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# On a fundamental variational lemma for extremal quasiconformal mappings

RICHARD FEHLMANN

### 1. Introduction

In [R2] E. Reich considers the following extremal problem in qc (quasiconformal) mappings. Given are a closed set  $\sigma$  on the boundary  $\partial D$  of the unit disk  $D = \{w \mid |w| < 1\}$  which contains at least four points and a measurable set E in D where  $D \setminus E$  has positive area-measure and where, if  $\sigma$  is an infinite set,  $\bar{E}$  is assumed to be compact in  $\bar{D} \setminus \sigma$ . Furthermore a quasisymmetric boundary mapping  $h: \partial D \to \partial D$  is given and a measurable non-negative function b(w) on E with ess  $\sup_{w \in E} b(w) < 1$  which is called the "dilatation bound function":  $Q(h, \sigma, E, b)$  then denotes the class of all qc mappings  $F: D \to D$  which satisfy the side-condition

$$F|_{\sigma} = h|_{\sigma}$$
 and  $|\kappa_F(w)| \le b(w)$  a.e. in E,

where  $\kappa_F = F_{\bar{w}}/F_w$  is the complex dilatation of F. In this class a mapping F is called *extremal* if it minimizes the value

$$\operatorname{ess\,sup}_{w\in D\setminus E}|\kappa_F(w)|$$

and is called uniquely extremal if it is the only such mapping.

In the case when E is the empty set a necessary and sufficient condition for extremality is the Hamilton-condition as has been shown in [H] and [RS]. In [R2] E. Reich has given a generalization of this condition which is necessary and sufficient for extremality in  $Q(h, \sigma, E, b)$  and by which extremal mappings can be characterized. But in his work an additional requirement had to be posed on b(w), namely that it is bounded away from zero. Later F. Gardiner succeeded in proving the analogous condition in the case when  $\sigma$  is finite and  $b(w) \equiv 0$  in E [G2]. He used a result from Teichmüller theory which he had proved in [G1].

In this note we use Gardiner's result to generalize a fundamental variational lemma which is needed in Reich's treatment. In its generalized form it turns out

to be adequate for the general case. In section 3 we apply it to handle the case where  $\sigma$  is infinite and  $b(w) \equiv 0$  in E. The proof then follows exactly the same pattern as the one in Reich's paper. In a forthcoming paper of K. Sakan [Sa] it then will be applied to arbitrary dilatation bound functions b(w).

In section 4 we give, based on Reich's treatment, alternative proofs of Gardiner's result in two special cases. Namely, if the area-measure of the boundary  $\partial E$  of the set E is zero, then this result follows immediately by approximation and if E is supposed to be a closed set, it can be proved similarly.

Finally, I want to add that the idea of setting variable dilatation bounds as a side-condition for extremal problems goes back to O. Teichmüller ([T], p. 15), and to my knowledge R. Kühnau has been the first one who attacked such problems successfully. In [K1] he solved a problem of this sort (Satz 1) which enables him in [K2] to give a complete solution of our extremal problem above in the case where  $\sigma$  consists of four points by an essentially different method. No requirements as  $b(w) \ge \varepsilon > 0$  had to be made except for some regularity assumptions on E and b(w).

## 2. Notations and the variational lemma

For a qc mapping F we denote its complex dilatation by  $\kappa_F$ , the dilatation of F at the point w by  $D_F(w) = (1 + |\kappa_F(w)|)/(1 - |\kappa_F(w)|)$  and its maximal dilatation by K[F]. We put  $\sigma' = h(\sigma)$ ,  $E_0 = \{w \in E \mid b(w) = 0\}$  and for a fixed element  $F \in Q(h, \sigma, E, b)$  we introduce

$$f = F^{-1}$$
,  $\kappa = \kappa_f$ ,  $k_F = \operatorname{ess sup}_{w \in D \setminus E} |\kappa_F(w)|$ 

and

$$\hat{\kappa}(z) = \begin{cases} \kappa(z) & z \in D \backslash F(E) \\ k_F \frac{\kappa(z)}{b(f(z))} & z \in F(E \backslash E_0) \\ 0 & z \in F(E_0) \end{cases}$$
(2.1)

We note that  $\|\hat{\kappa}\|_{\infty} := \operatorname{ess\,sup}_{z \in D} |\hat{\kappa}(z)| = k_F$ . Then the Banach-space  $B_{\sigma'} = \{\varphi \mid \varphi \text{ holomorphic in } D, \|\varphi\| < \infty, \varphi dz^2 \text{ real along } \partial D \setminus \sigma' \}$  over the field  $\mathbb R$  will be used, where

$$\|\varphi\| = \iint_D |\varphi(z)| dx dy, \quad (z = x + iy)$$

as well as the unit sphere in  $B_{\sigma'}$ 

$$B_{\sigma'1} = \{ \varphi \in B_{\sigma'} \mid ||\varphi|| = 1 \}.$$

For measurable sets A in D we will put

$$\|\varphi\|_A = \iint_A |\varphi(z)| \ dx \ dy.$$

If  $Q(h, \sigma, E, b)$  is not empty, then there exist extremal mappings in this class as follows by normality and the following result of Strebel [St]: If a sequence of qc mappings  $F_n$  converge locally uniformly in D to a qc mapping, F, then

$$|\kappa_F(w)| \leq \overline{\lim_{n \to \infty}} |\kappa_{F_n}(w)|$$
 a.e. in  $D$ .

The result of Gardiner then is the

THEOREM 2.1 [G2]. If  $\sigma$  is finite and  $b(w) \equiv 0$  in E, then  $F \in Q(h, \sigma, E, b)$  is extremal iff

$$\sup_{\substack{\varphi \in B_{\sigma'} \\ \|\varphi\|_{D \setminus F(E_{\sigma})} = 1}} \operatorname{Re} \iint_{D} \hat{k}(z) \varphi(z) \, dx \, dy = k_{F}.$$

Since  $\sigma$  is finite, the space  $B_{\sigma'}$  is finite dimensional and it is easy to see that the sup must be attained. Namely, if  $\varphi_n$  is a sequence in  $B_{\sigma'}$  with  $\|\varphi_n\|_{D \setminus F(E_0)} = 1$ , then the norms  $\|\varphi_n\|$  stay bounded. Otherwise, by normality of  $B_{\sigma'1}$ ,  $\psi_n := (\varphi_n/\|\varphi_n\|)$  would contain a subsequence which converges to zero locally uniformly in  $D \setminus F(E_0)$ , an impossibility because of the finite dimension of  $B_{\sigma'}$ . Hence  $\varphi_n$  is a normal sequence and if it is a maximizing sequence for the functional above, then the limit of a convergent subsequence maximizes the functional. Therefore this theorem implies the

COROLLARY 2.1 [G2]. If  $\sigma$  is finite,  $b(w) \equiv 0$  in E and F extremal in  $Q(h, \sigma, E, b)$ , then there is a  $\varphi \in B_{\sigma'} \setminus \{0\}$  with

$$\kappa(z) = \begin{cases} k_F \frac{\bar{\varphi}(z)}{|\varphi(z)|} & z \in D \backslash F(E) \\ 0 & z \in F(E). \end{cases}$$

Our main tool will be the Main Inequality of Reich and Strebel [RS], p. 380 (see also [R1], p. 110), or more precisely, two statements following from it. First

(M1) If  $\varphi \in B_{\sigma'1}$  and f and g are two qc mappings from D onto itself which agree on  $\sigma'$ , then

$$1 \leq \iint_{D} |\varphi(z)| \frac{\left|1 - \kappa_{f}(z) \frac{\varphi(z)}{|\varphi(z)|}\right|^{2}}{1 - |\kappa_{f}(z)|^{2}} D_{g^{-1}}(f(z)) dx dy.$$

Then, as is shown in [R1], p. 119, the Main Inequality applied to extremal n-gon Teichmüller mappings, yields

(M2) If  $\sigma'_n$  consists of n points on  $\partial D$  and  $f_n$  is a Teichmüller mapping with complex dilatation  $(K_n-1)/(K_n+1)(\bar{\varphi}_n/|\varphi_n|)$ , where  $\varphi_n \in B_{\sigma'_n 1}$ , then for every qc selfmapping g of D which agrees with  $f_n$  on  $\sigma'_n$ , we have

$$K_n \leq \iint_D |\varphi_n(z)| \frac{\left|1 + \kappa_g(z) \frac{\varphi_n(z)}{|\varphi_n(z)|}\right|^2}{1 - |\kappa_g(z)|^2} dx dy.$$

Before coming to the variational lemma we will derive the

LEMMA 2.1. Let h,  $\sigma$ ,  $\sigma'$  and E be as above and  $K \ge 1$  be a fixed number. Then there is a q < 1 such that

$$\|\varphi\|_{G(E)} \leq q$$

for every  $\varphi \in B_{\sigma'1}$  and every  $G \in Q_K(h, \sigma) := \{G \mid G : D \to D, K - qc, G|_{\sigma} = h|_{\sigma}\}.$ 

**Proof.** If this lemma were false, there would be a sequence  $\varphi_n$  in  $B_{\sigma'1}$  and  $G_n$  in  $Q_K(h, \sigma)$  with

$$\|\varphi_n\|_{G_n(E)} \to 1, \qquad n \to \infty.$$

The set  $B_{\sigma'1}$  is normal and, since  $\sigma$  contains at least three points, the set  $Q_K(h, \sigma)$  is normal and closed. So by passing to subsequences we may assume that there is a  $\varphi_{\infty} \in B_{\sigma'}$  and a  $G_{\infty} \in Q_K(h, \sigma)$  where

$$\varphi_n \xrightarrow{n \to \infty} \varphi_\infty$$
 locally uniformly in  $\bar{D} \setminus \sigma'$ .
$$G_n \xrightarrow{n \to \infty} G_\infty \text{ uniformly in } \bar{D}.$$

Taking into account that for infinite  $\sigma$  the set  $G_{\infty}(E)$  is supposed to be relatively compact in  $\bar{D} \setminus \sigma'$  we infer that

$$\|\varphi_{\infty}\|_{G_{\infty}(E)} = \lim_{n \to \infty} \|\varphi_n\|_{G_n(E)} = 1.$$
 (2.2)

(This equality seems to be obvious, but in lack of a precise reference we add a proof of it in the appendix). Clearly  $\|\varphi_{\infty}\| \le 1$  and hence

$$\iint\limits_{D\backslash G_{x}(E)} |\varphi_{\infty}(z)| \, dx \, dy = 0$$

which is a contradiction because  $D \setminus G_{\infty}(E)$  has positive measure and  $\varphi_{\infty} \neq 0$  is holomorphic.

FUNDAMENTAL VARIATIONAL LEMMA. Let E' be a measurable subset of D where  $D \setminus E'$  has positive measure and  $\sigma'$  be a closed set on  $\partial D$  which contains at least four points and where, if  $\sigma'$  is an infinite set,  $\overline{E'}$  is compact in  $\overline{D} \setminus \sigma'$ . If g is a qc mapping from D onto itself where its complex dilatation  $\kappa_g$  satisfies

$$\kappa_g(z) \equiv 0 \text{ in } E' \text{ and } \operatorname{Re} \iint_D \kappa_g \varphi \, dx \, dy = 0 \quad \forall \varphi \in B_{\sigma'},$$

then there is a qc mapping  $g^*: D \to D$  with  $g^* \circ g = id$  on  $\sigma'$  and with a complex dilatation  $\kappa_{g^*}$  that satisfies

$$\kappa_{g^*}(z) \equiv 0 \text{ in } g(E') \quad \text{and} \quad \|\kappa_{g^*}\|_{\infty} = O(\|\kappa_g\|_{\infty}^2) \quad \text{as} \quad \|\kappa_g\|_{\infty} \to 0.$$

*Proof.* The best choice for  $g^*$  is to be an extremal element in  $Q(g^{-1}|_{\partial D}, \sigma'', E'', 0)$  where  $\sigma'' = g(\sigma')$ , E'' = g(E'). If  $\sigma''$  is not finite, we choose  $\sigma''_n$  to consist of n points on  $\sigma''$  which become to be denser and denser as n tends to infinity. For every n there is an extremal mapping  $G_n$  in  $Q(g^{-1}|_{\partial D}, \sigma''_n, E'', 0)$  and by Corollary 2.1 there is a  $\varphi_n \in B_{\sigma'_n 1}$  ( $\sigma'_n = g^{-1}(\sigma''_n)$ ) such that the complex dilatation  $\kappa_n$  of  $g_n := G_n^{-1}$  satisfies

$$\kappa_n(z) = \begin{cases}
k_n \frac{\bar{\varphi}_n(z)}{|\varphi_n(z)|} & z \in G_n(D \setminus E'') \\
0 & z \in G_n(E'')
\end{cases}$$
(2.3)

Evidently  $k_n \le ||\kappa_{g^*}||_{\infty}$  and  $k_n$  is an increasing sequence. By normality we may assume that the qc mappings  $G_n$  converge locally uniformly to a qc mapping  $G_{\infty}: D \to D$  where obviously  $G_{\infty}|_{\sigma''} = g^{-1}|_{\sigma''}$  and by Strebel's result

$$|\kappa_{G_{\infty}}(z)| \leq \overline{\lim_{n \to \infty}} |\kappa_{G_n}(z)|$$
 a.e. in  $D$ .

We hence conclude that  $\kappa_{G_x}(z) = 0$  in E'' and  $\exp_{z \in D \setminus E''} |\kappa_{G_x}(z)| \le \lim_{n \to \infty} k_n \le \|\kappa_{g^*}\|_{\infty}$ . So  $G_{\infty} \in Q(g^{-1}|_{\partial D}, \sigma'', E'', 0)$  and  $G_{\infty}$  is therefore extremal in this class, i.e.,  $\|\kappa_{g^*}\|_{\infty} = \lim_{n \to \infty} k_n$  and we can take  $G_{\infty}$  for  $g^*$ .

For the purpose of estimating the numbers  $k_n$  we introduce the extremal Teichmüller n-gon mappings  $f_n: D \to D$  which agree on  $\sigma'_n$  with g. Their complex dilatations  $\tilde{\kappa}_n$  are equal to

$$\tilde{k}_n \frac{\bar{\tilde{\varphi}}_n}{|\tilde{\varphi}_n|}$$
 a.e. in  $D$ 

where  $\tilde{\varphi}_n \in B_{\sigma_n'1}$ .

We use the statement (M1) where we put the quadruple  $(\sigma'_n, \varphi_n, g_n, f_n)$  for  $(\sigma', \varphi, f, g)$ 

$$1 \leq \int_{D} \left| \varphi_n(z) \right| \frac{\left| 1 - \kappa_n(z) \frac{\varphi_n(z)}{|\varphi_n(z)|} \right|^2}{1 - |\kappa_n(z)|^2} \tilde{K}_n \, dx \, dy \quad \left( \tilde{K}_n = \frac{1 + \tilde{k}_n}{1 - \tilde{k}_n} \right).$$

Using (2.3), splitting up the integral and putting  $K_n = (1 + k_n)/(1 - k_n)$  we get

$$1 \leq \frac{\tilde{K}_n}{K_n} (1 - \|\varphi_n\|_{G_n(E'')}) + \tilde{K}_n \|\varphi_n\|_{G_n(E'')}.$$

We multiply with  $K_n$  and subtract  $K_n \|\varphi_n\|_{G_n(E^n)}$ , so

$$K_n(1-\|\varphi_n\|_{G_n(E'')}) \leq \tilde{K}_n(1-\|\varphi_n\|_{G_n(E'')}) + (\tilde{K}_n-1)K_n \|\varphi_n\|_{G_n(E'')}$$

and finally since  $\|\varphi_n\|_{G_n(E^n)} < 1$ ,  $K_n \le K[g^*]$ 

$$K_n \le \tilde{K}_n + (\tilde{K}_n - 1) \frac{K[g^*]}{1 - \|\varphi_n\|_{G_n(E^n)}}.$$
(2.4)

Next we estimate  $\tilde{K}_n$ . We use the statement (M2) and put the quadrupel  $(\sigma'_n, \tilde{\varphi}_n, f_n, g)$  for  $(\sigma'_n, \varphi_n, f_n, g)$ 

$$\tilde{K}_n \leq \iiint_D |\tilde{\varphi}_n(z)| \frac{\left|1 + \kappa_g(z) \frac{\tilde{\varphi}_n(z)}{|\tilde{\varphi}_n(z)|}\right|^2}{1 - |\kappa_g(z)|^2} dx dy.$$

Following Reich's calculation in [R2] the integral becomes

$$\iint_{D} |\tilde{\varphi}_{n}| \frac{1 + |\kappa_{g}|^{2}}{1 - |\kappa_{g}|^{2}} dx dy + 2 \operatorname{Re} \iint_{D} \frac{\kappa_{g} \tilde{\varphi}_{n}}{1 - |\kappa_{g}|^{2}} dx dy 
\leq \frac{1 + \|\kappa_{g}\|_{\infty}^{2}}{1 - \|\kappa_{g}\|_{\infty}^{2}} + 2 \operatorname{Re} \iint_{D} \left( \frac{\kappa_{g} \tilde{\varphi}_{n}}{1 - |\kappa_{g}|^{2}} - \kappa_{g} \tilde{\varphi}_{n} \right) dx dy$$

because of the hypothesis

$$\operatorname{Re} \iint_{D} \kappa_{g} \tilde{\varphi}_{n} \, dx \, dy = 0, \qquad \tilde{\varphi}_{n} \in B_{\sigma'_{n} 1} \subset B_{\sigma'}.$$

The second term can be estimated by

$$2\left|\iint_{D} \frac{|\kappa_{g}|^{2} \kappa_{g} \tilde{\varphi}_{n}}{1 - |\kappa_{g}|^{2}} dx dy\right| \leq \frac{2 \|\kappa_{g}\|_{\infty}^{3}}{1 - \|\kappa_{g}\|_{\infty}^{2}}.$$

Hence

$$\tilde{K}_n \le 1 + 2 \frac{\|K_g\|_{\infty}^2 + \|K_g\|_{\infty}^3}{1 - \|K_g\|_{\infty}^2} = 1 + 2 \frac{\|K_g\|_{\infty}^2}{1 - \|K_g\|_{\infty}}$$

With (2.4) finally

$$K_n \le 1 + 2 \frac{\|\kappa_g\|_{\infty}^2}{1 - \|\kappa_g\|_{\infty}} + 2 \frac{\|\kappa_g\|_{\infty}^2}{1 - \|\kappa_g\|_{\infty}} \frac{K[g^*]}{1 - \|\varphi_n\|_{G_n(E'')}}.$$

We let n tend to infinity, i.e.  $G_n \xrightarrow{n \to \infty} G_{\infty}$  locally uniformly in D and by normality of  $B_{\sigma'1}$  we may pass to a further subsequence such that

$$\varphi_n \xrightarrow{n \to \infty} \varphi_\infty$$
 locally uniformly in  $\bar{D} \setminus \sigma'$ 

where  $\varphi_{\infty} \in B_{\sigma'}$ ,  $\|\varphi_{\infty}\| \le 1$ .

This approximation takes place only if  $\sigma'$  is infinite, so E'' is relatively compact in  $\bar{D} \setminus \sigma''$  and hence

$$\|\varphi_n\|_{G_n(E'')} \xrightarrow{n \to \infty} \|\varphi_\infty\|_{G_\infty(E'')}.$$

Hence

$$K[g^*] = \lim_{n \to \infty} K_n \le 1 + \frac{2 \|\kappa_g\|_{\infty}^2}{1 - \|\kappa_g\|_{\infty}} \left(1 + \frac{K[g^*]}{1 - \|\varphi_{\infty}\|_{G_{\infty}(E'')}}\right).$$

Now we apply Lemma 2.1 with the quadrupel  $(id, \sigma', \sigma', E')$  instead of  $(h, \sigma, \sigma', E)$  for a fixed number  $K > K[g]^2$ . Namely,  $G_{\infty}(E'') = G_{\infty} \circ g(E')$  and  $G_{\infty} \circ g|_{\sigma'} = id|_{\sigma'}$ , so  $G_{\infty} \circ g \in Q_K(id, \sigma')$  and by  $||\varphi_{\infty}|| \le 1$  and  $(\varphi_{\infty}/||\varphi_{\infty}||) \in B_{\sigma'1}$ , there is a q < 1 which does not depend on g (only on K), such that

$$\|\varphi_{\mathbf{x}}\|_{G_{\mathbf{x}} \circ g(E')} \leq \left\|\frac{\varphi_{\mathbf{x}}}{\|\varphi_{\mathbf{x}}\|}\right\|_{G_{\mathbf{x}} \circ g(E')} \leq q$$

and hence

$$K[g^*] \le 1 + \left(1 + \frac{K}{1-q}\right) \frac{2 \|\kappa_g\|_{\infty}^2}{1 - \|\kappa_g\|_{\infty}^2}$$

which shows that

$$\|\kappa_{g^*}\|_{\infty} = O(\|\kappa_{g}\|_{\infty}^2)$$

for  $\|\kappa_g\|_{\infty} \to 0$ , since K can stay fixed as  $K[g] \to 1$ .

# 3. Application: The case $b(w) \equiv 0$

An elaboration of a technique employed by Krushkal [Kr] which is done in Reich's paper [R2] now yields the necessity part of the

THEOREM 3.1. A qc mapping F is extremal in  $Q(h, \sigma, E, 0)$  iff

$$\sup_{\substack{\varphi \in B_{\sigma} : \\ \|\varphi\|_{DV(E_{0})} = 1}} \operatorname{Re} \iint_{D} \hat{\kappa} \varphi \, dx \, dy = k_{F}$$

*Proof.* Let F be extremal. Since  $k_F = ||\hat{k}||_{\infty}$  we now assume that

$$a := \sup_{\substack{\varphi \in B_{\sigma'} \\ \|\varphi\|_{D \setminus F(E)} = 1}} \operatorname{Re} \iint_{D} \hat{\kappa} \varphi \, dx \, dy < k_{F}$$

and hence  $k_F > 0$ .

Then a is the norm of the linear operator

$$\varphi \mapsto \operatorname{Re} \iint_{D \setminus F(E)} \hat{\kappa} \varphi \, dx \, dy$$

 $(\hat{k}=0 \text{ on } F(E)!)$  defined on the Banach-space  $\{\varphi|_{D\setminus F(E)} \mid \varphi \in B_{\sigma'}\}$  which is a subspace of  $L_1(D\setminus F(E))$  over the field  $\mathbb{R}$ . By the Hahn-Banach Theorem there is an extension of this operator on  $L_1(D\setminus F(E))$  with norm a and by the Riesz-representation theorem there is a complex-valued function  $\beta$  on  $D\setminus F(E)$  with  $\|\beta\|_{\infty} = a$  which realizes this extension, i.e.,

$$\operatorname{Re} \iint\limits_{D \backslash F(E)} \hat{\kappa} \varphi \, dx \, dy = \operatorname{Re} \iint\limits_{D \backslash F(E)} \beta \varphi \, dx \, dy \quad \forall \varphi \in B_{\sigma'}.$$

We put

$$v(z) = \begin{cases} \hat{\kappa}(z) - \beta(z) & z \in D \backslash F(E) \\ 0 & z \in F(E) \end{cases}$$

and have  $||v||_{\infty} > 0$  ( $\hat{k} \neq \beta$ !) and

$$\operatorname{Re} \iint_{D} v \varphi \, dx \, dy = 0 \quad \text{for} \quad \varphi \in B_{\sigma'}.$$

For t,  $0 \le t < (1/||v||_{\infty})$  we put  $g:D \to D$  to be a qc mapping with g(1) = 1, g(i) = i, g(-1) = -1 and

$$\kappa_{g} = t \nu$$
.

Here we apply the Fundamental Variational Lemma on g,  $\sigma' = h(\sigma)$  and E' = F(E). Hence there is a qc mapping  $g^*: D \to D$  where  $g^* \circ g = id$  on  $\sigma'$ ,  $\kappa_{g^*} = 0$  in g(E') and

$$\|\kappa_{g^*}\|_{\infty} = O(t^2)$$
 as  $t \to 0$ .

We have  $g^* \circ g \circ F \in Q(h, \sigma, E, 0)$  and show that

$$\operatorname{ess sup}_{w \in D \setminus E} |\kappa_{g \cdot g \circ F}(w)| < k_F$$

for t>0, sufficiently small. This contradicts then the extremality of F. One computes for  $z \in D \setminus E'$ 

$$\left|\kappa_{f \circ g^{-1}}(g(z))\right| = \left|\frac{\kappa(1-t) + t\beta}{1 - t\bar{\nu}\kappa}\right|$$

and the computation in [R2], p. 109 and 110, assures the existence of numbers  $\delta > 0$ ,  $t_0 > 0$  with

$$|\kappa_{f \circ g^{-1}}(g(z))| \le k_F - \delta_1 t$$
 for  $0 \le t \le t_0$  and  $z \in D \setminus E'$ .

By  $\|\kappa_{g^*}\|_{\infty} = O(t^2)$  the values  $|\kappa_{f \circ g^{-1} \circ g^{*-1}}(g^*(g(z)))|$  can be estimated in the same manner in  $D \setminus E'$  and this yields the result.

The sufficiency part is immediate. We do not need any restriction on b(w) for it. Let  $F \in Q(h, \sigma, E, b)$  and

$$\sup_{\substack{\varphi \in B_{\sigma'} \\ \|\varphi\|_{D \setminus F(E_{\alpha})} = 1}} \operatorname{Re} \iint_{D} \hat{\kappa} \varphi \, dx \, dy = k_{F}.$$

If this sup is attained, then there is a  $\varphi \in B_{\sigma'}$ ,  $\|\varphi\|_{D \setminus F(E_0)} = 1$ , with

$$\hat{\kappa}(z) = k_F \frac{\bar{\varphi}(z)}{|\varphi(z)|}$$
 a.e. in  $D \setminus F(E_0)$ .

If  $k_F = 0$  we have extremality. If  $k_F > 0$  we conclude from (2.1)

$$\kappa(z) = \begin{cases} k_F \frac{\bar{\varphi}(z)}{|\varphi(z)|} & z \in D \backslash F(E) \\ b(f(z)) \frac{\bar{\varphi}(z)}{|\varphi(z)|} & z \in F(E) \end{cases}$$

Then by [R2], Theorem 5, F is even uniquely extremal in  $Q(h, \sigma, E, b)$ .

If the sup is not attained, and this can occur only if  $\sigma$  is infinite, then there is a sequence  $\varphi_n \in B_{\sigma'}$ ,  $\|\varphi_n\|_{D \setminus F(E_0)} = 1$  with  $\text{Re} \iint_D \hat{\kappa} \varphi_n \, dx \, dy \xrightarrow{n \to \infty} k_F$  and  $\varphi_n \xrightarrow{n \to \infty} 0$  locally uniformly in  $\bar{D} \setminus \sigma'$ .

From the relative compactness of E in  $\bar{D} \setminus \sigma'$  we conclude that  $\|\varphi_n\|_{F(E_0)} \xrightarrow{n \to \infty} 0$  and hence  $\|\varphi_n\| \xrightarrow{n \to \infty} 1$ . So if we put

$$\hat{\varphi}_n := \frac{\varphi_n}{\|\varphi_n\|}$$

we get a degenerating Hamilton sequence  $\hat{\varphi}_n$  for the complex dilatation  $\hat{k}$ , this is a sequence  $\hat{\varphi}_n \in B_{\sigma'1}$  where  $\text{Re} \int \int_D \hat{k} \hat{\varphi}_n \, dx \, dy \xrightarrow{n \to \infty} \|\hat{k}\|_{\infty} = k_F$  and  $\hat{\varphi}_n \xrightarrow{n \to \infty} 0$  locally uniformly in  $\bar{D} \setminus \sigma'$ . We denote by  $\hat{f}$  a qc selfmapping from D with complex dilatation  $\hat{k}$ . By the sufficiency of Hamilton's condition  $\hat{f}$  is extremal for its own boundary values on  $\sigma'$ , and by Satz 5.2 in [F] there exists a substantial boundary point on  $\sigma'$ . i.e., a point with local dilatation equal to  $K[\hat{f}] = (1 + k_F) \mid (1 - k_F)$  (for the boundary values  $\hat{f}|_{\sigma'}$ ). Since  $\hat{f} \circ F$  is conformal in  $D \setminus E$  which contains a neighborhood of  $\sigma$  and since local dilatations of the boundary mapping are preserved under conformal mapping we conclude that there is a point on  $\sigma$  with local dilatation  $K[\hat{f}]$  for the boundary values  $h|_{\sigma}$ . Hence every mapping in  $Q(h, \sigma, E, b)$  needs to have its dilatation near that point at least as large as  $K[\hat{f}] = (1 + k_F)/(1 - k_F)$ , in particular, F is extremal.

# 4. Alternative proofs of Theorem 2.1 in two special cases

Let  $\sigma$  be finite and  $b(w) \equiv 0$  in E. We put  $b_n(w) \equiv 1/n$  in E and denote by  $F_n$  an extremal qc mapping in  $Q(h, \sigma, E, 1/n)$ . By Reich's result there is a  $\varphi_n \in B_{\sigma'1}$  where the complex dilatation  $\kappa_n$  of  $f_n := F_n^{-1}$  is

$$\kappa_n(z) = \begin{cases}
k_n \frac{\bar{\varphi}_n(z)}{|\varphi_n(z)|} & z \in D \setminus F_n(E) \\
\frac{1}{n} \frac{\bar{\varphi}_n(z)}{|\varphi_n(z)|} & z \in F_n(E)
\end{cases}$$

Furthermore, let F be extremal in  $Q(h, \sigma, E, 0)$ , hence  $k_n \le k_F$ . Passing to subsequences we find a qc mapping  $F_{\infty}$  and a function  $\varphi_{\infty}$  where

$$F_n \xrightarrow{n \to \infty} F_{\infty}$$
 locally uniformly in D

and

 $\varphi_n \xrightarrow{n \to \infty} \varphi_\infty$  locally uniformly in  $\bar{D} \backslash \sigma'$ 

As above we conclude that  $F_{\infty} \in Q(h, \sigma, E, 0)$ , and therefore  $k_F \le k_{F_{\infty}} \le \lim_{n \to \infty} k_n \le k_F$ , and since  $\sigma$  is finite we have  $\varphi_{\infty} \in B_{\sigma'_1}$ . Furthermore

$$f_n \xrightarrow{n \to \infty} f_{\infty} := F_{\infty}^{-1}$$
 locally uniformly in  $D$ .

 $F_{\infty}$  is hence extremal in  $Q(h, \sigma, E, 0)$  and the complex dilatations  $\kappa_n$  converge pointwise a.e. in the interior of  $F_{\infty}(E)$  or in the interior of  $D \setminus F_{\infty}(E)$  to zero or  $k_F(\overline{\varphi_{\infty}}/|\varphi_{\infty}|)$  respectively. Since qc mappings preserve sets of area-measure zero we infer from Theorem 5.2 in [LV], p. 187, that if the area-measure of  $\partial E$  is zero, then a.e.

$$\kappa_{f_{x}}(z) = \begin{cases}
k_{F} \frac{\overline{\varphi_{x}}(z)}{|\varphi_{x}(z)|} & z \in D \backslash F_{x}(E) \\
0 & z \in F_{x}(E).
\end{cases}$$
(4.1)

By Theorem 5 in [R2]  $F_{\infty}$  is uniquely extremal in  $Q(h, \sigma, E, 0)$  and hence  $F = F_{\infty}$ . Clearly

$$\sup_{\substack{\varphi \in B_{\sigma'} \\ \|\varphi\|_{D(MF)} = 1}} \operatorname{Re} \iint_{D} \kappa_{f_{x}} \varphi \ dx \ dy = k_{F}$$

and we thus have proved the first part of

PROPOSITION 4.1. Let F be extremal in  $Q(h, \sigma, E, 0)$  where  $\sigma$  is finite. If the area-measure of  $\partial E$  is zero or if E is a closed set, then

$$\sup_{\substack{\varphi \in B_{\sigma'} \\ \|\varphi\|_{D \setminus F(E)} = 1}} \operatorname{Re} \iint_{D} \hat{\kappa} \varphi \, dx \, dy = k_{F}.$$

For the second part the reasoning in the proof above has to be slightly changed since we do not know if  $\kappa_n$  is convergent a.e. in D. For this purpose we change to the w-plane. First we observe that by  $|\kappa_{Fx}(w)| \leq \overline{\lim}_{n\to\infty} |\kappa_{F_n}(w)|$  a.e. in D we conclude that  $\kappa_{F_x}(w) = 0$  a.e. in E, hence  $\kappa_{f_x}(z) = 0$  a.e. in  $F_x(E)$ . Next we use the fact that E is closed. Let  $z_0 \in D \setminus F_x(E)$ . There is a neighborhood  $U_{z_0}$  of  $z_0$  with  $U_{z_0}cD \setminus F_x(E)$  and by the local uniform convergence of  $F_n$  to  $F_x$  we find an open disk  $D_{z_0}$  with center  $z_0$  in  $U_{z_0}$  such that for a number  $n_0 \in \mathbb{N}$ 

$$D_{r_0}cD\backslash F_n(E) \quad \forall n\geq n_0.$$

We hence infer that

$$\kappa_n(z) = k_n \frac{\bar{\varphi}_n(z)}{|\varphi_n(z)|} \xrightarrow{n \to \infty} k_F \frac{\overline{\varphi_\infty}(z)}{|\varphi_\infty(z)|}$$
 a.e. in  $D_{z_0}$ .

Again by Theorem 5.2 in [LV] we get

$$\kappa_{f_{\infty}}(z) = k_F \frac{\overline{\varphi_{\infty}}(z)}{|\varphi_{\infty}(z)|}$$
 a.e. in  $D_{z_0}$ 

and since  $z_0$  was arbitrary in  $D \setminus F_{\infty}(E)$ , we again have (4.1) from which the result follows.

## 5. Appendix

As has been pointed out to me by K. Sakan, the equation (2.2) in Lemma 2.1 is not at all a triviality. So let me add a proof here. There are several ways to do it, e.g. one could infer this statement from results on the weak-convergence of Jacobians of qc mappings (see [L]). I prefer here to use a consequence of a result on area-distortion by Gehring and Reich [GR].

Let us denote the Jacobian of a qc mapping f by  $J_f$ . For a number  $K \ge 1$  let F be the set of K-qc mappings of the unit disk D onto itself which fix the origin. From [GR] it then follows that the integrals  $\iint_D J_f dx dy$  are uniformly absolutely continuous, i.e., for every  $\varepsilon > 0$  there is a  $\delta > 0$  such that

$$\iint_{E} J_f \, dx \, dy < \varepsilon$$

for every  $f \in F$  and every measurable set E in the disk with  $|E| := \iint_E dx \, dy < \delta$ . This property will imply the

THEOREM 5.1. Let  $f_n$  and f be K-qc selfmappings of the unit disk D where  $f_n \xrightarrow{n \to \infty} f$  locally uniformly in D. Then we have for every measurable and bounded function  $\varphi$  in D

$$\lim_{n\to\infty}\iint_D \varphi(z)J_{f_n}(z)\ dx\ dy = \iint_D \varphi(z)J_f(z)\ dx\ dy.$$

In case that  $\varphi_0$  is continuous in  $\bar{D} \setminus \sigma$  and E relatively compact in  $\bar{D} \setminus \sigma$  we choose  $\varphi = \varphi_0 \chi_E$  ( $\chi_E$  denotes the characteristic function of E), hence  $\varphi$  is bounded and from this theorem we derive that

$$\iint_{E} \varphi_0 J_{f_n} \, dx \, dy \xrightarrow{n \to \infty} \iint_{E} \varphi_0 J_f \, dx \, dy. \tag{5.1}$$

The equation (2.2) claims that

$$\iiint_{E} |\varphi_{n} \circ G_{n}| J_{G_{n}} dx dy \xrightarrow{n \to \infty} \iiint_{E} |\varphi_{\infty} \circ G_{\infty}| J_{G_{\infty}} dx dy$$

which clearly follows from (5.1) by putting  $\varphi_0 = \varphi_\infty \circ G_\infty$  because  $\varphi_n \circ G_n \to \varphi_\infty \circ G_\infty$  locally uniformly in  $\bar{D} \setminus \sigma$ .

Proof of Theorem 5.1. By [GR] we conclude that obviously also the integrals  $\iint_D J_{f_n} dx dy$  are uniformly absolutely continuous. We first choose  $\varphi = \chi_R$  where R is a rectangle whose closure is contained in D. Then the statement follows from Lebesgue's dominated convergence theorem since  $\chi_{f_n(R)} \to \chi_{f(R)}$  a.e. in D. Hence for step-functions  $s = \sum_{i=1}^N c_i \chi_{R_i}$  we have

$$\iint_{D} s J_{f_n} dx dy \xrightarrow{n \to \infty} \iint_{D} s J_f dx dy.$$
 (5.2)

Finally let  $\varphi$  be measurable and bounded in D. Let  $s_m$  be a sequence of step-functions with  $s_m(z) \xrightarrow{m \to \infty} \varphi(z)$  a.e. in D and M be a number with  $|\varphi| \le M$  and  $|s_m| \le M$  for all m. Let  $\varepsilon > 0$  be given. By the uniform absolute continuity of the integrals  $\iint_D J_{f_m} dx dy$ , there is an  $\eta > 0$  such that

$$\iint_{\mathcal{E}} J_{f_n} \, dx \, dy < \varepsilon \quad \forall n \quad \text{and} \quad \iint_{\mathcal{E}} J_f \, dx \, dy < \varepsilon$$

whenever E is a measurable set in D with  $|E| < \eta$ . By Egoroff's theorem there is a set  $E_{\eta} \subset D$  with  $|E_{\eta}| < \eta$  such that  $s_m \xrightarrow{m \to \infty} \varphi$  uniformly on  $D \setminus E_{\eta}$ . We choose  $m_0$  such that

$$|s_m(z) - \varphi(z)| < \varepsilon \quad \forall z \in D \setminus E_n, \quad \forall m \ge m_0.$$

For an  $m \ge m_0$  we then have

$$\iint_{D} \varphi J_{f_n} dx dy - \iint_{D} \varphi J_f dx dy = \iint_{D \setminus E_{\eta}} (\varphi - s_m) J_{f_n} + \iint_{E_{\eta}} (\varphi - s_m) J_{f_n}$$
$$+ \iint_{D} s_m (J_{f_n} - J_f) + \iint_{D \setminus E_{\eta}} (s_m - \varphi) J_f + \iint_{E_{\eta}} (s_m - \varphi) J_f.$$

By

$$\iint\limits_{D\setminus E_{\eta}} |\varphi - s_{m}| J_{f_{n}} dx dy \le \varepsilon \pi \quad \text{and} \quad \iint\limits_{E_{\eta}} |\varphi - s_{m}| J_{f_{n}} \le 2M\varepsilon$$

(also with  $J_f$  instead of  $J_{f_n}$ ) we conclude from (5.2)

$$\overline{\lim_{n\to\infty}}\left|\iint\limits_{D}\varphi(J_{f_n}-J_f)\,dx\,dy\right|\leq 2\varepsilon\pi+4M\varepsilon$$

and  $\varepsilon \rightarrow 0$  proves the theorem.

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