \varphi(z) \, dx \, dy = k_F.$$

Since  $\sigma$  is finite, the space  $B_{\sigma'}$  is finite dimensional and it is easy to see that the sup must be attained. Namely, if  $\varphi_n$  is a sequence in  $B_{\sigma'}$  with  $\|\varphi_n\|_{D \setminus F(E_0)} = 1$ , then the norms  $\|\varphi_n\|$  stay bounded. Otherwise, by normality of  $B_{\sigma'1}$ ,  $\psi_n := (\varphi_n / \|\varphi_n\|)$  would contain a subsequence which converges to zero locally uniformly in  $D \setminus F(E_0)$ , an impossibility because of the finite dimension of  $B_{\sigma'}$ . Hence  $\varphi_n$  is a normal sequence and if it is a maximizing sequence for the functional above, then the limit of a convergent subsequence maximizes the functional. Therefore this theorem implies the

**COROLLARY 2.1 [G2].** *If  $\sigma$  is finite,  $b(w) \equiv 0$  in  $E$  and  $F$  extremal in  $Q(h, \sigma, E, b)$ , then there is a  $\varphi \in B_{\sigma'} \setminus \{0\}$  with*

$$K(z) = \begin{cases} k_F \frac{\bar{\varphi}(z)}{|\varphi(z)|} & z \in D \setminus F(E) \\ 0 & z \in F(E). \end{cases}$$



Our main tool will be the Main Inequality of Reich and Strebel [RS], p. 380 (see also [R1], p. 110), or more precisely, two statements following from it. First

(M1) If  $\varphi \in B_{\sigma'1}$  and  $f$  and  $g$  are two qc mappings from  $D$  onto itself which agree on  $\sigma'$ , then

$$1 \leq \iint_D |\varphi(z)| \frac{\left| 1 - \kappa_f(z) \frac{\varphi(z)}{|\varphi(z)|} \right|^2}{1 - |\kappa_f(z)|^2} D_{g^{-1}}(f(z)) dx dy.$$

Then, as is shown in [R1], p. 119, the Main Inequality applied to extremal  $n$ -gon Teichmüller mappings, yields

(M2) If  $\sigma'_n$  consists of  $n$  points on  $\partial D$  and  $f_n$  is a Teichmüller mapping with complex dilatation  $(K_n - 1)/(K_n + 1)(\bar{\varphi}_n/|\varphi_n|)$ , where  $\varphi_n \in B_{\sigma'_n1}$ , then for every qc selfmapping  $g$  of  $D$  which agrees with  $f_n$  on  $\sigma'_n$ , we have

$$K_n \leq \iint_D |\varphi_n(z)| \frac{\left| 1 + \kappa_g(z) \frac{\varphi_n(z)}{|\varphi_n(z)|} \right|^2}{1 - |\kappa_g(z)|^2} dx dy.$$

Before coming to the variational lemma we will derive the

**LEMMA 2.1.** *Let  $h$ ,  $\sigma$ ,  $\sigma'$  and  $E$  be as above and  $K \geq 1$  be a fixed number. Then there is a  $q < 1$  such that*

$$\|\varphi\|_{G(E)} \leq q$$

for every  $\varphi \in B_{\sigma'1}$  and every  $G \in Q_K(h, \sigma) := \{G \mid G: D \rightarrow D, K - qc, G|_\sigma = h|_\sigma\}$ .

*Proof.* If this lemma were false, there would be a sequence  $\varphi_n$  in  $B_{\sigma'1}$  and  $G_n$  in  $Q_K(h, \sigma)$  with

$$\|\varphi_n\|_{G_n(E)} \rightarrow 1, \quad n \rightarrow \infty.$$

The set  $B_{\sigma'1}$  is normal and, since  $\sigma$  contains at least three points, the set  $Q_K(h, \sigma)$  is normal and closed. So by passing to subsequences we may assume that there is a  $\varphi_\infty \in B_{\sigma'}$  and a  $G_\infty \in Q_K(h, \sigma)$  where

$$\varphi_n \xrightarrow{n \rightarrow \infty} \varphi_\infty \text{ locally uniformly in } \bar{D} \setminus \sigma'$$

$$G_n \xrightarrow{n \rightarrow \infty} G_\infty \text{ uniformly in } \bar{D}.$$

Taking into account that for infinite  $\sigma$  the set  $G_\infty(E)$  is supposed to be relatively compact in  $\bar{D} \setminus \sigma'$  we infer that

$$\|\varphi_\infty\|_{G_\infty(E)} = \lim_{n \rightarrow \infty} \|\varphi_n\|_{G_n(E)} = 1. \quad (2.2)$$

(This equality seems to be obvious, but in lack of a precise reference we add a proof of it in the appendix). Clearly  $\|\varphi_\infty\| \leq 1$  and hence

$$\iint_{D \setminus G_\infty(E)} |\varphi_\infty(z)| \, dx \, dy = 0$$

which is a contradiction because  $D \setminus G_\infty(E)$  has positive measure and  $\varphi_\infty \neq 0$  is holomorphic.

**FUNDAMENTAL VARIATIONAL LEMMA.** *Let  $E'$  be a measurable subset of  $D$  where  $D \setminus E'$  has positive measure and  $\sigma'$  be a closed set on  $\partial D$  which contains at least four points and where, if  $\sigma'$  is an infinite set,  $\overline{E'}$  is compact in  $\bar{D} \setminus \sigma'$ . If  $g$  is a qc mapping from  $D$  onto itself where its complex dilatation  $\kappa_g$  satisfies*

$$\kappa_g(z) \equiv 0 \text{ in } E' \quad \text{and} \quad \operatorname{Re} \iint_D \kappa_g \varphi \, dx \, dy = 0 \quad \forall \varphi \in B_{\sigma'},$$

*then there is a qc mapping  $g^*: D \rightarrow D$  with  $g^* \circ g = \operatorname{id}$  on  $\sigma'$  and with a complex dilatation  $\kappa_{g^*}$  that satisfies*

$$\kappa_{g^*}(z) \equiv 0 \text{ in } g(E') \quad \text{and} \quad \|\kappa_{g^*}\|_\infty = O(\|\kappa_g\|_\infty^2) \quad \text{as} \quad \|\kappa_g\|_\infty \rightarrow 0.$$

*Proof.* The best choice for  $g^*$  is to be an extremal element in  $Q(g^{-1}|_{\partial D}, \sigma'', E'', 0)$  where  $\sigma'' = g(\sigma')$ ,  $E'' = g(E')$ . If  $\sigma''$  is not finite, we choose  $\sigma''_n$  to consist of  $n$  points on  $\sigma''$  which become to be denser and denser as  $n$  tends to infinity. For every  $n$  there is an extremal mapping  $G_n$  in  $Q(g^{-1}|_{\partial D}, \sigma''_n, E'', 0)$  and by Corollary 2.1 there is a  $\varphi_n \in B_{\sigma'_n}$  ( $\sigma'_n = g^{-1}(\sigma''_n)$ ) such that the complex dilatation  $\kappa_n$  of  $g_n := G_n^{-1}$  satisfies

$$\kappa_n(z) = \begin{cases} k_n \frac{\bar{\varphi}_n(z)}{|\varphi_n(z)|} & z \in G_n(D \setminus E'') \\ 0 & z \in G_n(E'') \end{cases} \quad (2.3)$$

Evidently  $k_n \leq \|\kappa_{g^*}\|_\infty$  and  $k_n$  is an increasing sequence. By normality we may assume that the qc mappings  $G_n$  converge locally uniformly to a qc mapping  $G_\infty: D \rightarrow D$  where obviously  $G_\infty|_{\sigma''} = g^{-1}|_{\sigma''}$  and by Strebel's result

$$|\kappa_{G_\infty}(z)| \leq \overline{\lim_{n \rightarrow \infty}} |\kappa_{G_n}(z)| \quad \text{a.e. in } D.$$

We hence conclude that  $\kappa_{G_\infty}(z) = 0$  in  $E''$  and  $\text{ess sup}_{z \in D \setminus E''} |\kappa_{G_\infty}(z)| \leq \lim_{n \rightarrow \infty} k_n \leq \|\kappa_{g^*}\|_\infty$ . So  $G_\infty \in Q(g^{-1}|_{\partial D}, \sigma'', E'', 0)$  and  $G_\infty$  is therefore extremal in this class, i.e.,  $\|\kappa_{g^*}\|_\infty = \lim_{n \rightarrow \infty} k_n$  and we can take  $G_\infty$  for  $g^*$ .

For the purpose of estimating the numbers  $k_n$  we introduce the extremal Teichmüller  $n$ -gon mappings  $f_n: D \rightarrow D$  which agree on  $\sigma'_n$  with  $g$ . Their complex dilatations  $\tilde{\kappa}_n$  are equal to

$$\tilde{\kappa}_n \frac{\bar{\varphi}_n}{|\bar{\varphi}_n|} \quad \text{a.e. in } D$$

where  $\tilde{\varphi}_n \in B_{\sigma'_n, 1}$ .

We use the statement (M1) where we put the quadruple  $(\sigma'_n, \varphi_n, g_n, f_n)$  for  $(\sigma', \varphi, f, g)$

$$1 \leq \iint_D |\varphi_n(z)| \frac{\left| 1 - \kappa_n(z) \frac{\varphi_n(z)}{|\varphi_n(z)|} \right|^2}{1 - |\kappa_n(z)|^2} \tilde{K}_n \, dx \, dy \quad \left( \tilde{K}_n = \frac{1 + \tilde{\kappa}_n}{1 - \tilde{\kappa}_n} \right).$$

Using (2.3), splitting up the integral and putting  $K_n = (1 + k_n)/(1 - k_n)$  we get

$$1 \leq \frac{\tilde{K}_n}{K_n} (1 - \|\varphi_n\|_{G_n(E'')}) + \tilde{K}_n \|\varphi_n\|_{G_n(E'')}.$$

We multiply with  $K_n$  and subtract  $K_n \|\varphi_n\|_{G_n(E'')}$ , so

$$K_n(1 - \|\varphi_n\|_{G_n(E'')}) \leq \tilde{K}_n(1 - \|\varphi_n\|_{G_n(E'')}) + (\tilde{K}_n - 1)K_n \|\varphi_n\|_{G_n(E'')}$$

and finally since  $\|\varphi_n\|_{G_n(E'')} < 1$ ,  $K_n \leq K[g^*]$

$$K_n \leq \tilde{K}_n + (\tilde{K}_n - 1) \frac{K[g^*]}{1 - \|\varphi_n\|_{G_n(E'')}}. \quad (2.4)$$

Next we estimate  $\tilde{K}_n$ . We use the statement (M2) and put the quadrupel  $(\sigma'_n, \tilde{\varphi}_n, f_n, g)$  for  $(\sigma'_n, \varphi_n, f_n, g)$

$$\tilde{K}_n \leq \iint_D |\tilde{\varphi}_n(z)| \frac{\left| 1 + \kappa_g(z) \frac{\tilde{\varphi}_n(z)}{|\tilde{\varphi}_n(z)|} \right|^2}{1 - |\kappa_g(z)|^2} dx dy.$$

Following Reich's calculation in [R2] the integral becomes

$$\begin{aligned} & \iint_D |\tilde{\varphi}_n| \frac{1 + |\kappa_g|^2}{1 - |\kappa_g|^2} dx dy + 2 \operatorname{Re} \iint_D \frac{\kappa_g \tilde{\varphi}_n}{1 - |\kappa_g|^2} dx dy \\ & \leq \frac{1 + \|\kappa_g\|_\infty^2}{1 - \|\kappa_g\|_\infty^2} + 2 \operatorname{Re} \iint_D \left( \frac{\kappa_g \tilde{\varphi}_n}{1 - |\kappa_g|^2} - \kappa_g \tilde{\varphi}_n \right) dx dy \end{aligned}$$

because of the hypothesis

$$\operatorname{Re} \iint_D \kappa_g \tilde{\varphi}_n dx dy = 0, \quad \tilde{\varphi}_n \in B_{\sigma'_n 1} \subset B_{\sigma'}.$$

The second term can be estimated by

$$2 \left| \iint_D \frac{|\kappa_g|^2 \kappa_g \tilde{\varphi}_n}{1 - |\kappa_g|^2} dx dy \right| \leq \frac{2 \|\kappa_g\|_\infty^3}{1 - \|\kappa_g\|_\infty^2}.$$

Hence

$$\tilde{K}_n \leq 1 + 2 \frac{\|\kappa_g\|_\infty^2 + \|\kappa_g\|_\infty^3}{1 - \|\kappa_g\|_\infty^2} = 1 + 2 \frac{\|\kappa_g\|_\infty^2}{1 - \|\kappa_g\|_\infty}.$$

With (2.4) finally

$$K_n \leq 1 + 2 \frac{\|\kappa_g\|_\infty^2}{1 - \|\kappa_g\|_\infty} + 2 \frac{\|\kappa_g\|_\infty^2}{1 - \|\kappa_g\|_\infty} \frac{K[g^*]}{1 - \|\varphi_n\|_{G_n(E'')}}.$$

We let  $n$  tend to infinity, i.e.  $G_n \xrightarrow{n \rightarrow \infty} G_\infty$  locally uniformly in  $D$  and by normality of  $B_{\sigma' 1}$  we may pass to a further subsequence such that

$$\varphi_n \xrightarrow{n \rightarrow \infty} \varphi_\infty \text{ locally uniformly in } \bar{D} \setminus \sigma'$$

where  $\varphi_\infty \in B_{\sigma'}$ ,  $\|\varphi_\infty\| \leq 1$ .

This approximation takes place only if  $\sigma'$  is infinite, so  $E''$  is relatively compact in  $\bar{D} \setminus \sigma''$  and hence

$$\|\varphi_n\|_{G_n(E'')} \xrightarrow{n \rightarrow \infty} \|\varphi_\infty\|_{G_\infty(E'')}.$$

Hence

$$K[g^*] = \lim_{n \rightarrow \infty} K_n \leq 1 + \frac{2 \|\kappa_g\|_\infty^2}{1 - \|\kappa_g\|_\infty} \left( 1 + \frac{K[g^*]}{1 - \|\varphi_\infty\|_{G_\infty(E'')}} \right).$$

Now we apply Lemma 2.1 with the quadrupel  $(id, \sigma', \sigma', E')$  instead of  $(h, \sigma, \sigma', E)$  for a fixed number  $K > K[g]^2$ . Namely,  $G_\infty(E'') = G_\infty \circ g(E')$  and  $G_\infty \circ g|_{\sigma'} = id|_{\sigma'}$ , so  $G_\infty \circ g \in Q_K(id, \sigma')$  and by  $\|\varphi_\infty\| \leq 1$  and  $(\varphi_\infty / \|\varphi_\infty\|) \in B_{\sigma', 1}$ , there is a  $q < 1$  which does not depend on  $g$  (only on  $K$ ), such that

$$\|\varphi_\infty\|_{G_\infty \circ g(E')} \leq \left\| \frac{\varphi_\infty}{\|\varphi_\infty\|} \right\|_{G_\infty \circ g(E')} \leq q$$

and hence

$$K[g^*] \leq 1 + \left( 1 + \frac{K}{1 - q} \right) \frac{2 \|\kappa_g\|_\infty^2}{1 - \|\kappa_g\|_\infty}$$

which shows that

$$\|\kappa_{g^*}\|_\infty = O(\|\kappa_g\|_\infty^2)$$

for  $\|\kappa_g\|_\infty \rightarrow 0$ , since  $K$  can stay fixed as  $K[g] \rightarrow 1$ .

### 3. Application: The case $b(w) \equiv 0$

An elaboration of a technique employed by Krushkal [Kr] which is done in Reich's paper [R2] now yields the necessity part of the

**THEOREM 3.1.** *A qc mapping  $F$  is extremal in  $Q(h, \sigma, E, 0)$  iff*

$$\sup_{\substack{\varphi \in B_{\sigma'} \\ \|\varphi\|_{D \setminus \sigma'(E_0)} = 1}} \operatorname{Re} \int_D \hat{\kappa} \varphi \, dx \, dy = k_F$$

*Proof.* Let  $F$  be extremal. Since  $k_F = \|\hat{k}\|_\infty$  we now assume that

$$a := \sup_{\substack{\varphi \in B_{\sigma'} \\ \|\varphi\|_{D \setminus F(E)} = 1}} \operatorname{Re} \iint_D \hat{k} \varphi \, dx \, dy < k_F$$

and hence  $k_F > 0$ .

Then  $a$  is the norm of the linear operator

$$\varphi \mapsto \operatorname{Re} \iint_{D \setminus F(E)} \hat{k} \varphi \, dx \, dy$$

( $\hat{k} = 0$  on  $F(E)$ !) defined on the Banach-space  $\{\varphi|_{D \setminus F(E)} \mid \varphi \in B_{\sigma'}\}$  which is a subspace of  $L_1(D \setminus F(E))$  over the field  $\mathbb{R}$ . By the Hahn–Banach Theorem there is an extension of this operator on  $L_1(D \setminus F(E))$  with norm  $a$  and by the Riesz-representation theorem there is a complex-valued function  $\beta$  on  $D \setminus F(E)$  with  $\|\beta\|_\infty = a$  which realizes this extension, i.e.,

$$\operatorname{Re} \iint_{D \setminus F(E)} \hat{k} \varphi \, dx \, dy = \operatorname{Re} \iint_{D \setminus F(E)} \beta \varphi \, dx \, dy \quad \forall \varphi \in B_{\sigma'}.$$

We put

$$v(z) = \begin{cases} \hat{k}(z) - \beta(z) & z \in D \setminus F(E) \\ 0 & z \in F(E) \end{cases}$$

and have  $\|v\|_\infty > 0$  ( $\hat{k} \neq \beta$ !) and

$$\operatorname{Re} \iint_D v \varphi \, dx \, dy = 0 \quad \text{for } \varphi \in B_{\sigma'}.$$

For  $t$ ,  $0 \leq t < (1/\|v\|_\infty)$  we put  $g: D \rightarrow D$  to be a qc mapping with  $g(1) = 1$ ,  $g(i) = i$ ,  $g(-1) = -1$  and

$$\kappa_g = tv.$$

Here we apply the Fundamental Variational Lemma on  $g$ ,  $\sigma' = h(\sigma)$  and  $E' = F(E)$ . Hence there is a qc mapping  $g^*: D \rightarrow D$  where  $g^* \circ g = id$  on  $\sigma'$ ,  $\kappa_{g^*} = 0$  in  $g(E')$  and

$$\|\kappa_{g^*}\|_\infty = O(t^2) \quad \text{as } t \rightarrow 0.$$

We have  $g^* \circ g \circ F \in Q(h, \sigma, E, 0)$  and show that

$$\operatorname{ess\,sup}_{w \in D \setminus E} |\kappa_{g^* \circ g \circ F}(w)| < k_F$$

for  $t > 0$ , sufficiently small. This contradicts then the extremality of  $F$ . One computes for  $z \in D \setminus E'$

$$|\kappa_{f \circ g^{-1}}(g(z))| = \left| \frac{\kappa(1-t) + t\beta}{1 - t\tilde{\nu}\kappa} \right|$$

and the computation in [R2], p. 109 and 110, assures the existence of numbers  $\delta > 0$ ,  $t_0 > 0$  with

$$|\kappa_{f \circ g^{-1}}(g(z))| \leq k_F - \delta_1 t \quad \text{for } 0 \leq t \leq t_0 \quad \text{and } z \in D \setminus E'.$$

By  $\|\kappa_{g^*}\|_\infty = O(t^2)$  the values  $|\kappa_{f \circ g^{-1} \circ g^*}(g^*(g(z)))|$  can be estimated in the same manner in  $D \setminus E'$  and this yields the result.

The sufficiency part is immediate. We do not need any restriction on  $b(w)$  for it. Let  $F \in Q(h, \sigma, E, b)$  and

$$\sup_{\substack{\varphi \in B_{\sigma'} \\ \|\varphi\|_{D \setminus F(E_0)} = 1}} \operatorname{Re} \iint_D \hat{\kappa} \varphi \, dx \, dy = k_F.$$

If this sup is attained, then there is a  $\varphi \in B_{\sigma'}$ ,  $\|\varphi\|_{D \setminus F(E_0)} = 1$ , with

$$\hat{\kappa}(z) = k_F \frac{\bar{\varphi}(z)}{|\varphi(z)|} \quad \text{a.e. in } D \setminus F(E_0).$$

If  $k_F = 0$  we have extremality. If  $k_F > 0$  we conclude from (2.1)

$$\kappa(z) = \begin{cases} k_F \frac{\bar{\varphi}(z)}{|\varphi(z)|} & z \in D \setminus F(E) \\ b(f(z)) \frac{\bar{\varphi}(z)}{|\varphi(z)|} & z \in F(E) \end{cases}$$

Then by [R2], Theorem 5,  $F$  is even uniquely extremal in  $Q(h, \sigma, E, b)$ .

If the sup is not attained, and this can occur only if  $\sigma$  is infinite, then there is a sequence  $\varphi_n \in B_{\sigma'}$ ,  $\|\varphi_n\|_{D \setminus F(E_0)} = 1$  with  $\operatorname{Re} \iint_D \hat{\kappa} \varphi_n \, dx \, dy \xrightarrow{n \rightarrow \infty} k_F$  and  $\varphi_n \xrightarrow{n \rightarrow \infty} 0$  locally uniformly in  $\bar{D} \setminus \sigma'$ .

From the relative compactness of  $E$  in  $\bar{D} \setminus \sigma'$  we conclude that  $\|\varphi_n\|_{F(E_0)} \xrightarrow{n \rightarrow \infty} 0$

and hence  $\|\varphi_n\| \xrightarrow{n \rightarrow \infty} 1$ . So if we put

$$\hat{\varphi}_n := \frac{\varphi_n}{\|\varphi_n\|}$$

we get a degenerating Hamilton sequence  $\hat{\varphi}_n$  for the complex dilatation  $\hat{k}$ , this is a sequence  $\hat{\varphi}_n \in B_{\sigma',1}$  where  $\operatorname{Re} \iint_D \hat{k} \hat{\varphi}_n dx dy \xrightarrow{n \rightarrow \infty} \|\hat{k}\|_\infty = k_F$  and  $\hat{\varphi}_n \xrightarrow{n \rightarrow \infty} 0$  locally uniformly in  $\bar{D} \setminus \sigma'$ . We denote by  $\hat{f}$  a qc selfmapping from  $D$  with complex dilatation  $\hat{k}$ . By the sufficiency of Hamilton's condition  $\hat{f}$  is extremal for its own boundary values on  $\sigma'$ , and by Satz 5.2 in [F] there exists a substantial boundary point on  $\sigma'$ , i.e., a point with local dilatation equal to  $K[\hat{f}] = (1 + k_F) / (1 - k_F)$  (for the boundary values  $\hat{f}|_{\sigma'}$ ). Since  $\hat{f} \circ F$  is conformal in  $D \setminus E$  which contains a neighborhood of  $\sigma$  and since local dilatations of the boundary mapping are preserved under conformal mapping we conclude that there is a point on  $\sigma$  with local dilatation  $K[\hat{f}]$  for the boundary values  $h|_\sigma$ . Hence every mapping in  $Q(h, \sigma, E, b)$  needs to have its dilatation near that point at least as large as  $K[\hat{f}] = (1 + k_F) / (1 - k_F)$ , in particular,  $F$  is extremal.

#### 4. Alternative proofs of Theorem 2.1 in two special cases

Let  $\sigma$  be finite and  $b(w) \equiv 0$  in  $E$ . We put  $b_n(w) \equiv 1/n$  in  $E$  and denote by  $F_n$  an extremal qc mapping in  $Q(h, \sigma, E, 1/n)$ . By Reich's result there is a  $\varphi_n \in B_{\sigma',1}$  where the complex dilatation  $\kappa_n$  of  $f_n := F_n^{-1}$  is

$$\kappa_n(z) = \begin{cases} k_n \frac{\bar{\varphi}_n(z)}{|\varphi_n(z)|} & z \in D \setminus F_n(E) \\ \frac{1}{n} \frac{\bar{\varphi}_n(z)}{|\varphi_n(z)|} & z \in F_n(E) \end{cases}$$

Furthermore, let  $F$  be extremal in  $Q(h, \sigma, E, 0)$ , hence  $k_n \leq k_F$ . Passing to subsequences we find a qc mapping  $F_\infty$  and a function  $\varphi_\infty$  where

$$F_n \xrightarrow{n \rightarrow \infty} F_\infty \quad \text{locally uniformly in } D$$

and

$$\varphi_n \xrightarrow{n \rightarrow \infty} \varphi_\infty \quad \text{locally uniformly in } \bar{D} \setminus \sigma'$$



As above we conclude that  $F_\infty \in Q(h, \sigma, E, 0)$ , and therefore  $k_F \leq k_{F_\infty} \leq \lim_{n \rightarrow \infty} k_n \leq k_F$ , and since  $\sigma$  is finite we have  $\varphi_\infty \in B_{\sigma'1}$ . Furthermore

$$f_n \xrightarrow{n \rightarrow \infty} f_\infty := F_\infty^{-1} \text{ locally uniformly in } D.$$

$F_\infty$  is hence extremal in  $Q(h, \sigma, E, 0)$  and the complex dilatations  $\kappa_n$  converge pointwise a.e. in the interior of  $F_\infty(E)$  or in the interior of  $D \setminus F_\infty(E)$  to zero or  $k_F(\overline{\varphi_\infty}/|\varphi_\infty|)$  respectively. Since qc mappings preserve sets of area-measure zero we infer from Theorem 5.2 in [LV], p. 187, that if the area-measure of  $\partial E$  is zero, then a.e.

$$\kappa_{f_\infty}(z) = \begin{cases} k_F \frac{\overline{\varphi_\infty}(z)}{|\varphi_\infty(z)|} & z \in D \setminus F_\infty(E) \\ 0 & z \in F_\infty(E). \end{cases} \quad (4.1)$$

By Theorem 5 in [R2]  $F_\infty$  is uniquely extremal in  $Q(h, \sigma, E, 0)$  and hence  $F = F_\infty$ . Clearly

$$\sup_{\substack{\varphi \in B_{\sigma'} \\ \|\varphi\|_{D \setminus F(E)} = 1}} \operatorname{Re} \iint_D \kappa_{f_\infty} \varphi \, dx \, dy = k_F$$

and we thus have proved the first part of

**PROPOSITION 4.1.** *Let  $F$  be extremal in  $Q(h, \sigma, E, 0)$  where  $\sigma$  is finite. If the area-measure of  $\partial E$  is zero or if  $E$  is a closed set, then*

$$\sup_{\substack{\varphi \in B_{\sigma'} \\ \|\varphi\|_{D \setminus F(E)} = 1}} \operatorname{Re} \iint_D \hat{\kappa} \varphi \, dx \, dy = k_F.$$

For the second part the reasoning in the proof above has to be slightly changed since we do not know if  $\kappa_n$  is convergent a.e. in  $D$ . For this purpose we change to the  $w$ -plane. First we observe that by  $|\kappa_{F_\infty}(w)| \leq \lim_{n \rightarrow \infty} |\kappa_{F_n}(w)|$  a.e. in  $D$  we conclude that  $\kappa_{F_\infty}(w) = 0$  a.e. in  $E$ , hence  $\kappa_{f_\infty}(z) = 0$  a.e. in  $F_\infty(E)$ . Next we use the fact that  $E$  is closed. Let  $z_0 \in D \setminus F_\infty(E)$ . There is a neighborhood  $U_{z_0}$  of  $z_0$  with  $U_{z_0} \cap D \setminus F_\infty(E)$  and by the local uniform convergence of  $F_n$  to  $F_\infty$  we find an open disk  $D_{z_0}$  with center  $z_0$  in  $U_{z_0}$  such that for a number  $n_0 \in \mathbb{N}$

$$D_{z_0} \cap D \setminus F_n(E) \quad \forall n \geq n_0.$$

We hence infer that

$$\kappa_n(z) = k_n \frac{\bar{\varphi}_n(z)}{|\varphi_n(z)|} \xrightarrow{n \rightarrow \infty} k_F \frac{\bar{\varphi}_\infty(z)}{|\varphi_\infty(z)|} \quad \text{a.e. in } D_{z_0}.$$

Again by Theorem 5.2 in [LV] we get

$$\kappa_{f_\infty}(z) = k_F \frac{\bar{\varphi}_\infty(z)}{|\varphi_\infty(z)|} \quad \text{a.e. in } D_{z_0}$$

and since  $z_0$  was arbitrary in  $D \setminus F_\infty(E)$ , we again have (4.1) from which the result follows.

## 5. Appendix

As has been pointed out to me by K. Sakan, the equation (2.2) in Lemma 2.1 is not at all a triviality. So let me add a proof here. There are several ways to do it, e.g. one could infer this statement from results on the weak-convergence of Jacobians of qc mappings (see [L]). I prefer here to use a consequence of a result on area-distortion by Gehring and Reich [GR].

Let us denote the Jacobian of a qc mapping  $f$  by  $J_f$ . For a number  $K \geq 1$  let  $F$  be the set of  $K$ -qc mappings of the unit disk  $D$  onto itself which fix the origin. From [GR] it then follows that the integrals  $\iint_D J_f dx dy$  are uniformly absolutely continuous, i.e., for every  $\varepsilon > 0$  there is a  $\delta > 0$  such that

$$\iint_E J_f dx dy < \varepsilon$$

for every  $f \in F$  and every measurable set  $E$  in the disk with  $|E| := \iint_E dx dy < \delta$ .

This property will imply the

**THEOREM 5.1.** *Let  $f_n$  and  $f$  be  $K$ -qc selfmappings of the unit disk  $D$  where  $f_n \xrightarrow{n \rightarrow \infty} f$  locally uniformly in  $D$ . Then we have for every measurable and bounded function  $\varphi$  in  $D$*

$$\lim_{n \rightarrow \infty} \iint_D \varphi(z) J_{f_n}(z) dx dy = \iint_D \varphi(z) J_f(z) dx dy.$$

In case that  $\varphi_0$  is continuous in  $\bar{D} \setminus \sigma$  and  $E$  relatively compact in  $\bar{D} \setminus \sigma$  we choose  $\varphi = \varphi_0 \chi_E$  ( $\chi_E$  denotes the characteristic function of  $E$ ), hence  $\varphi$  is bounded and from this theorem we derive that

$$\iint_E \varphi_0 J_{f_n} dx dy \xrightarrow{n \rightarrow \infty} \iint_E \varphi_0 J_f dx dy. \quad (5.1)$$

The equation (2.2) claims that

$$\iint_E |\varphi_n \circ G_n| J_{G_n} dx dy \xrightarrow{n \rightarrow \infty} \iint_E |\varphi_\infty \circ G_\infty| J_{G_\infty} dx dy$$

which clearly follows from (5.1) by putting  $\varphi_0 = \varphi_\infty \circ G_\infty$  because  $\varphi_n \circ G_n \rightarrow \varphi_\infty \circ G_\infty$  locally uniformly in  $\bar{D} \setminus \sigma$ .

*Proof of Theorem 5.1.* By [GR] we conclude that obviously also the integrals  $\iint_D J_{f_n} dx dy$  are uniformly absolutely continuous. We first choose  $\varphi = \chi_R$  where  $R$  is a rectangle whose closure is contained in  $D$ . Then the statement follows from Lebesgue's dominated convergence theorem since  $\chi_{f_n(R)} \rightarrow \chi_{f(R)}$  a.e. in  $D$ . Hence for step-functions  $s = \sum_{i=1}^N c_i \chi_{R_i}$  we have

$$\iint_D s J_{f_n} dx dy \xrightarrow{n \rightarrow \infty} \iint_D s J_f dx dy. \quad (5.2)$$

Finally let  $\varphi$  be measurable and bounded in  $D$ . Let  $s_m$  be a sequence of step-functions with  $s_m(z) \xrightarrow{m \rightarrow \infty} \varphi(z)$  a.e. in  $D$  and  $M$  be a number with  $|\varphi| \leq M$  and  $|s_m| \leq M$  for all  $m$ . Let  $\varepsilon > 0$  be given. By the uniform absolute continuity of the integrals  $\iint_D J_{f_n} dx dy$ , there is an  $\eta > 0$  such that

$$\iint_E J_{f_n} dx dy < \varepsilon \quad \forall n \quad \text{and} \quad \iint_E J_f dx dy < \varepsilon$$

whenever  $E$  is a measurable set in  $D$  with  $|E| < \eta$ . By Egoroff's theorem there is a set  $E_\eta \subset D$  with  $|E_\eta| < \eta$  such that  $s_m \xrightarrow{m \rightarrow \infty} \varphi$  uniformly on  $D \setminus E_\eta$ . We choose  $m_0$  such that

$$|s_m(z) - \varphi(z)| < \varepsilon \quad \forall z \in D \setminus E_\eta, \quad \forall m \geq m_0.$$

For an  $m \geq m_0$  we then have

$$\begin{aligned} \iint_D \varphi J_{f_n} dx dy - \iint_D \varphi J_f dx dy &= \iint_{D \setminus E_\eta} (\varphi - s_m) J_{f_n} + \iint_{E_\eta} (\varphi - s_m) J_{f_n} \\ &+ \iint_D s_m (J_{f_n} - J_f) + \iint_{D \setminus E_\eta} (s_m - \varphi) J_f + \iint_{E_\eta} (s_m - \varphi) J_f. \end{aligned}$$

By

$$\iint_{D \setminus E_\eta} |\varphi - s_m| J_{f_n} dx dy \leq \varepsilon \pi \quad \text{and} \quad \iint_{E_\eta} |\varphi - s_m| J_{f_n} \leq 2M\varepsilon$$

(also with  $J_f$  instead of  $J_{f_n}$ ) we conclude from (5.2)

$$\overline{\lim}_{n \rightarrow \infty} \left| \iint_D \varphi (J_{f_n} - J_f) dx dy \right| \leq 2\varepsilon \pi + 4M\varepsilon$$

and  $\varepsilon \rightarrow 0$  proves the theorem.

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