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## On the volume of a unit vector field on the three-sphere

HERMAN GLUCK and WOLFGANG ZILLER

A unit vector field on a compact Riemannian manifold  $M$  can be pictured as a cross-section, and hence submanifold, of the unit tangent bundle  $T_1 M$ . We define the *volume of the vector field* to be the volume of this submanifold, measured in the natural Riemannian metric which  $T_1 M$  inherits from  $M$ . It can be expressed by the formula

$$\text{vol } V = \int_M \sqrt{\det(I + (\nabla V)(\nabla V)')} d \text{vol}_M,$$

in which we view the covariant derivative  $\nabla V$  as a linear transformation of the tangent space  $TM_x$  to itself.

One hopes that the “visually best organized” unit vector fields on  $M$  are rewarded with minimum possible volume. For example, it is clear that on the flat torus, the unit vector fields of minimum volume are precisely those of constant slope. But on the round three-sphere  $S^3$ , the story becomes more involved.

Consider the Hopf fibration  $H$  of  $S^3$ , whose fibres are the unit circles on the complex lines in  $R^4 = C^2$ . Any fibration congruent to this is also called a Hopf fibration, and a unit vector field  $V_H$  tangent to the fibres will be called a *Hopf vector field*. It is natural to regard these as visually the best organized unit vector fields on  $S^3$ . We will prove

**THEOREM.** *The unit vector fields of minimum volume on  $S^3$  are precisely the Hopf vector fields, and no others.*

The proof is by the method of “calibrated geometries” of Federer [F] and Harvey–Lawson [H–L], and is a one-time-deal which fails on the 5-sphere.

To carry out the argument, we will find a smooth closed 3-form  $\mu$  on the unit tangent bundle  $T_1 S^3$ , such that

$$\mu(u \wedge v \wedge w) \leq \text{vol}(u \wedge v \wedge w), \quad (*)$$

with equality holding for any properly oriented tangent 3-plane to a Hopf vector field  $V_H$ , viewed as a submanifold of  $T_1 S^3$ .

It will follow immediately that the Hopf vector fields are absolutely volume minimizing in their homology classes in  $T_1 S^3$ . For if  $M^3$  is a 3-manifold in the same homology class as  $V_H$ , then

$$\text{vol } V_H = \int_{V_H} \mu = \int_{M^3} \mu \leq \text{vol } M^3,$$

by equality in (\*), Stokes' theorem, and inequality in (\*), respectively. If  $V$  is another unit vector field on  $S^3$ , then it is easy to see that it is in the same homology class as  $V_H$  when viewed as a 3-dimensional submanifold of  $T_1 S^3$ , since the projection  $T_1 S^3 \rightarrow S^3$  is an isomorphism on 3-dimensional homology. Hence  $\text{vol } V_H \leq \text{vol } V$ , so the Hopf vector fields on  $S^3$  minimize volume.

Call an oriented 3-dimensional submanifold of  $T_1 S^3$  a  $\mu$ -submanifold if the equality  $\mu(u \wedge v \wedge w) = \text{vol}(u \wedge v \wedge w)$  holds for each of its tangent planes. By examining the 3-planes for which this equality holds, we will find other  $\mu$ -submanifolds besides the  $V_H$ . But they lie in other homology classes in  $T_1 S^3$  and hence do not come from vector fields. Furthermore, they all have volumes  $> \text{vol } V_H$ . Since the  $\mu$ -submanifolds are the *only* volume minimizing submanifolds in their homology classes, it will follow that the Hopf vector fields are the only volume minimizing unit vector fields on  $S^3$ , completing the proof of the theorem.

The family  $\{V_H\}$  of Hopf vector fields is invariant under the group of isometries of the unit tangent bundle  $T_1 S^3$ . Hence if there is any form  $\mu$  on  $T_1 S^3$  which "calibrates" the Hopf vector fields, as above, then we can average it over the group and obtain an isometry-invariant form which does the same. Hence there is no loss in restricting our search for  $\mu$  to the isometry-invariant forms. The advantage is that such forms are explicitly calculable. It turns out that there is, up to constant multiple, just one isometry-invariant closed 3-form, and it does the job.

The drawback to using the method of calibrated geometries for this problem is that we must prove a little more than we want: the  $V_H$  have minimum volume among all 3-manifolds in the same homology class in  $T_1 S^3$  whether or not these 3-manifolds come from unit vector fields on  $S^3$ . As a result, the *method* will fail on the five-sphere  $S^5$ , because there is a 5-manifold in  $T_1 S^5$  in the same homology class as  $2V_H$ , but with less volume. The corresponding isometry invariant closed 5-form  $\mu$  on  $T_1 S^5$  provides a calibrated geometry which distinguishes these submanifolds instead of the Hopf vector fields. And likewise on  $S^7, S^9, S^{11}, \dots$

Whether the theorem itself remains true on these higher dimensional spheres, we do not know. We can, however, use the method of calibrated geometries to see a little in this direction.

If  $V$  is a parallel vector field on a compact Riemannian manifold  $M$ , then

$\text{vol } V = \text{vol } M$ . It is natural to ask: if  $M$  admits no parallel vector fields, is  $\text{vol } V$  bounded away from  $\text{vol } M$  for any unit vector field  $V$  on  $M$ ? We will observe that

$$\text{vol } V \geq 2 \text{ vol sphere}$$

for any unit vector field on a unit sphere. By contrast,

$$\text{vol } V_H = 2^n \text{ vol } S^{2n+1},$$

so that starting on  $S^5$  the above inequality is much weaker than the expected one. Nevertheless, this inequality reports that all unit vector fields on an odd-dimensional round sphere fail to be parallel by at least a certain amount.

When trying to show that nicely organized submanifolds minimize volume in their homology classes, it is good to keep in mind the following simple example, which shows that higher dimensions can frustrate the attempt.

The diagonal in  $S^1 \times S^1$  has length equal to  $\sqrt{2}$  times that of  $S^1$ , and certainly minimizes length in its homology class. The diagonal in  $S^2 \times S^2$  has linear dimensions multiplied by  $\sqrt{2}$ , and hence

$$\text{area diag } (S^2 \times S^2) = 2 \text{ area } S^2.$$

The diagonal still minimizes area in its homology class, but now the union

$$S^2 \times \text{point} \cup \text{point} \times S^2,$$

which lies in the same homology class, has the same area. Moving up one more dimension, we get

$$\text{vol diag } (S^3 \times S^3) = 2\sqrt{2} \text{ vol } S^3,$$

and now it is

$$S^3 \times \text{point} \cup \text{point} \times S^3,$$

and no longer the diagonal, which minimizes volume in its homology class.

Exactly this phenomenon is at work in the following circumstance. Define the *volume of a map*  $f: M \rightarrow N$  between Riemannian manifolds to be the volume of its graph, considered as a submanifold of  $M \times N$ . It follows from the work of Walter Wei [W] that the Hopf map  $h: S^3 \rightarrow S^2$  does *not* have minimum volume in its homotopy class. Indeed, we will derive in section 4 a general inequality for the

volume of a fibre bundle map over a surface, and use it to display a large family of mutually homotopic maps from  $S^3 \rightarrow S^2$ , amongst which the Hopf map has *maximum* volume.

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## 1. Finding the 3-form $\mu$

We divide the proof of the main theorem into two parts. In this section we find the closed invariant 3-form  $\mu$  on the unit tangent bundle  $T_1 S^3$ . In the next we will complete the argument by finding the 3-dimensional submanifolds calibrated by  $\mu$  and noting that the Hopf vector fields, and no others, have minimum volume among them.

We begin by summarizing the geometry of the situation. The points of the unit tangent bundle  $T_1 S^3$  may be regarded as pairs  $(x, y)$  of orthogonal unit vectors from  $S^3$ . The same is true for the Stiefel manifold  $V_2 R^4$  of orthonormal two-frames  $(x, y)$  in  $R^4$ . Hence as sets, these five-dimensional manifolds are identical. As topological spaces, they are homeomorphic to  $S^3 \times S^2$ , since the 3-sphere is parallelizable.

The natural Riemannian metric on  $T_1 S^3$ , defined in terms of covariant derivatives of vector fields, is the same as the one it inherits as the homogeneous space  $SO(4)/SO(2)$ . The natural Riemannian metric on  $V_2 R^4$  is the one it inherits as a subspace of  $R^8$ . These two metrics are not identical; we will compare them in a moment. But neither one is the product metric on  $S^3 \times S^2$ .

The *geodesic flow*  $g_t$  on the unit tangent bundle  $T_1 S^3$  is defined by

$$g_t(x, y) = (x \cos t + y \sin t, -x \sin t + y \cos t).$$

It is an  $SO(2)$  action by isometries in either metric.

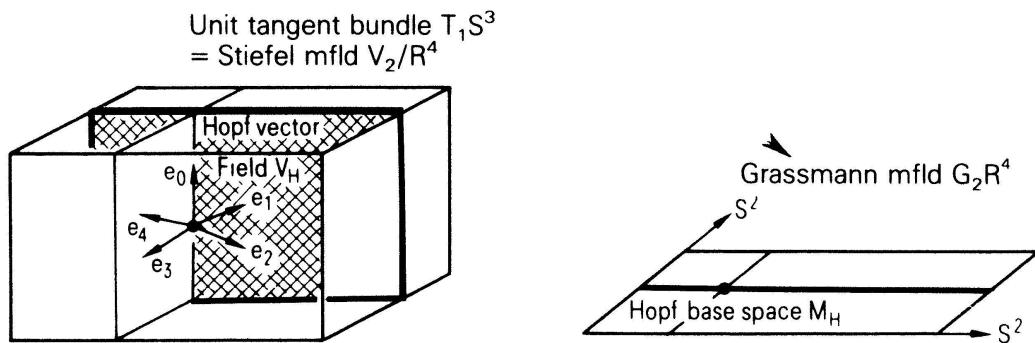
One passes from the natural Riemannian metric on  $T_1 S^3$  to that on  $V_2 R^4$  by multiplying lengths in the direction of the geodesic flow by  $\sqrt{2}$ , with no change in directions orthogonal to this. Our main theorem is true regardless of which of these two metrics is used to measure the volume of vector fields.

The Stiefel manifold  $V_2 R^4$  sits as a circle bundle over the Grassmann manifold  $G_2 R^4$  of oriented two-planes in four-space, with each orthonormal two-frame  $(x, y)$  sitting over the oriented two-plane  $x \wedge y$  which it spans. The projection map is a Riemannian submersion, that is, its differential preserves lengths of tangent vectors orthogonal to fibres. The Grassmann manifold  $G_2 R^4$  is isometric to  $S^2 \times S^2$ , with each factor a round two-sphere of radius  $1/\sqrt{2}$ .

In the Grassmann manifold, the base spaces  $M_H$  of the various Hopf fibrations  $H$  of  $S^3$  appear as  $S^2 \times \text{point}$  and as  $\text{point} \times S^2$ . There is just one of each kind passing through any given point of the Grassmann manifold. See [G-W] for details.

Up in the Stiefel manifold, the Hopf vector fields  $V_H$  appear as totally geodesic round three-spheres of radius  $\sqrt{2}$ , sitting over the base spaces  $M_H$  in the Grassmann manifold.

We try to summarize much of this information in the following figure, which also includes an orthonormal set of tangent vectors at one point, to be used in a moment.



Now we set about finding our isometry-invariant closed 3-form  $\mu$  on the unit tangent bundle. Each isometry  $g: S^3 \rightarrow S^3$  has an induced action on  $T_1 S^3 = V_2 R^4$ ,

$$(x, y) \rightarrow (g(x), g(y)),$$

which is an isometry in either metric. Such isometries preserve both the  $S^2$  fibres of the unit tangent bundle and the  $S^1$  fibres of the Stiefel bundle. It will be sufficient for our purposes to restrict attention to those  $g$  which are orientation preserving. So far this gives us an  $SO(4)$  action.

In addition, the circle group  $SO(2)$  acts on  $T_1 S^3 = V_2 R^4$  by the geodesic flow, again isometries in either metric. Such isometries preserve the  $S^1$  fibres of the Stiefel bundle, but do *not* preserve the  $S^2$  fibres of the unit tangent bundle.

Since these two actions commute, we get an action of

$$G = SO(4) \times SO(2)$$

on  $T_1 S^3 = V_2 R^4$  by isometries in either metric.  $G$  is simply the identity component of the full isometry group.

We next find the  $G$ -invariant differential forms on  $T_1 S^3 = V_2 R^4$ .

Since  $G$  acts transitively, we simply calculate those linear forms on the tangent

space to  $T_1 S^3 = V_2 R^4$  at a single point which are invariant under the action of the isotropy subgroup of  $G$ . This isotropy subgroup must be two-dimensional, since  $G$  is seven-dimensional and  $T_1 S^3 = V_2 R^4$  is five-dimensional. In fact, it is isomorphic to  $SO(2) \times SO(2)$ , and operates by independently spinning the  $e_1 e_2$ -plane and the  $e_3 e_4$ -plane, while keeping the  $e_0$ -axis fixed. See figure above.

By abuse of language, we use the symbols  $e_i$  to denote both tangent vectors and dual one-forms. We easily get the following table.

Table of invariant forms and their exterior derivatives

Dimension 1	$e_0$	$\overrightarrow{-d}$	$2(e_1 \wedge e_2 + e_3 \wedge e_4)$
Dimension 2	$e_1 \wedge e_2$	$\overrightarrow{-d}$	0
	$e_3 \wedge e_4$	$\overrightarrow{-d}$	0
Dimension 3	$e_0 \wedge e_1 \wedge e_2$	$\overrightarrow{-d}$	$2e_1 \wedge e_2 \wedge e_3 \wedge e_4$
	$e_0 \wedge e_3 \wedge e_4$	$\overrightarrow{-d}$	$2e_1 \wedge e_2 \wedge e_3 \wedge e_4$
Dimension 4	$e_1 \wedge e_2 \wedge e_3 \wedge e_4$	$\overrightarrow{-d}$	0
Dimension 5	$e_0 \wedge e_1 \wedge e_2 \wedge e_3 \wedge e_4$	$\overrightarrow{-d}$	0

Note that the invariant one-form  $e_0$  represents inner product with a unit vector tangent to the Stiefel fibres, and is hence the *connection form*  $\nu$  of the Stiefel bundle.

Note that the invariant forms on  $T_1 S^3 = V_2 R^4$  which occur in dimensions two and four are missing the  $e_0$ -factor. They represent the pullbacks to the Stiefel manifold of the corresponding invariant forms down on the Grassmann manifold  $G_2 R^4$ . Down there,  $e_1 \wedge e_2$  and  $e_3 \wedge e_4$  are the volume forms of  $S^2 \times \text{point}$  and  $\text{point} \times S^2$ , respectively, while  $e_1 \wedge e_2 \wedge e_3 \wedge e_4$  is the volume form of  $S^2 \times S^2 = G_2 R^4$ .

Embed the Grassmann manifold  $G_2 R^4$  in  $CP^3$  in the usual way by sending  $x \wedge y$ , where  $x$  and  $y$  are orthonormal, to the complex line through  $x + iy$  in  $C^4$ . The image is the complex hyperquadric

$$z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0,$$

where  $z = x + iy$ . In this way the Grassmann manifold inherits the complex structure  $J$  and the corresponding Kähler 2-form  $\omega$  from  $CP^3$ . It is easy to check that

$$J(e_1) = e_2 \quad \text{and} \quad J(e_3) = e_4.$$

Hence

$$\omega = e_1 \wedge e_2 + e_3 \wedge e_4$$

is the Kähler 2-form on the Grassmann manifold.

Refer again to the above table and note that the even-dimensional forms, which are pulled back from the Grassmann manifold, are already closed. This happens because the Grassmann manifold is a symmetric space, and hence every invariant form is closed. By contrast, the invariant forms on the Stiefel manifold in dimensions one and three are not all closed. But clearly the cohomology computed from the invariant forms is the same as the deRham cohomology.

Given the preceding table of  $G$ -invariant forms and their derivatives on  $T_1 S^3 = V_2 R^4$ , we naturally choose

$$\mu = e_0 \wedge e_1 \wedge e_2 - e_0 \wedge e_3 \wedge e_4.$$

This form is closed and generates the 3-dimensional cohomology. We can write

$$\mu = v \wedge \lambda,$$

where  $v = e_0$  is the connection form of the Stiefel bundle and where  $\lambda = e_1 \wedge e_2 - e_3 \wedge e_4$  generates the 2-dimensional cohomology of the Stiefel manifold, and is the pullback of a closed form which together with the Kähler form  $\omega$  generates the 2-dimensional cohomology of the Grassmann manifold. Finally, note that

$$dv = 2\omega$$

up in the Stiefel manifold.

## 2. Finding what $\mu$ calibrates

Having selected the closed  $G$ -invariant 3-form

$$\mu = e_0 \wedge e_1 \wedge e_2 - e_0 \wedge e_3 \wedge e_4$$

on  $T_1 S^3 = V_2 R^4$ , we face the following tasks:

A) Show that  $\mu(u \wedge v \wedge w) \leq \text{vol}(u \wedge v \wedge w)$ . Then we will know that  $\mu$  is a calibrating form.

- B) Find out what  $\mu$  calibrates infinitesimally. This means finding those oriented 3-planes in 5-space for which the above inequality is actually an equality.
- C) Find out what  $\mu$  calibrates globally. This means finding those oriented 3-manifolds in  $T_1 S^3 = V_2 R^4$  which are tangent to such 3-planes at each point.

To begin, we write

$$\mu = e_0 \wedge (e_1 \wedge e_2 - e_3 \wedge e_4) = \nu \wedge \lambda,$$

as before.

It is straightforward linear algebra to check that  $\mu$  is calibrating, and that infinitesimally it calibrates precisely the 3-planes which contain the  $e_0$ -axis and which meet the  $e_1 e_2 e_3 e_4$ -space in the graph of an anticonformal map from the  $e_1 e_2$ -plane to the  $e_3 e_4$ -plane (including the  $e_3 e_4$ -plane itself). After all, except for the minus sign,  $\lambda = e_1 \wedge e_2 - e_3 \wedge e_4$  is the usual Kähler 2-form in real 4-space, and multiplication by the new variable  $e_0$  has the expected effect.

Suppose the oriented 3-manifold  $M^3$  in  $T_1 S^3 = V_2 R^4$  is calibrated by our form  $\mu = e_0 \wedge e_1 \wedge e_2 - e_0 \wedge e_3 \wedge e_4$ . Infinitesimally, this means that each tangent space to  $M^3$  contains the  $e_0$ -axis, which is itself tangent to the Stiefel fibres. Globally, this means that  $M^3$  is a union of Stiefel fibres, and hence the inverse image of a submanifold  $M^2$  down in the Grassmannian  $G_2 R^4$ . And this submanifold  $M^2$  must in turn be calibrated by the invariant 2-form  $\lambda = e_1 \wedge e_2 - e_3 \wedge e_4$  on  $G_2 R^4$ .

The usual complex structure  $J$  on the Grassmann manifold  $G_2 R^4$  is defined by  $J(e_1) = e_2$  and  $J(e_3) = e_4$ . Define another complex structure  $J^*$  there by  $J^*(e_1) = e_2$  and  $J^*(e_3) = -e_4$ . Then the 2-form  $\lambda$  is the Kähler form of the complex structure  $J^*$ , and hence calibrates the  $J^*$ -complex submanifolds of  $G_2 R^4$ . Each such  $J^*$ -complex submanifold  $M^2$  minimizes area in its homology class. Its inverse image  $M^3$  in the Stiefel manifold is calibrated by our 3-form  $\mu$  and minimizes volume in its homology class. In fact, the volume of  $M^3$  is simply the length of a Stiefel fibre times the area of  $M^2$ .

So we come to the conclusion: *our 3-form  $\mu$  calibrates those oriented 3-manifolds in  $T_1 S^3 = V_2 R^4$  which are inverse images under the Stiefel projection of the  $J^*$ -complex submanifolds of  $G_2 R^4$ .*

It is clear that the submanifold  $M^2$  of  $G_2 R^4$  has minimum area (over all nontrivial homology classes) precisely when it equals  $S^2 \times \text{point}$  or  $\text{point} \times S^2$ , in which case its inverse image  $M^3$  is a Hopf vector field  $V_H$ . Since all unit vector fields  $V$  on  $S^3$  represent the same non-trivial 3-dimensional homology class when

viewed as submanifolds of  $T_1 S^3 = V_2 R^4$ , this gives the desired result:

*The unit vector fields of minimum volume on  $S^3$  are the Hopf vector fields, and no others.*

### 3. Why the method of calibrated geometries fails in higher dimensions

In this section we will see that there is a 5-dimensional submanifold of  $T_1 S^5 = V_2 R^6$  in the same homology class as  $2V_H$ , but with less volume. If there were any closed 5-form on the unit tangent bundle  $T_1 S^5$  (isometry-invariant or not) which calibrated the Hopf vector fields  $V_H$ , then automatically  $kV_H$  would be the “manifold” of minimum volume in the homology class  $k[V_H]$ . Since this is not the case for  $k = 2$ , the method of calibrated geometries can not be used to show that the Hopf vector fields on  $S^5$  have minimum volume. The same holds on  $S^7$ ,  $S^9$ ,  $S^{11}$ , . . . .

To produce this 5-manifold inside  $T_1 S^5$ , start with a single fibre  $F^4$  of the unit tangent bundle  $T_1 S^5 \rightarrow S^5$ . It is a totally geodesic round 4-sphere of radius 1. Flow it by the geodesic flow  $g_t$  to produce the 5-dimensional submanifold  $L^5$  of  $T_1 S^5$ .

We can see  $L^5$  another way. Take the fibre  $F^4$  in  $T_1 S^5$  and view it in  $V_2 R^6$ , where it now appears horizontal. Use the Stiefel projection to project it to a totally geodesic round 4-sphere  $L^4$  of radius 1 in the Grassmann manifold  $G_2 R^6$ .  $L^4$  represents the set of all oriented 2-planes in 6-space which can be obtained from a given one by rotating it about a given line therein. Then  $L^5$  is simply the inverse image of  $L^4$  under the Stiefel projection, because the orbits of the geodesic flow on  $T_1 S^5$  are the same as the fibres of the Stiefel bundle  $V_2 R^6$ .

In the Stiefel manifold,  $L^5$  is isometric to  $S^4(1) \times S^1(\sqrt{2})$ . In the unit tangent bundle, it is isometric to  $S^4(1) \times S^1(1)$ . These isometries follow immediately from the parametrization of  $L^5$  given below.

We claim that  $L^5$ , properly oriented, represents the 5-dimensional homology class  $2[V_H]$  in the unit tangent bundle  $T_1 S^5$ .

Suppose that  $F^4$  is the fibre of the unit tangent bundle over the point  $x_0$  on  $S^5$ . Thus

$$F^4 = \{(x_0, y) : y \in S^5, \langle x_0, y \rangle = 0\}.$$

Applying the geodesic flow,  $L^5$  can be viewed as the image of  $S^4 \times S^1$  under the map

$$(y, \theta) \rightarrow (x_0 \cos \theta + y \sin \theta, -x_0 \sin \theta + y \cos \theta).$$

The projection  $T_1 S^5 \rightarrow S^5$  is an isomorphism on 5-dimensional homology, so we simply need to check the degree of the map

$$(y, \theta) \rightarrow x_0 \cos \theta + y \sin \theta.$$

Clearly the 5-sphere is covered once for  $0 \leq \theta \leq x$ . Note that the above map takes  $(y, \theta)$  and  $(y, \theta + \pi)$  to antipodal points. Since the antipodal map on  $S^5$  has degree 1, our map must have degree 2. The corresponding map for a Hopf vector field  $V_H$  in place of  $L^5$  has degree 1, and the claim follows:  $[L^5] = 2[V_H]$ .

In contrast to this, we claim that

$$\text{vol } L^5 < 2 \text{ vol } V_H.$$

First we compute  $\text{vol } V_H$ . For each complex structure  $J$  on  $R^6$  we have a Hopf fibration of  $S^5$  by the unit circles on the corresponding complex lines, and a Hopf vector field  $V_H = \{(x, Jx) : x \in S^5\}$ . Since  $J$  is an isometry,  $V_H$  is a round 5-sphere of radius  $\sqrt{2}$  in the Stiefel manifold  $V_2 R^6$ . One easily calculates that the unit 5-sphere has volume  $\pi^3$ , and hence  $\text{vol } V_H = 4\sqrt{2} \pi^3$ .

Next we compute  $\text{vol } L^5$ . Viewed in the Stiefel manifold, this submanifold is isometric to  $S^4(1) \times S^1(\sqrt{2})$ . Since  $\text{vol } S^4 = (8/3)\pi^2$ , we have

$$\text{vol } L^5 = (8/3)\pi^2 \times 2\pi\sqrt{2} = 5\frac{1}{3}\sqrt{2} \pi^3,$$

verifying the claim.

Thus we have found a submanifold  $L^5$  of  $T_1 S^5$  in the same homology class as  $2V_H$ , but with less volume. Hence the method of calibrated geometries can not be used to show that the Hopf vector fields on  $S^5$  have minimum volume. The same holds on  $S^7, S^9, S^{11}, \dots$

Nevertheless we can carry out the search for  $G$ -invariant forms on  $T_1 S^{2n+1} = V_2 R^{2n+2}$  for all  $n$ , where now  $G$  is the group  $SO(2n+2) \times SO(2)$  of isometries.

In our earlier problem on  $S^3$ , there was no ambiguity (except for sign) in the choice of  $G$ -invariant calibrating 3-form

$$\mu = \nu \wedge \lambda = e_0 \wedge e_1 \wedge e_2 - e_0 \wedge e_3 \wedge e_4.$$

The same thing happens on  $S^{2n+1}$ : there is a unique (up to sign)  $G$ -invariant calibrating  $2n+1$ -form  $\mu$  on the Stiefel manifold  $V_2 R^{2n+2}$ , and it too can be written as  $\nu \wedge \lambda$ , where  $\nu$  is the connection form of the Stiefel manifold and where  $\lambda$  is the pullback of a  $G$ -invariant  $2n$ -form (also written  $\lambda$ ) from the Grassmann manifold  $G_2 R^{2n+2}$ . Down there,  $\lambda$  represents the “other” generator in

the middle dimensional cohomology  $H^{2n}(G_2 R^{2n+2}) \cong \mathbb{Z} + \mathbb{Z}$ , that is, other than the  $n$ th power  $\omega^n$  of the Kähler form. This becomes precise when we ask in addition that  $\lambda$  be in the kernel of exterior multiplication by the Kähler form  $\omega$ . And this in turn is what makes the  $2n + 1$  form  $\mu = \nu \wedge \lambda$  closed:

$$d\mu = d(\nu \wedge \lambda) = d\nu \wedge \lambda = 2\omega \wedge \lambda = 0.$$

The  $2n + 1$  form  $\mu$  provides a calibrated geometry on the Stiefel manifold, while the  $2n$ -form  $\lambda$  provides one on the Grassmann manifold. We know from our previous discussion that  $\mu$  can not calibrate the Hopf vector fields when  $n \geq 2$ . Defining the submanifolds  $L^{2n+1} = S^{2n}(1) \times S^1(\sqrt{2})$  of the Stiefel manifold  $V_2 R^{2n+2}$ , and  $L^{2n} = S^{2n}(1)$  of the Grassmann manifold  $G_2 R^{2n+2}$  just as we did above for  $n = 2$ , we will prove in [G–M–Z] the

**PROPOSITION.** *For  $n \geq 2$ : The  $2n + 1$  form  $\mu$  on the Stiefel manifold calibrates the submanifolds  $L^{2n+1}$  and nothing else. The  $2n$ -form  $\lambda$  on the Grassmann manifold calibrates the submanifolds  $L^{2n}$  and nothing else.*

Note that the subgroup  $G' = SO(2n + 2)$  of  $G = SO(2n + 2) \times SO(2)$  still acts transitively on  $T_1 S^{2n+1} = V_2 R^{2n+2}$ . If we look for  $G'$ -invariant closed  $2n + 1$  forms, the choice is much wider, and includes e.g. the pullback to the unit tangent bundle of the volume form on  $S^{2n+1}$ . Choosing an appropriate  $G'$ -invariant calibrating  $2n + 1$  form, one easily obtains

$$\text{vol } V \geq 2 \text{ vol sphere}$$

for a unit vector field  $V$  on a unit sphere.

We can do a little better. Let  $V$  be a unit vector field on  $S^{2n+1}$  and  $\mu$  the calibrating  $2n + 1$  form on  $T_1 S^{2n+1}$  mentioned above. Then

$$\text{vol } V \geq \int_V \mu = \int_{V_H} \mu = c(n) \text{ vol } S^{2n+1}.$$

It follows from an explicit formula for  $\mu$  given in [G–M–Z] that

$$c(n) = \sum_{k=0}^n \binom{n}{k}^2 / \binom{2n}{2k}$$

For example,  $c(1) = 2$ ,  $c(2) = 2^2_3$ ,  $c(3) = 3^1_5$ , . . . .

By contrast

$$\text{vol } V_H = 2^n \text{ vol } S^{2n+1},$$

so that starting on  $S^5$ , the above inequalities are much weaker than the expected ones.

Nevertheless, the inequalities report that all unit vector fields on a round sphere of any dimension fail to be parallel by at least a certain minimum amount.

#### 4. An inequality for the volume of a fibre bundle map over a surface

Our goal here is to prove the following

**PROPOSITION.** *Suppose  $f: M^m \rightarrow N^2$  is a fibration of the compact Riemannian manifold  $M^m$  over the compact surface  $N^2$ . Then*

$$\text{vol } f \geq \text{vol } M + (\text{average vol fibre}) (\text{area } N),$$

*with equality if and only if  $f$  is a conformal submersion.*

Recall that we defined the *volume* of a map  $f: M \rightarrow N$  between Riemannian manifolds to be the volume of its graph in  $M \times N$ .

**EXAMPLE.** Among all maps  $f: S^2 \rightarrow S^2$  of nonzero degree, the conformal and anticonformal homeomorphisms have minimum volume.

**EXAMPLE.** Among all maps  $f: S^3 \rightarrow S^3$  of nonzero degree, there are *none* of minimum volume. All have volume  $> 2 \text{ vol } S^3$ . Some maps have volumes approaching this lower limit, but none equal it. The identity map  $S^3 \rightarrow S^3$  has volume  $= 2\sqrt{2} \text{ vol } S^3$ .

Homotopically nontrivial maps  $f: S^3 \rightarrow S^2$  behave as in the previous example: there are *none* of minimum volume. All have volume  $> \text{vol } S^3$ . Using the *equality* in the above proposition, we will see some very beautiful maps which have volumes approaching this lower limit. By contrast, the Hopf map  $h: S^3 \rightarrow S^2$  has volume  $= 2 \text{ vol } S^3$ .

Let  $f: M^m \rightarrow N^n$  again be a smooth map between Riemannian manifolds. For each  $x \in M$ , we define a pseudo-norm  $|df_x|$  for the differential of  $f$  at  $x$ , as follows. Pick an orthonormal basis  $e_1, e_2, \dots, e_m$  for the tangent space  $TM_x$ . If  $m \leq n$ ,

define

$$|df_x| = |df_x(e_1) \wedge \cdots \wedge df_x(e_m)|,$$

the usual norm in  $\bigwedge^m TN_f(x)$ . But if  $m > n$ , pick the basis for  $TM_x$  so that  $e_{n+1}, \dots, e_m$  belong to the kernel of  $df_x$ , which is at least  $m - n$  dimensional. Then define

$$|df_x| = |df_x(e_1) \wedge \cdots \wedge df_x(e_n)|.$$

Note that  $|df_x|$  is nonzero when the dimension of  $\ker df_x$  is exactly  $m - n$ , and is zero when the dimension of the kernel is  $> m - n$ .

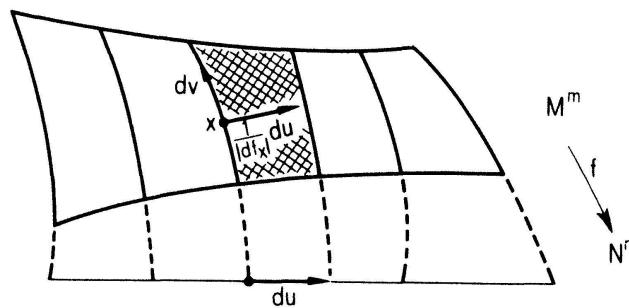
Now define the *image volume* of the map  $f$  by integrating the above pseudo-norm of its differential over the domain  $M$ :

$$\text{image vol } f = \int_M |df_x| d \text{ vol}.$$

EXAMPLE. If  $f: M \rightarrow N$  is an embedding, then the image volume of  $f$  is simply the volume of  $f(M)$  as a submanifold of  $N$ .

LEMMA. Suppose  $f: M^m \rightarrow N^n$  is a fibration between compact Riemannian manifolds. Then

$$\text{image vol } f = (\text{average vol fibre}) (\text{vol } N).$$



Referring to the picture above, we have

$$d \text{ vol}_x = |df_x|^{-1} du dv,$$

where  $du$  and  $dv$  are the volume forms on base and fibre, respectively.

Integrating over  $m$ , we get

$$\begin{aligned}\text{image vol } f &= \int_M |df_x| d\text{vol}_x = \int_M du dv \\ &= \int_N \left( \int_{\text{fibre}} dv \right) du = \int_N (\text{vol of fibre}) du \\ &= (\text{average vol of fibre}) (\text{vol } N),\end{aligned}$$

as claimed.

Again let  $f: M^m \rightarrow N^n$  be a smooth map between compact Riemannian manifolds, with  $m \geq n$ . Given  $x \in M$ , let  $e_1, \dots, e_n, e_{n+1}, \dots, e_m$  be an orthonormal basis for the tangent space  $TM_x$ , chosen so that  $e_{n+1}, \dots, e_m$  belong to the kernel of  $df_x$ . Suppose that on the  $n$ -plane spanned by  $e_1, \dots, e_n$ , the map  $df_x$  is conformal, and suppose this is true for each  $x \in M$ . Then we call  $f$  a *conformal submersion*. If the constant of conformality is never zero, then  $f$  is a submersion in the usual sense, and hence a fibration.

**EXAMPLE.** A Riemannian submersion  $f: M^n \rightarrow N^n$  is a submersion whose differential is an isometry on subspaces orthogonal to the fibres. An example is the Hopf fibration  $h: S^3 \rightarrow S^2(1/2)$ . Any Riemannian submersion is also a conformal submersion.

**EXAMPLE.** Consider the composite map

$$S^3 \xrightarrow{g} S^3 \xrightarrow{h} S^2 \xrightarrow{g'} S^2,$$

where  $g$  is a conformal homeomorphism of the three-sphere,  $h$  is the Hopf map, and  $g'$  is a conformal homeomorphism of the two-sphere. Any such map is a conformal submersion.

**LEMMA.** *Let  $f: M^m \rightarrow N^2$  be a smooth map of a compact Riemannian manifold to a compact surface. Then*

$$\text{vol } f \geq \text{vol } M^m + \text{image vol } f,$$

*with equality if and only if  $f$  is a conformal submersion.*

Given  $x \in M^m$ , we choose an orthonormal basis  $e_1, e_2, e_3, \dots, e_m$  for the tangent space  $TM_x$  so that  $e_3, \dots, e_m$  belong to the kernel of  $df_x$ . Then the volume element of the graph of  $f$  is

$$\begin{aligned} & |(e_1 + df_x e_1) \wedge (e_2 + df_x e_2) \wedge e_3 \wedge \cdots \wedge e_n| \\ &= \sqrt{1 + |df_x e_1|^2 + |df_x e_2|^2 + |df_x e_1 \wedge df_x e_2|^2} \\ &\geq 1 + |df_x e_1 \wedge df_x e_2|, \end{aligned}$$

by elementary linear algebra, with equality if and only if  $df_x$  is a conformal map of the  $e_1 e_2$ -plane to the tangent plane  $TN_{f(x)}$ .

Integrating this inequality over  $M^m$  proves the lemma.

Putting the preceding two lemmas together, we get the proposition stated at the beginning of this section.

If we apply the *equality* in the proposition to the conformal submersion

$$f = g' h g : S^3 \rightarrow S^2$$

defined above, we get

$$\text{vol } f = \text{vol } S^3 + (\text{average length fibre}) (\text{area } S^2).$$

If  $g$  and  $g'$  are the identity maps, then  $f$  is the Hopf map  $h$  and we get

$$\text{vol } h = 2 \text{ vol } S^3.$$

Now choose the conformal homeomorphism  $g$  of  $S^3$  so that it takes a very small circle to one of the Hopf circles. Then choose the conformal homeomorphism  $g'$  of  $S^2$  so that it spreads a small neighborhood of the point corresponding to this Hopf circle over most of the two-sphere. The composite map  $f = g' h g$  then has average fibre length very small. As a result,  $\text{vol } f$  is very close to  $\text{vol } S^3$ . If we keep the homeomorphisms  $g$  and  $g'$  orientation preserving, then  $f$  is in the same homotopy class as the Hopf map  $h$ , yet has smaller volume. Amusingly,  $\text{vol } h$  is the *maximum* volume among all maps  $f$  of this type. The limiting value of  $\text{vol } f$ , namely  $\text{vol } S^3$ , can never be achieved for a map homotopic to  $h$ .

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