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Autor: Brodmann, Markus / Rung, Josef

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Local cohomology and the connectedness dimension in algebraic varieties

MARKUS BRODMANN and JOSEF RUNG

0. Introduction

In this paper, we use local cohomology to prove some connectedness results for algebraic varieties. What is used from local cohomology are the Mayer–Vietoris sequence, the Hartshorne–Lichtenbaum vanishing theorem and Hartshorne's bound on the cohomological dimension of an ideal by its arithmetical rank. This method already was successfully applied by the second author in [10] and also in [11], where a proof of the connectedness theorem of Fulton–Hansen [3] (Barth [1], Faltings [2]) for the intersection of two projective varieties was given.

Besides the connectedness of intersections, the connectedness of fibres under projective morphisms plays an important role in algebraic geometry. We will show that – using essentially the same tools from local cohomology – it is possible to get results in the latter field, too. In particular we may prove a generalized and sharpened version of Zariskis' connectedness theorem [12] for projective birational morphisms, (3.4). This result has some applications to blowing-up and tangent cones, which will be published elsewhere.

According to the two types of questions mentioned above, our paper is divided into two parts.

The first part concerns connectedness of closed subsets in a variety and of their intersections. Here we give a bound on the (local) formal connectedness-dimension of a closed subset in terms of its arithmetical rank and the formal connectedness-dimension of the ambient variety, (2.4). From this we get a bound on formal connectedness-dimensions of intersections, (2.5). Passing to affine cones we immediately get a bound on the connectedness-dimension of two intersecting projective varieties, (2.6).

The second part of the paper concerns the connectedness of fibres under projective (in fact even under quasi-homogeneous) morphisms. First we prove a connectedness criterion for the fibres under such morphisms, (3.1). This criterion has some similarity with Grotendieck's version of the Zariski connectedness theorem [4]. But instead of Stein-factors our criterion uses the formal extension of the given morphism. Combining (3.1) with the results of the first part, we get a lower bound for the connectedness-dimension of the fiber, (3.4).

The connecteness-criterion of the second part relies on (3.2). To prove this result, we also use the method of [11], but in a refined form. Instead of vanishing and non-vanishing of local cohomology we have to look at vanishing and non-vanishing of graded local cohomology at different degrees. This requests to consider the "graded versions" of the previously mentioned results on local cohomology.

1. Preliminaries

We now list a few definitions and results which will be used.

Let X be a noetherian scheme and let $Z \subseteq X$ be a closed subset. We define the *connectedness-dimension* $c(Z)$ of Z as the largest integer n such that $Z - W$ is connected for all closed subsets $W \subset Z$ with $\dim(W) < n$. Thereby the empty set $\emptyset$ is considered as disconnected. So, using the convention $\dim(\emptyset) = -1$, we may write

(1.1) $c(Z) = \min\{\dim(W) \mid W \subseteq Z \textit{ closed, } Z - W \textit{ disconnected}\}.$

In particular Z is connected iff $c(Z)$ is ≥ 0. Another description of $c(Z)$ is:

(1.2) $c(Z) = \min\{\dim(Z_1 \cap Z_2)\}$, *where* Z_1, Z_2 *are unions of irreducible components of* Z *such that* $Z_1 \cup Z_2 = Z$.

Let R be a noetherian ring. Let $I \subseteq R$ be an ideal. The *arithmetical rank* $r(I)$ of I is defined as the minimal number of elements in I which span an ideal with the same radical as I:

(1.3) $r(I) := \min\{r \mid \exists a_1, \ldots, a_r \in I \textit{ with } \sqrt{(a_1, \ldots, a_r)} = \sqrt{I}\}.$

H_I^i denotes the i-th local cohomology functor with respect to the ideal I. We now list the results on these functors which will be used in our proofs (for reference see [6], [7]).

If $J \subseteq R$ is another ideal and if M is an R-module, we have the following natural exact *Mayer–Vietoris sequence*:

(1.4) $\cdots \to H_{I\cap J}^{i-1}(M) \to H_{I+J}^i(M) \to H_I^i(M) \oplus H_J^i(M) \to H_{I\cap J}^i(M) \to \cdots$

We will use the following form of the *Hartshorne–Lichtenbaum vanishing theorem* (cf. [8]):

(1.5) *Let* R *be a local complete domain and let* $\dim(R/I) > 0$. *Then* $H_I^i(R)$ *vanishes for all* $i \geq \dim(R)$.

Parallel to (1.5) we shall use

(1.6) *Let* $(R, \mathfrak{m})$ *be local of dimension* $d>0$. *Then* $H^d_{\mathfrak{m}}(R)$ *is not finitely generated as an R-module.*

We recall Hartshorne's bound for the cohomological dimension of an ideal I (cf. [8]):

(1.7) $i > r(I) \Rightarrow H^i_I \equiv 0$.

To treat projective morphisms, we must use the *homogeneous versions* of (1.4–7). More precisely, let $R = R_0 \oplus R_1 \cdots$ be a graded noetherian ring, let $I \subset R$ be a homogeneous ideal and let $M = \oplus_{n \in \mathbb{Z}} M_n$ be a graded R-module. Then the modules $H^i_I(M)$ are naturally graded over R. If $J \subset R$ is another homogeneous ideal, the natural homomorphisms in (1.4) are homogeneous of degree 0. The vanishing result (1.5) then has the following homogeneous version:

(1.8) *Let* $R = R_0 \oplus R_1 \cdots$ *be a graded noetherian domain such that* R_0 *is local and complete. Let* $I \subset R$ *be a homogeneous ideal with* $\dim(R/I) > 0$. *Then* $H^i_I(R)$ *vanishes for all* $i \geq \dim(R)$.

(1.6) has the following homogeneous version:

(1.9) *Let* $R = R_0 \oplus R_1 \oplus \cdots$ *be graded, noetherian of dimension* $d > 0$ *and such that* R_0 *is local. Let* $\mathfrak{m}$ *be the homogeneous maximal ideal of R. Then the graded module* $H^d_{\mathfrak{m}}(R)$ *has non-vanishing homogeneous parts in almost all negative degrees.*

Finally we will use the following fact, which is an easy consequence of the base ring independence of local cohomology:

(1.10) *Let* $R = R_0 \oplus R_1 \oplus \cdots$ *be graded noetherian and let* $I_0 \subset R_0$ *be an ideal. Then the graded modules* $H^i_{I_0R}(R)$ *vanish in all negative degrees.*

2. Local connectedness-dimensions

If R is a noetherian ring, we write $c(R)$ instead of $c(\mathrm{Spec}\,(R))$.

(2.1) PROPOSITION. (cf. [11], Satz 1). *Let* $(R, \mathfrak{m})$ *be a local noetherian complete ring and let* $J, L \subset \mathfrak{m}$ *be ideals such that* $\dim(R/J)$, $\dim(R/L) > \dim(R/(J+L))$. *Then*

$\dim(R/(J+L)) \geq c(R) - r(J \cap L) - 1.$

Proof. (Induction on $\mathrm{d} = \dim(R/(J+L))$). Let $\mathrm{d} = 0$. We must show

$r(J \cap L) \geq c(R)-1$. First we treat the case where at least one of the ideals $J+\mathfrak{q}$ or $L+\mathfrak{q}$ is $\mathfrak{m}$-primary for each minimal prime ideal $\mathfrak{q}$ of R. Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_n$ be the minimal primes of R. After an appropriate reordering we find a non-negative integer $s \leq n$ such that $J+\mathfrak{q}_1, \ldots, J+\mathfrak{q}_s$ and $L+\mathfrak{q}_{s+1}, \ldots, L+\mathfrak{q}_n$ are $\mathfrak{m}$-primary. As neither J nor L is $\mathfrak{m}$-primary we must have $0<s<n$. Put $Z_1=V(\mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_s)$, $Z_2=V(\mathfrak{q}_{s+1} \cap \cdots \cap \mathfrak{q}_n)$. Then $\operatorname{Spec}(R)-(Z_1 \cap Z_2)$ is disconnected. This shows that $\dim(Z_1 \cap Z_2) \geq c(R)$. Consequently there is a minimal prime $\mathfrak{s}$ of $\mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_s+\mathfrak{q}_{s+1} \cap \cdots \cap \mathfrak{q}_n$ such that $\dim(R/\mathfrak{s}) \geq c(R)$. Now we may choose indices i, j with $i \leq s \leq j \leq n$ and such that $\mathfrak{q}_i, \mathfrak{q}_j \subseteq \mathfrak{s}$. Therefore $J \cap L+\mathfrak{s}$ is $\mathfrak{m}$-primary. This induces $r((J \cap L)/\mathfrak{s})=\dim(R/\mathfrak{s})$. As $r(J \cap L) \geq r((J \cap L)/\mathfrak{s})$ we get the requested equality.

So we may assume that neither $J+\mathfrak{q}$ nor $L+\mathfrak{q}$ is $\mathfrak{m}$-primary for an appropriately chosen minimal prime ideal $\mathfrak{q}$ of R. Put $\bar{R}=R/\mathfrak{q}$, $\bar{\mathfrak{m}}=\mathfrak{m}/\mathfrak{q}$, $\bar{J}=J\bar{R}$, $\bar{L}=L\bar{R}$ and $t=\dim(\bar{R})$. Then neither $\bar{J}$ nor $\bar{L}$ is $\bar{\mathfrak{m}}$-primary. Moreover it holds $r(\bar{J} \cap \bar{L}) \leq r(J \cap L)$. As $\operatorname{Spec}(\bar{R})$ is an irreducible component of $\operatorname{Spec}(R)$ we have $t \geq c(R)$. So, replacing R by $\bar{R}$ we may assume that R is a domain of dimension $t>0$ and restrict ourselves to prove $r(J \cap L) \geq t-1$. To do so, consider the following piece of the Mayer–Vietoris sequence (1.4):

$$H^{t-1}_{J \cap L}(R) \to H^t_{J+L}(R) \to H^t_J(R) \oplus H^t_L(R).$$

Here the last term vanishes by the theorem of Hartshorne–Lichtenbaum. As $J+L$ is $\mathfrak{m}$-primary, the middle term of our sequence may be written as $H^t_{\mathfrak{m}}(R)$ and thus does not vanish, (1.6). It follows $H^{t-1}_{J \cap L}(R) \neq 0$, thus $r(J \cap L) \geq t-1$, (1.7).

Finally let $d>0$. Put $r=r(J \cap L)$ and let $a_1, \ldots, a_r \in J \cap L$ be such that $\sqrt{(a_1, \ldots, a_r)}=\sqrt{J \cap L}$. As $J+L$ is not $\mathfrak{m}$-primary we find an element $b \in \mathfrak{m}$ which lies outside of all minimal prime divisors of $J+L$. Setting $J'=(J, b)$, $L'=(L, b)$ we obtain $\dim(R/(J'+L'))=d-1$, $\dim(R/J')=\dim(R/J)-1$, $\dim(R/L')=\dim(R/L)-1$, $r(L' \cap J') \leq r+1$. By induction we have

$$\dim(R/(J'+L')) \geq c(R)-r(J' \cap L')-1.$$

Altogether this proves our claim.

Let X be a locally noetherian scheme, $Z \subset X$ a closed set, $x \in Z$ and let $\mathcal{J} \subseteq \mathcal{O}_X$ be the ideal of sections vanishing at Z. We define the *local connectedness-dimension of Z at x* resp. the *formal connectedness-dimension of Z at x* by

(2.2) (i) $c_x(Z):=c(\mathcal{O}_{X,x}/\mathcal{J}_x)$,
(ii) $\hat{c}_x(Z):=c((\mathcal{O}_{X,x}/\mathcal{J}_x)^{\wedge})$.

Obviously $\hat{c}_x(Z) \leq c_x(Z)$.

The *local analytical rank of* Z *at* x is defined as

(2.3) $r_{X,x}(Z) = r(\mathcal{J}_x)$.

In these notations we have

(2.4) THEOREM. *Let* X *be a noetherian scheme,* $Z \subset X$ *a closed set,* $x \in Z$. *Then*

$$c_x(Z) \geq \hat{c}_x(X) - r_{X,x}(Z) - 1.$$

Proof. Passing to local rings at x and then taking completions we may assume that $X = \operatorname{Spec}(R)$, $x = \mathfrak{m}$, where $(R, \mathfrak{m})$ is local and complete. Let $I \subseteq R$ be the vanishing ideal of Z and put $c = c(Z)$. We must show that $c \geq c(R) - r(I) - 1$.

If Z is irreducible, this is obvious. Otherwise we may write $Z = Z_1 \cup Z_2$, where Z_1 and Z_2 are both unions of irreducible components of Z such that $\dim(Z_1 \cap Z_2) = c$. But then Z_1 and Z_2 have no common components (by the minimality of $\dim(Z_1 \cap Z_2)$). Therefore $\dim(Z_1)$, $\dim(Z_2) > \dim(Z_1 \cap Z_2) = c$. Let J, $L \subseteq R$ be the vanishing ideals of Z_1 and Z_2 respectively. We have $J \cap L = I$, $\dim(R/J)$, $\dim(R/L) > c = \dim(R/(J + L))$. Now we get our claim by (2.1).

Denoting the *embedding dimension* of a noetherian scheme X at a point $x \in X$ by $\operatorname{edim}_x(X)$ we obtain

(2.5) COROLLARY. *Let* X *be of finite type over an algebraically closed field* k *and let* S, $T \subseteq X$ *be two closed sets which have in common a closed point* p. *Then*

$$\hat{c}_p(S \cap T) \geq \hat{c}_p(S) + \hat{c}_p(T) - \operatorname{edim}_p(\mathrm{X}) - 1.$$

Proof. Put $e = \operatorname{edim}_p(X)$. The formal space germ $(X, p)^\wedge$ is closed in $(\mathbb{A}_k^e, p)^\wedge$, where $p \in \mathbb{A}_k^e$ is closed. As our statement is concerned only with closed subsets of $(X, p)^\wedge$ we thus may replace X by $\mathbb{A}_k^e = \mathbb{A}^e$.

Let $D \subset \mathbb{A}^e \times \mathbb{A}^e$ be the diagonal. Then the canonical isomorphism $\delta : \mathbb{A}^e \to D$ gives rise to a isomorphism $S \cap T \cong (S \times T) \cap D$ and maps p to $p \times p$. So, applying (2.4) to the pair $(S \times T) \cap D \subset S \times T$ and observing that

$$r_{S \times T,\, p \times p}((S \times T) \cap D) \leq e$$

we obtain $\hat{c}_p(S \cap T) \geq \hat{c}_{p \times p}(S \times T) - e - 1$. It thus remains to prove $\hat{c}_p(S) + \hat{c}_p(T) \leq \hat{c}_{p \times p}(S \times T)$. This may be done easily by decomposing the germs $(S, p)^\wedge$, $(T. p)^\wedge$ in irreducible components and using the description (1.2) of the connectedness-dimension.

(2.6) COROLLARY. *Let k be an algebraically closed field and let $V, W \subset \mathbb{P}_k^d$ be closed. Then $c(V \cap W) \geq c(V) + c(W) - d - 1$.*

Proof. Denote the origin of $\mathbb{A}_k^{d+1}$ by p. If $Z \subset \mathbb{P}_k^d$ is closed let $K(Z) \subset \mathbb{A}_k^{d+1}$ be its affine cone. Obviously $c(Z) = c_p(K(Z)) - 1$. As the irreducible components of $K(Z)$ are analytically irreducible in their vertex, we also have $c_p(K(Z)) = \hat{c}_p(K(Z))$. Now apply (2.5) to $K(V)$ and $K(W)$ and observe that $K(V \cap W) = K(V) \cap K(W)$.

Choosing V and W irreducible we immediately see that (2.6) contains the connectedness theorem of Fulton–Hansen mentioned in the introduction.

Recall that the connectedness-part of Bertinis theorem claims that a hyperplane section of an irreducible projective variety of dimension ≥ 2 is connected. We close the section by another application of (2.5), which generalizes this result:

(2.7) COROLLARY. *Let k be an algebraically closed field. Let $V \subset \mathbb{P}_k^d$ be closed and let $H \subset \mathbb{P}_k^d$ be a hypersurface. Then $c(H \cap V) \geqslant c(V) - 2$.*

Proof. Using the notations of the previous proof we have $r_{K(V),p}(K(H \cap V)) \leqslant 1$. As observed above we have $c(Z) = \hat{c}_p(K(Z)) - 1$ for $Z = V$ and for $Z = H \cap V$. So, applying (2.4) to the pair $K(H \cap V) \subset K(V)$ we get our claim.

3. Projective morphisms

Throughout this section let $(R, \mathfrak{m})$ be a local noetherian ring, $(\hat{R}, \hat{\mathfrak{m}})$ its completion. Moreover let $S = \bigoplus_{n \geq 0} S_n$ be a graded noetherian R-algebra with $R = S_0$. Put $\hat{S} = \hat{R} \otimes_R S$. We consider the canonical morphism

(3.1) $\pi : \operatorname{Proj}(S) \to \operatorname{Spec}(R)$.

(3.2) PROPOSITION. *The special fiber $\pi^{-1}(\{\mathfrak{m}\})$ is connected iff* $\operatorname{Proj}(\hat{S})$ *is connected.*

Proof. Considering the canonical diagram

$$\begin{array}{ccc} \operatorname{Proj}(S) & \xrightarrow{\pi} & \operatorname{Spec}(R) \\ \uparrow{\scriptstyle \alpha} & & \uparrow{\scriptstyle \alpha_0} \\ \operatorname{Proj}(\hat{S}) & \xrightarrow{\hat{\pi}} & \operatorname{Spec}(\hat{R}) \end{array}$$

and observing that $\pi^{-1}(\{\mathfrak{m}\})$ and $\pi^{-1}(\{\hat{\mathfrak{m}})$ are homeomorphic by means of α, we may replace S by $\hat{S}$, and thus assume that R is complete. Put $Z = \pi^{-1}(\{\mathfrak{m}\})$.

Assume that Z is connected. As π is a closed map and as R is local, each non-empty closed subset of Proj (S) meets Z. So Proj (S) is connected.

Assume that Z is disconnected. We must show that $X :=$ Proj (S) is disconnected. If $Z = \emptyset$, this is clear as then $X = \emptyset$ (by the fact that any non-empty closed subset of X meets Z). So we may assume $Z \neq \emptyset$. We find two non-empty closed subsets $V, W \subset Z$ such that $Z = V \dot{\cup} W$. Assume first that no irreducible component of X meets both V and W. Let $X_1, \ldots, X_n$ be the irreducible components of X. Then, after an appropriate reordering we find a non-negative integer $s \leq n$ such that $V \cap X_1, \ldots, V \cap X_s$ and $W \cap X_{s+1}, \ldots, W \cap X_n$ are empty. As $V \cup W \subset \bigcup_{i=1}^n X_i$ we must have $0 < s < n$. Put $X' = \bigcup_{i \leq s} X_i$, $X'' = \bigcup_{i > s} X_i$. As $Z \cap (X' \cap X'') = (V \cap X') \cup (W \cap X'') = \emptyset$ we get $X' \cap X = \emptyset$, thus a closed disjoint decomposition $X = X' \dot{\cup} X''$. Therefore X is disconnected. So it remains to treat the case where $V \cap \bar{X}$, $W \cap \bar{X} \neq \emptyset$ for an irreducible component $\bar{X}$ of X. We show that this case cannot occur as it would lead to a contradiction. To do so, write $\bar{X} =$ Proj $(\bar{S})$, where $\bar{S} = \bigoplus_{n \geq 0} \bar{S}_n$ is a graded homomorphic image domain of S. $\bar{S}_0 =: \bar{R}$ is a homomorphic image domain of R, whose maximal ideal will be denoted by $\bar{\mathfrak{m}}$. *If* $\bar{\pi}$: Proj $(\bar{S}) \to$ Spec $(\bar{R})$ is the canonical map, we have the closed disjoint decomposition $\bar{Z} := \bar{\pi}^{-1}(\{\bar{\mathfrak{m}}\}) = (V \cap \bar{X}) \cup (W \cap \bar{X})$. So, replacing $\bar{S}$ by S we are in the situation where S is a domain. Let $J, L \subset S$ be the homogeneous vanishing ideals of V and W and let $\mathfrak{n} \subset S$ be the homogeneous maximal ideal. Put $t = \dim(S_{\mathfrak{n}})$ and consider the following piece of the graded Mayer–Vietoris sequence

$$H^{t-1}_{J \cap L}(S) \to H^t_{J+L}(S) \to H^t_J(S) \oplus H^t_L(S).$$

As $V \neq \emptyset$, J is not $\mathfrak{n}$-primary, and so $H^t_J(S)$ vanishes by Hartshorne–Lichtenbaum (1.8). By the same reason we have $H^t_L(S) = 0$. As $V \cap W = \emptyset$ and $V, W \subseteq \pi^{-1}(\{\mathfrak{m}\})$, $J + L$ is $\mathfrak{n}$-primary. So the middle term of our sequence may be replaced by $H^t_{\mathfrak{n}}(S)$ and thus contains non-zero homogeneous elements of negative degree, (1.9). As $V \cup W = \pi^{-1}(\{\mathfrak{m}\})$ we have $J \cap L = \sqrt{\mathfrak{m}S}$. So the first term of the sequence coincides with $H^{t-1}_{\mathfrak{m}S}(S)$ and thus vanishes in all homogeneous parts of negative degree, (1.10). Altogether this gives a contradiction.

For a closed subset Z of a noetherian scheme X we introduce the notation

(3.3) $r_X(Z) := \max \{r_{X,x}(Z) \mid x \in Z\}$

Using this notation we have

(3.4) THEOREM. *Let* $Z \subset$ Spec (R) *be a closed set. Assume that* Proj $(\hat{S})$ *is* *connected. Then* $\pi^{-1}(Z)$ *is connected and satisfies* $c(\pi^{-1}(Z)) \geq c(\operatorname{Proj}(\hat{S})) - r_{\operatorname{Proj}(\hat{S})}(\pi^{-1}(Z)) - 1$.

Proof. We put $Y = \pi^{-1}(Z)$. By (3.2) $\pi^{-1}(\{\mathfrak{m}\})$ is connected. By the closedness of π we obtain that Y is connected.

To prove the remaining claim we first want to replace S by $\hat{S}$, which allows to assume that R is complete. To be able to perform this relacement we must show (in view of the previous diagram (*)) the inequalities $c(Y) \geq c(\alpha^{-1}(Y))$, $r_{\operatorname{Proj}(S)}(Y) \geq r_{\operatorname{Proj}(\hat{S})}(\alpha^{-1}(Y))$. The second inequality is an immediate consequence of the fact that α is surjective. The first inequality is shown using in addition that α gives a bijection between the special fibres $\hat{\pi}^{-1}(\{\hat{\mathfrak{m}}\})$ and $\pi^{-1}(\{\mathfrak{m}\})$: Indeed, let $Y = Y_1 \cup Y_2$, where Y_1 and Y_2 are unions of irreducible components of Y such that $\dim(Y_1 \cap Y_2) = c(Y)$. Then by the surjectivity of α, $\alpha^{-1}(Y_1)$, $\alpha^{-1}(Y_2)$ are both unions of irreducible components of $\alpha^{-1}(Y)$ and it holds $\alpha^{-1}(Y) = \alpha^{-1}(Y_1) \cup \alpha^{-1}(Y_2)$. So we have to prove the inequality $\dim(\alpha^{-1}(Y_1) \cap \alpha^{-1}(Y_2)) \leq \dim(Y_1 \cap Y_2)$, hence $\dim(\alpha^{-1}(Y_1 \cap Y_2)) \leq \dim(Y_1 \cap Y_2)$. The mentioned bijection between the special fibres means that α maps closed points to closed points and that the preimage $\alpha^{-1}(x)$ of a closed point $x \in Y$ consists of a single closed point. From this the above inequality is clear.

Next we want to show that in addition S may be supposed to be a domain. We may restrict ourselves to the case $c(Y) < c(\operatorname{Proj}(S))$, as otherwise (3.4) holds trivially. By (1.2) we find two closed sets $Y_1, Y_2 \subset Y$ such that $Y_1 \cup Y_2 = Y$ and $\dim(Y_1 \cap Y_2) = c(Y)$ and such that $Y - Y_1 \cap Y_2$ is disconnected. There is an irreducible component $\bar{X}$ of $\operatorname{Proj}(S)$ with $\bar{X} \cap Y_1, \bar{X} \cap Y_2 \neq \emptyset$. (Otherwise, collecting the components which avoid Y_1 and those which avoid Y_2 we get a closed decomposition $\operatorname{Proj}(S) = X_1 \cup X_2$ with $X_1 \cap X_2 \subseteq Y_1 \cap Y_2$, hence the contradiction $c(Y) \geq c(\operatorname{Proj}(S))$). It follows $c(\bar{X} \cap Y) \leq c(Y)$, $c(\bar{X}) = \dim(\bar{X}) \geq c(\operatorname{Proj}(S))$ and $r_x(\bar{X} \cap Y) \leq r_{\operatorname{Proj}(S)}(Y)$. This allows to replace S by its homomorphic image domain $\bar{S}$ for which $\operatorname{Proj}(\bar{S}) = \bar{X}$.

Finally we want to show that $\operatorname{Proj}(S)$ moreover may be assumed to be normal. To do so let $S' = \bigoplus S'_n$ be the graded integral closure of S. As S is of finite type over the complete local domain R, it is excellent. So S' is a finite integral extension domain of S [5]. In particular $R' := S'_0$ is a local complete domain and S'_n vanishes for $n < 0$. We have the canonical diagram

$$\begin{array}{ccc}
\operatorname{Proj}(S) & \xrightarrow{\quad\pi\quad} & \operatorname{Spec}(R) \\
\Big\uparrow{\scriptstyle\beta} & & \Big\uparrow{\scriptstyle\beta_0} \\
\operatorname{Proj}(S') & \xrightarrow{\qquad} & \operatorname{Spec}(R')
\end{array}$$

Thereby β is finite birational and Proj (S') is normal. From the finiteness of β we get the inequalities

$$c(Y) \geq c(\beta^{-1}(Y)), \qquad r_{\mathrm{Proj}\,(S)}(Y) \geq r_{\mathrm{Proj}\,(S')}(\beta^{-1}(Y)),$$

which allow to replace S by S'.

So let R be complete, let S be a domain and assume that $X := \mathrm{Proj}\,(S)$ is normal. Consider a closed decomposition $Y = Y_1 \cap Y_2$, where Y_1 and Y_2 are unions of irreducible components of Y such that $\dim(Y_1 \cap Y_2) = c(Y)$. Let x be a generic point of $Y_1 \cap Y_2$ such that $\dim \overline{\{x\}} = c(Y)$. If x is a generic point of Y_1 or of Y_2 it is a generic point of Y and so we get $\dim(\mathcal{O}_{X,x}) = r_{X,s}(Y) \leq r_X(Y) \leq r_X(Y) + 1$, thus the inequality

$$(^{**}) \quad -\dim(\mathcal{O}_{X,x}) \geq -r_X(Y) - 1.$$

If x is neither a generic point of Y_1 nor of Y_2, let $J, L \subset \mathcal{O}_{X,x} = R$ be the vanishing ideals of Y_1 and Y_2. Then J and L satisfy the hypotheses of (2.1) and we obtain $-c(\mathcal{O}_{X,x}) \geq r(J \cap L) - 1 = -r_{X,x}(Y) - 1 \geq -r_X(Y) - 1$. As X is normal and excellent, $\hat{\mathcal{O}}_{X,x}$ is a domain [5], and thus satisfies $c(\hat{\mathcal{O}}_{X,x}) = \dim(\mathcal{O}_{X,x})$. So we get again the inequality (**). Finally, as S is of finite type over the local excellent domain R, we have $c(Y) = \dim \overline{\{x\}} = \dim(X) - \dim(\mathcal{O}_{X,x})$, [9]. Now, by (**) we obtain $c(Y) \geq \dim(X) - r_X(Y) - 1$. This proves our claim.

Let R be analytically normal and assume that $\pi : X := \mathrm{Proj}\,(S) \to \mathrm{Spec}\,(R)$ is birational. Then Proj $(\hat{S})$ obviously is irreducible. By (3.4) $\pi^{-1}(Z)$ is connected. So (3.4) contains Zariski's connectedness theorem for projective birational morphisms. As $c(\mathrm{Proj}\,(S)) = \dim(R)$ we get in addition the estimate $c(\pi^{-1}(Z)) \geq \dim(R) - r_X(\pi^{-1}(Z)) - 1$.

BIBIOGRAPHY

[1] BARTH, W., *Verallgemeinerung des Bertinischen Theorems in Abelschen Mannigfaltigkeiten*, Ann. Sc. Norm. Sup. Pisa *23* (1969) 317–330.
[2] FALTINGS, G., *Algebraization of Some Vector Bundles*, Ann. Math. *110* (1979) 501–514.
[3] FULTON, and W. HANSEN, J., *A Connectedness Theorem for Projective Varieties with Application to Intersections and Singularity of Mappings*, Ann. Math. *110* (1979) 159–166.
[4] GROTHENDIECK, A., *EGA III*, Publ. Math. I.H.E.S., *11* (1961).
[5] GROTHENDIECK, A., *EGA IV*, Publ. Math. I.H.E.S., *24* (1968).
[6] GROTHENDIECK, A., *Local Cohomology*, Springer Lecture Notes *41* (1966).
[7] HARTSHORNE, R., *Residues and Duality*, Springer Lecture Notes *20* (1960).
[8] HARTSHORNE, R., *Cohomological Dimension of Algebraic Varieties*, Ann. Math. *88* (1968) 403–450.
[9] MATSUMURA, H., Commutative Algebra, Benjamin, New York (1970).

[10] RUNG, J., *Mengentheoretische Durchschnitte und Zusammenhang*, Regensburger Mathematische Schriften *3*, 1978.

[11] RUNG, J., *Mengentheoretische Durchschnitte und Zusammenhang*, Preprint 1979.

[12] ZARISKI, O., *A Simple Analytical Proof of a Fundamental Property of Birational Transformations*, Proc. Nat. Acad. Sci. USA *32* (1949) 62–66.

M. Brodmann
Mathematisches Institut der
Universität Zürich,
Rämistrasse 74
CH-8001 Zuerich
Switzerland

J. Rung
Bayerisches Staatsministerium
für Unterricht und Kultus
Salvatorstrasse 2
D-8000 Muenchen
W. Germany

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