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## Cyclic homology of groups and the Bass conjecture

BENO ECKMANN

### 0. Introduction

**0.1.** The cyclic homology  $HC_i(\mathbb{Q}G)$  of a group algebra  $\mathbb{Q}G$  decomposes into a direct sum indexed by the conjugacy classes  $[x]$  in  $G$ , as shown by Burghlea [7] (see Section 1.3 below). We will consider certain classes of groups of finite homology dimension over  $\mathbb{Q}$ ,  $hd_{\mathbb{Q}}G = n$ , and show that for  $i \geq n$  the terms in  $HC_i(\mathbb{Q}G)$  corresponding to conjugacy classes  $[x]$  of elements of infinite order vanish. The groups  $G$  with  $hd_{\mathbb{Q}}G = n < \infty$  for which this will be done are

- (a) Nilpotent groups  $G$ ,
- (b) Torsion-free solvable groups,
- (c) Linear groups  $G \subset GL_r(F)$  where  $F$  is a field of characteristic 0,
- (d) Groups of cohomology dimension  $cd_{\mathbb{Q}}G \leq 2$  (here  $n \leq 2$ ).

We recall (Serre [10]) that if in (c)  $F$  is a number field and  $G$  finitely generated, then  $G$  is always of finite virtual cohomology dimension, whence  $hd_{\mathbb{Q}}G = n < \infty$ . The case (b) actually falls under (c), but we prefer to give a simple direct argument, cf. Remark 2.3' below.

As an immediate consequence of that vanishing result it follows that the character maps from  $K$ -theory of  $\mathbb{Q}G$  to  $HC_*(\mathbb{Q}G)$ , see Karoubi [8], have vanishing components in the summands indexed by  $[x]$  with  $x$  of infinite order—for all the groups listed above.

In particular, the character map  $Ch_0^0: K_0(\mathbb{Q}G) \rightarrow HC_0(\mathbb{Q}G)$  can easily be seen to be the “Hattori–Stallings rank”  $r_P$  of finitely generated projective  $\mathbb{Q}G$ -modules  $P$  (representing elements of  $K_0(\mathbb{Q}G)$ ), see Section 3.2. For the groups above it thus follows that  $r_P$  is concentrated on the conjugacy classes  $[x]$  of elements  $x$  of finite order; hence on [1] if  $G$  is torsion-free. This is a contribution towards the strong Bass conjecture [3, p. 156]. Note that the case (c) yields a weaker statement than Bass’ result on linear groups [3, p. 156/57]; but our method is entirely different and stems from a result more general in another direction. The result establishing the Bass conjecture over  $\mathbb{Q}G$  in the other cases seems to be new.

## 1. Cyclic homology of groups

**1.1.** Let  $G$  be a group,  $\mathbb{Q}G$  its rational group algebra, and  $HC_i(\mathbb{Q}G)$ ,  $i \in \mathbb{Z}$ , the cyclic homology of  $\mathbb{Q}G$  in the sense of Connes; we will call it here in short the cyclic homology of  $G$ . It is related to the Hochschild homology  $HH_i(\mathbb{Q}G)$  of  $\mathbb{Q}G$ , with bimodule-coefficients in  $\mathbb{Q}G$  by left and right multiplication, through the “Connes–Gysin exact sequence”

$$\cdots \rightarrow HH_i(\mathbb{Q}G) \rightarrow HC_i(\mathbb{Q}G) \xrightarrow{S} HC_{i-2}(\mathbb{Q}G) \rightarrow HH_{i-1}(\mathbb{Q}G) \rightarrow \cdots \quad (1.1)$$

It is a standard fact (see [9]) that Hochschild homology of  $\mathbb{Q}G$  with bimodule coefficients can be expressed as homology of  $G$  with the same coefficient module turned into a right  $G$ -module; in the present case this is  $\mathbb{Q}G$  with  $G$ -action by conjugation in  $G$ . It thus follows that for groups  $G$  of finite homology dimension  $hd_{\mathbb{Q}}G = n < \infty$  over  $\mathbb{Q}$  (i.e., for all  $\mathbb{Q}G$ -module coefficients) the cyclic homology of  $G$  stabilized above  $n$ :

$$HC_{n+2k}(\mathbb{Q}G) = HC_n(\mathbb{Q}G),$$

$$HC_{n+2k+1}(\mathbb{Q}G) = HC_{n+1}(\mathbb{Q}G)$$

for  $k = 0, 1, 2, \dots$

**1.2.** The conjugation module  $\mathbb{Q}G$  obviously decomposes into a direct sum of right  $\mathbb{Q}G$ -module indexed by the conjugacy classes  $[x]$  of  $G$ ;  $x$  is an arbitrary but fixed representative of  $[x]$ :

$$\mathbb{Q}G = \bigoplus_{[x]} D_x$$

where  $D_x$  is the  $\mathbb{Q}$ -module over the elements  $x' \in [x]$  as basis, and with  $\mathbb{Q}G$ -action by  $x' \mapsto y^{-1}x'y$ ,  $y \in G$ . If  $C_x$  is the centralizer of  $x$  in  $G$ ,  $D_x$  is isomorphic to  $\mathbb{Q}(G/C_x)$ , the right  $\mathbb{Q}G$ -module generated by the right cosets modulo  $C_x$ ; the isomorphism is given by  $x' \mapsto C_x z$  where  $z \in G$  is such that  $z^{-1}xz = x'$ . Thus

$$\mathbb{Q}G = \bigoplus_{[x]} \mathbb{Q}(G/C_x) = \bigoplus_{[x]} (\mathbb{Q} \otimes_{C_x} \mathbb{Q}G),$$

and finally

$$HH_i(\mathbb{Q}G) = \bigoplus_{[x]} H_i(G; \mathbb{Q} \otimes_{C_x} \mathbb{Q}G) = \bigoplus_{[x]} H_i(C_x; \mathbb{Q}) \quad (1.2)$$

with trivial  $G$ -module coefficients  $\mathbb{Q}$ .

*Remark 1.1.*  $HH_0(\mathbb{Q}G)$  is the  $\mathbb{Q}$ -module having the conjugacy classes  $[x]$  in  $G$  as basis. This can also be seen directly from the well-known fact that  $HH_0(\mathbb{Q}G)$  is  $\mathbb{Q}G/\{\lambda\mu - \mu\lambda\}$ , where  $\{\lambda\mu - \mu\lambda\}$  denotes the  $\mathbb{Q}$ -submodule generated by all  $\lambda\mu - \mu\lambda$ ,  $\lambda, \mu \in \mathbb{Q}G$ ; i.e., the  $\mathbb{Q}$ -submodule generated by all  $xy - yx$ ,  $x, y \in G$ . We write  $\overline{\mathbb{Q}G}$  for this factor- $\mathbb{Q}$ -module of  $\mathbb{Q}G$ ,  $T: \mathbb{Q}G \rightarrow \overline{\mathbb{Q}G}$  for the canonical map, with  $T(\lambda\mu) = T(\mu\lambda)$ ,  $\lambda, \mu \in \mathbb{Q}G$ .

**1.3.** A direct sum decomposition of  $HC_i(\mathbb{Q}G)$ , with terms indexed by the conjugacy classes  $[x]$  in  $G$ , has been given by Burghlea [7] using topological (simplicial) constructions:

$$HC_i(\mathbb{Q}G) = \bigoplus'_{[x]} [H_*(C_x/\langle x \rangle; \mathbb{Q}) \otimes H_*(\mathbb{C}P^\infty; \mathbb{Q})]_i \oplus \bigoplus''_{[x]} H_i(C_x/\langle x \rangle; \mathbb{Q}). \quad (1.3)$$

Here  $\langle x \rangle$  denotes the cyclic subgroup of  $G$  generated by  $x$ , and  $\bigoplus'$  is summation over all  $[x]$  with finite  $\langle x \rangle$ ,  $\bigoplus''$  over all  $[x]$  with infinite  $\langle x \rangle$ .

The methods of [7] also yield the  $\bigoplus$ -decomposition (1.2) of  $HH_i(\mathbb{Q}G)$  and shows that the Connes–Gysin sequence (1.1) decomposes into exact sequences of the same type, one for each  $[x]$  in  $G$ .

We will consider in Section 2 groups of finite dimension  $hd_{\mathbb{Q}}G = n < \infty$  and show that, for certain classes of such groups, one has  $\bigoplus''_{[x]} = 0$  in the stable value

$HC_n(\mathbb{Q}G)$  and  $HC_{n+1}(\mathbb{Q}G)$ ; in other words, *the stable value is concentrated on the conjugacy classes  $[x]$  of elements  $x$  of finite order* – hence for torsion-free groups on the conjugacy class  $[1]$ . This will be done for the classes (a)–(d) listed in the introduction. Immediate consequences (Section 3) concern the character maps from  $K$ -theory of  $\mathbb{Q}G$  to cyclic homology and, in particular, the Hattori–Stallings rank as mentioned in the introduction.

## 2. Groups of finite dimension

**2.1.** “Finite dimension” for groups  $G$  will refer here, unless otherwise specified, to the homology dimension  $hd_{\mathbb{Q}}G$  over  $\mathbb{Q}$ , i.e., with respect to



$\mathbb{Q}G$ -module coefficients. For any subgroup  $S \subset G$ , in particular for the centralizers  $C_x$ , we have  $hd_{\mathbb{Q}}S \leq hd_{\mathbb{Q}}G = n < \infty$ .

In our context we are thus interested in the homology of factor groups  $G/\langle x \rangle$  where  $hd_{\mathbb{Q}}G = n < \infty$  and  $x$  is a central element; actually in homology with trivial  $\mathbb{Q}$ -coefficients only, and in its vanishing above  $n$ . In other words, we are looking at  $thd_{\mathbb{Q}}G/\langle x \rangle$ , the *trivial homology dimension over  $\mathbb{Q}$* ; i.e., defined exactly as  $hd_{\mathbb{Q}}$  but referring to trivial  $\mathbb{Q}$ -module coefficients only. One always has  $thd_{\mathbb{Q}} \leq hd_{\mathbb{Q}}$ . For that type of dimension we recall the following very simple but useful sum formula (Bieri [6]):

**LEMMA 2.1.** *Let  $U$  be a central subgroup of the group  $V$ , and  $W = V/U$ . If both  $thd_{\mathbb{Q}}U$  and  $thd_{\mathbb{Q}}W$  are finite then*

$$thd_{\mathbb{Q}}V = thd_{\mathbb{Q}}U + thd_{\mathbb{Q}}W.$$

A further preliminary remark concerns the case where  $\langle x \rangle$  is finite: then the spectral sequence

$$H_i(G/\langle x \rangle; H_j(\langle x \rangle; \mathbb{Q})) \Rightarrow H_{i+j}(G; \mathbb{Q})$$

shows that  $thd_{\mathbb{Q}}G/\langle x \rangle = thd_{\mathbb{Q}}G$ , hence  $\leq n$ . In all what follows we therefore restrict attention to central elements  $x$  of *infinite* order. In that case the spectral sequence does *not* imply that  $thd_{\mathbb{Q}}G/\langle x \rangle$  is finite; however, if it is finite then the sum formula yields

$$thd_{\mathbb{Q}}G/\langle x \rangle \leq n - 1.$$

**2.2. Nilpotent groups.** We recall (Stammbach [11]) that if  $G$  is nilpotent then  $hd_{\mathbb{Q}}G$  is equal to the Hirsch number  $hG$  (the sum of the torsion-free ranks of the factors of any normal series of  $G$  with Abelian factors); this holds, more generally, for solvable groups. We thus assume  $hG = n < \infty$ .

Let  $x \in G$  be a central element of infinite order,  $S$  a finitely generated subgroup of  $G/\langle x \rangle$ , and  $T$  the preimage of  $S$  in  $G$ ,  $T/\langle x \rangle = S$ . Since  $S$  is finitely generated nilpotent it is polycyclic, and therefore  $hd_{\mathbb{Q}}S = hs$  is finite (equal to the number of infinite cyclic factors in a normal series with cyclic factors). The sum formula now yields

$$thd_{\mathbb{Q}}S = thd_{\mathbb{Q}}T - 1 \leq n - 1.$$

$G/\langle x \rangle$  is the direct limit of its finitely generated subgroups  $S$ ; and since homology commutes with direct limits it follows that  $thd_{\mathbb{Q}}G/\langle x \rangle$  is  $\leq n - 1$ :

**THEOREM 2.2.** *Let  $G$  be a nilpotent group of finite dimension  $hd_{\mathbb{Q}}G = n$ . Then one has for any central element  $x \in G$  of infinite order*

$$H_i(G/\langle x \rangle; \mathbb{Q}) = 0 \quad \text{for } i \geq n.$$

**2.3. Torsion-free solvable groups.** Let  $G$  be torsion-free solvable with  $hd_{\mathbb{Q}}G = hG = n < \infty$  (or equivalently, solvable with  $hd_{\mathbb{Z}}G < \infty$ ). We consider the Hirsch–Plotkin radical  $R$  of  $G$ , i.e., the maximal locally nilpotent normal subgroup of  $G$ . For any Abelian subgroup  $S$  of  $G$  the torsion-free rank  $hS = hd_{\mathbb{Q}}S$  is  $\leq n$ . As  $G$  is torsion-free (actually a weaker condition would do) we can apply a theorem of Baer–Heineken [2] which tells that

( $\alpha$ )  $R$  is nilpotent

( $\beta$ )  $G/R$  is finitely generated

( $\gamma$ )  $G/R$  contains an Abelian subgroup  $A$  of finite index.

From (b) and (c) we infer that  $hd_{\mathbb{Q}}G/R = hd_{\mathbb{Q}}A$  is finite, say  $= m$ . If  $x \in G$  is central it must lie in  $R$ , and if it is of infinite order Theorem 2.2 tells that  $thd_{\mathbb{Q}}R/\langle x \rangle = thd_{\mathbb{Q}}R - 1 \leq n - 1$ . From the spectral sequence for  $G/R \cong G/\langle x \rangle / R/\langle x \rangle$ ,

$$H_i(G/R; H_j(R/\langle x \rangle; \mathbb{Q})) \Rightarrow H_{i+j}(G/\langle x \rangle; \mathbb{Q})$$

we see that  $H_k(G/\langle x \rangle; \mathbb{Q}) = 0$  for  $k > m + n - 1$ ; i.e.,  $thd_{\mathbb{Q}}G/\langle x \rangle$  is finite and hence  $\leq n - 1$ .

**THEOREM 2.3.** *Let  $G$  be a torsion-free solvable group of finite dimension  $hd_{\mathbb{Q}}G = n < \infty$ . Then one has for any central element  $x \in G$  of infinite order*

$$H_i(G/\langle x \rangle; \mathbb{Q}) = 0 \quad \text{for } i \geq n.$$

*Remark 2.3'.* The groups  $G$  above admit faithful linear representations over  $\mathbb{Q}$  (cf. [13], p. 25) and thus are included in 2.4 below. However, the proof of the linear embedding starts precisely from the structure properties ( $\alpha$ ), ( $\beta$ ), ( $\gamma$ ); thus the simple direct argument seems preferable.

**2.4. Linear groups.** We now consider a linear group  $G \subset GL_r(F)$ , where  $F$  is a field of characteristic 0, with  $hd_{\mathbb{Q}}G = n < \infty$ . Let  $Z$  be the center of  $G$ ,  $\pi: G \rightarrow G/Z$  the canonical map. Since  $Z$  is closed in the Zariski topology,  $G/Z$  is again a linear group over the same field  $F$ . We are going to apply the R. Alperin–Shalen criterion [1] to finitely generated subgroups  $S$  of  $G/Z$ , in order to prove that they are virtually of finite cohomology dimension.

For this let first  $U$  be a finitely generated unipotent subgroup of  $G/Z$  and put  $\pi^{-1}U = V$ ,  $V/Z = U$ . Then  $U$  being torsion-free finitely generated nilpotent (polycyclic),  $hd_{\mathbb{Q}}U = thd_{\mathbb{Q}}U = hU$  (Hirsch number, see 2.2). By Lemma 2.1,

$$thd_{\mathbb{Q}}Z + thd_{\mathbb{Q}}U = thd_{\mathbb{Q}}V.$$

Now  $hd_{\mathbb{Q}}Z \leq n$  and  $hd_{\mathbb{Q}}V \leq n$ ; let  $m = thd_{\mathbb{Q}}Z$ . Thus  $thd_{\mathbb{Q}}U \leq n - m$ ; i.e., we get a uniform bound for all finitely generated unipotent subgroups of  $G/Z$ , the Hirsch numbers  $hU$  being  $\leq n - m$ .

If  $S$  is any finitely generated subgroup of  $G/Z$ , it follows by [1] that its virtual cohomology dimension, and hence  $hd_{\mathbb{Q}}S$ , is finite. Putting  $\pi^{-1}S = T \subset G$ ,  $T/Z = S$ , Lemma 2.1 tells that

$$thd_{\mathbb{Q}}Z + thd_{\mathbb{Q}}S = thd_{\mathbb{Q}}T \leq n,$$

and thus  $thd_{\mathbb{Q}}S \leq n - m$ . The direct limit argument then yields  $thd_{\mathbb{Q}}G/Z \leq n - m$ .

**THEOREM 2.4.** *Let  $G$  be a linear group of finite dimension  $hd_{\mathbb{Q}}G = n$ , over a field of characteristic 0,  $Z$  is center and  $thd_{\mathbb{Q}}Z = m$ . Then*

$$H_i(G/Z; \mathbb{Q}) = 0 \quad \text{for } i > n - m.$$

We are looking for a similar result concerning  $G/\langle x \rangle$  where  $x$  is a central element of infinite order. Since  $G/Z = G/\langle x \rangle / Z/\langle x \rangle$  we can apply the spectral sequence (with trivial  $\mathbb{Q}$ -module coefficients) together with Theorem 2.2 on  $Z/\langle x \rangle$ . This immediately yields  $H_i(G/\langle x \rangle; \mathbb{Q}) = 0$  for  $i > (n - m) + (m - 1)$ :

**THEOREM 2.4'.** *Let  $G$  be as in Theorem 2.4. and  $x$  a central element of  $G$  of infinite order. Then*

$$H_i(G/\langle x \rangle; \mathbb{Q}) = 0 \quad \text{for } i \geq n.$$

**2.5. Groups of cohomology dimension  $\leq 2$ .** We write as usual  $cd_{\mathbb{Q}}G$  for the cohomology dimension of the group  $G$  over  $\mathbb{Q}$ ; i.e., with respect to all  $\mathbb{Q}G$ -module coefficients. The assumption  $cd_{\mathbb{Q}}G \leq 2$  includes all groups which are virtually of cohomology dimension 2 (over  $\mathbb{Z}$ ), but is more general; it of course implies  $hd_{\mathbb{Q}}G \leq 2$ .

The case  $cd_{\mathbb{Q}} \leq 1$  is easily dealt with: it means that  $G$  contains a free (normal) subgroup  $F$  of finite index. If  $x$  is a central element of infinite order then  $\langle x \rangle \cap F \neq 1$ ; hence  $F$  having non-trivial center must be cyclic  $= \langle c \rangle$ . Then both

$G/\langle x, F \rangle$  and  $\langle x, F \rangle/\langle x \rangle \cong F/F \cap \langle x \rangle$  are finite, and so is  $G/\langle x \rangle$ . Thus  $hd_{\mathbb{Q}} G/\langle x \rangle = 0$ . We thus restrict attention to the case  $cd_{\mathbb{Q}} G = 2$ .

**THEOREM 2.5.** *Let  $G$  be a group with  $cd_{\mathbb{Q}} G = 2$ , and  $x$  a central element of infinite order. Then*

$$H_i(G/\langle x \rangle; \mathbb{Q}) = 0 \quad \text{for } i \geq 2.$$

This is a consequence of various known facts concerning group (co-) homology, finiteness properties, and structure theorems (cf. Bieri [4], Corollary 8.7, and [5]). We give a short outline of the proof for our somewhat more special situation. It suffices to prove the claim for finitely generated  $G$ : If the subgroup  $S \subset G/\langle x \rangle$  is finitely generated so is its preimage  $T \subset G$ , and from the result for  $S = T/\langle x \rangle$  the direct limit argument yields the claim for  $G/\langle x \rangle$ .

As a first step one shows that  $G$  is of type  $FP_{\mathbb{Q}}$ ; i.e., that there exists a finitely generated projective resolution over  $\mathbb{Q}G$  (of length 2 since  $cd_{\mathbb{Q}} G = 2$ )

$$0 \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow \mathbb{Q} \rightarrow 0.$$

To prove this we use R. Strebel's finiteness criterion [11]:  $G$  is of type  $FP_{\mathbb{Q}}$  if and only if  $cd_{\mathbb{Q}} G$  is finite and the canonical map  $H^i(G; \bigoplus \mathbb{Q}G) \rightarrow \bigoplus H^i(G; \mathbb{Q}G)$  is an isomorphism for all  $i$  and all direct sums  $\bigoplus$ . In our case the spectral sequence for  $G/\langle x \rangle$  yields

$$H^1(G; \mathbb{Q}G) \cong H^0(G/\langle x \rangle; H^1(\langle x \rangle; \mathbb{Q}G))$$

and

$$H^2(G; \mathbb{Q}G) \cong H^1(G/\langle x \rangle; H^1(\langle x \rangle; \mathbb{Q}G))$$

(the action of  $G$  on  $H^1(\langle x \rangle; \mathbb{Q}G)$  is induced by the trivial conjugation of  $G$  on  $\langle x \rangle$ ). For the infinite cyclic group  $\langle x \rangle$  one has  $H^i(\langle x \rangle; \mathbb{Q}\langle x \rangle) = 0$  for  $i \neq 1$ , and  $H^1(\langle x \rangle; \mathbb{Q}\langle x \rangle) = \mathbb{Q}$  with trivial action; and  $H^1(\langle x \rangle; \mathbb{Q}G) = 0$  for  $i \neq 1$ ,  $H^1(\langle x \rangle; \mathbb{Q}G) = \mathbb{Q} \otimes_{\mathbb{Q}\langle x \rangle} \mathbb{Q}G \cong \mathbb{Q}(G/\langle x \rangle)$ .

For  $A = \bigoplus \mathbb{Q}G$  we get  $H^1(\langle x \rangle; A) = \bigoplus H^1(\langle x \rangle; \mathbb{Q}G)$ , and since  $G/\langle x \rangle$  is finitely generated, we see that  $G$  fulfills the Strebel criterion for  $i = 1$  and 2; in dimensions  $i \neq 1, 2$  this is trivially the case since  $H^i(G; \bigoplus \mathbb{Q}G) = 0 = \bigoplus H^i(G; \mathbb{Q}G)$ . Thus  $G$  is of type  $FP_{\mathbb{Q}}$ .

As a second step one draws more conclusions from the above formula for  $H^2(G; \mathbb{Q}G)$ . We note that

$$H^2(G; \mathbb{Q}G) = H^1(G/\langle x \rangle; \mathbb{Q}(G/\langle x \rangle)).$$

As  $G$  is of type  $FP_{\mathbb{Q}}$  with  $cdG = 2$ ,  $H^2(G; \mathbb{Q}G)$  is  $\neq 0$  and finitely generated as a right  $\mathbb{Q}G$ -module, and so is  $H^1(G/\langle x \rangle; \mathbb{Q}(G/\langle x \rangle))$  over  $\mathbb{Q}(G/\langle x \rangle)$ . This implies that  $G/\langle x \rangle$  has more than one end and is *accessible*; in other words,  $G/\langle x \rangle$  is the fundamental group of a finite graph of groups with finite edge groups, and with vertex groups  $V$  satisfying  $H^1(V; \mathbb{Q}V) = 0$  (1 or 0 ends).  $V$  is finitely generated, and so is its preimage  $W$  in  $G$ ,  $W/\langle x \rangle = V$ . As before we get  $H^2(W; \mathbb{Q}W) = H^1(V; \mathbb{Q}V)$ ; but now this is  $= 0$ , whence  $cd_{\mathbb{Q}} W = 1$ . The above formula for  $H^1(G; \mathbb{Q}G)$  applied to  $W$  and to  $W/\langle x \rangle = V$  yields  $H^1(W; \mathbb{Q}W) \cong H^0(V; \mathbb{Q}V) \neq 0$ . This implies that  $V$  is finite.

We thus have proved that  $G/\langle x \rangle$  is the fundamental group of a finite graph of finite edge *and* vertex groups. Such a group is well-known to contain a (normal) free subgroup of finite index; from the corresponding spectral sequence we obtain the required result

$$H_i(G/\langle x \rangle; \mathbb{Q}) = 0$$

for  $i \geq 2$ .

**2.6.** From Theorems 2.2, 2.3, 2.4', 2.5 we immediately obtain the result claimed in 1.3 for cyclic group homology:

**COROLLARY 2.6.** *Let  $G$  be a group with  $hd_{\mathbb{Q}} G = n < \infty$  and belonging to one of the classes (a), (b), (c), or (d),  $n = 2$  in the case (d). Then for  $i \geq n$  the cyclic homology  $HC_i(\mathbb{Q}G)$  vanishes on the conjugacy classes of elements of infinite order.*

### 3. Cyclic homology characters of $\mathbb{Q}G$

**3.1.** The Connes character  $Ch'_0$  of  $\mathbb{Q}G$  (cf. Karoubi [8]) is a homomorphism of  $K_0(\mathbb{Q}G)$  to  $HC_{2l}(\mathbb{Q}G)$ ,  $l = 0, 1, 2, \dots$ ; we will write  $Ch^l$  for  $Ch'_0$  since we will not consider here the higher characters  $Ch'_i$  (see however Remark 3.2). The  $Ch^l$  are compatible with the map  $S$  in the Connes–Gysin sequence (1.1), i.e.,

$$\begin{array}{ccc} & & HC_{2l}(\mathbb{Q}G) \\ & \nearrow Ch^l & \downarrow S \\ K_0(\mathbb{Q}G) & \xrightarrow{Ch^{l-1}} & HC_{2l-2}(\mathbb{Q}G) \end{array}$$

is commutative. Corollary 2.6 immediately yields

**THEOREM 3.1.** *Let  $G$  be a group of finite dimension  $hd_{\mathbb{Q}}G = n$  belonging to one of the classes (a), (b), (c) or (d). Then the characters  $Ch^l$ ,  $l = 0, 1, 2, \dots$  all have 0-components in the summands  $\bigoplus_{[x]}''$  corresponding to elements  $x$  of infinite order. In particular, if  $G$  is torsion-free, the  $Ch^l$  are concentrated on the  $[1]$ -summand, i.e. lie in  $[H_*(G; \mathbb{Q}) \otimes H_*(\mathbb{C}P^\infty; \mathbb{Q})]_{2l}$ .*

*Remark 3.2.* A similar result holds, of course, for the higher characters  $Ch_i^l: K_i(\mathbb{Q}G) \rightarrow HC_{2l+i}(\mathbb{Q}G)$ .

**3.2.** A look at the definitions shows that  $Ch^0: K_0(\mathbb{Q}G) \rightarrow HC_0(\mathbb{Q}G)$  is the same as the Hattori–Stallings rank, as follows.

By (1.2)  $HC_0$  is isomorphic to Hochschild homology  $HH_0$ . For any  $\mathbb{Q}$ -algebra  $\Lambda$  the latter, with  $\Lambda$  as bimodule by left- and right-multiplication for coefficients,  $HH_0(\Lambda)$  is well-known to be  $\Lambda/\{\lambda\mu - \mu\lambda\}$ , where  $\{\lambda\mu - \mu\lambda\}$  is the  $\mathbb{Q}$ -sub-module generated by all  $\lambda\mu - \mu\lambda$ ,  $\lambda, \mu \in \Lambda$ . In Remark 1.1 we have written  $\overline{\mathbb{Q}G}$  for  $HH_0(\mathbb{Q}G)$  and  $T: \mathbb{Q}G \rightarrow \overline{\mathbb{Q}G}$  for the canonical map. Similarly, for the matrix algebra  $\mathbb{M}_k(\mathbb{Q}G)$ , we have  $HH_0(\mathbb{M}_k(\mathbb{Q}G)) = \overline{\mathbb{M}_k(\mathbb{Q}G)}$  with  $T: \mathbb{M}_k(\mathbb{Q}G) \rightarrow \overline{\mathbb{M}_k(\mathbb{Q}G)}$ . The trace of matrices  $\text{tr}: \mathbb{M}_k(\mathbb{Q}G) \rightarrow \mathbb{Q}G$  induces an isomorphism  $\text{tr}: \overline{\mathbb{M}_k(\mathbb{Q}G)} \rightarrow \overline{\mathbb{Q}G}$ , and clearly  $T \circ \text{tr} = \text{tr} \circ T: \mathbb{M}_k(\mathbb{Q}G) \rightarrow \overline{\mathbb{Q}G}$ .

Now  $Ch^0$  is defined, on a finitely generated projective  $\mathbb{Q}G$ -module  $P$  representing an element of  $K_0(\mathbb{Q}G)$ , as follows: Let  $p$  be an idempotent matrix  $\in \mathbb{M}_k(\mathbb{Q}G)$ , for suitable  $k$ , describing  $P$  as a direct summand of a free  $\mathbb{Q}G$ -module  $M$ , and put  $Ch^0 p = \text{tr} \circ T(p) \in \overline{\mathbb{Q}G}$ , i.e.,  $= T \circ \text{tr}(p)$ . This is precisely the definition of the Hattori–Stallings rank  $r_p \in \overline{\mathbb{Q}G}$ , independent of choices and of bases in  $M$ . We recall that  $\overline{\mathbb{Q}G}$  is the  $\mathbb{Q}$ -module having the conjugacy classes  $[x]$  as basis.

**THEOREM 3.3.** *For the groups  $G$  of finite dimension  $hd_{\mathbb{Q}}G$  belonging to one of the classes (a), (b), (c), (d), the Hattori–Stallings rank  $r_p$  of a finitely generated projective  $\mathbb{Q}G$ -module  $P$  vanishes on the conjugacy classes of elements of infinite order.*

*Remark 3.4.* The vanishing of the character map  $Ch^l: K_0(\mathbb{Q}G) \rightarrow HC_{2l}(\mathbb{Q}G)$  on the conjugacy classes of elements of infinite order, in particular of the Hattori–Stallings rank  $Ch^0$ , would of course follow from properties much weaker than those established in Section 2 for certain classes of groups. Indeed it suffices that, for a group  $G$  under consideration, some iteration  $S^k: HC_{2l+2k}(\mathbb{Q}G) \rightarrow HC_{2l}(\mathbb{Q}G)$ ,  $1 \leq k \leq \infty$ , of  $S$  in (1.1) is zero on the conjugacy classes of elements of infinite order;  $k = \infty$  refers to the inverse limit. It has been conjectured, for

example, that this is the case for  $k = \infty$  and for all groups having a finite Eilenberg–MacLane complex, cf. [7].

*Note Added in Proof.* The proof of Theorem 2.3, without assuming  $G$  to be torsion-free (and hence also of Theorem 2.2), becomes much simpler if one uses the fact that the Hirsch number of a factor group of  $G$  is less or equal to that of  $G$ , combined with Stammbach's theorem [11].

## REFERENCES

- [1] R. ALPERIN and P. SHALEN, *Linear groups of finite cohomological dimension*, Bull. Am. Math. Soc. (New Series) 4, 339–341; Inventiones Math. 66, 89–98 (1982).
- [2] R. BAER and H. HEINEKEN, *Radical groups of finite Abelian subgroup rank*, Illinois J. of Math. 16, 533–586 (1972).
- [3] H. BASS, *Euler characteristics and characters of discrete groups*, Inventiones Math. 35, 155–196 (1976).
- [4] R. BIERI, *Homological dimension of discrete groups*, Queen Mary College Lecture Notes (1976).
- [5] R. BIERI, *On groups of cohomology dimension 2*, in Topology and Algebra (Enseignement Math. 1978), 55–62.
- [6] R. BIERI, *A connection between the integral homology and the centre of a rational linear group*, Math. Z. 170, 263–266 (1980).
- [7] D. BURGHELEA, *The cyclic homology of the group rings*, Comment. Math. Helv. 60, 354–365 (1985).
- [8] M. KAROUBI, *Homologie cyclique et K-théorie algébrique*, C.R. Acad. Sc. Paris, 297 S.I., 447–450 (1983).
- [9] S. MACLANE, *Homology* (Springer Verlag Berlin 1975), Chapt. X.
- [10] J.-P. SERRE, *Cohomologie des groupes discrets*, Annals of Math. Studies 70 (Princeton University Press 1971).
- [11] U. STAMMBACH, *On the weak homological dimension of the group algebra of solvable groups*, J. London Math. Soc. (2) 2, 567–570 (1970).
- [12] R. STREBEL, *A homological finiteness criterion*, Math. Z. 151, 263–275 (1976).
- [13] B. A. F. WEHRFRITZ, *Infinite linear groups* (Springer Verlag Berlin 1973).

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