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Homology of classical Lie groups made discrete, I.[1] Stability theorems and Schur multipliers

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Let G^δ denote the Lie group G equipped with the discrete topology. As mentioned in Milnor [18], the (integral) homology of the classifying space BG^δ (equivalently, the (integral) Eilenberg–MacLane homology $H_*(G)$ of the abstract group G) is of interest in at least the following areas: algebraic K-theory, the study of bundles with flat connections, foliation theory, and the study of scissors congruences of polyhedra.

In all such cases, the group G usually belongs to one of the known infinite series of classical groups. According to the general philosophy of algebraic K-theory, the study of the homology groups may be divided into several steps. The first step is to prove some sort of stability theorems for groups belonging to each of the infinite series. The second step is to give some sort of descriptions of the stable groups. These are then followed by interpretations and/or applications of the results found. In the case of stability theorems, techniques of algebraic K-theory appear to have reached their limits and improvements seem to be quite difficult. We use the special nature of classical Lie groups and a more naive approach to obtain somewhat sharper stability results. In the case of the determination of the stable groups, we concentrate our efforts on the description of the Schur multipliers. Our results are closely related to algebraic K-theory (in particular, to K_2 of fields). The improvement obtained in the stability results is such that we can cut out K_2-calculations in most cases. When it becomes unavoidable, we only perform them in the manner first described by Milnor [17]. Interpretations and/or applications are limited to abbreviated comparisons of our results with known results. The principal motivation of the present work is the problem of scissors congruence of polyhedra. The sharpness of our results is partly due to our insistence on making some headway in this old problem. The principal results in this work are:

Let $\mathbb{F}$ denote one of the three classical division algebras $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.

THEOREM 1.1. *Let $n \geq 0$. The inclusion map from $U(n, \mathbb{F})$ to $U(n+1, \mathbb{F})$*

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induces a surjection from $H_i(U(n, \mathbb{F}))$ *to* $H_i(U(n+1, \mathbb{F}))$ *for* $i \leq n$. *This map is a bijection when* $i < n$.

THEOREM 2.1. *Under the natural inclusion maps, we have bijections:*
(a) $H_2(SPin(3)) \to H_2(SPin(n)) \to H_2(SPin(n+1))$, $n \geq 5$.
(b) $H_2(SU(2)) \to H_2(SU(n)) \to H_2(SU(n+1))$, $n \geq 3$.
(c) $H_2(Sp(1)) \to H_2(Sp(n)) \to H_2(Sp(n+1))$, $n \geq 2$.

THEOREM 3.8. *Let* $G(p, q) = U(p, q, \mathbb{F})$, $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$ *or* $\mathbb{H}$. *Fix* $p \geq 0$ *and consider the inclusion of* $G(p, q)$ *into* $G(p, q+1)$. *The induced map from* $H_i(G(p, q))$ *to* $H_i(G(p, q+1))$ *is then surjective for* $i \leq q$ *and bijective for* $i < q$.

THEOREM 4.1. *Let G be a connected, simply-connected, absolutely simple real Lie group. Assume that G is noncompact and of classical type. Then* $H_2(G) \cong K_2(\mathbb{C})^+$ *under the universal complexification homomorphism.*

Theorem 3.8 includes Theorem 1.1 by taking $p = 0$; however, the proof of Theorem 3.8 uses Theorem 1.1 both to get it started and to keep the induction going. Theorems 2.1 and 4.1 exhibit the distinction between the compact and the noncompact cases. It is known that $H_2(SU(2))$ is divisible and maps onto $K_2(\mathbb{C})^+$ under the universal complexification map. (In fact, a more difficult argument shows that the map is bijective, see the later work by Dupont–Parry–Sah [7].) In the case of complex simply-connected Lie groups, the Schur multiplier is known to be $K_2(\mathbb{C})$ which is a $\mathbb{Q}$-vector space with continuum dimension. If we combine these with a few other known results, then we have the following:

COROLLARY. *Let G be any Lie group whose Levi components do not involve one of* 10 *exceptional simple Lie groups of types E and F* (3 *are compact,* 7 *are noncompact and all are nonsplit over* $\mathbb{R}$). *Then* $H_i(BG^\delta, \mathbb{F}_p)$ *is isomorphic to* $H_i(BG, \mathbb{F}_p)$ *for* $i \leq 2$.

The preceding corollary confirms a conjecture of Milnor (also called the Friedlander–Milnor conjecture, see Milnor [18] and Friedlander–Mislin [9]) under the restrictions named. It should be noted that the inclusion of the compact groups of type G_2 and F_4 in the preceding corollary requires the unpublished work of John Hurley (Stony Brook Dissertation, 1983) and of Johan Dupont (1984). In the stable range, the Friedlander–Milnor conjecture is now known to hold for $GL(n)$, $SL(n)$ over $\mathbb{R}$ and $\mathbb{C}$ by using the recent results of Suslin [31–34] which confirmed the Lichtenbaum–Quillen conjecture. This result has been extended by Karoubi by using his Hermitian K-theory [13]. However, in spite of the connotation, our results are mostly outside of the stability range results from algebraic K-theory. The main point is that these results use coefficients $\mathbb{F}_p$ and kill off unknown $\mathbb{Q}$-vector spaces at the very beginning. In the range of interest to us, these unknown $\mathbb{Q}$-vector spaces are connected with the scissors congruence problem. These connections will be discussed in greater detail in later works.

The present work is organized in the following manner. Section 0 gives a rapid review of the relevant background materials and fixes the notation. It is a bit lengthy. Except for checking over the notations, readers with some familiarities of the basic results of geometric algebra, homological algebra, Lie groups, Lie algebras, Schur multipliers of algebraic groups and the K_2 functor (associated to fields) of Milnor, can skip this section. Cartan's classification of simple Lie algebras will not be reviewed but will be invoked. Section 1 deals with Theorem 1.1 and sets the tone for Section 3. Section 2 deals with Theorem 2.1 and sets the tone for Section 4. Some technical results needed to complete the proof of Theorem 4.1 are relegated to Appendix A and Appendix B. In particular, Appendix B contains an easier proof of a stronger result that gives a part of the stability theorem of Suslin [32].

We thank Wu-Chung Hsiang, Michio Kuga, Dusa McDuff, John Milnor and Walter Parry for many helpful conversations. We also thank Johan Dupont and Jack Wagoner for collaborations leading to many valuable ideas. In paricular, the idea that a chain complex based on unit vectors might be of interest was first suggested to us by Kuga in a conversation (c. 1972) concerning finite orthogonal groups acting on finite vector spaces. In an earlier version of the present work, a weaker form of Theorem 1.1 was obtained by adapting an idea of Hsiang which was based on a combinatorial topological approach to tackle the conjecture of Milnor for $SL(n, \mathbb{R})$. A variation of this adaptation is implicitly used in Appendix A. A similar idea was suggested by the referee. Neither of these ideas makes an appearance in the present work. Nevertheless, both Hsiang and the referee deserve the credits for providing inspirations. A number of the technical ideas (such as the dimension filtration and the spectral sequence machine) have already made their appearances in Dupont [6]. Aside from all these, we would like to express our deepest appreciation to the unknown referee who toiled heroically through the several drafts of the present work and made valuable suggestions and caught many errors. Needless to say, we take the credit for the remaining errors (hopefully, they are of the trivial kind).

0. Preliminaries

This section reviews some basic results in geometric algebra, homological algebra, Lie groups, Lie algebras, Schur multipliers of algebraic groups and the

K_2-functor of fields defined by Milnor. Part of this will serve to fix our notations and part of this will serve to outline our arguments.

$\mathbb{F}$ denotes one of the three classical division algebras $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. A careful proof analysis shows that our results are valid for division algebras of the same three types over any Pythagorean field – any ordered field R such that each positive element of R is a square in R. The involutions (or antiautomorphisms of order dividing 2) of $\mathbb{F}$ over $\mathbb{R}$ are known. The less wellknown ones for $\mathbb{H}$ can be normalized to be $**$ which sends i, j, k into i, $-j$, k respectively. We indiscriminately take $*$ to be one of these involutions (including Id over $\mathbb{R}$ or $\mathbb{C}$) for the time being. Later on, the choice will be clear from context. $\mathbb{F}^n$ denotes the right $\mathbb{F}$-vector space formed by all the column vectors with n entries from $\mathbb{F}$. For $\varepsilon = \pm$, the nondegenerate ε-hermitian $*$-sesquilinear forms on $\mathbb{F}^n$ have all been classified. They are viewed as inner products:

$$\langle\ ,\ \rangle : \mathbb{F}^n \times \mathbb{F}^n \to \mathbb{F}^n.$$

The unitary groups of these forms together with the general linear groups account for all the simple Lie groups of classical types after we pass to groups that are locally isomorphic to the commutator subgroups of the connected components of the groups mentioned. The Theorem of Witt can be stated in the following form:

Let $U = \sum_{0 \le i \le j} u_i \mathbb{F}$ and $V = \sum_{0 \le i \le j} v_i \mathbb{F}$ be nondegenerate $\mathbb{F}$-subspaces of $\mathbb{F}^n$. Suppose that σ is unitary with respect to $\langle\ ,\ \rangle$ such that $\sigma(u_i) = v_i$, $0 \le i \le j$. Then there exists unitary ρ on $\mathbb{F}^n$ with respect to $\langle\ ,\ \rangle$ such that $\rho(u_i) = v_i$ for $0 \le i \le j$ and such that $\rho(w) = w$ holds for all w orthogonal to $U + V$. If $\dim U = \dim V$ and if $\langle u_s, u_t \rangle = \langle v_s, v_t \rangle$ holds for $0 \le s, t \le j$, then σ can be found.

With some mild restrictions when dealing with characteristic 2, Witt's Theorem actually holds in general.

Spectral sequences will be one of the basic tools in our investigation. These arise from double complexes in the first quadrant. In most cases, they are homology spectral sequences with the first (or column) index as well as the second (or row) index filtration. To avoid possible confusion, we adopt redundant notation as needed. For example, a complete notation for a homology spectral sequence with second index filtration (often called the transposed spectral sequence) will have its E^2-terms denoted by $''E^2_{i,j}$ and we write:

$$H_{i+j} \Leftarrow {}''E^2_{i,j}, \qquad {}''d^2$$

The broken arrow is used to remind us of the usual "staircase" description of $''d^2$. The horizontal doubled arrow signifies that $''E^{\infty}_{n,0}$ injects into H_n while H_n surjects onto $''E^{\infty}_{0,n}$. Typically, our transposed spectral sequences arise as follows:

Consider any exact sequence of (left) G-modules for the group G:

$$\cdots \to M_j \to M_{j-1} \to \cdots \to M_0 \to A \to 0. \tag{0.1}$$

This will be called an acyclic G-chain complex M_* with augmentation A. The G-homomorphisms in (0.1) are denoted by ∂_M. We next take any $\mathbb{Z}G$-free (or $\mathbb{Z}G$-projective) resolution of the G-trivial module $\mathbb{Z}$:

$$\cdots \to C_i(G) \to C_{i-1}(G) \to \cdots \to C_0(G) \to \mathbb{Z} \to 0. \tag{0.2}$$

For example, $C_*(G)$ can be the standard bar resolution, either homogeneous or nonhomogeneous, either normalized or nonnormalized. We then form the double complex $C_*(G) \otimes_G M_*$ with (i, j)-th term $C_i(G) \otimes_G M_j$ and with total boundary $\partial = \partial_G \otimes 1 + \epsilon \otimes \partial_M$ where ϵ is $(-1)^i$ on $C_i(G)$. This yields:

$$'E^2_{*,0} = H_*(G, A) \quad \text{and} \quad 'E^2_{*,j} = 0 \text{ for } j > 0. \tag{0.3}$$

Comparison of spectral sequences then yields:

$$H_{i+j}(G, A) \Leftarrow ''E^1_{i,j} = H_i(G, M_j), \quad \downarrow ''d^1 = \epsilon\partial_M. \tag{0.4}$$

We often ignore the sign ϵ in (0.4) with the understanding that some of our maps may have an ambiguous factor of ± 1 that cause no problem on our assertions. Contrary to some conventions, we do not transpose the terms of our spectral sequences but do transpose the differentials. This is consistent with our emphasis on the comparison feature of the two different filtrations. In our applications, M_* may be viewed as the complex of cellular chains on a suitable cell complex equipped with a cellular group action from G (however, the action is usually not properly discontinuous). In such cases, $H_i(G, M_j)$ in (0.4) can be described through Shapiro's Lemma. Namely, if K is any subgroup of G and if N is any (left) $\mathbb{Z}K$-module, then Shapiro's Lemma reads:

$$H_*(G, \operatorname{ind}^G_K N) \cong H_*(K, N).$$

The interesting case for us usually occurs with K acting trivially on N. In the setting of transformation groups, Shapiro's Lemma is just the "Principle of Reduction to Isotropy". In the topological setting, the group action is usually required to be proper. An explicit map describing (0.5) may be found in Dupont–Sah [8].

The next result is formal. Let $\mathcal{F}^i$, $i \geq 0$, be an increasing filtration of a chain complex C_* by subcomplexes. It then leads to a spectral sequence with first index filtration given by the $\mathcal{F}^i$'s.

BOOT-STRAPPING LEMMA. *Suppose that $\mathcal{F}^i$ is $(i-1)$-acyclic for $i = n$, $n+1$, and that $'E^1_{n+1,0} = 0$. Then $\mathcal{F}^n$ is in fact n-acyclic.*

Proof. Look at the long exact sequence:

$$\cdots \to H_{n+1}(\mathcal{F}^{n+1}, \mathcal{F}^n) \to H_n(\mathcal{F}^n) \to H_n(\mathcal{F}^{n+1}) \to \cdots$$

$H_{n+1}(\mathcal{F}^{n+1}, \mathcal{F}^n)$ is just $'E^1_{n+1,0}$. □

Remark. In the preceding lemma, the spectral sequence is merely a convenient way to describe the relative groups $H_{n+1}(\mathcal{F}^{n+1}, \mathcal{F}^n) = H_{n+1}(\mathcal{F}^{n+1}/\mathcal{F}^n)$. The spectral sequence formulation is retained partly because $'E^1_{n+1,0}$ is the scissors congruence group in the homological algebraic approach to the scissors congruence problem formulated by Dupont [6]. With trivial coefficients in place of the twisted coefficients used by Dupont, we have killed off the scissors congruence groups.

Let G denote any (real) Lie group with connected component G^0. There is a "functorial" definition of the universal complexification $G_{\mathbb{C}}$ of G, see Bourbaki [4; III.6.10]. $(G^0)_{\mathbb{C}}$ and $(G_{\mathbb{C}})^0$ are then isomorphic under the functorial homomorphism and $G/G^0 \cong G_{\mathbb{C}}/G^0_{\mathbb{C}}$. In general, the universal complexification homomorphism does not have to be injective. However, its kernel is always a discrete normal subgroup of G^0 so that it is part of the center of G^0 (though not necessarily part of the center of G).

Suppose that G is a connected, simply-connected, simple, nonabelian real Lie group. $G_{\mathbb{C}}$ is then connected and simply-connected. $G_{\mathbb{C}}$ is either simple or isomorphic $G \times G$. In the first case, G is said to be absolutely simple (when there is no chance of confusion, any Lie group locally isomorphic to such a G is also called absolutely simple). The second case occurs exactly when G is a complex Lie group viewed as a real Lie group. In general, the Lie algebra of G is the fixed point set of a real involution of the (real) Lie algebra of $G_{\mathbb{C}}$. This real Lie algebra involution can be integrated into an involution η of the real Lie group $G_{\mathbb{C}}$. Under the universal complexification homomorphism, G is then mapped onto the connected component $(G^\eta_{\mathbb{C}})^0$ of the fixed point set $G^\eta_{\mathbb{C}}$ of η. When G is complex but viewed as a real Lie group, then η exchanges the two simple factors G of $G_{\mathbb{C}}$

and $G \cong G_{\mathbb{C}}^{\eta}$ where the latter represents the diagonal subgroup of $G \times G$. Since G is always a perfect group, the relation between $H_2(G)$ and $H_2(G_{\mathbb{C}})$ is clear when G is complex. Namely, $H_2(G) \cong H_2(G_{\mathbb{C}})^{\eta}$. Similar relations can be obtained by using Künneth's Theorem as long as we have some control on the Tor terms. Our interests will be concentrated on the cases where G is absolutely simple. The general outline is that $H_2(G)$ will be determined by using the map induced by the universal complexification homomorphism. It turns out that this program is best divided into two parts according to G is compact or not. In both cases, we restrict ourselves to those G that are of classical types and replace each G by the appropriate group of all "isometries" of appropriate "inner products". We therefore need to know something about $H_2(G_{\mathbb{C}})$ before the replacements are carried out.

Assume that G is absolutely simple so that $G_{\mathbb{C}}$ is a simple, connected, simply-connected, complex Lie group. In this case, $G_{\mathbb{C}}$ is an example of a universal Chevalley group $G(\Phi, F)$ associated to the connected (or irreducible) root system Φ over a field F by taking $F = \mathbb{C}$ and letting Φ be the root system associated to the Dynkin diagram of $G_{\mathbb{C}}$. Explicit descriptions of $H_2(G(\Phi, F))$ are known from the works of Steinberg [29, 30], Moore [19] and Matsumoto [15]. With the exception of a small number of cases (when both $\#(F)$ and rank (Φ) are small), the descriptions are given by generators and defining relations, see Steinberg [30; p. 86]. In all cases, $H_2(G(\Phi, F))$ is the homomorphic image of $H_2(SL(2, F))$ by means of a "long root homomorphism" of $SL(2, F)$ into $G(\Phi, F)$ as long as we avoid the finite number of exceptional cases. When Φ is of type C_n, $n \geq 1$ (note: $A_1 = C_1$ and $B_2 = C_2$), the homomorphism is actually an isomorphism. When Φ is not of type C_n, $n \geq 1$, we obtain the same quotient group denoted by $K_2(F)$. These techniques can be summarized as "reduction to rank 1 though algebraic group theory". With a large amount of work, this process has been extended by Deodhar [5] to the "quasi-split" cases. In general, the cases of $G(\Phi, F)$ cover the "split" case. For our purposes, the technical definitions are not important. It is enough to know that $SL(n, F)$ and $Sp(2n, F)$ are "split" while $SO(p, q, \mathbb{R})$ is "$\mathbb{R}$-split" if and only if $|p-q| \leq 1$ and is "$\mathbb{R}$-quasi-split" if and only if $|p-q| \leq 2$. Moreover, the compact cases are the analogues of the "anisotropic" cases in algebraic group theory where the procedures from algebraic group theory give no information. This comes about because the algebraic group theory is based on having enough unipotent elements (in particular, having sufficiently rich supply of $SL(2, F)$'s) in the group. Our procedure is roughly that of replacing the "reduction of rank 1 by algebraic group theory" by the "stability results via geometric algebra". In the compact cases, we end up with $H_2(SU(2))$ and in the noncompact cases, we end up with $K_2(\mathbb{C})^+$ – the fixed points under the action of complex conjugation automorphism of $\mathbb{C}$. For our purpose, the following result will be used a number of times. It can be deduced from Milnor [18; Lemma 6]:

Assume that G is the universal covering group of G' and that G is connected and semisimple. Then $H_2(G') \cong H_2(G)\, r\amalg\, \pi_1(G')$. (0.6)

In practice, $H_2(G')$ will be seen (in the cases of interest to us) to be a direct sum of $\pi_1(G')$ and a divisible group so that $H_2(G)$ is uniquely determined as the divisible part of $H_2(G')$. Variations of this theme will be used in some cases where G and G' are not connected but are semidirect products by a compatibly defined group whose actions are known.

We end this section with a short review of the K_2-functor of Milnor [17]. It is a covariant functor from associative rings with unit to abelian groups. We restrict ourselves to the case of fields (also some division rings in Appendix B) where additional properties are available, see also Kervaire [14] and Suslin [32]. By definition, $K_2(F)$ is the Schur multiplier $H_2(SL(F))$ where $SL(F)$ is the union of all $SL(n, F)$ under the stabilization homomorphisms. If we leave out a few small finite fields, then $H_2(SL(F))$ stabilizes surjectively at $H_2(SL(2, F))$ and bijectively at $H_2(SL(3, F))$. In all cases, $K_2(F)$ is generated by the "K_2-symbols" $\{u, v\}$, $u, v \in F^\times$ with defining relations: $\{u, v\}$ is bimultiplicative in u, v and $\{u, 1-u\} = 1$ if $u \neq 0, 1$. Here $K_2(F)$ is viewed as a multiplicative abelian group. We note that these relations are simpler than the ones given in [30; p. 86]. The symbol $\{u, v\}$ is interpreted below.

Let G be any abstract group. When G is perfect, $H_2(G)$ can be described as the kernel of a universal central extension. For a lucid exposition, see Milnor [17; §5]. For a general G, we can follow Schur [27, 28]. Let C_* denote the standard nonhomogeneous complex that computes $H_*(G)$ so that C_* is G-trivial. We can construct the following central exact sequence:

$$0 \to C_2/\partial C_3 \to G^* \to G \to 1$$

The construction is canonical in the sense that we use the 2-cocycle f defined by the rule: $f(x, y) = [x \mid y] \bmod \partial C_3$. In the Hochschild–Serre spectral sequence associated to the preceding exact sequence with trivial coefficient $\mathbb{Z}$, $'d^2_{2,0}: H_2(G) \to C_2/\partial C_3$ is just the obvious injection that identified $H_2(G)$ as the kernel of ∂: $C_2/\partial C_3 \to C_1$. We therefore obtain the exact sequence from the spectral sequence:

$$0 \to H_2(G) \to C_2/\partial C_3 \to H_1(G^*) \to H_1(G) \to 0.$$

This shows that $H_2(G) \cong (C_2/\partial C_3) \cap [G^*, G^*]$. When G is perfect, the commutator subgroup $[G^*, G^*]$ is the universal central extension of G. In general, we note that $C_2/\partial C_3 \cong H_2(G) \amalg \partial C_2$ because C_1 is a free abelian group based on the symbols $[x]$, $x \in G$ (or $x \in G - \{1\}$ if C_* is normalized). This splitting is not canonical. If we select such a splitting of $C_2/\partial C_3$ and quotient out the factor corresponding to ∂C_2, we then have a central extension:

$$0 \to H_2(G) \to \bar{G} \to G \to 1$$

In this exact sequence $H_2(G) \subset [\bar{G}, \bar{G}]$ and the determination of $H_2(G)$ (up to isomorphism) can be carried out in the manner described in Milnor [17]. Except when G is perfect, the preceding central extension is usually not universal and $\bar{G}$ is usually not unique up to isomorphism. Nevertheless, this was the method used by Schur in the case of finite symmetric groups. Topologically, C_* is the complex of cellular chains of an Eilenberg–MacLane $K(G, 1)$-space BG^δ. Elements of $H_2(X)$ for any CW-complex X can be realized as the image of the fundamental class of a map of a compact orientable surface S_g of genus g into X. This depends on the fact that we are dealing with H_2 and the fact that compact orientable 2-manifolds are classified by the genus. Since BG^δ has trivial higher homotopy groups, we can assume $g > 0$ when dealing with $H_2(G) \cong H_2(BG^\delta)$. S_g is also an Eilenberg–MacLane space when $g > 0$. $\pi_1(S_g)$ can be generated by $2g$ elements x_i, y_i, $1 \leq i \leq g$, with the single defining relation $\prod_i [x_i, y_i] = 1$, where $[x, y] = xyx^{-1}y^{-1}$. As a result, elements of $H_2(G)$ can be described in terms of group homomorphisms of $\pi_1(S_g)$ into G, $g > 0$. This explains the topological significance of the group theoretic results. By means of commutator manipulations in G, it is sometimes possible to show that $H_2(G)$ can be generated by the images of the fundamental class of $\pi_1(S_g)$ with $g = 1$. This process can be called "reduction to genus 1". The success of this procedure often depends on knowing a good presentation of G (or of some candidate for $\bar{G}$). This is the procedure for the description of $K_2(F)$ when F is not too small so that $K_2(F) \cong H_2(SL(3, F))$ is a quotient of $H_2(SL(2, F))$. In particular, let $T = S_1$ so that $\pi_1(T)$ is the free abelian group of rank 2 based on the generators x and y. $H_2(T)$ is generated by the class of the fundamental 2-cycle $[x \mid y] - [y \mid x]$. If $\sigma: T \to G$, then the element of $H_2(G)$ determined by $[\sigma(x) \mid \sigma(y)] - [\sigma(y) \mid \sigma(x)]$ is denoted by $\sigma(x) \star \sigma(y)$. In terms of the fixed choice $\bar{G}$, $\sigma(x) \star \sigma(y)$ is just the commutator $[\overline{\sigma(x)}, \overline{\sigma(y)}]$ in $\bar{G}$ where $\overline{\sigma(x)}$ and $\overline{\sigma(y)}$ are arbitrary lifts of $\sigma(x)$ and $\sigma(y)$ from G to $\bar{G}$. We can avoid the noncanonical choice of $\bar{G}$ by working with G^*. As shown in Milnor [17; §8], $\star$-product is defined on pairs of elements of G that commute. As such, it is skew-symmetric and bimultiplicative. We note that the bimultiplicativity only requires each of the factors on the left side to commute with each of the factors on the right side; it does not require factors on the same side to commute with each other. When $K_2(F) \cong H_2(SL(3, F))$, the symbol $\{u, v\}$ is just $\operatorname{diag}(u, u^{-1}, 1) \bigstar \operatorname{diag}(v, 1, v^{-1})$. We note that these factors do not make sense in $SL(2, F)$ in that we can not simultaneously conjugate them into $SL(2, F)$ by an element of $SL(3, F)$. Upon stabilization and simple computation, $\operatorname{diag}(u, u^{-1}) \bigstar \operatorname{diag}(v, v^{-1})$ is mapped onto $\{u, v\}$. $\{u^{-1}, v^{-1}\} = \{u, v\}^2$.

Suppose that $F^\times = (F^\times)^2$. The discussion at the end of the preceding paragraph already implies that $H_2(F^\times)$ maps surjectively onto $K_2(F)$ where $F^\times$ denotes the diagonal subgroup of $SL(2, F)$. In fact, it is known that $K_2 \cong H_2(SL(2, F))$ is uniquely 2-divisible, see Matsumoto [15], Bass–Tate [3]. In such cases, $K_2(F)$ may be viewed as a quotient of $\Lambda^2_{\mathbb{Z}}(F^\times)$ with $u \wedge v$ mapped onto $\{u, v\}$. This ignores a factor of 2 (unimportant because $K_2(F)$ is uniquely 2-divisible). However, such factors of 2 cannot be dismissed so lightly for a general F or in dealing with subgroups of $SL(n, F)$ in the absence of prior information on Schur multipliers. Every so often, such factors of 2 may be mentioned. These usually arise because they have some connections (not explained) with 2-torsion problems in scissors congruence groups. We only note (without detailed explanations) that extraneous 2-torsions in appropriate places would provide simultaneous counterexamples to the proposed solution of the third problem of Hilbert (appropriately modified) as well as the conjecture of Friedlander–Milnor.

In the case of $F = \mathbb{C}$ (or any algebraically closed field), $K_2(\mathbb{C})$ is a $\mathbb{Q}$-vector space by a theorem of Bass–Tate [3]. Since K_2 is a covariant functor, we have a natural map of $K_2(\mathbb{R})$ into $K_2(\mathbb{C})$. This map is in fact induced by the complexification homomorphism. The image is $K_2(\mathbb{C})^+$ and the kernel is $\mathbb{Z}/2\mathbb{Z} = \pi_i(SO(n))$, $n \geq 3$. In fact, $K_2(\mathbb{R}) \cong K_2(\mathbb{C})^+ \amalg \mathbb{Z}/2\mathbb{Z}$. Similarly, $H_2(SL(2, \mathbb{R})) \cong K_2(\mathbb{C})^+ \amalg \mathbb{Z}$ where $\mathbb{Z} \cong \pi_1(SO(2))$. We note that $\pi_1(SL(n, \mathbb{R})) \cong \pi_1(SO(N))$ because $SO(n)$ is a maximal compact subgroup of $SL(N, \mathbb{R})$ and $SL(n, \mathbb{R})/SO(n)$ is contractible as a topological space. By (0.6), we see that $H_2(G) \cong K_2(\mathbb{C})^+$ where G is the universal covering group of $SL(n, \mathbb{R})$, $n \geq 2$. These examples serve as illustrations on the use of (0.6). In particular, in our determination of the Schur multipliers, we will exclude both $SL(n, \mathbb{R})$, $Sp(2n, \mathbb{R})$ as well as $G(\Phi, \mathbb{C})$.

1. Homology stability in the compact cases

This section deals with our basic example. Let $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. For $n \geq 0$, let $\mathbb{F}^{n+1} = \sum_j e_j\mathbb{F}$ be equipped with the standard *-hermitian symmetric positive definite inner product $\langle\ ,\ \rangle$ given by the rule:

$$\left\langle \sum_j e_j\alpha_j, \sum_j e_j\beta_j \right\rangle = \sum_j \alpha_j^*\beta_j, \text{ where } * \text{ denotes the standard involution.}$$

Let $U(n+1, \mathbb{F})$ denote the associated unitary group acting on the left of $\mathbb{F}^{n+1}$ through matrix multiplication. e_j is the standard unit column vector with entry 1 in the j-th position, $0 \le j \le n$. $SU(n+1, \mathbb{F})$ is the perfect commutator subgroup of $U(n+1, \mathbb{F})$ and $U(n+1, \mathbb{F})$ splits over $SU(n+1, \mathbb{F})$ with quotient groups $O(1)$, $U(1)$ and 1 respectively. When precision is needed, we use the conventional notation of $O(n+1)$, $U(n+1)$ and $Sp(n+1)$ respectively.

Let $S(\mathbb{F}^{n+1})$ be the space of all unit vectors in $\mathbb{F}^{n+1}$ so that $S(\mathbb{F}^{n+1})$ is topologically a sphere of real dimension $(n+1)\cdot|\mathbb{F}:\mathbb{R}|-1$. Let $C_* = C_*(S(\mathbb{F}^{n+1}))$ be the normalized Eilenberg–MacLane chain complex based on the set $S(\mathbb{F}^{n+1})$. C_t is therefore the free abelian group based on the set of all *ordered* $(t+1)$-tuples $(v_0, \ldots, v_t)$ of points of $S(\mathbb{F}^{n+1})$ with the understanding that such a t-cell is 0 if $v_i = v_{i-1}$ holds for some i. We use the usual boundary operator:

$$\partial_C(v_0, \ldots, v_t) = \sum_j (-1)^j \cdot (\ldots, \hat{v}_j, \ldots)$$

When there is no chance of confusion, ∂_C is written as ∂. C_* can be identified as the complex of cellular chains on a CW-complex, see May [16]. For our purposes, it is just as easy to work formally. As is wellknown, C_* is acyclic with augmentation $\mathbb{Z}$. We can filter C_* by subcomplexes $\mathcal{F}^i$, $0 \le i \le n$, where a t-cell $(v_0, \ldots, v_t)$ belongs to $\mathcal{F}^i$ if and only if $\dim_{\mathbb{F}} \sum_{0\le j\le t} v_j\mathbb{F} \le i+1$. Evidently, $C_j \subset \mathcal{F}^i$ holds for $j \le i$ and $\mathcal{F}^i/\mathcal{F}^{i-1}$ begins in degree i as a free abelian group spanned by the independent i-cells $(v_0, \ldots, v_i)$. In the present context, independent means the $v_0, \ldots, v_i$ are $\mathbb{F}$-linearly independent. The filtration $\mathcal{F}^i$, $0 \le i \le n$, will be called the ($\mathbb{F}$-) dimension filtration. Evidently, for any group G acting on C_* through the action of $U(n+1, \mathbb{F})$ on $\mathbb{F}^{n+1}$, the action will be compatible with ∂_C and will preserve the dimension filtration. Moreover, $\mathcal{F}^i$ is a G-direct summand of C_* when C_* is viewed as a G-module. In view of the discussion in Section 0, C_* can be used to compute $H_*(G)$ by means of a transposed spectral sequence with $''E^1_{i,j} \cong H_i(G, C_j)$. The main result of the present section is:

THEOREM 1.1. *Let $n \ge 0$. The inclusion map from $U(n, \mathbb{F})$ to $U(n+1, \mathbb{F})$ induces a surjection from $H_i(U(n, \mathbb{F}))$ to $H_i(U(n+1, \mathbb{F}))$ for $i \le n$. This map is a bijection when $i < n$.*

By the transposed spectral sequence described above for $G = U(n+1, \mathbb{F})$, Theorem 1.1 follows formally from the next assertion.

THEOREM 1.2. *In the transposed spectral sequence* $''E^1$, *the i-th column* $''E^1_{i,*}$ *is* $(n-i)$*-acyclic with augmentation* $H_i(U(n, \mathbb{F}))$, $0 \le i \le n$.

We begin our proofs with the observation:

$$''E^1_{i,0} \cong H_i(U(n, \mathbb{F})), \qquad i \ge 0. \tag{1.3}$$

This is just Shapiro's Lemma because $U(n+1, \mathbb{F})$ acts transitively on $S(\mathbb{F}^{n+1})$ and the stability subgroup of e_0 is $U(n, \mathbb{F})$. More generally, if $(v_0, \ldots, v_t)$ spans an $\mathbb{F}$-subspace of dimension $j+1 \le t+1$, then the stability subgroup of the t-cell $(v_0, \ldots, v_t)$ is conjugate to $U(n-j, \mathbb{F})$ in $U(n+1, \mathbb{F})$. Since conjugation is an abstract group G induces the identity automorphism on its homology groups (taken with trivial coefficients), all these stability embeddings are compatible and we can always select a particular embedding. This is the meaning of the inclusion map in Theorem 1.1. We assert next that:

$$''d^1_{i,1} = 0 \quad \text{so that} \quad ''E^2_{i,0} = ''E^1_{i,0}, \qquad i \ge 0. \tag{1.4}$$

By Shapiro's Lemma, $''E^1_{i,1} \cong \coprod_{(a,b)} H_i(G_{(a,b)}) \oplus (a, b)$ where (a, b) ranges over the distinct $U(n+1, \mathbb{F})$-orbits of 1-cells and $G_{(a,b)}$ denotes the stability subgroup in $U(n+1, \mathbb{F})$ of the representative 1-cell (a, b). If c is an i-cycle of $G_{(a,b)}$, then $(-1)^{i}\,''d^1_{i,1} = \partial_C$ sends $c \otimes (a, b)$ onto $c \otimes (b) - c \otimes (a)$. By Shapiro's Lemma (in reverse), each of these two terms represents an element of $H_i(U(n+1, \mathbb{F}), C_0)$. By Witt's Theorem, we can find σ in $U(n+1, \mathbb{F})$ so that σ centralizes $G_{(a,b)}$ and $\sigma(a) = b$. Since conjugation by σ on $U(n+1, \mathbb{F})$ together with application of σ on C_0 induce the identity automorphism on $H_i(U(n+1, \mathbb{F}), C_0)$, we may conclude that $c \otimes (a)$ and $c \otimes (b)$ are homologous so that their difference is 0 in $H_i(U(n+1, \mathbb{F}), C_0) = ''E^1_{i,0}$. This gives us (1.4) and shows that we have the correct augmentation as described in Theorem 1.2.

We next observe that the dimension filtration $\mathcal{F}^*$ induces a filtration on the double complex $C_*(G) \otimes_G C_*$, $G = U(n+1, \mathbb{F})$. This in turn induces a filtration $\mathcal{F}^*_i$ on the i-th column $''E^1_{i,*}$. We begin the attack on Theorem 1.2 by taking $i = 0$ and observing that we have $''E^1_{0,*} = H_0(G, C_*) = C_* \otimes_G \mathbb{Z}$ and $''d^1_{0,*} = \partial_C$. Since $\mathcal{F}^n_0 = ''E^1_{0,*}$, we can use Witt's Theorem to identify $\mathcal{F}^n_0$ with a subcomplex of $\mathcal{F}^{n+1}_0$. To be precise, we take the obvious embedding of $\mathbb{F}^n$ into $\mathbb{F}^{n+1}$ and pass to the direct limit $\mathbb{F}^\infty$. This induces an embedding of $C_*(S(\mathbb{F}^n))$ into $C_*(S(\mathbb{F}^{n+1}))$. The power of Witt's Theorem is such that the other embeddings are "conjugate" to the one named. As long as G is transitive on the set of all possible embeddings, there is no problem with our identification process. We assert that:

$$\mathcal{F}_0^n = {}''E^1_{0,*} \text{ is } (n-1)\text{-acyclic}, \qquad n \geq 0. \tag{1.5}$$

Remark. As it will become clear, (1.5) is valid as long as $G \supset SU(n+1, \mathbb{F})$.

The proof of (1.5) is based on the Orthogonal Join Construction. We prove a stronger result. Let c be any t-cycle of $C_* \otimes_G \mathbb{Z}$ lying in $\mathcal{F}^{n-1}$. Any t-cell appearing in c can be moved by G to the hyperplane $\sum_{i>0} e_i\mathbb{F}$. We may therefore assume that c is supported on $\sum_{i>0} e_i\mathbb{F}$. This allows us to form the orthogonal join $e_0 \# c$, namely each t-cell $(v_0, \ldots, v_t)$ in c is replaced by $(e_0, v_0, \ldots, v_t)$. $\partial_C(e_0 \# c) = c - e_0 \# \partial_C c$. Since c is a t-cycle in $C_* \otimes_G \mathbb{Z}$, the cells appearing in $\partial_C c$ must cancel out in pairs under the action of G. Since these cells actually lie on the chosen hyperplane orthogonal to e_0, Witt's Theorem tells us that the cancellation phenomenon can be assumed to be realized by elements of G that fix e_0. In other words, $e_0 \# \partial_C c$ represents 0 in $C_* \otimes_G \mathbb{Z}$. When $t \leq n-1$, any t-cycle of $C_* \otimes_G \mathbb{Z}$ automatically lies in $\mathcal{F}^{n-1}$. We therefore have (1.5). In fact, $\partial_C c$ automatically lies in $\mathcal{F}^{n-2}$ so that we even have enough room to adjust for the determinant to take care of the remark.

(1.5) can be improved to the following assertion:

$$\mathcal{F}_0^n = {}''E^1_{0,*} \text{ is } n\text{-acyclic}, \qquad n \geq 0. \tag{1.6}$$

Namely, the preceding argument actually gives us short exact sequences:

$$0 \to H_t(\mathcal{F}_0^n) \to H_t(\mathcal{F}_0^n/\mathcal{F}_0^{n-1}) \to H_{t-1}(\mathcal{F}_0^{n-1}) \to 0, \qquad t > 1. \tag{1.7}$$

It is enough to show that $H_n(\mathcal{F}_0^n/\mathcal{F}_0^{n-1}) = 0$. $\mathcal{F}_0^n/\mathcal{F}_0^{n-1}$ begins in degree n and is a free abelian group based on the set of all G-equivalence classes of independent n-cells $(v_0, \ldots, v_n)$ in $\mathbb{F}^{n+1}$. We shall show that each such n-cell represents a boundary in $\mathcal{F}_0^n/\mathcal{F}_0^{n-1}$. Our argument is based on the Circumcenter Construction. (There is a similar, less inductive, but "simpler" Inscribed Center Construction.)

Since $v_0, \ldots, v_n$ form an $\mathbb{F}$-basis of $\mathbb{F}^{n+1}$, the orthogonal complement of $\sum_{j>0} (v_j - v_0)\mathbb{F}$ is 1-dimensional over $\mathbb{F}$ and can be taken to be $z\mathbb{F}$ with $\langle z, z\rangle = 1$. It follows that $\langle z, v_j\rangle$ is a constant independent of j, $0 \leq j \leq n$. Since our inner product is nondegenerate, this constant is not 0. If we multiply z by a suitable element of norm 1 in $\mathbb{F}$, we may assume that $\langle z, v_j\rangle = r$ is a positive real number. From Schwarz's inequality, $r < 1$. This uniquely determined z is called the circumcenter of the independent n-cell $(v_0, \ldots, v_n)$. Modulo $\partial_C(z, v_0, \ldots, v_n)$,

$(v_0, \ldots, v_n)$ becomes an integral linear combination of cells of the form:

$$(z, w_1, \ldots, w_n),\ \langle z, w_j\rangle \text{ is a constant independent of } j, \qquad 1 \le j \le n. \tag{1.8}$$

If we let z_1 be the circumcenter of the independent $(n-1)$-cell $(w_1, \ldots, w_n)$ in $\sum_j w_j\mathbb{F}$ and look at the boundary of $(z, z_1, w_1, \ldots, w_n)$, then the independent n-cell in (1.8) becomes an integral linear combination of independent n-cells of the form $(z_0, z_1, y_2, \ldots, y_n)$ where $\langle z_i, y_j\rangle$ depends on i but not on j, $2 \le j \le n$. We note that a number of dependent n-cells have been absorbed by $\mathcal{F}_0^{n-1}$ and the given n-cell has been modified by a boundary. This process can clearly be continued until we reach the stage of an independent n-cell of the form:

$$(z_0, \ldots, z_{n-2}, u_{n-1}, u_n), \qquad \langle z_i, u_{n-1}\rangle = \langle z_i, u_n\rangle, \qquad 0 \le i \le n-2. \tag{1.9}$$

If z_{n-1} denotes the "midpoint" or the circumcenter of (u_{n-1}, u_n), then Witt's Theorem and inner product computations imply that $(z_0, \ldots, z_{n-1}, u_{n-1})$ and $(z_0, \ldots, z_{n-1}, u_n)$ are G-congruent. The independent n-cell in (1.9) is just $(-1)^{n-1}\,\partial_C(z_0, \ldots, z_{n-1}, u_{n-1}, u_n)$ in $\mathcal{F}_0^n/\mathcal{F}_0^{n-1}$. This proves (1.6) via (1.7).

We now tackle Theorems 1.1 and 1.2 in tandem by complete induction. The induction hypothesis is that we have proved Theorem 1.1 for all integers less than n. To complete the induction, we only have to verify Theorem 1.2. In view of (1.4) and (1.6), the index i in Theorem 1.2 may be restricted to the range $1 \le i \le n-1$. For this range, $''E^1_{i,*}$ can be replaced by its subcomplex $\mathcal{F}_i^{n-i}$ because $\mathcal{F}_i^{n-i} \supset {}''E^1_{i,j}$ for $j \le n-i$. By Shapiro's Lemma, $\mathcal{F}_i^{n-i}$ is a direct sum of abelian groups of the form $H_i(G_c, \mathbb{Z}c) = H_i(G_c) \otimes \mathbb{Z}c$ where c ranges over the distinct G-orbits of cells $(v_0, \ldots, v_t)$ lying in $\mathcal{F}^{n-i}$, and G_c is the stability subgroup of $(v_0, \ldots, v_t)$. By Witt's Theorem, $(v_0, \ldots, v_t)$ can be assumed to lie on a fixed $\mathbb{F}$-subspace $\mathbb{F}^{n+1-i}$. If $t \le n-i$, it does not lie in $\mathcal{F}^{n-i-1}$ if and only if $t = n-i$ and $(v_0, \ldots, v_t)$ is independent. In this special case, the coefficient group is the unstable group $B(i) = H_i(U(i, \mathbb{F}))$. In all other cases, the coefficient groups are the stable group $C(i) = H_i(U(i+1, \mathbb{F}))$. At this point, we have invoked the induction hypothesis and the restrictions on the range of i. We let $A(i)$ denote the kernel of the stabilization map carrying $B(i)$ onto $C(i)$. An inspection shows that we have the following exact sequence of chain complexes:

$$0 \to A(i) \otimes \mathcal{F}_0^{n-i-1} \to B(i) \otimes \mathcal{F}_0^{n-i} \to \mathcal{F}_i^{n-i} \to 0, \qquad 1 \le i \le n-1. \tag{1.10}$$

The usual long exact homology exact sequences follows from (1.10). We note that $\mathcal{F}_0^j$ is a $\mathbb{Z}$-free chain complex. By (1.6), $\mathcal{F}_0^j$ is j-acyclic. By the universal coefficient theorem, $\mathcal{F}_i^{n-i}$ is $(n-i)$-acyclic. As indicated before, this gives Theorem 1.2 and completes the inductive proof of Theorem 1.1.

Remark 1.11. The independent $(n+1)$-cells of the form $(z_0, \ldots, z_n, u_n)$ described after (1.9) are the "orthoschemes" of L. Schäfli when $\mathbb{F} = \mathbb{R}$. The circumcenter argument is an adaptation of a classical argument used to show that the scissors congruence groups are 2-divisible, see Sah [24]. As shown in Dupont [6], if $\mathbb{F} = \mathbb{R}$ and if twisted coefficients are used, then $H_i(\mathcal{F}_0^i/\mathcal{F}_0^{i-1}, \mathbb{Z}^t)$ can be identified with a quotient of the scissors congruence group in i-dimensional spherical space. This group has a rather complicated structure. With the use of trivial coefficient groups, we have effectively killed off this scissors congruence group. To retain this group, we can use SO in place of O.

Remark 1.12. In the special cases considered here, Theorem 1.1 extends the work of Vogtmann [36]. Our more precise range of stability is a consequence of the special nature of our division algebras. Some of our surjectivity statements can be improved to bijectivity statements. Milnor raised the question: Is it true that the stability results are valid with the $\mathbb{F}$-dimension replaced by the appropriate topological dimension?

Remark 1.13. There is no problem proving the corresponding stability results for trivial coefficient groups other than $\mathbb{Z}$. The point to note is that our stability result is based on acyclicity results. The universal coefficient theorem may be applied in these cases. However, the corresponding results for nontrivial coefficient groups would require more care. The problem rests with the inductive nature of our arguments involving the use of Shapiro's Lemma. In the study of the scissors congruence problem, nontrivial coefficient groups do appear, see Dupont [6]. In special cases, the "center kills" lemma gets around the difficulty.

It would be desirable to have some sort of general theory of the Eilenberg–MacLane homology (or cohomology) of algebraic groups with algebraic coefficient modules. Such a theory does exist over absolutely algebraic fields of positive characteristics. The main point is the fact that finite fields are Galois extensions of the prime fields with Galois groups generated by the Frobenius automorphism. Conceivably, this theory can be extended to cyclic extensions of the rational numbers. In any case, one would like to have precise results rather than stable results. In this last aspect, beyond the knowledge on the level of Schur multipliers, our knowledge is very fragmentary.

2. Schur multipliers of compact classical groups

The principal result in the present section is the following theorem:

THEOREM 2.1. *Under the natural inclusion maps, we have bijections:*
(a) $H_2(SPin(3)) \to H_2(SPin(n)) \to H_2(SPin(n+1))$, $n \geq 5$;

(b) $H_2(SU(2)) \to H_2(SU(n)) \to H_2(SU(n+1))$, $n \geq 3$; and
(c) $H_2(Sp(1)) \to H_2(Sp(n)) \to H_2(Sp(n+1))$, $n \geq 2$.

All these maps are compatible with the identification of our compact groups as the maximal compact subgroup of the corresponding complex Lie groups. On the level of Schur multipliers, all of our groups are then mapped onto $K_2(\mathbb{C})^+$. (In fact, these maps are bijective, see Dupont–Parry–Sah [7].)

Remark 2.2. There is a functorial procedure to complexify a real Lie group, see Bourbaki [4; III.6.10]. Since the functorial approach do not always give injective maps, care must be exercised. Our approach is more naive and concrete. To avoid possible confusion, we will usually use complexification to indicate that we have mapped our Lie groups into some similarly defined complex Lie group. Theorem 2.1 is also valid for compact groups of type G_2 and F_4 in the sense that their Schur multipliers are isomorphic to $H_2(SU(2))$. The case of G_2 is due to John Hurley in his 1983 Ph.D. dissertation at SUNY, Stony Brook. The case of F_4 is due to Johan Dupont (1984, unpublished). Both are based on the geometries of suitable compact symmetric spaces associated to the corresponding groups.

We begin the proof of Theorem 2.1 by recalling the split exact sequence:

$$1 \to SU(n, \mathbb{F}) \to U(n, \mathbb{F}) \rightleftarrows U(n, \mathbb{F})/SU(n, \mathbb{F}) \to 1, \qquad n \geq 1 \tag{2.3}$$

The quotient map is given by the determinant (trivial when $\mathbb{F} = \mathbb{H}$). $SU(n, \mathbb{F})$ is connected and is simply-connected when $\mathbb{F} = \mathbb{C}$ or $\mathbb{H}$. The universal covering group of $SO(n)$ is $SPin(n)$, $n \geq 3$, with kernel $\pi_1(SO(n))$ of order 2. We have the following "exotic isomorphisms", see Helgason [12; p. 519]:

$$\begin{aligned} &Sp(1) \cong SU(2) \cong Spin(3),\ Sp(2) \cong SPin(5), \\ &SPin(4) \cong SPin(3) \times SPin(3),\ SU(4) \cong SPin(6). \end{aligned} \tag{2.4}$$

In view of (2.4) and Theorem 2.1, we have the immediate corollary:

COROLLARY 2.5. *If G is a nontrivial, connected, simply-connected, compact Lie group of classical type, then $H_2(G)$ is naturally isomorphic to $H_2(SU(2))$ under an inclusion map of $SU(2)$ into G. (This corresponds to a* suitable *choice of roots in the Dynkin diagram.)*

Each of the three cases of Theorem 2.1 has some special feature of its own. We therefore treat them separately.

CASE 1. $\mathbb{F} = \mathbb{R}$. The proof of Theorem 1.1 depended strongly on the

n-acyclicity of $C_* \otimes_G \mathbb{Z}$, $G = O(n+1)$. If we use the pair $SO(n)$ and $SO(n+1)$, then (1.5) is still valid. Namely, we have just enough room to adjust for the determinant in the orthogonal join construction. This observation easily gives:

$$H_2(SO(n)) \cong H_2(SO(n+1)), \qquad n \geq 5. \tag{2.5}$$

In general, we may examine the Hochschild–Serre spectral sequence associated to the universal covering sequence. In the present case, we obtain the exact sequence:

$$0 \to H_2(SPin(n)) \to H_2(SO(n)) \to \pi_1(SO(n)) \to 0,\ n \geq 3. \tag{2.6}$$

As shown by Milnor [18], (2.6) always splits so that we have:

$$H_2(SPin(n)) \amalg \pi_1(SO(n)) \cong H_2(SO(n)),\ n \geq 3. \tag{2.7}$$

By (2.5) and (2.7), we have (a) of Theorem 2.1 for $n \geq 5$. We next observe that $O(2t+1) \cong SO(2t+1) \times I_{2t+1}$, $t \geq 1$. We obtain from Künneth's Theorem:

$$H_2(O(2t+1)) \cong H_2(SO(2t+1)), \qquad t \geq 1. \tag{2.8}$$

By Theorem 1.1, we obtain: $H_2(SO(3)) \cong H_2(O(3)) \cong H_2(O(5)) \cong H_2(SO(5))$. (a) of Theorem 2.1 now follows from putting together the preceding isomorphisms with (2.7) and the compatibility of the universal covering sequences.

Remark 2.9. Assertion (a) with $n \geq 5$ has also been obtained by Roger Alperin (unpublished) along the line of arguments used in Alperin [1]. If we carry out our argument for $n = 2$, then we obtain easily the surjectivity of the following map:

$$H_2(U(1)) \to H_2(SU(2)),\ H_2(U(1)) \cong \Lambda^2_{\mathbb{Z}}(\mathbb{R}/\mathbb{Z}) \text{ is a } \mathbb{Q}\text{-vector space.} \tag{2.10}$$

Here $U(1)$, $SU(2)$ doubly cover $SO(2)$ and $SO(3)$ respectively. (2.10) was first proved by Mather (unpublished but see Alperin–Dennis [2]) by looking at the geometry of $SU(2)$. Our argument uses the geometry of the 2-sphere.

Remark 2.11. The skipping of $n = 4$ in (a) of Theorem 2.1 is necessary. To see this, we identify $\mathbb{H}$ and $\mathbb{R}^4$ so that $\langle u, v\rangle = \mathrm{tr}_{\mathbb{H}/\mathbb{R}}(u^* \cdot v)/2$. $Sp(1)$ is the group of unit quaternions. $Sp(1) \times Sp(1)$ acts on $\mathbb{H}$ through the rule:

$$(u, v)\{q\} = u \cdot q \cdot v^{-1} = u \cdot q \cdot v^*, \qquad q \in \mathbb{H}, \qquad u, v \in Sp(1). \tag{2.12}$$

This defines a homomorphism of $Sp(1) \times Sp(1)$ into $SO(4)$ with kernel $\langle(-1, -1)\rangle$ of order 2. Computation of dimensions together with connectivity show that this map is surjective. The involution * on $\mathbb{H}$ defines a complement of $SO(4)$ in $O(4)$. This involution clearly exchanges the two $Sp(1)$ factors. Evidently, we have a split exact sequence:

$$1 \to SPin(4) \to SPin(4) \cdot \langle * \rangle \rightleftarrows \langle * \rangle \to 1. \tag{2.13}$$

$SPin(4) \cdot \langle * \rangle$ doubly covers $O(4)$ and its subgroup $SPin(3) \times \langle * \rangle$ doubly covers the subgroup $O(3)$ pf $O(4)$. Here $SPin(3)$ denotes the diagonal of $Sp(1) \times Sp(1)$. As indicated in the beginning of Case 1, we can replace $O(3)$ and $O(4)$ by the pair just described. Theorem 1.1 then gives:

$$H_2(SPin(3)) \cong H_0(\langle * \rangle, H_2(SPin(4))). \tag{2.14}$$

The right hand side of (2.14) is obtained through the Hochschild–Serre spectral sequence associated to (2.13). It can also be obtained by using Künneth's Theorem. Since $H_1(Sp(1)) = 0$, it is immediate that the right hand side of (2.14) is also isomorphic to $H_2(Sp(1))$ through either one of the two factors embedded in $SPin(4)$. Since $SPin(3)$ is embedded in $SPin(4)$ through the diagonal, (2.14) contains the assertion:

$$H_2(SPin(3)) \cong H_2(SU(2)) \cong H_2(Sp(1)) \text{ is uniquely 2-divisible.} \tag{2.15}$$

Assertion (2.15) is implicitly contained in Dupont [6]. It is now clear that (a) of Theorem 2.1 must skip over $n = 4$. If we use the Hochschild–Serre spectral sequence associated to (2.3), then (2.5) and (2.8) imply that:

$$O(n) \text{ acts trivially on } H_2(SO(n)), \qquad n \geq 5 \text{ or } n = 3.$$
$$H_2(O(n)) \cong H_2(SO(n)) \cong H_2(O(4)), \qquad n \geq 5 \text{ or } n = 3. \tag{2.16}$$

CASE 2. $\mathbb{F} = \mathbb{C}$. The quotient group in (2.3) can be identified with $U(1)$. Since $\mathbb{C}$ is divisible, each element of $U(1)$ acts on $SU(n)$ through an inner automorphism of $SU(n)$. It follows that:

$$H_0(U(1), H_i(SU(n))) \cong H_i(SU(n)), \qquad i \geq 0. \tag{2.17}$$

By the Hochschild–Serre spectral sequence associated to (2.3), we have:

$$H_2(U(n)) \cong H_2(SU(n)) \amalg H_2(U(1)), \qquad n \geq 1. \tag{2.18}$$

If we view $\mathbb{C}^n$ as $\mathbb{R}^{2n}$, then we have the commutative diagram below:

$$\begin{array}{ccc} SU(n) & \to & SU(n+1) \\ \downarrow & & \downarrow \\ O(2n) & \to & O(2n+2) \end{array}, \quad n \geq 2. \tag{2.19}$$

The map in (2.19) from $SU(n)$ to $O(2n)$ factors through $SO(2n)$. In terms of the description in Remark 2.11 and the assertion (2.15), $H_2(SO(4))$ is the direct sum $H_2(SU(2)) \amalg H_2(SO(3))$ where $SU(2)$ is either one of the two factor embeddings on the universal covering level; alternately, $H_2(SO(4))$ is the direct sum of the $\pm$ eigenspaces under the action of $O(4)/SO(4) \cong \langle \pm 1 \rangle$. Here $H_2(SO(3))$ is the $+$ eigenspace while $H_2(SU(2))$ is isomorphic to the $-$ eigenspace under the projection map and is mapped injectively into the $+$ eigenspace also under the projection map. For these purposes, (2.15) is essential. In particular, $H_2(O(4)) \cong H_2(SU(2)) \amalg \mathbb{Z}/2\mathbb{Z}$. By (2.5) and (2.16) we can go the other way in (2.19) to conclude that:

$$H_2(SU(n)) \to H_2(SU(n+1)) \text{ is injective}, \quad n \geq 2. \tag{2.20}$$

(b) of Theorem 2.1 now follows from (2.18), (2.20) and Theorem 1.1.

Remark. (b) of Theorem 2.1 improves the result of Alperin [1] and answers a question of Milnor [18].

CASE 3. $\mathbb{F} = \mathbb{H}$. We have the following commutative diagram of maps:

$$\begin{array}{ccc} Sp(n) & \to & Sp(n+1) \\ \downarrow & & \downarrow \\ SU(2n) & \to & SU(2n+2) \end{array}, \quad n \geq 1 \tag{2.21}$$

Consider $n = 1$. The column on the left is just $Sp(1) \cong SU(2)$ as described in (2.4). By (b), $H_2(Sp(1)) \cong H_2(SU(n))$, $n \geq 2$, and $H_2(Sp(1)) \to H_2(Sp(2))$ is injective. By (2.4), $Sp(2) \cong SPin(5)$. This arises from looking at the induced action on $\Lambda^2_{\mathbb{C}}(\mathbb{C}^4)) \cong \mathbb{C}^6$ and noticing that $Sp(4, \mathbb{C})$ is mapped onto $SO(5, \mathbb{C})$. Even though $Sp(1) \cong SPin(3)$, the map of Sp(1) into Sp(2) does not correspond to the universal covering map of $SO(3)$ into $SO(5)$. Instead, it corresponds to one of the two factor embeddings of $SU(2)$ into $SO(4)$. Since $O(4)$ can be embedded in $SO(5)$, the discussions in Case 2 and Remark 2.11 imply that $H_2(Sp(1)) \to H_2(Sp(2))$ is bijective. In particular, $H_2(Sp(2)) \cong H_2(SU(4))$ holds in (2.21). This gets us started in (2.21) for $n \geq 2$ and we can repeat the argument as in Case 2 to get (c) of Theorem 2.1.

The rest of the assertions are straightforward. In particular, the assertion about the image being $K_2(\mathbb{C})^+$ was proved in Sah–Wagoner [26] □

COROLLARY 2.22. *Let G be any compact Lie group without a simple component of exceptional types E, F or G. Then,*

$H_*(BG^\delta, \mathbb{F}_p) \to H_*(BG, \mathbb{F}_p)$ *is bijective for* $* \leq 2$.

Namely, our arguments are valid with $\mathbb{F}_p$ in place of $\mathbb{Z}$ when G is as in Theorem 2.1. The general case follows as in Milnor [18]. In fact, if we quote the unpublished results of Hurley and Dupont, components may include types F and G.

Remark 2.23. In Dupont–Parry–Sah [7], it will be shown that $H_2(SU(2))$ is isomorphic to $K_2(\mathbb{C})^+$ under the inclusion of $SU(2)$ into $SL(2, \mathbb{C})$. If $\mathbb{R}$ is replaced by the real closure $\mathbb{R} \cap \bar{\mathbb{Q}}$ of $\mathbb{Q}$, then we can conclude from the vanishing of $K_2(\bar{\mathbb{Q}}) = 0$ that $H_2(SU(2, \bar{\mathbb{Q}})) = 0$. This also uses the deep results of Garland [10], Quillen [22], Bass–Tate [3]. See also Harris [11] for related results.

3. Homology stability in some noncompact cases

We are interested in some noncompact analogues of Theorem 1.1. Let $\mathbb{F}^{p,q}$ denote the right $\mathbb{F}$-vector space of all column vectors with $p+q$ entries from $\mathbb{F}$ together with the $*$-hermitian symmetric inner product $\langle\ ,\ \rangle_{p,q}$ of signature (p, q) defined by the rule:

$$\langle u, v\rangle_{p,q} = {}^t u^* \cdot I_{p,q} \cdot v, \qquad u, v \in \mathbb{F}^{p,q}.$$

Here $I_{p,q}$ is the diagonal matrix with first p eigenvalues -1 and last q eigenvalues $+1$ and $*$ denotes the standard involution on $\mathbb{F}$. $U(p, q, \mathbb{F})$ is the subgroup of all $\mathbb{F}$-linear automorphisms of $\mathbb{F}^{p,q}$ preserving $\langle\ ,\ \rangle_{p,q}$. This group is compact if and only if $\min(p, q) = 0$. The commutator subgroup of $U(p, q, \mathbb{F})$ is denoted by $SU(p, q, \mathbb{F})$, Since $-\langle\ ,\ \rangle_{p,q}$ is equivalent to $\langle\ ,\ \rangle_{q,p}$, $U(p, q, \mathbb{F})$ and $SU(p, q, \mathbb{F})$ are both symmetric in (p, q) in the sense of isomorphism. We have the split exact sequence:

$$1 \to SU(p, q, \mathbb{F}) \to U(p, q, \mathbb{F}) \rightleftarrows U(p, q, \mathbb{F})/SU(p, q, \mathbb{F}) \to 1. \tag{3.1}$$

We concentrate our attention on the noncompact cases where $1 \le \min(p, q)$. However, the compact subgroups will play a role. The commutator quotient groups are respectively $O(1) \times O(1)$, $U(1)$ and 1. The quotient maps are given by spinor norm ($\mathbb{F} = \mathbb{R}$) and determinant ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$). The commutator subgroup $SU(p, q, \mathbb{F})$ is always connected and is pefect except when $p = q = 1$ and $\mathbb{F} = \mathbb{R}$. $SU(1, 1, \mathbb{R}) \cong \mathbb{R}$. When $\mathbb{F} = \mathbb{R}$, $O^1(p, q)$ denotes the kernel of the spinor norm map so that $SO^1(p, q) \cong SU(p, q, \mathbb{R})$. In general, $O^1(p, q)$ is not symmetric in (p, q).

For $\epsilon = \pm$, let $S^\epsilon(p, q, \mathbb{F}) = S^\epsilon(\mathbb{F}^{p,q})$ denote the set of all $u \in \mathbb{F}^{p,q}$ with $\langle u, u\rangle_{p,q} = \epsilon 1$. $U(p, q, \mathbb{F})$ is transitive on $S^\epsilon(p, q, \mathbb{F})$ with isotropy subgroups $U(p-1, q, \mathbb{F})$ and $U(p, q-1, \mathbb{F})$ corresponding to $\epsilon = -$ and $+$ respectively. Except when $1 = \min(p, q)$ and $\mathbb{F} = \mathbb{R}$, $S^\epsilon(p, q, \mathbb{F})$ is connected. $S^-(1, n, \mathbb{R})$ has two connected components and we use $S^-_+(1, n, \mathbb{R})$ to denote the "forward" cone. $S^-_+(1, n, \mathbb{R})$ is then a model for the real hyperbolic n-space.

In analogy with Section 1, let $C^\epsilon_* = C^\epsilon_*(p, q, \mathbb{F})$ denote the normalized chain complex so that C^ϵ_t is the free abelian group based on the set of all ordered $(t+1)$-tuples $(u_0, \ldots, u_t)$ of elements of $S^\epsilon(p, q, \mathbb{F})$ satisfying the normalization condition as well as the following condition:

For any subset of $\{u_0, \ldots, u_t\}$, the $\mathbb{F}$-linear subspace spanned by this subset is nondegenerate with respect to $\langle\ ,\ \rangle_{p,q}$. (3.2)

Similarly, let $C^{-,+}_* = C^{-,+}_*(p, q, \mathbb{F})$ denote the normalized chain complex so that $C^{-,+}_t$ is the free abelian group based on the set of all ordered $(r+s+2)$-tuples $(v_0, \ldots, v_r, w_0, \ldots, w_s)$ with $v_i \in S^-(p, q, \mathbb{F})$, $w_j \in S^+(p, q, \mathbb{F})$ and such that both the normalization condition and the condition (3.2) hold for the faces of $(v_0, \ldots, v_r, w_0, \ldots, w_s)$, $-1 \le r, s$, and $0 \le r+s+1 = t$. When $p = 1$ and $\mathbb{F} = \mathbb{R}$, we define ${}_+C^-_*$ and ${}_+C^{-,+}_*$ by using $S^-_+(1, q, \mathbb{R})$ in place of $S^-(1, q, \mathbb{R})$. We note that ${}_+C^-_*$ and ${}_+C^{-,+}_*$ admit action from $O^1(1, q)$ only. These are used in Appendix A only. (3.2) allows us to speak of the signature (a, b) of a cell. More generally, a cell is said to be supported on a $\mathbb{F}^{a,b}$ when the $\mathbb{F}$-subspace spanned by the vertices of the cell is isometric to a subspace of $\mathbb{F}^{a,b}$. $C^{-,+}_*$ is a restricted form of the ordered join of C^-_* and C^+_*. We note that,

(3.2) is automatically satisfied by all subsets of $S^-(1, q, \mathbb{F})$.
$C^{-,+}_*(1, q, \mathbb{F})$ and ${}_+C^{-,+}_*(1, q, \mathbb{R})$ are respectively the ordered join of $C^-_*(1, q, \mathbb{F})$, ${}_+C^-_*(1, q, \mathbb{R})$ and the corresponding $C^+_*(1, q, \mathbb{F})$. (3.3)

We assert that:

C^-_*, $+C^-_*$, C^+_*, $C^{-,+}_*$, and ${}_+C^{-,+}_*$ are all acyclic with augmentation $\mathbb{Z}$. (3.4)

The proof of (3.4) is the same for all five complexes. Let $u_0, \ldots, u_t$ span a nondegenerate subspace U of $\mathbb{F}^{p,q}$. Let $z \in S^{\epsilon}(p, q, \mathbb{F})$ or $S^{-}_{+}(1, q, \mathbb{R})$ and let z' denote the components of z in $U^{\perp}$. If $z\mathbb{F} + U$ is degenerate, then $z' \neq 0$ and $\langle z', z' \rangle_{p,q} = 0$. Degeneracy therefore forces z to lie in a proper Zariski closed (over $\mathbb{R}$) subset of $S^{\epsilon}(p, q, \mathbb{F})$ or $S^{-}_{+}(1, q, \mathbb{R})$. A t-cycle in each of our five chain complexes determines a finite number of nondegenerate subspaces of $\mathbb{F}^{p,q}$. The preceding discussion therefore assures us of the existence of a suitable z so that one of the joins $z * c$ or $c * z$ will make sense in the appropriate complex. Up to a sign, c is the boundary of one of $z * c$ or $c * z$ (whichever makes sense). (3.4) then follows.

With (3.4) at hand, we proceed as in Section 1 and prove a number of acyclicity results under the assumption $1 \leq \min(p, q)$.

Let $G = U(p, q, \mathbb{F})$ or $SU(p, q, \mathbb{C})$, $1 \leq \min(p, q)$. Then $C^{-}_{*} \otimes_G \mathbb{Z}$ and $C^{+}_{*} \otimes_G \mathbb{Z}$ are respectively $(p-1)$- and $(q-1)$-acyclic with augmentation $\mathbb{Z}$. (3.5)

By symmetry, we only need to consider $C^{+}_{*} \otimes_G \mathbb{Z}$. The assertion on augmentation follows from (3.4). Let $(w_0, \ldots, w_t)$ be a t-cell appearing in a t-cycle of $C^{+}_{*} \otimes_G \mathbb{Z}$, $t \leq q-1$. Except when $t = q-1$ and $(w_0, \ldots, w_{q-1})$ has signature $(0, q)$, $(w_0, \ldots, w_t)$ is supported on the orthogonal complement of a suitable $z \in S^{+}(p, q, \mathbb{F})$. In the exceptional case, we note that $U(0, q, \mathbb{F})$ can be embedded in G because $1 \leq \min(p, q)$ (this is needed when $G = SU(p, q, \mathbb{C})$). The circumcenter construction of Section 1 allows us to modify c by boundaries and replace $(w_0, \ldots, w_{q-1})$ of signature $(0, q)$ by an integral linear combination of cells supported on an $\mathbb{F}$-subspace of signature $(0, q-1)$. By means of the transitivity of G on $S^{+}(p, q, \mathbb{F})$, c may be assumed to be supported on the orthogonal complement of a single $z \in S^{+}(p, q, \mathbb{F})$. As in Section 1, we can use Witt's Theorem to show that c is the boundary of $c \# z$ up to a sign. (3.5) follows.

Remark 3.6. (3.5) depends on the use of the circumcenter construction. It would save some work if the acyclicity results in (3.5) can be strengthened to something like $(p+q-1)$-acyclicity as in the compact cases. Unfortunately, this is false for $C^{-}_{*}(1, 2, \mathbb{C}) \otimes_G \mathbb{Z}$, $G = U(1, 2, \mathbb{C})$ or $SU(1, 2, \mathbb{C})$. Namely, 2-acyclicity in these cases would imply the surjectivity of the map $H_2(SU(2)) \rightarrow H_2(SU(1, 2))$. This latter is false since $H_2(SU(2))$ is divisible while $H_2(SU(1, 2))$ has a direct summand $\mathbb{Z} \cong \pi_1(SU(1, 2))$ so that it is not divisible.

Since $O(0, q)$, $SO(p, 0)$ can b embedded in $O^1(p, q)$, we can imitate (3.5) and show:

Let $G = O^1(p, q)$, $1 \le \min(p, q)$. Then $C_*^- \otimes_G \mathbb{Z}$ and $C_*^+ \otimes_G \mathbb{Z}$ are respectively $(p-2)$- and $(q-1)$-acyclic with augmentation $\mathbb{Z}$. (3.7)

We now extend Theorem 1.1.

THEOREM 3.8. *Let $G(p, q) = U(p, q, \mathbb{F})$, $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. Fix $p \ge 0$ and consider the inclusion of $G(p, q)$ into $G(p, q+1)$. The induced map from $H_i(G(p, q))$ to $H_i(G(p, q+1))$ is then surjective for $i \le q$ and bijective for $i < q$.*

Proof. Theorem 1.1 takes care of $p = 0$. The general case proceeds by induction on p. We restrict ourselves to the case $p = 1$ and leave the detailed induction to the careful reader.

We compute the homology of $G(1, q+1)$ by using $C_*^+ = C_*^+(1, q+1, \mathbb{F})$. This leads to a transposed spectral sequence with $''E^1_{i,j} = H_i(G(1, q+1), C_j^+)$. As in Theorem 1.1, $''E^1_{i,*}$ has augmentation $''E^1_{i,0} \cong H_i(G(1, q)) \cong {''E^2_{i,0}}$. By (3.5), $''E^1_{*},0 \cong C_*^+ \otimes_{G(1,q+1)} \mathbb{Z}$ is q-acyclic. It is enough to show that $''E^1_{i,*}$ is $(q-i)$-acyclic for $1 \le i \le q-1$. We note that the isotropy subgroup of a cell is conjugate in $G(1, q+1)$ to either $G(1, s)$ or to $U(s, \mathbb{F})$.

Let $P_{0,*}$ denote the subcomplex of $''E^1_{0,*}$ spanned by cells of signatures $(0, t)$, $1 \le t \le q+1$. The quotient complex $Q_{0,*}$ is therefore spanned by cells of signatures $(1, t)$, $1 \le t \le q+1$. Both of these are $\mathbb{Z}$-free and our description shows that we have exact sequences of chain complexes that are $\mathbb{Z}$-free and $\mathbb{Z}$-split. As in (3.5) or as in the proof of Theorem 1.1, $P_{0,*}$ is q-acyclic. By means of the long homology exact sequence associated to a short exact sequence of chain complexes, $Q_{0,*}$ is also q-acyclic and augments to 0 (since it begins in degree 1). For $i > 0$, let $P_{i,*}$ and $Q_{i,*}$ denote the subcomplex and quotient complex of $''E^1_{i,*}$ induced respectively by $P_{0,*}$ and $Q_{0,*}$. By Shapiro's Lemma, $Q_{i,j}$ is a direct sum of terms spanned by j-cells from $Q_{0,j}$ together with a coefficient group isomorphic to $H_i(U(q+1-t, \mathbb{F}))$, $t \le j$. Since $Q_{0,*}$ is q-acyclic, we can use Theorem 1.1 and the universal coefficient theorem to conclude that $Q_{i,*}$ is $(q-i)$-acyclic with augmentation 0, $1 \le i \le q$ (in fact, it is even $(q+1-i)$-acyclic). The $(q-i)$-acyclicity of $''E^1_{i,*}$ is therefore reduced to the $(q-i)$-acyclicity of $P_{i,*}$, $1 \le i \le q-1$. A j-cell of $P_{i,*}$ has coefficient group $H_i(G(1, q-t))$, $1 \le t \le j$. At this point, the argument proceeds formally as in the compact cases treated in Section 1.

For general $p > 1$, each of the columns can be filtered by using the index p. The desired acyclicity results on the quotients are obtained by induction while the "bottom" subcomplex is treated by the formal argument used in Theorem 1.1 just as the case of $p = 1$. We omit further details. □

Remark 3.9. We can state Theorem 3.8 in the symmetric form:

Let $1 \leq p \leq p'$, $1 \leq q \leq q'$. The inclusion of $U(p, q, \mathbb{F})$ into $U(p', q', \mathbb{F})$ induces surjective maps on H_i for $i \leq \min(p, q)$ and the map is bijective when $i < \min(p, q)$. In the case of $SU(p, q, \mathbb{C})$, $\min(p, q)$ has to be replaced by $\min(p, q) - 1$ in the conclusions. In the case of $SO^1(p, q, \mathbb{R})$, $\min(p, q)$ has to be replaced by $\min(p, q) - 2$ in the conclusions. (3.10)

We note that both C_*^- and C_*^+ are used in the preceding symmetric versions. In the cases of SU or SO^1, we need to have enough room to adjust for determinants as well as spinor norms.

4. Schur multipliers of noncompact classical groups

The principal result in the present section is:

THEOREM 4.1. *Let G be a connected, simply-connected, absolutely simple real Lie group. Assume that G is noncompact and of classical type. Then $H_2(G) \cong K_2(\mathbb{C})^+$ under the universal complexification homomorphism.*

Simple Lie algebras were classified by E. Cartan. We follow the notation of Helgason [12; p. 519] (or the "dictionary" [37; p. 1412]). The classical ones over $\mathbb{C}$ are $sl(n, \mathbb{C})$, $n \geq 2$, $so(n, \mathbb{C})$, $n \geq 5$, and $sp(2n, \mathbb{C})$, $n \geq 3$. (Our $sp(2n, \mathbb{C})$ corresponds to $sp(n, \mathbb{C})$ in Helgason [12].) The real analogues $sl(n, \mathbb{R})$ and $sp(2n, \mathbb{R})$ are "$\mathbb{R}$-split" and absolutely simple. The remaining noncompact ones are among the ones listed below:

$su(p, q, \mathbb{F})$, $1 \leq \min(p, q)$ and $3 \leq p + q$; $su^*(2n) \cong sl(n, \mathbb{H})$, $n > 1$; and $so^*(2n) \cong so^*(n, \mathbb{H})$, $n > 2$.

In the case of $su(p, q, \mathbb{F})$, we drop $\mathbb{F}$ and use so, su and sp respectively. Among these noncompact cases, there are some "exotic" isomorphisms:

$$\begin{aligned}
&so(1, 2) \cong su(1, 1) \cong sl(2, \mathbb{R}) \cong sp(2, \mathbb{R});\ so(1, 3) \cong sl(2, \mathbb{C}) \cong sp(2, \mathbb{C}).\\
&su(1, 3) \cong so^*(6);\ so(1, 4) \cong sp(1, 1);\ so(1, 5) \cong su^*(4) \cong sl(2, \mathbb{H});\\
&so(2, 2) \cong sl(2, \mathbb{R}) \times sl(2, \mathbb{R});\ so(2, 3) \cong sp(4, \mathbb{R});\ so(2, 4) \cong su(2, 2);\\
&so(2, 6) \cong so^*(8);\ so(3, 3) \cong sl(4, \mathbb{R});\ so^*(4) \cong su(2) \times sl(2, \mathbb{R}).
\end{aligned} \tag{4.2}$$

An inspection of (4.2) together with the outline described in Section 2 indicate that $su(1, n, \mathbb{F})$ is at the heart of the matter. Each of the three cases of $\mathbb{F}$ has some special features. As in the compact cases, we treat them separately.

CASE 1. $\mathbb{F}=\mathbb{R}$. We first note that $O(p, q) \cong O^1(p, q) \times \pm I_{p+q}$ when p is odd. The exotic isomorphisms in (4.2) lead to:

$$O^1(1,2) \cong PSU(1,1) \cdot \langle\rho\rangle;\ O^1(1,3) \cong PSL(2,\mathbb{C}) \cdot \langle\rho\rangle;$$
$$O^1(1,4) \cong PSp(1,1) \cdot \langle\rho\rangle;\ \text{and } O^1(1,5) \cong PSL(2,\mathbb{H}) \cdot \langle\rho\rangle. \tag{4.3}$$

In (4.3), ρ is an automorphism of order 2 induced by the following map:

$$\sigma(A) = w \cdot ({}^tA^*)^{-1} \cdot w^{-1}, \qquad A \in GL(2, \mathbb{F}), \qquad w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

In all cases, $O^1(1, n)$ is the full isometry group of the real hyperbolic n-space $\mathcal{H}^n(\mathbb{R})$ and $SO^1(1, n)$ is the subgroup of index 2 consisting of all the orientation preserving isometries.

We first consider $H_2(O(p, q))$. Since we are only concerned with the noncompact cases, $\min(p, q) \geq 1$. Let us first consider the case where $\min(p, q) \geq 2$. Since $SO^1(2, 2)$ is not simple, see (4.2), we may assume that $\max(p, q) \geq 3$. By the symmetrized version of Theorem 3.8, we see that $H_2(O(p, q))$ stabilizes to $H_2(O(2, 3))$. Now, $O(2, 3) \cong SO(2, 3) \times \pm I_5$ and $SO(2, 3)$ is the semidirect product of $SO^1(2, 3)$ and $\mathbb{Z}/2\mathbb{Z}$. It is then easy to describe $H_2(O(2, 3))$ by means of Künneth's Theorem and the Hochschild–Serre spectral sequence associated to a split exact sequence of groups. By (4.2), $SO^1(2, 3) \cong PSp(4, \mathbb{R})$. This exotic isomorphism is seen to be induced by the action of $Sp(4, \mathbb{R})$ on $\Lambda^2_{\mathbb{R}}(\mathbb{R}^4)$ together with the symmetric inner product induced by exterior product because $\Lambda^4_{\mathbb{R}}(\mathbb{R}^4) \cong \mathbb{R}$ is trivial under $Sp(4, \mathbb{R})$. There is no problem seeing that $H_2(O(2, 3))$ is the direct sum of $K_2(\mathbb{C})^+$ and a finitely generated abelian group (in fact, $(\mathbb{Z}/2\mathbb{Z})^2$). The divisible part $K_2(\mathbb{C})^+$ can be detected either directly by complexification or indirectly by enlarging $Sp(4, \mathbb{R})$ to $SL(4, \mathbb{R})$ first. We note that the fundamental group of $PSp(4, \mathbb{R}) \cong SO^1(2, 3)$ is $\mathbb{Z}$ and it is negated by a suitable element of $O(2, 3)$. These are straightened out by a careful examination of the spectral sequence and we get the desired result on the universal covering group of $SO^1(p, q)$ when $\min(p, q) \geq 2$ and $\max(p, q) \geq 3$.

By symmetry, we are left with $O(1, q)$, $q \geq 2$. As indicated before, $O(1, q) \cong O^1(1, q) \times \pm I_{q+1}$. By Theorem 3.8, we only have to consider the two cases where $q = 2$ and $q = 3$. By (4.2) and (4.3), $O^1(1, 2) \cong PGL(2, \mathbb{R})$ and $O^1(1, 3) \cong PSL(2, \mathbb{C}) \cdot \langle\rho\rangle$, $\rho =$ complex conjugation. As in the preceding cases, there is no problem showing that $H_2(O(1, q)) \cong K_2(\mathbb{C})^+ \amalg (\mathbb{Z}/2\mathbb{Z})^2$, $q \geq 2$. We again have the desired result on the universal covering group of $SO^1(1, q)$, $q \geq 2$ and $q \neq 3$. We note that $SL(2, \mathbb{C})$ is not absolutely simple; nevertheless, we can use it in analyzing $O^1(1, q)$, $q \geq 3$. A crucial role is played by the Galois automorphism ρ of $\mathbb{C}$ over $\mathbb{R}$ in this analysis.

For ease of reference, we summarize the results:

$$H_2(SO^1(p, q)) \cong K_2(\mathbb{C})^+ \amalg \pi_1(SO^1(p, q)), \min(p, q) \geq 1, \qquad \text{and } p + q = 3 \text{ or } \geq 5. \tag{4.4}$$

CASE 2. $\mathbb{F} = \mathbb{C}$. $U(p, q)$ is the semidirect product of $SU(p, q)$ and $U(1)$ through the determinant map. $U(1)$ acts on $SU(p, q)$ through inner automorphisms of $SU(p, q)$. As in the compact case, we have:

$$H_2(U(p, q)) \cong H_2(SU(p, q)) \amalg H_2(U(1)). \tag{4.5}$$

Compatibility shows that stability theorems for $H_2(U(p, q))$ are equivalent to stability theorems for $H_2(SU(p, q))$. By (4.2), $SU(1, 1) \cong SL(2, \mathbb{R})$ so that $H_2(SU(1, 1)) \cong K_2(\mathbb{C})^+ \amalg \pi_1$. Similarly, $SU(2, 2) \cong SO^1(2, 4)$ so that $H_2(SU(2, 2)) \cong K_2(\mathbb{C})^+ \amalg \pi_1$. In these two cases, both π_1 are isomorphic to $\mathbb{Z}$ and are matched. An examination of Case 1 shows that we in fact have $H_2(SU(1, 1)) \cong H_2(SU(2, 2))$ under the inclusion of $SU(1, 1)$ into $SU(2, 2)$. By Theorem A.5 of Appendix A, the inclusion $SU(1, 1) \subset SU(1, 2) \subset SU(2, 2)$ must therefore induce isomorphsms on H_2. This assertion is by far the most difficult case. Once we have this, we note that $H_2(SU(2, 2))$ maps surjectively to the stable group $H_2(SU(p, q))$ where $\min(p, q) \geq 2$ and $\max(p, q) \geq 3$. Since $K_2(\mathbb{C})^+$ is not touched when $SU(p, q)$ is complexified to $SL(p + q, \mathbb{C})$, we in fact have the desired statement on the level of the universal covering groups. The argument is similar to Case 1 and the compact cases of Section 1. We summarize the results:

$$H_2(SU(p, q)) \cong K_2(\mathbb{C})^+ \amalg \pi_1(SU(p, q)), \qquad 1 \leq \min(p, q). \tag{4.6}$$

CASE 3. $\mathbb{F} = \mathbb{H}$. By (4.2), $Sp(1, 1)$ is the universal covering group of $SO^1(1,4)$. By (0.6) and (4.4), $H_2(Sp(1, 1)) \cong K_2(\mathbb{C})^+$ and the isomorphism is induced by the universal complexification homomorphism. By Theorem A.3 of Appendix A together with complexification, we have $H_2(Sp(1, 1)) \cong H_2(Sp(1, 2))$. By adapting the proofs of (1.6), (3.5) and Theorem A.3, we may show the surjectivity of the map from $H_2(Sp(2, 1))$ onto $H_2(Sp(2, 2))$. Since $K_2(\mathbb{C})^+$ is not disturbed under complexification, we obtain the injectivity.

At this point, the argument of Case 2 can be imitated word by word. We simply record the result:

$$H_2(Sp(p, q)) \cong K_2(\mathbb{C})^+, \qquad 1 \leq \min(p, q). \tag{4.7}$$

We next consider the infinite series $SL(n, \mathbb{H})$, $n \geq 2$. We note that $SL(1, \mathbb{H}) = Sp(1) \cong SU(2)$ is compact. We also note that $GL(n, \mathbb{H}) \cong SL(n, \mathbb{H}) \times \mathbb{R}^+$. Thus, we have:

$$H_2(GL(n, \mathbb{H})) \cong H_2(SL(n, \mathbb{H})) \amalg H_2(\mathbb{R}^+), \qquad n \geq 1. \tag{4.8}$$

By compatibility, the stability theorem in Appendix B for $H_i(GL(n, \mathbb{H}))$ is in fact equivalent to stability theorem for $H_i(SL(n, \mathbb{H}))$. By (4.2) and (4.3), $SL(2, \mathbb{H})$ is the universal covering group of $SO^1(1, 5)$. By (4.4) and Theorem B.1 of Appendix B, we obtain the result:

$$H_2(SL(n, \mathbb{H})) \cong K_2(\mathbb{C})^+, n \geq 2. \tag{4.9}$$

Remark 4.10. (4.9) also follows from the result of Alperin–Dennis [2] where is was shown that $K_2(\mathbb{H}) \cong K_2(\mathbb{C})^+$. The main point is that Theorem B.1 improves the stability range of Alperin–Dennis and does not involve any "K_2-calculations". Ultimately, (4.9) will be improved to $n \geq 1$, see Dupont–Parry–Sah [7]. This latter improvement is much more difficult.

The last series of Lie groups is $SO^*(2n)$ or $SO(n, \mathbb{H})$, $n \geq 2$. It is the group of all $\mathbb{H}$-linear automorphisms of $\mathbb{H}^n$ preserving a $**$-hermitian symmetric inner product $\langle\ ,\ \rangle^{**}$ on $\mathbb{H}^n$. Here, $**$ denotes the involution of $\mathbb{H}$ fixing 1, i, k and mapping j onto $-j$. $\langle\ ,\ \rangle^{**}$ is defined by the rule:

$$\langle u, v\rangle^{**} = {}^t u^{**} \cdot v, \qquad u, v \in \mathbb{H}^n.$$

We proceed as in the case of $U(p, q)$ and define $C_*^+ = C_*^+(\mathbb{H}^n, **)$ so as to satisfy the analogue of (3.2). The involution $**$ can be used to define a norm from $\mathbb{H}$ to $\mathbb{R}1 + \mathbb{R}i + \mathbb{R}k$. It is easy to see that this map is surjective. Essentially, this amounts to showing that quadratic equations over $\mathbb{C}$ can be solved. The argument leading to (3.5) can be adapted to show that $C_*^+ \otimes_G \mathbb{Z}$ is $(n-2)$-acyclic. This then leads to a stability theorem similar to Theorem 1.1 with stability range roughly $n/2$. However, there is no problem showing:

$$H_2(SO(n, \mathbb{H})) \to H_2(SO(n+1, \mathbb{H})) \text{ is surjective for } n \geq 3 \text{ and bijective for } n \geq 4. \tag{4.11}$$

The point to note is that $SO(n, \mathbb{H})$ is perfect for $n \geq 2$. By (4.2), (4.4), (4.6), (4.11), we obtain the result:

$$H_2(SO(n, \mathbb{H})) \cong K_2(\mathbb{C})^+ \amalg \pi_1(SO(n, \mathbb{H})), \qquad n \geq 3. \tag{4.12}$$

These results complete the proof of Theorem 4.1.

Appendix A

LEMMA A.0. *Let $G = O^1(1, q)$, $q \geq 1$. Then ${}_+C_*^- \otimes_G \mathbb{Z}$ and ${}_+C_*^{-,+} \otimes_G \mathbb{Z}$ are respectively 1- and q-acyclic. If $G = SO^1(1, q)$, $q \geq 2$, then we have 1- and $(q-1)$-acyclicity respectively. Similarly, $C_*^+ \otimes_G \mathbb{Z}$ is $(q-2)$-acyclic. In all cases, the augmentations are $\mathbb{Z}$.*

Proof. We recall from (3.3) that (3.2) automatically holds for hyperbolic q-space. The assertion about $C_*^+ \otimes_G \mathbb{Z}$ follows from (3.7). We first examine the case of ${}_+C_*^- \otimes_G \mathbb{Z}$. Every 1-cell (v_0, v_1) represents a 1-cycle. (v_0, v_1) is the boundary of (v_0, v_1, v_2) for the unique midpoint v_2 of (v_0, v_1) because (v_0, v_2) and (v_1, v_2) are congruent under a hyperbolic reflection when $q = 1$ and under a hyperbolic rotation when $q \geq 2$. This argument is valid for any symmetric space.

We now consider ${}_+C_*^{-,+} \otimes_G \mathbb{Z}$ for $G = O^1(1, q)$. The case of $G = SO^1(1, q)$ is similar and will be omitted. The proof will proceed by induction on q. Let c be a t-cycle of ${}_+C_*^{-,+} \otimes_G \mathbb{Z}$, $t \leq q$.

Suppose that $c = (v) \# c_1 + c_2 \# (w)$, $v \in S_+^-(1, q, \mathbb{R})$, $w \in S^+(1, q, \mathbb{R})$, where c_1 is a $(t-1)$-chain of $C_*^+(1, q, \mathbb{R})$ supported on the orthogonal complement of v, and where c_2 is a $(t-1)$-chain of ${}_+C_*^{-,+}(1, q, \mathbb{R})$ supported on the orthogonal complement of w. This case covers the situation when $t < q$ or when $t = q$ and c is made up from dependent q-cells. Namely, we can modify each t-cell appearing in c by the boundary of an orthogonal join (with (v) or with (w) at the appropriate end). In doing this, we may further assume that every cell appearing in c_1 has signature $(0, q)$ (in particular, $c_1 = 0$ when $t < q$) and is not an orthogonal join with (w). Since $0 = \partial c = c_1 + (-1)^t c_2 - (v) \# \partial c_1 + \partial c_2 \# (w)$, we may take $c_1 = 0$ in all cases and we can assume $c = c' \# (w') \# (w)$. If $t = 1$, then c is a boundary though the midpoint argument. If $t > 1$, then $\partial c' = 0$. If Lemma A.0 were proven for smaller values of q or if $q = 2$, then $c' = \partial a$ for some chain a supported on the orthogonal complement of w and w' and $c = \partial(a \# (w') \# (w))$ as desired. Thus, to complete the proofs of the case $q = 1$ as well as the inductive step, we must consider the case $t = q$ and some independent q-cells appear in c. Let $(v_0, \ldots, v_r, w_0, \ldots, w_s)$ be an independent q-cell appearing in c with $r, s \geq -1$,

$r+s=q-1$, $v_i \in S_+^-(1, q, \mathbb{R})$, and $w_j \in S^+(1, q, \mathbb{R})$. If $s>0$, then we shall reduce the problem to one with smaller s. If $s=q$, we randomly select $v_0 \in S_+^-(1, q, \mathbb{R})$ and modify c by $\partial(v_0, w_0, \ldots, w_q)$. We may therefore assume that $s<q$ and $r \geq 0$.

CASE 1. Suppose that $(w_0, \ldots, w_s)$ has signature $(1, s)$. Let $v_{r+1} \in S_+^-(\sum_j w_j\mathbb{R})$ be arbitrary and modify c by $\partial(v_0, \ldots, v_{r+1}, w_0, \ldots, w_s)$. $(v_0, \ldots, v_r, w_0, \ldots, w_s)$ is then replaced by an integral linear combination of independent q-cells with smaller s and a number of dependent q-cells.

CASE 2. Suppose that $(w_0, \ldots, w_s)$ has signature $(0, s+1)$. For each $i \leq r$, let $v_i' \in S_+^-(1, q, \mathbb{R})$ so that $v_i'\mathbb{R} \perp (\sum_j w_j \mathbb{R}) = v_i\mathbb{R} + (\sum_j w_j\mathbb{R})$. This is possible because of the independence and the signature conditions. We then successively modify c by the boundaries of $(v_0', \ldots, v_n', v_n, \ldots, v_r, w_0, \ldots, w_s)$, $0 \leq n \leq r$, so that $(v_0, \ldots, v_r, w_0, \ldots, w_s)$ is replaced by the orthogonal join $(v_0', \ldots, v_r') \# (w_0, \ldots, w_s)$ and an integral linear combination of independent q-cells with small s and some dependent q-cells. If $s>0$, then the circumscribed center construction on $(w_0, \ldots, w_s)$ can be performed to replace our orthogonal join by an integral linear combination of dependent q-cells. If $s=-1$, then $r>0$ and it is classically known that a geodesic q-simplex in real hyperbolic q-space has an inscribed center. By dropping perpendiculars from the inscribed center to the codimensional 1 faces, the independent q-cell $(v_0, \ldots, v_q)$ is replaced modulo boundaries by an integral linear combination of dependent q-cells and differences of independent q-cells that are congruent under hyperplane reflections. These cancel out under $G=O^1(1, q)$. Finally, for $s=0$, our cycle c has been reduced to a linear combination of dependent q-cells and cells of the form $(v_0, \ldots, v_{q-1}) \# (w)$. This type of cycle was handled at the beginning of our proof. We have concluded the proof of Lemma A.0. □

LEMMA A.1. *Let* $F=\mathbb{C}$ *or* $\mathbb{H}$ *and let* $G=U(1, q, \mathbb{F})$.
(a) *If* $q \geq 1$, *then* $C_*^-(1, q, \mathbb{F}) \otimes_G \mathbb{Z}$ *is 1-acyclic with augmentation* $\mathbb{Z}$.
(b) *If* $1 \leq q \leq 2$, *then* $C_*^{-,+}(1, q, \mathbb{F}) \otimes_G \mathbb{Z}$ *is q-acyclic with augmentation* $\mathbb{Z}$.

Proof. The idea is to show that a cycle is necessarily homologous to an orthogonal join, hence to a boundary as in Lemma A.0.

We begin with the proof of (a) and note that every 1-cell (v_0, v_1) of $C_*^- \otimes_G \mathbb{Z}$ is a 1-cycle. Modulo the boundary of (v_0, v_1, v_2) for a generic choice of v_2 in $S^-(1, q, \mathbb{F})$, we can assume that (v_0, v_1) is an independent 1-cell and that $q=1$. Since $U(1, q, \mathbb{F})$ is transitive on $S^-(1, q, \mathbb{F})$, we can assume $v_0=e_0$ and $v_1=e_0\alpha+e_1\beta$ with $-|\alpha|^2+|\beta|^2=-1$, $\beta \neq 0$. (v_0, v_1) will have a midpoint v_2 (no longer uinique) if and only if $0 < \langle v_0 - v_1, v_0 - v_1 \rangle_{1,1} = -|1-\alpha|^2 + |\beta|^2 = -2 + \operatorname{tr}(\alpha)$ where $\operatorname{tr}(\alpha) = \alpha + \alpha^*$ and $|\alpha|^2 = \alpha^*$. α is the multiplicative norm from $\mathbb{F}$ to $\mathbb{R}^+$. Since every subfield of $\mathbb{F}$ over $\mathbb{R}$ is conjugate to $\mathbb{R}$ or to $\mathbb{C}$, we can assume $\alpha \in \mathbb{C}$. Since $|\alpha|^2 = 1 + |\beta|^2 > 1$, it is evident that for a fixed $|\beta| > 0$, a midpoint v_2 of (v_0, v_1) will exist when $\arg(\alpha)$ is sufficiently close to 0. If we use the boundary of 2-cells of the form $(v_0, v_1 \exp(\iota\theta_1), v_0 \exp(\iota\theta_2))$ and $(v_0, v_0 \exp(\iota\theta_2), v_1)$, (v_0, v_1) can be replaced by the following sum:

$$(v_0, v_1 \exp(\iota\theta_1)) + (v_1 \exp(\iota\theta_1), v_0 \exp(\iota\theta_2)) + (v_0 \exp(\iota\theta_2), v_1).$$

We note that $|\beta|$ is unchanged in each of these new independent 1-cells. It is easy to see that θ_1 and θ_2 can be selected so that all three new 1-cells have arguments at most $|\arg(\alpha)|/2$. In a finite number of steps, any 1-cell becomes homologous to an integral linear combination of 1-cells so that each of them has a midpoint. As in Lemma A.0, (v_0, v_1) is then homologous to 0.

We now go to the proof of (b) and let c be a t-cycle of $C_*^{-,+} \otimes_G \mathbb{Z}$. By Theorem 1.2 or by (3.5), we only have to consider the case of $t = q$. If $t = 1$, then the argument used to prove Lemma A.0 can be adapted to show that $c = c' + m \cdot (v_0) \# (w_0)$ for a suitable integer m and a suitable 1-chain c' of $C_*^-(1, 1, \mathbb{F})$. Since c is a cycle, m must be 0 and c is then a boundary by (a).

Suppose $t = 2 = q$. The proof of Lemma A.0 can be adapted to show that $c = c' + c_1 \# (w_0) + (v_0) \# c_2$ with $c' \in C_2^-(1, 2, \mathbb{F})$, $c_1 \in C_1^-(1, 1, \mathbb{F})$ and $c_2 \in C_1^+(0, 2, \mathbb{F})$. Since $C_*^-(1, 1, \mathbb{F}) \otimes_G \mathbb{Z}$ and $C_*^+(0, 2, \mathbb{F}) \otimes_G \mathbb{Z}$ are both 1-acyclic (note: G has a different interpretation in these two cases and these are meant to be identified with their images in $C_*^{-,+}(1, 2, \mathbb{F})$), we can assume $c_1 = 0 = c_2$ because each 1-cell in each of these complexes is a 1-cycle, hence a boundary. By the orthogonal join construction, this shows that all appearances of dependent 2-cells in c can be ignored and that c can be assumed to be made up from independent 2-cells of $C_2^-(1, 2, \mathbb{F})$. Let (v_0, v_1, v_2) be such an independent 2-cell that appears in c. We can find $\alpha_i \in U(1, \mathbb{F})$ so that $\langle v_0, v_i\alpha_i \rangle_{1,2} \in \mathbb{R}$, $i = 1, 2$. If we modify c successively by the boundaries of $(v_0, v_1, v_2, v_2\alpha_2)$ and $(v_0, v_1, v_1\alpha_1, v_2\alpha_2)$, then (v_0, v_1, v_2) is replaced by $(v_0, v_1\alpha_1, v_2\alpha_2)$ and a linear combination of dependent 2-cells that can be ignored. This means that the independent 2-cells making up c can be assumed to satisfy the added condition that $\langle v_0, v_i \rangle_{1,2} \in \mathbb{R}$ for $i = 1, 2$. If we collect all such cells with a fixed value of $\langle v_1, v_2 \rangle_{1,2} \notin \mathbb{R}$ and if we then use the fact that c is a cycle, it follows that these cells must occur in pairs and each pair must have the form:

$$(v_0', v_1, v_2) - (v_0, v_1, v_2), \qquad \langle v_1, v_2 \rangle_{1,2} \notin \mathbb{R}.$$

We note that it may be necessary to apply elements of $U(1, 2, \mathbb{F})$ to the cells. The stability subgroup of the independent 1-cell (v_1, v_2) is isomorphic to $U(1, \mathbb{F})$. If we apply a suitable element of this stability subgroup to (v_0', v_1, v_2), we can arrange to have $\langle v_0', v_0\rangle_{1,2} \in \mathbb{R}$. If we modify c by the boundary of (v_0', v_0, v_1, v_2), the above pair is then replaced by a pair of independent 2-cells where all the inner products belong to $\mathbb{R}$. Such 2-cells can be viewed as 2-cells of $S^-(1, 2, \mathbb{R})$. If we now multiply each of the vertices by ± 1 and repeat the argument involving the multiplication of vertices by $\alpha_i \in U(1, \mathbb{F})$, we can even assume that the 2-cells have vertices in $S_+^-(1, 2, \mathbb{R})$. Such 2-cells have inscribed centers. If we use the inscribed center construction as in Lemma A.0, each such 2-cell can be replaced by a linear combination of dependent 2-cells modulo boundaries. In this manner, c itself becomes a linear combination of dependent 2-cells after a finite number of modifications. As mentioned before, c is then a boundary and we have proved (b). □

Remark A.2. For higher acyclicity, extension of Lemma A.1 runs into bookkeeping problems. The first part of the preceding proof resembles "mountain climbing" while the second part of the preceding proof is an adaptation of an argument used in a preliminary version of a weak form of Theorem 1.1. $C_*^-(1, 2, \mathbb{C}) \otimes_G \mathbb{Z}$ is definitely not 2-acyclic while $C_*^-(1, 2, \mathbb{H}) \otimes_G \mathbb{Z}$ happens to be 2-acyclic. Roughly, the difference is that $S^-(1, 2, \mathbb{C})$ is homotopic to S^1 while $S^-(1, 2, \mathbb{H})$ is homotopic to S^3.

THEOREM A.3. *Under the stabilization map, $H_2(Sp(1, 1)) \to H_2(Sp(1, 2))$ is surjective.*

Proof. We compute the homology of $Sp(1, 2) = U(1, 2, \mathbb{H})$ by using the transposed spectral sequence associated to $C_*^{-,+}(1, 2, \mathbb{H})$. From Lemma A.1, $''E^1_{0,*} \cong C_*^{-,+}(1, 2, \mathbb{H}) \otimes_G \mathbb{Z}$ is 2-acyclic with augmentation $\mathbb{Z}$. Since all the stability subgroups of cells are either trivial or perfect, $''E^1_{1,*} = 0$. $''E^1_{2,0} \cong H_2(Sp(1, 1)) \cdot (w_0) \amalg H_2(Sp(2)) \cdot (v_0)$. $''d^1_{2,1}$ is nonzero only on summands of the form $H_2(Sp(1)) \cdot (v_0, w_0)$. We note that cells of the form (v_0, w_0) must be independent and have signatures $(1, 1)$. By Theorem 2.1, the image of $''d^1_{2,1}$ is (up to a sign) the graph of the map from $H_2(Sp(2)) \cong H_2(Sp(1))$ into $H_2(Sp(1, 1))$. Thus, $''E^2_{2,0} \cong H_2(Sp(1, 1)) \cdot (w_0)$ because the graph is not changed (actually, it matters little because it is enough to know that $''E^2_{2,0}$ is a quotient of $H_2(Sp(1. 1)) \cdot (w_0)$). Since the spectral sequence converges to $H_*(Sp(1, 2))$, the desired assertion follows. □

LEMMA A.4. *Under the stabilization maps, the image of $H_2(SU(2))$ in $H_2(SU(1, 2))$ is contained in the image of $H_2(SU(1, 1))$ in $H_2(SU(1, 2))$.*

Proof. Our argument is based on "K_2-type calculations". By (2.10), $H_2(U(1)) \to H_2(SU(2))$ is surjective when $U(1)$ is mapped into $SU(2)$ along the diagonal. It is therefore enough to replace $SU(2)$ by the diagonal subgroup $U(1)$ of $SU(2)$. If we use the basis e_0, e_1, e_2 of $\mathbb{C}^{1,2}$, $SU(2)$ is the stability subgroup of e_0 while $SU(1, 1)$ is the stability subgroup of e_2. $U(1)$ is the subgroup consisting of the matrices:

$$\operatorname{diag}(1, u, u^{-1}), \qquad u \in U(1).$$

By (4.2), $SU(1, 1) \cong SL(2, \mathbb{R})$ so that $H_2(SU(1, 1)) \cong H_2(SL(2, \mathbb{R})) \cong K_2(\mathbb{C})^+ \amalg \pi_1$. As shown in Sah–Wagoner [26], the symbolic part $K_2(\mathbb{C})^+$ of $H_2(SL(2, \mathbb{R}))$ is covered by $H_2(SO(2))$. Up to conjugation in $SU(1, 1)$, $SO(2)$ can be identified with:

$$\operatorname{diag}(u, u^{-1}, 1), \qquad u \in U(1).$$

Since e_2 and e_0 are not equivalent under $SU(1, 2)$, this $SO(2)$ is not conjugate to the $U(1)$ inside $SU(2)$. However, $\operatorname{diag}(1, v, v^{-1}) = \operatorname{diag}(v^{-2}, v, v) \cdot \operatorname{diag}(v^2, 1, v^{-2})$ holds in $SU(1, 2)$, Since $\operatorname{diag}(v^{-2}, v, v)$ commutes with the perfect subgroup $SU(2)$ in $SU(1, 2)$, we know that $\operatorname{diag}(1, u, u^{-1}) \star \operatorname{diag}(v^{-2}, v, v)$ is 0 in $H_2(SU(1, 2))$. As a result, we have:

$$\operatorname{diag}(1, u, u^{-1}) \star \operatorname{diag}(1, v, v^{-1}) = \operatorname{diag}(1, u, u^{-1}) \star \operatorname{diag}(v^2, 1, v^{-2}).$$

Similarly, we have:

$$\operatorname{diag}(x, x^{-1}, 1) \star \operatorname{diag}(y, y^{-1}, 1) = \operatorname{diag}(x, x^{-1}, 1) \star \operatorname{diag}(1, y^{-2}, y^2).$$

Moreover, $SU(1, 2)$ contains P that maps e_0 onto $-e_0$ and exchanges e_1 and e_2. Since conjugation by P induces the identity on $H_2(SU(1, 2))$, we see that

$$\operatorname{diag}(x, x^{-1}, 1) \star \operatorname{diag}(y, y^{-1}, 1) = \operatorname{diag}(x, 1, x^{-1}) \star \operatorname{diag}(1, y^2, y^{-2}).$$

Since $\star$ is a skew-symmetric and bimultiplicative (when our abelian groups are written multiplicatively), the preceding equalities imply that the generators of the image of $H_2(SU(2))$ are contained in the image of $H_2(SU(1, 1))$ in $H_2(SU(1, 2))$ because $U(1)$ is a divisible group and u, v, x, y range over $U(1)$. We note that the roles of $SU(2)$ and $SU(1, 1)$ cannot be interchanged because of the presence of π_1. $\square$

THEOREM A.5. *Under the stabilization map, $H_2(SU(1, 1)) \to H_2(SU(1, 2))$ is surjective.*

Proof. As noted after (4.5), it is enough to show the surjectivity of $H_2(U(1, 1)) \to H_2(U(1, 2))$. The proof of Theorem A.3 can be imitated with a difference occurring with $''d^1_{2,1}$. The map $H_2(U(1)) \to H_2(U(2))$ has image exactly equal to the complement of $H_2(SU(2))$ in $H_2(U(2))$. As a result, we can only conclude that $H_2(U(1, 2))$ is the image of $H_2(U(1, 1)) \amalg H_2(SU(2))$. Since $H_2(U(1, j)) = H_2(U(1)) \amalg H_2(SU(1, j))$, $j = 1, 2$, the compatibility of the determinant maps together with Lemma A.4 imply the surjectivity of the map $H_2(SU(1, 1)) \to H_2(SU(1, 2))$. □

Remark A.6. Lemma A.0 covers more ground than Lemma A.1. The main point is that geodesic simplices in real hyperbolic spaces have inscribed centers (but may not have circumscribed centers). Complex and quaternionic hyperbolic geometries are more complicated, see Mostow [20] for complex hyperbolic plane. $S^\epsilon(1, 2, \mathbb{C})$ fibers over the complex hyperbolic plane and its geometry is even more complicated. The arguments used to prove Lemma A.1 bypasses the geometric difficulties. It is possible to show that ${}_+C^-(1, q, \mathbb{R}) \otimes_G \mathbb{Z}$ is q-acyclic when $q \leq 3$ and $G = O^1(1, q)$, see Dupont–Parry–Sah [7]. Obvious extensions to $q > 3$ are open.

Appendix B

Homology stability for general linear groups over division rings.

THEOREM B.1. *Let D be a division ring with an infinite center. Then,*

(a) *$H_i(GL(n, D)) \to H_i(GL(n+1, D))$ is bijective for $i \leq n$;*

(b) *the inclusion of $GL(n-1, D) \times GL(1, D)$ into $GL(n, D)$ "along the diagonal" induces a surjective map from $H_{n-1}(GL(n-1, D)) \otimes H_1(GL(1. D))$ onto the quotient group $H_n(GL(n, D))/\mathrm{im}\, H_n(GL(n-1, D))$;*

(c) *the inclusion of $GL(1, D)^{\times n}$ "along the diagonal" of $GL(n, D)$ induces a surjective map from $H_1(GL(1, D))^{\otimes n}$ onto $H_n(GL(n, D))/\mathrm{im}\, H_n(GL(n-1, D))$.*

Remark B.2. For finite fields with more than 2 elements, a result of this type was first obtained by Quillen as a prelude to his K-theory. For infinite fields, a more precise form of the preceding theorem was found by Suslin [32] as a prelude of his resolution of the Lichtenbaum–Quillen conjecture. In fact, Suslin identified the quotient $H_n(GL(n, F))/\mathrm{im}\, H_n(GL(n-1, F))$ with the Milnor K-group $K_n^M(F)$

of the infinite field F. An extensive discussions of the basic properties of $K_*^M(F)$ can be found in Bass–Tate [3]. We note that Milnor's definition of $K_*^M(F)$ makes sense for any division ring (its significance is not clear when the division ring has infinite dimension over its center). Namely, for any division ring D (no restriction on the size of its center), we can define $K_*^M(D)$ to be the universal associate ring (with unit) generated by the symbols $l(a)$, $a \in D^\times = GL(1, D)$, and satisfying the defining relations:

(R_1) $l(ab) = l(a) + l(b)$, $a, b \in D^\times$;

(R_2) $l(a) \cdot l(b) = 0$ if $a, b \in D^\times$ satisfy $a + b = 1$.

There is no difficulty deriving the further relations (see Bass–Tate [3]):

(R_3) $l(a)l(-a) = 0$; equivalently, $l(a)^2 = l(a)l(-1)$, $a \in D^\times$;

(R_4) $l(a)l(b) = -l(b)l(a)$;

(R_5) $l(a_1) \cdots l(a_t) = 0$ if $a_1 + \cdots + a_t = 0$ or 1, $t \geq 2$, $a_i \in D^\times$.

In particular, $K_*^M(D)$ is a graded, graded commutative ring and is covariant with respect to homomorphisms of division rings. $K_0^M(D) \cong K_0(D) \cong \mathbb{Z}$ under the augmentation map sending all $l(a)$ onto 0. By Dieudonne's theory of noncommutative determinant, $K_1^M(D) \cong K_1(D) \cong H_1(GL(n, D))$, $n \geq 1$, is naturally isomorphic to the commutator quotient group of $D^\times$ with $l(a)$ mapped onto the coset $a[D^\times, D^\times]$ in $D^\times/[D^\times, D^\times]$. For a field F, $K_2^M(F)$ is isomorphic to $K_2(F)$. Both $K_*^M(F)$ and $K_*(F)$ are generalizations of the K_2-functor of Milnor when the associative ring is restricted to be a field. In the case of a commutative ring A, Quillen's K-theory admits a graded commutative product so that $K_*(A)$ is a graded and graded commutative ring. It follows that we have a natural ring homomorphism from $K_*^M(F)$ to $K_*(F)$ in the case of a field F. In general, $K_n(F)$ and $K_n^M(F)$ are different when $n > 2$. For example, when $F = \mathbb{F}_q$ is a finite field when q elements, Quillen showed that $K_{2i}(\mathbb{F}_q) = 0$ and $K_{2i-1}(\mathbb{F}_q) \cong \mathbb{Z}/(q^i - 1) \cdot \mathbb{Z}$ for $i > 0$. In contrast, $K_i^M(\mathbb{F}_q) = 0$ for $i > 1$. In the case of an infinite field F, Suslin [32] exhibited a natural group homomorphism from $K_n(F)$ to $K_n^M(F)$ such that,

$$K_n^M(F) \to K_n(F) \to K_n^M(F) \text{ compose to multiplication by } (-1)^{n-1} \cdot (n-1)!. \tag{B.3}$$

This shows that $K_n^M(F)$ is the "decomposable part of $K_n(F)$ up to factorial". The composition in (B.3) is not a ring homomorphism. For a general division ring D, $K_*(D)$ no longer has a ring structure. As a result, no analogue to the first map in (B.3) is known. However, the second map in (B.3) has an analogue in the form of a group homomorphism from $K_n(D)$ to the quotient $H_n(GL(n, D))/\operatorname{im} H_n(GL(n-1, D))$ when D has an infinite center. Namely, we begin with the Hurewicz homomorphism from $K_n(D) = \pi_n(BGL^+(D))$ to $H_n(BGL^+(D))$. The functorial properties of the Quillen plus-construction identifies $H_n(BGL^+(D))$

with $H_n(BGL(D))$. (a) of Theorem B.1 then identifies $H_n(BGL(D))$ with $H_n(GL(n, D))$. The desired group homomorphism is then obtained by composing the preceding maps with the natural projection map. The presence of the Hurewicz homomorphism prevents us from having a ring homomorphism in (B.3). It appears natural to ask:

Let D be a division ring with an infinite center. How is $K_n^M(D)$ related to $H_n(GL(n, D))/\operatorname{im} H_n(GL(n-1, D))$, $n > 1$? (B.4)

For a commutative ring R, stability theorems for $GL(n, R)$ are usually phrased in terms of the Krull dimension d of R. A very general discussion of results in this direction can be found in van der Kallen [35]. They are based on acyclicity results described in a uniform manner in terms of n and d. In the case of an arbitrary division ring D, we will show that a sharper acyclicity result exists when $n > 1$.

Let D be an arbitrary division ring (no assumption on the size of its center is made). Let $G(n) = GL(n, D)$. Let $C_*(n)$ denote the normalized Eilenberg–MacLane chain complex based on the set D^n-$\{0\}$. For $1 \le i \le n$, let $\mathscr{F}^i$ denote the D-dimension filtration on $C_*(n)$ as well as on $C_*(n) \otimes_{G(n)} \mathbb{Z}$. Namely, a t-cell $(v_0, \ldots, v_t)$ belongs to $\mathscr{F}^i$ if and only if $\dim_D \sum_j v_j D \le i$. We observe the elementary result:

Let W be any D-subspace of D^n. Let $u_1, \ldots, u_i$ and $v_1, \ldots, v_i$ be elements of D^n spanning D-subspaces U and V respectively. Let $\sigma: U \to V$ be a D-linear isomorphism such that $\sigma(u_j) = v_j$, $1 \le j \le i$. Let $\rho \in \operatorname{Aut}_D(W)$. σ and ρ are the restrictions of a common element of $GL(n, D)$ if and only if ρ and σ coincide in $W \cap U$. (B.5)

We note that the D-linear independence of $u_1, \ldots, u_i$ and of $v_1, \ldots, v_i$ are not needed; however, $W \cap U$ and $W \cap V$ must be D-linearly isomorphic. We note also that Witt's Theorem is a weak form of (B.5). We finally note that $C_*(n)$ is an acyclic $G(n)$-resolution of the $G(n)$-trivial module $\mathbb{Z}$ and this resolution is $G(n)$-free when $n = 1$. Thus,

If $n = 1$, then $H_*(C_*(1) \otimes_{G(1)} \mathbb{Z}) \cong H_*(D^\times)$, $D^\times = G(1) = GL(1, D)$. (B.6)

We assert:

THEOREM B.7. *If $n > 1$, then $C_*(n) \otimes_{G(n)} \mathbb{Z}$ is n-acyclic.*

Proof. Let c be any t-cycle of $C_*(n) \otimes_{G(n)} \mathbb{Z}$. Suppose that $t \in \mathcal{F}^{n-1}$. By (B.5), c may be assumed to be supported on $\sum_{j>1} e_j D$ so that c becomes the boundary of $e_1 * c$. This involves another application of (B.5) and is analogous to the argument used in Section 1.

By an abuse of notation, set $\mathcal{F}^i = \mathcal{F}^i(C_*(n) \otimes_{G(n)} \mathbb{Z})$, $i \leq n$. The preceding argument yields the following exact sequences of abelian groups:

$$\begin{aligned} &0 \to H_t(\mathcal{F}^n) \to H_t(\mathcal{F}^n/\mathcal{F}^{n-1}) \to H_{t-1}(\mathcal{F}^{n-1}) \to 0, \; t > 1; \\ &0 \to H_1(\mathcal{F}^n) \to H_1(\mathcal{F}^n/\mathcal{F}^{n-1}) \to 0. \end{aligned} \tag{B.8}$$

The quotient complex $\mathcal{F}^n/\mathcal{F}^{n-1}$ begins in degree $n-1$ as $\mathbb{Z} \cdot (e_1, \ldots, e_n)$. For $n > 1$, $(e_1, \ldots, e_n) = \partial(e_1 + e_2, e_1, \ldots, e_n)$ in $\mathcal{F}^n/\mathcal{F}^{n-1}$. This gives us the $(n-1)$-acyclicity of $\mathcal{F}^n$. By (B.8), Theorem B.7 is equivalent to:

$$H_n(\mathcal{F}^n/\mathcal{F}^{n-1}) = 0 \text{ for } n \geq 3. \tag{B.9}$$

For any t-cell $(w_0, \ldots, w_t)$, its rank is defined to be $\dim_D \sum_j w_j D$. The following assertion is easy to prove:

Let $(v_0, \ldots, v_n)$ be any n-cell of rank n. There is a unique s-face F of $(v_0, \ldots, v_n)$ such that,
(a) F has rank s, $1 \leq s \leq n$;
(b) every codimension 1 face of F also has rank s. (B.10)

The uniquely determined face in F in (B.10) will be called the singular face of $(v_0, \ldots, v_n)$. It is possible that F has repeated adjacent vertices (this forces s to be 1). We next note that (B.9) follows formally from the assertion below:

Let $R(n) = (e_1 + e_2, e_1, \ldots, e_n) \otimes_{G(n)} \mathbb{Z} + \partial(C_{n+1} \otimes_{G(n)} \mathbb{Z}) + \mathcal{F}^{n-1}$.
Then every n-cell $(v_0, \ldots, v_n)$ of rank n belongs to $R(n)$, $n \geq 3$. (B.11)

The proof of (B.11) is broken down to several steps.

Step 1. If $\operatorname{rank}(v_2, \ldots, v_n) = n-2$, then $(v_0, \ldots, v_n) \in R(n)$. For this, we look at $\partial(v_0 + v_1, v_0, \ldots, v_n)$ and use (B.5).

Step 2. If $v_{i+2} = v_i$, then $(v_0, \ldots, v_n) \in R(n)$.
If $i > 1$, this follows from Step 1. If $i = 0$, then $\partial(v_0 + v_1, v_0, v_1, v_0, v_3, \ldots, v_n)$ together with (B.5) and Step 1 take care of this case. (recall that $C_*(n)$ is normalized). Assume $i = 1$. By (B.5) and look at $\partial(v_0 + v_1 + v_2, v_0, v_1, v_2, \ldots)$,

we see that $(v_0, v_1, v_2, v_0, \ldots) \in R(n)$. By combining this with the case $i=0$ and looking at $\partial(v_0, v_1, v_0, v_2, v_0, \ldots)$, we obtain the case of $i=1$.

Step 3. $(\ldots, v_i, v_{i+1}, \ldots) + (\ldots, v_{i+1}, v_i, \ldots) \in R(n)$.
For this, use $\partial(\ldots, v_i, v_{i+1}, v_i, \ldots)$ and Step 2.

Step 4. $(\ldots, v_i, \ldots) - (\ldots, v_i\alpha, \ldots) \in R(n)$.
For this, use $\partial(\ldots, v_i, v_i\alpha, \ldots)$ and Steps 1 and 3.

With Steps 3 and 4 at hand, the general case of an n-cell $(v_0, \ldots, v_n)$ of rank n with a singular face of rank s can be studied under the added assumption that $s = n-1$ or n. Moreover, in the relation among the vertices of the singular face, the coefficients can be modified at will over $D^\times$. In particular, we may assume that $(v_0, \ldots, v_s)$ is the singular face and the case of $s=2$ is already taken care of. We can take $s \geq 3$ and assume $v_s = \sum_{0 \leq j \leq s-1} v_j$. By (B.5) and $\partial(v_0 + v_1, v_0, \ldots, v_n)$, the general case is then reduced to the case $s=2$. □

Proof of Theorem B.1. We imitate the argument used in Section 1. In the transposed spectral sequence associated to $C_*(n+1)$ and $G(n+1) = GL(n+1, D)$, we have $''E^1_{*,0} \cong H_*(GL(n, D))$ through the use of the "center kills" lemma, see Suslin [32]. This depends on the assumption that D has an infinite center. In general, the homology of the stability subgroup of a t-cell $(v_0, \ldots, v_t)$ of rank $r \leq t+1$ has the form of $H_*(GL(n+1-r, D))$. By induction on (a) of Theorem B.1, $''E^1_{i,*}$ has a subcomplex isomorphic to:

$$H_i(GL(i, D)) \otimes (C_*(n+1-i) \otimes_{G(n+1-i)} \mathbb{Z}), \ i > 0. \tag{B.12}$$

This subcomplex is spanned by all the cells of rank at most $n+1-i$. The form of the coefficient groups arises from Shapiro's Lemma, the lemma on "center kills" and the induction hypothesis. By Theorem B.7 and the universal coefficient theorem, this subcomplex is $(n+1-i)$-acyclic for $0 < i \leq n-1$ and the augmentation is just $H_i(GL(n, D))$. The quotient complex begins in degree $n+1-i$ and is spanned by $(e_1, \ldots, e_{n+2-i})$. When $i \leq n-1$, this cell has the same coefficient group $H_i(GL(i-1, D))$ as $(e_1 + e_2, e_1, \ldots, e_{n+2-i})$. In the quotient complex, $(e_1, \ldots, e_{n+2-i}) = \partial(e_1 + e_2, e_1, \ldots, e_{n+2-i})$. It follows that $''E^2_{i,j} = 0$ holds for $1 \leq j \leq n+1-i$, $0 \leq i \leq n-1$. The same analysis also shows that $''E^2_{n,1}$ is a suitable quotient group of $H_n(GL(n, D)) \otimes H_1(D^\times)$ by the use of (B.5) and (B.6). Assertions (a) and (b) are therefore formal consequences of the form of $''E^2_{i,j}$, $i+j \leq n+1$, in our inductive argument. Assertion (c) follows from (b) together with induction.

COROLLARY B.13. *Let D be a division ring with an infinite center. Suppose that $H_2(GL(1, D)) \to H_2(GL(2, D))$ is surjective. (For example, $D = \mathbb{H}$, $\bar{\mathbb{Q}}$; or any* *infinite algebraic extension of* $\mathbb{F}_p$). *Then, for* $n \geq 1$,

$$H_i(GL(n, D)) \to H_i(GL(n+1, D)) \text{ is surjective for } i \leq n+1, \text{ and bijective for } i \leq n. \tag{B.14}$$

When $D = \mathbb{H}$, (B.14) *is valid also for* SL *in place of* GL.

Proof. When $n = 1$, (B.14) follows either by hypothesis ($i = 2$), by Dieudonne's theory of noncommutative determinants ($i = 1$), or by definition ($i = 0$). The bijectivity assertion of (B.14) is part of Theorem B.1. We only need to show surjectivity of $H_{n+1}(GL(n, D)) \to H_{n+1}(GL(n+1, D))$ for $n \geq 1$. This is done by induction. The hypothesis takes care of $n = 1$. By the proof of Theorem B.1, it is enough to show that $''E^2_{n,1} = 0$ for $n \geq 2$ in the transposed spectral sequence associated to $C_*(n+1)$ and $GL(n+1, D)$. $''E^1_{n,1}$ is the direct sum of $H_n(GL(n, D)) \otimes (e_1, e_1\alpha)$, with $\alpha \in D^\times\text{-}\{1\}$ and $H_n(GL(n-1, D)) \otimes (e_1, e_2)$. If we inductively assume that $H_n(GL(n-1, D))$ maps surjectively onto $H_n(GL(n, D))$, then $''d^1_{n,2}$ carries $c \otimes (e_1, e_1\alpha, e_2)$ onto $c \otimes (e_1, e_1\alpha)$ for $c \in H_n(GL(n-1, D))$ and the image ranges over all of $H_n(GL(n, D)) \otimes (e_1, e_1\alpha)$. Evidently, $''d^1_{n,2}$ carries $H_n(GL(n-1, D)) \otimes (e_1 + e_2, e_1, e_2)$ on $H_n(GL(n-1, D)) \otimes (e_1, e_2)$. These imply $''E^2_{n,1} = 0$ and complete the inductive step. By Künneth's Theorem and the fact that $GL(n, \mathbb{H}) = SL(n, \mathbb{H}) \times \mathbb{R}^+$, the last assertion follows from (B.14) with $D = \mathbb{H}$. □

Remark B.15. It is not difficult to see that $K^M_n(\mathbb{H}) = 0$ for $n \geq 2$. The critical case is $n = 2$ and the proof is essentially contained in Sah–Wagoner [26; Proposition 1.23]. The main point is that $l(q) = l(|q|)$ holds for $q \in \mathbb{H}^\times$. We note that (B.4) is answered by Suslin [32] for infinite fields. When D has finite dimension over its center, some information can be obtained by using the "transfer homomorphism". When D has infinite dimension over its center, we have very little information on (B.4).

For small values of i, a better picture of $H_i(GL(n, D))$ can be obtained with more work by an examination of the action of $GL(n, D)$ on the projective space $\mathbb{P}^{n-1}(D)$. In special cases, it is just as easy to work with $SL(n, D)$. Theorem B.7 was first observed in this setting for $n = 2$ by using the 3-transitivity of $GL(2, D)$ on $\mathbb{P}^1(D)$. The more detailed results will be reported elsewhere. For example, in Dupont–Parry–Sah [7], the detailed study of the action of $SL(2, \mathbb{H})$ on $\mathbb{P}^1(\mathbb{H})$ leads to $H_2(SU(2)) \cong K_2(\mathbb{C})^+$ under the complexification homomorphism. With the help of this and some other results, it is then possible to show that $H_3(SL(2, \mathbb{R})) \cong H_3(SL(2, \mathbb{C}))^+$ while $H_3(SU(2))$ maps surjectively onto $H_3(SL(2, \mathbb{C}))^+$. Both maps are induced by the complexification homomorphisms.

In fact, $K_3(\mathbb{C}) \cong K_3^M(\mathbb{C}) \amalg H_3(SL(2, \mathbb{C}))$ so that $H_3(SL(2, \mathbb{C}))$ is the direct sum of $\mathbb{Q}/\mathbb{Z}$ and a suitable $\mathbb{Q}$-vector space (of at least countably infinite dimension). This last result will be reported in Sah [25]. All of these depend on the present work as well as the various works of Dupont–Parry–Sah [6, 7, 8, 21] and Suslin [31–34].

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