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# Changes of sign of error terms related to Euler's function and to divisor functions

Y.-F. S. PÉTERMANN

## 1. Introduction

Let

$$R(x) := \Phi(x) - \frac{3}{\pi^2}x^2 \quad (x \geq 1), \quad (1)$$

where  $\Phi(x) := \sum_{n \leq x} \phi(n)$  and  $\phi(n)$  is Euler's function. If one computes values of  $R(n)$  and of

$$R(n-) := \sum_{m < n} \phi(m) - \frac{3}{\pi^2}n^2 = R(n) - \phi(n),$$

one comes to suspect that  $R(x)$  changes sign very frequently between consecutive integers, but that there are very few integers  $n$  for which  $R(n) < 0$ .

Sylvester even conjectured in 1883 ([32] and [33]; the reference to [31] in [7] and [20] is mistaken) that  $R(n) > 0$  for all positive integers  $n$ . But [33] contains a table of  $\phi(n)$ ,  $\Phi(n)$  and  $3n^2/\pi^2$  for  $1 \leq n \leq 1000$ ; Sylvester does not seem to have noticed that the entries  $\Phi(820) = 204376$  and  $3 \cdot 820^2/\pi^2 = 204385.09 \dots$  disprove his conjecture. Sarma [23] (attributing the conjecture to Pillai and Chowla) rediscovered this counterexample in 1931.

Let  $X_R(x)$  denote the number of changes of sign of  $R(t)$  in the interval  $1 < t < x$ , and  $N_R(x)$  the number of changes of sign of  $R(n)$  on the integers  $n$  with  $1 < n < x$  (i.e. the number of integers  $n$ ,  $1 < n < x$ , such that  $R(n)R(n-1) < 0$ ).

In 1967 Erdős conjectured [5] that

$$N_R(x) = Cx + o(x) \quad (x \rightarrow \infty) \quad (2a)$$

for some positive constant  $C$ ; in 1985 he proposed [6] the weaker

$$N_R(x) = \Omega(x) \quad (x \rightarrow \infty). \quad (2b)$$

In 1951 Erdős and Shapiro [7] proved that

$$R(x) = \Omega_{\pm}(x \log \log \log \log x), \quad (3)$$

and hence that

$$N_R(x) \rightarrow \infty \quad (x \rightarrow \infty).$$

The only other result in the literature is due to Proschan (1971, [20]):

$$N_R(x) \geq IL(x) + O(1) \quad (x \rightarrow \infty), \quad (4)$$

where  $IL(x)$  is the smallest integer  $k$  such that  $\log_{4k}(x)$ , the  $4k$ -fold iterated logarithm of  $x$  in a sufficiently large basis, is either smaller than 2 or undefined.

We show in Section 3 of this paper that

$$X_R(x) = Cx + o(x) \quad (x \rightarrow \infty), \quad (5)$$

where

$$C \geq \frac{8}{3} \left(1 - \frac{\pi^2}{24}\right) = 1.57004 \dots, \quad (6)$$

and in another article [18] that

$$N_R(x) \geq \left(\frac{2}{\log 2} - \varepsilon\right) \log \log x + O_{\varepsilon}(1), \quad \text{for any } \varepsilon > 0. \quad (7)$$

For the divisor functions  $\sigma_k(n) := \sum_{d|n} d^k$ , we consider the error term

$$F_k(x) := S_k(x) - T_k(x), \quad (8)$$

where

$$S_k(x) := \sum_{n \leq x} \sigma_k(n) \quad (9)$$

and

$$T_k(x) := \begin{cases} \frac{\pi^2}{6}x - \frac{\log x}{2} - \frac{(\gamma + \log 2\pi)}{2} & (k = -1) \\ \frac{\zeta(1+k)}{1+k}x^{1+k} + \zeta(1-k)x - \frac{\zeta(-k)}{2} & (k \neq 0, -1) \\ x \log x + (2\gamma - 1)x + \frac{1}{4} & (k = 0) \end{cases} \quad (10)$$

( $\gamma$  is Euler's constant and  $\zeta$  is Riemann's zeta function). Let  $X_{F_k}(x)$  denote the number of changes of sign of  $F_k(t)$  in  $1 < t < x$ , and  $N_{F_k}(x)$  the number of changes of sign of  $F_k(n)$  on the integers  $n$ ,  $1 < n < x$ . It follows from a result of Steinig's [28] that

$$X_{F_k}(x) \geq 4\sqrt{x} + O_k(1), \quad \text{for all } k \in \mathbb{R}; \quad (11)$$

there is no result in the literature concerning  $N_{F_k}(x)$ .

We show in Section 4 that

$$X_{F_k}(x) \geq \frac{8}{3} \left( 1 - \frac{\zeta(2|k|)}{4\zeta(2+2|k|)} \right) x + o_k(x), \quad \text{for all } k \in \mathbb{R}, \quad (12)$$

and in [18] that

$$N_{F_{\pm 1}}(x) \geq \left( \frac{2}{\log 2} - \varepsilon \right) \log \log x + O_\varepsilon(1), \quad \text{for all } \varepsilon > 0. \quad (13)$$

Estimate (12) improves (11) when it is non-trivial, that is for

$$|k| > k_0 = 0.6236622 \dots \quad (14)$$

In Section 5 we consider error terms associated with the lattice points in certain four-dimensional ellipsoids, which are closely related to the error terms  $F_{-1}$  and  $F_1$ . The author wishes to thank Prof. J. Steinig for the time he spent to read the manuscript of this article and for his many useful suggestions.



## 2. Two general theorems

Let us first define what we mean by the number of changes of sign of a real-valued function  $f$  in a non-empty interval  $I$ .

DEFINITION.

- 1) We say that  $f$  is of constant sign in  $I$  if either  $f \geq 0$  or  $f \leq 0$  throughout  $I$ .
- 2) We say that  $f$  has  $N$  changes of sign in  $I$  if  $I$  can be partitioned into  $N + 1$  subintervals  $I_i$ ,  $i = 0, 1, \dots, N$  ( $I_i$  and  $I_{i+1}$  being consecutive), with the following properties:
  - i)  $f$  is not identically zero in any  $I_i$ ;
  - ii)  $f$  is of constant sign in each  $I_i$ ;
  - iii)  $f$  is of opposite signs in  $I_i$  and  $I_{i+1}$ .
- 3) We say that  $f$  has a finite number of sign changes in  $I$  if there is an  $N \geq 0$  such that  $f$  has  $N$  changes of sign in  $I$ .

Throughout this article, we consider functions  $f : [1, \infty) \rightarrow \mathbb{R}$  which have a finite number of sign changes in  $(1, x)$  for all  $x > 1$ , and we denote this number by  $X_f(x)$ .

We also set  $I_n = (n, n + 1)$  and  $\bar{I}_n = [n, n + 1)$  for each integer  $n \geq 1$ , and  $\{x\} := x - [x]$  if  $x$  is real. If  $E$  is a finite set,  $|E|$  denotes its cardinality.

THEOREM 1. Let  $f : [1, \infty) \rightarrow \mathbb{R}$  be such that for each  $n \geq 1$ ,

$$f(x) = f(n) - C\{x\} + \theta(x) \quad \text{if } x \in \bar{I}_n, \quad (15)$$

where  $C$  is a constant,  $C \neq 0$ , and

$$\theta(x) = o(1) \quad (x \rightarrow \infty). \quad (16)$$

Suppose further that there is a constant  $K > 0$  such that

$$\int_1^x f^2(u) du \leq Kx + o(x) \quad (x \rightarrow \infty). \quad (17)$$

Then, as  $x \rightarrow \infty$ ,

$$X_f(x) \geq \frac{8}{3} \left(1 - \frac{3K}{C^2}\right)x + o(x). \quad (18)$$

If in addition the distribution function for  $f$ ,

$$D_f(u) := \lim_{x \rightarrow \infty} \frac{|\{n \leq x, f(n) \geq u\}|}{x} \quad (19)$$

exists and is continuous, and if  $f$  itself is monotonic on each interval  $\bar{I}_n$  (decreasing if  $C > 0$ , increasing if  $C < 0$ ), then as  $x \rightarrow \infty$

$$X_f(x) = 2 |D_f(0) - D_f(C)| x + o(x). \quad (20)$$

**THEOREM 2.** Let  $f : [1, \infty) \rightarrow \mathbb{R}$  satisfy conditions (15) through (17) of Theorem 1. Let  $h : [1, \infty) \rightarrow \mathbb{R}$  be positive, and  $g : [1, \infty) \rightarrow \mathbb{R}$  be such that as  $x \rightarrow \infty$ ,

$$g(x) = h(x)(f(x) + o(1)). \quad (21)$$

Then as  $x \rightarrow \infty$ ,

$$X_g(x) \geq \frac{8}{3} \left(1 - \frac{3K}{C^2}\right) x + o(x). \quad (22)$$

If in addition  $f$  satisfies condition (19), and if the function  $g/h$  is monotonic on each  $\bar{I}_n$  (decreasing if  $C > 0$ , increasing if  $C < 0$ ), then as  $x \rightarrow \infty$ ,

$$X_g(x) = 2 |D_f(0) - D_f(C)| x + o(x). \quad (23)$$

*Proof of Theorem 1.* We may suppose  $C > 0$  (if  $C < 0$ , consider  $-f$  instead of  $f$ ). We may also restrict ourselves to the case where  $x$  is an integer. For  $r > 0$ , set

$$A_r(x) = \{n \leq x, |f(n) - C/2| < r\},$$

$$B_r(x) = \{n \leq x, |f(n) - C/2| \geq r\}.$$

From (16), (17) and Cauchy's inequality,

$$\int_1^x \theta(u) f(u) du = o(x); \quad (24)$$

then from (15) and (16),

$$\begin{aligned}
 \int_1^x f^2(u) du &= \int_1^x ((f(u) - \theta(u))^2 + 2\theta(u)f(u) - \theta^2(u)) du \\
 &= \sum_{n=1}^{x-1} \int_0^1 (f(n) - Ct^2) dt + \int_1^x (2\theta(u)f(u) - \theta^2(u)) du \\
 &= \sum_{n=1}^{x-1} ((f(n) - C/2)^2 + C^2/12) + o(x),
 \end{aligned}$$

whence

$$\int_1^x f^2(u) du \geq r^2 |B_r(x)| + \frac{C^2}{12}x + o(x),$$

that is

$$\int_1^x f^2(u) du \geq (r^2 + C^2/12)x - r^2 |A_r(x)| + o(x). \quad (25)$$

From (17) and (25) we have

$$|A_r(x)| \geq \left(1 - \frac{K}{r^2} + \frac{C^2}{12r^2}\right)x + o(x). \quad (26)$$

Now take  $r = C/2 - \varepsilon$ , with  $0 < \varepsilon < C/2$ . Condition (15) implies that  $f$  decreases by  $C + o(1)$  on  $\tilde{I}_n$ . Hence by definition of  $A_r(x)$  there is an  $N = N(\varepsilon)$  such that  $f$  changes sign from  $+$  to  $-$  on  $I_n$  whenever  $n \geq N$  and  $n \in A_r(x)$ . This means that the number of sign changes of  $f$  from  $+$  to  $-$  on  $(1, x)$ , say  $X_f^+(x)$ , is at least

$$\left(1 - \frac{K}{(C/2 - \varepsilon)^2} + \frac{C^2}{12(C/2 - \varepsilon)^2}\right)x - N + p(x),$$

where  $p(x) = o(x)$  as  $x \rightarrow \infty$ . Hence if  $x$  is large enough to ensure that  $x \geq N/\varepsilon$  and  $|p(x)| < \varepsilon x$ , then

$$X_f^+(x) \geq \left(\frac{4}{3} - 4K/C^2\right)x - \delta(\varepsilon)x - 2\varepsilon x, \quad (27)$$

where  $\delta(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0+$ . Since  $\varepsilon$  can be arbitrarily small, and as between two changes of sign from  $+$  to  $-$  there must be one from  $-$  to  $+$ , we have proved (18).

Suppose now that we also have (19) and that  $f$  is monotonically decreasing on each  $\tilde{I}_n$ . For  $r > 0$ , let

$$D_r := D_f(C/2 - r) - D_f(C/2 + r); \quad (28)$$

then we have

$$|A_r(x)| = D_r x + o(x). \quad (29)$$

With the same argument we used to deduce (18) from (26) we obtain from (29) and the continuity of  $D_f$

$$X_f(x) \geq 2D_{C/2}x + o(x). \quad (30)$$

We will now show that

$$X_f(x) \leq 2D_{C/2}x + o(x); \quad (31)$$

(20) then follows from (28), (30), and (31).

*Proof of (31).* As we pointed out above,  $X_f(x) \geq 2X_f^+(x) - 1$ . Since  $f$  decreases on each  $\tilde{I}_n$ ,  $f$  changes sign at most once there (necessarily from  $+$  to  $-$ ). And since  $f(n) - f(n+1^-) = C + o(1)$ , there is for each  $\varepsilon > 0$  an  $N = N(\varepsilon)$  such that if  $f$  changes sign on  $I_n$  and  $n \geq N$ , then  $f(n) \in (0, C + \varepsilon)$ . So we have

$$\begin{aligned} X_f^+(x) &\leq (D_f(0) - D_f(C + \varepsilon))x + N + o(x) \\ &\leq (D_f(0) - D_f(C + \varepsilon))x + \varepsilon x, \end{aligned} \quad (32)$$

for  $x$  sufficiently large; (31) now follows from (32) and the continuity of  $D_f$ . ■

The proof of Theorem 2 is straightforward, since Theorem 1 can be applied to the function  $f^* := g/h$ . Indeed, if  $D_f$  exists and is continuous, then  $D_{f^*}$  also exists and  $D_f = D_{f^*}$ . ■

### 3. Error terms associated with Euler's function

We first define the summatory functions  $\Phi$  and  $\Phi'$  and the corresponding error terms  $R$  and  $H$ : for  $x \geq 1$ ,

$$\Phi'(x) := \sum_{n \leq x} \frac{\phi(n)}{n} =: \frac{6}{\pi^2}x + H(x) \quad (33)$$

and

$$\Phi(x) := \sum_{n \leq x} \phi(n) =: \frac{3}{\pi^2} x^2 + R(x). \quad (34)$$

We consider the changes of sign of  $H$  and  $R$ , and prove

**THEOREM 3.** *As  $x \rightarrow \infty$ ,*

$$X_H(x) \geq \frac{8}{3} \left(1 - \frac{\pi^2}{24}\right)x + o(x) = (1.57004 \dots)x + o(x), \quad (35)$$

$$X_R(x) \geq \frac{8}{3} \left(1 - \frac{\pi^2}{24}\right)x + o(x), \quad (36)$$

$$X_H(x) = 2(D_H(0) - D_H(6/\pi^2))x + o(x), \quad (37)$$

$$X_R(x) = 2(D_H(0) - D_H(6/\pi^2))x + o(x). \quad (38)$$

*Proof.* The hypotheses of Theorem 1 are satisfied by  $f(x) = H(x)$ , with  $C = 6/\pi^2$  and  $K = 1/2\pi^2$ . Indeed

$$\int_1^x H^2(u) du \sim \frac{1}{2\pi^2} x \quad (x \rightarrow \infty) \quad (39)$$

is a theorem of Chowla's [3, (48)] (see Remark 2 in Section 6). And (33) shows that

$$H(x) = H(n) - \frac{6}{\pi^2} \{x\} \quad \text{for } x \in \bar{I}_n. \quad (40)$$

This proves (35).

Estimate (37) follows from Theorem 1 by using the existence and continuity of  $D_H$ , proved by Erdős and Shapiro [8].

For (36) and (38) we use the estimate

$$R(x) = xH(x) + o(x) \quad (41)$$

due to Pillai and Chowla [19, p. 99] (see Remark 1). As it is easy to see that  $R(x)/x$  is decreasing on each  $\bar{I}_n$ , the hypotheses of Theorem 2 are satisfied if we take  $f(x)$ ,  $C$  and  $K$  as above,  $g(x) = R(x)$  and  $h(x) = x$ . ■

Theorems 1 and 2 can also be applied to a class of error terms including  $H$  and  $R$ , first studied by Proschan [20], and for which Sivaramasarma [27, (7.1.45)] determined the constant  $K$  of (17). This is done in [17, II.2].

#### 4. Error terms associated with divisor functions

Let  $F_k$  be defined as in (8). We shall prove the following theorem about its changes of sign.

**THEOREM 4.** *Let  $k_0$  be the solution of the equation*

$$\zeta(2k_0) = 4\zeta(2 + 2k_0) \quad (42)$$

*in the interval  $(1/2, \infty)$ . Then if  $k_0 < |k|$ , we have, as  $x \rightarrow \infty$*

$$X_{F_k}(x) \geq \frac{8}{3} \left( 1 - \frac{\zeta(2|k|)}{4\zeta(2 + 2|k|)} \right) x + o_k(x) \quad (43)$$

(Using a variant of Simpson's approximation method, B. Gisin computed  $k_0 = 0.6236622010 \dots$ ).

In order to deduce Theorem 4 from Theorems 1 and 2 we need three lemmata

**LEMMA 1.** *As  $x \rightarrow \infty$ ,*

$$F(x) := xF_{-1}(x) - F_1(x) = o(x), \quad (44)$$

$$F_k(x) = O(x^{(1+k)/2}) \quad \text{for } -1 < k < -\frac{1}{2}, \quad (45)$$

$$F_k(x) = x^k F_{-k}(x) + o(x^k) \quad \text{for } \frac{1}{2} < k \leq 1. \quad (46)$$

*Proof.* Estimate (44) is classical (see Remark 4 in Section 6). For (45) see [3, (112)]. An estimate implying (46) can be found in [13, (6)] (see Remark 5). ■

**LEMMA 2.** *With  $F$  as in (44), we have*

$$F(x) = \int_1^x F_{-1}(t) dt + O(1) \quad (x \rightarrow \infty). \quad (47)$$

*Proof.* On the one hand,

$$\sum_{n \leq x} \sigma_{-1}(n)(x - n) = \int_1^x S_{-1}(t) dt$$

and on the other,

$$\sum_{n \leq x} \sigma_{-1}(n)(x - n) = xS_{-1}(x) - S_1(x).$$

By using (8), (10) and  $\zeta(0) = -\frac{1}{2}$ , we get (47). ■

LEMMA 3. As  $x \rightarrow \infty$ ,

$$\int_1^x F_{-1}^2(t) dt \sim \frac{5\pi^2}{144} x \quad (48)$$

and for  $-1 < k < -\frac{1}{2}$ ,

$$\int_1^x F_k^2(t) dt \sim \frac{\zeta(-2k)\zeta^2(1-k)}{12\zeta(2-2k)} x. \quad (49)$$

*Proof.* (48) is due to Walfisz [36, (I)] and (49) to Chowla [3, (7)]. They considered an error term slightly different from  $F_k$  (see Remark 3) and proved, respectively, that for  $k = -1$ ,

$$\int_1^x \left( F_{-1}(t) - \frac{(\gamma + \log 2\pi)}{2} \right)^2 dt = \left( \frac{(\gamma + \log 2\pi)^2}{4} + \frac{5\pi^2}{144} \right) x + O(x^{1/2}) \quad (50)$$

and that for  $-1 < k < -\frac{1}{2}$ ,

$$\int_1^x \left( F_k(t) - \frac{\zeta(-k)}{2} \right)^2 dt = \left( \frac{\zeta^2(-k)}{4} + \frac{\zeta(-2k)\zeta^2(1-k)}{12\zeta(2-2k)} \right) x + O(x^{k+3/2} \log x). \quad (51)$$

(48) follows from (50) with (47) and (44), and (49) from (51) with

$$\int_1^x F_k(t) dt = O(x^{1+k/2}) \quad \text{if } -1 < k \leq -\frac{1}{2}, \quad (52)$$

which we proceed to prove. For  $-1 < k \leq -\frac{1}{2}$ , we have

$$F_k(x) = - \sum_{n \leq \sqrt{x}} n^k \Psi(x/n) - x^k \sum_{n \leq \sqrt{x}} n^{-k} \Psi(x/n) + O(x^{k/2}), \quad (53)$$

where  $\Psi(y) := \{y\} - \frac{1}{2}$  [3, (65)], whence

$$\begin{aligned} \int_1^x F_k(t) dt &= - \sum_{n \leq \sqrt{x}} n^k \int_{n^2}^x \Psi(t/n) dt - \sum_{n \leq \sqrt{x}} n^{-k} \int_{n^2}^x t^k \Psi(t/n) dt + O(x^{1+k/2}) \\ &= - \sum_{n \leq \sqrt{x}} n^{k+1} \int_n^{x/n} \Psi(u) du - \sum_{n \leq \sqrt{x}} n \int_n^{x/n} u^k \Psi(u) du + O(x^{1+k/2}) = O(x^{1+k/2}). \quad \blacksquare \end{aligned}$$

After this preparation, we pass to the proof of Theorem 4. We shall restrict ourselves to the case  $|k| \leq 1$  (for the case  $|k| > 1$ , see Remark 6). We consider four subcases.

a)  $k = -1$ : if  $n \leq x < n+1$ ,

$$F_{-1}(x) = F_{-1}(n) - \frac{\pi^2}{6} \{x\} + O(1/x), \quad (54)$$

whence with (48), conditions (15) through (17) of Theorem 1 are satisfied by  $f(x) = F_{-1}(x)$ , with  $C = \pi^2/6$  and  $K = 5\pi^2/144$ .

b)  $k = +1$ : with (44), we see that  $g(x) = F_1(x)$  and  $h(x) = x$  satisfy condition (21) of Theorem 2, if  $f(x)$  is as in Case (a).

c)  $k \in (-1, -k_0)$ : we have by (10), if  $x \in \bar{I}_n$

$$F_k(x) = F_k(n) - \zeta(1-k)\{x\} + O(x^k), \quad (55)$$

whence with (49), conditions (15) through (17) of Theorem 1 are satisfied by  $f = F_k$ , with  $C = \zeta(1-k)$  and  $K = \zeta(-2k)\zeta^2(1-k)/12\zeta(2-2k)$ .

d)  $k \in (k_0, 1)$ : with (46) we see that if  $f = F_{-k}$ , and  $C, K$  are as in Case (c), condition (21) of Theorem 2 is satisfied by  $g(x) = F_k(x)$  and  $h(x) = x^k$ .  $\blacksquare$

## 5. Error terms associated with the lattice points in certain four-dimensional ellipsoids

Arnold Walfisz considered in [36] and [37] the quadratic forms

$$\begin{cases} Q_0 = n_1^2 + n_2^2 + n_3^2 + n_4^2, \\ Q_1 = n_1^2 + n_2^2 + 2n_3^2 + 2n_4^2, \\ Q_2 = n_1^2 + 2n_2^2 + 2n_3^2 + 4n_4^2, \\ Q_3 = n_1^2 + 2n_2^2 + 4n_3^2 + 8n_4^2, \end{cases} \quad (56)$$



the associated four-dimensional ellipsoids

$$0 \leq Q_k \leq x \quad (k = 0, 1, 2, 3) \quad (57)$$

of respective volumes

$$W_k(x) = \frac{\pi^2}{2^{k+1}} x^2 \quad (k = 0, 1, 2, 3), \quad (58)$$

and the corresponding error terms

$$P_k(x) = \sum_{Q_k \leq x} 1 - W_k(x) \quad (k = 0, 1, 2, 3). \quad (59)$$

He showed that

$$\begin{cases} P_0(x) = 8E_1(x) - 32E_1(x/4), \\ P_1(x) = 4E_1(x) - 4E_1(x/2) + 8E_1(x/4) - 32E_1(x/8), \\ P_2(x) = 2E_1(x) - 2E_1(x/2) + 8E_1(x/8) - 32E_1(x/16), \\ P_3(x) = E_1(x) - E_1(x/2) + 8E_1(x/16) - 32E_1(x/32) + O(x^{5/6}), \end{cases} \quad (60)$$

(where  $E_1(x) = F_1(x) - x/2 - \zeta(-1)/2$ ; see Remark 3) and that

$$\int_1^x P_k^2(t) dt = \frac{\pi^2}{3 \cdot 2^{2k+1}} x^3 + \begin{cases} O(x^{5/2}) (k = 0, 1, 2; [36]) \\ O(x^{5/2} \log^2 x) (k = 3; [37]). \end{cases} \quad (61)$$

Using

$$F(x) = O(x^{5/6}), \quad (62)$$

where  $F$  is as in (44) (see Remark 4), we can rewrite (60) as

$$P_k(x) = xR_k(x) + O(x^{5/6}) \quad (k = 0, 1, 2, 3), \quad (63)$$

where

$$\begin{cases} R_0(x) = 8F_{-1}(x) - 8F_{-1}(x/4), \\ R_1(x) = 4F_{-1}(x) - 2F_{-1}(x/2) + 2F_{-1}(x/4) - 4F_{-1}(x/8), \\ R_2(x) = 2F_{-1}(x) - F_{-1}(x/2) + F_{-1}(x/8) - 2F_{-1}(x/16), \\ R_3(x) = F_{-1}(x) - \frac{1}{2}F_{-1}(x/2) + \frac{1}{2}F_{-1}(x/16) - F_{-1}(x/32). \end{cases} \quad (64)$$

Integrating by parts in (61) and using (63) we obtain

$$\int_1^x R_k^2(t) dt = \frac{\pi^2}{2^{2k-1}} x + O(x^{5/6} \log x) \quad (k = 0, 1, 2, 3). \quad (65)$$

It is not difficult, using (64) and (54), to show that for  $x \in \bar{I}_n$  we have

$$R_k(x) = R_k(n) - \frac{\pi^2}{2^k} \{x\} + O(1/x) \quad (k = 0, 1, 2, 3). \quad (66)$$

We see with (63) through (66) that Theorems 1 and 2 can be applied; we obtain

**THEOREM 5.** *For  $P_k$  as in (59) ( $k = 0, 1, 2, 3$ ) we have*

$$X_{P_k}(x) \geq \frac{8}{3} \left(1 - \frac{6}{\pi^2}\right) x + o(x) = (1.045527 \dots) x + o(x). \quad (67)$$

*For  $k = 0$ , this improves*

$$X_{P_0}(x) \geq 2\sqrt{x} + O(1), \quad (68)$$

*which is implied by a general result of Steinig's [28, (4.5)].*

## 6. Remarks

*Remark 1.* If  $f$  is strictly monotonic on each  $I_n$ , we have the trivial upper bound

$$X_f(x) \leq 2x + 1. \quad (69)$$

This, with the example below, shows that (18) can be sharp: if

$$\Psi(x) := \{x\} - \frac{1}{2},$$

(15) holds with  $C = -1$ , and we have

$$\int_1^x \Psi^2(t) dt \sim x/12 \quad (x \rightarrow \infty);$$

thus by Theorem 1

$$X_{\psi}(x) \geq 2x + o(x).$$

(69) should also be compared with (83).

*Remark 2.* Chowla's estimate of the error term in (39) was  $O(x/\log^4 x)$ . For better estimates, and also for estimates of  $R(x) - xH(x)$ , see [26], [30], [27]; [30] also gives estimates subject to the truth of the Riemann hypothesis.

One can obtain a simpler proof of (39) than in [3] by adapting the arguments of Lemmata 3.2. and 3.3 of [8]. One gains the advantage of not having to prove Lemma 7 of [3] (Hilfssatz 6 of [34]).

*Remark 3.* Some authors (Walfisz [34–37], Chowla [3]) considered another error term  $E_k$  defined by

$$S_1(x) =: \frac{\pi^2}{12} x^2 + E_1(x) \quad (70)$$

$$S_k(x) =: \frac{\zeta(1+k)}{1+k} x^{1+k} + \zeta(1-k)x + E_k(x) \quad (-1 < k < 1, k \neq 0) \quad (71)$$

$$S_{-1}(x) =: \frac{\pi^2}{6} x - \frac{1}{2} \log x + E_{-1}(x) \quad (72)$$

(hence the estimates (50) and (51)). This is a more natural choice than  $F_k$ , in the sense that

$$E_k(x) = \begin{cases} o(x^2) & (k = 1) \\ o(x) & (0 < k < 1) \\ o(x^{1+k}) & (-1 < k < 0) \\ o(\log x) & (k = -1), \end{cases} \quad (73)$$

whereas

$$F_k(x) \neq o(1) \quad (-1 \leq k \leq 1). \quad (74)$$

$F_k$  is the error term one obtains when dealing with  $S_k$  by the complex variable methods developed by Chandrasekharan and Narasimhan to exploit the

representation

$$\sum_{n \geq 1} \sigma_k(n) n^{-s} = \zeta(s-k)\zeta(s) \quad (\operatorname{Re} s > \max(1, k+1)) \quad (75)$$

and the functional equation satisfied by  $\zeta(s-k)\zeta(s)$  (see [1], [2], [9], [10]). It seems to be the “right” error term to consider if one is interested in the change of sign problems. To be concrete, let us say that a good point in favor of  $F_k$  for these problems is that for  $k < 0$ , we have

$$\int_1^x F_k(t) dt = o(x), \quad (76)$$

which shows that the mean value of  $F_k(t)$  is 0. As for  $\Omega$  or 0 estimates, since

$$E_k(x) - F_k(x) = O(1) \quad \text{for } k < 0, \quad (77)$$

the results one obtains for any one of these error terms are also true for the other.

*Remark 4.* 0-estimates of the error term in (44) were successively improved in [38], [12], [13], [35], [16]. The current record-holder is Recknagel [22] with

$$F(x) = O(x^{109/382}). \quad (78)$$

A special case of a result of Segal’s [24] reads

$$\sum_{n \leq x} F_{-1}(n) = \frac{\pi^2}{12} x + O(x^{1/4}), \quad (79)$$

which is equivalent to

$$F(x) = O(x^{1/4}) \quad (80)$$

(use (47) and (54)). Segal pointed out in [25] that his proof of (79) is incorrect. In fact, (79) itself is incorrect: see [17, Appendix ]. (However, [25] was sometimes overlooked, as in [14] and [29]).

*Remark 5.* To our knowledge, the best 0-estimate to date of  $F_k(x) - x^k F_{-k}(x)$  for  $\frac{1}{2} < k \leq 1$  comes from using [22] instead of the weaker [15] in [11, Corollary 1

p. 403]. One obtains

$$F_k(x) = x^k F_{-k}(x) + O(x^{\theta(1-k)}) \quad (\tfrac{1}{2} < k \leq 1), \quad (81)$$

where

$$\theta(t) = \begin{cases} \frac{109}{382} + (\frac{75}{191})t & (0 \leq t < \frac{5}{93}) \\ \frac{49}{172} + (\frac{69}{172})t & (\frac{5}{93} \leq t < \frac{1}{110}) \\ \frac{211}{744} + (\frac{77}{186})t & (\frac{1}{10} \leq t < \frac{41}{224}) \\ \frac{209}{742} + (\frac{45}{106})t & (\frac{41}{224} \leq t < \frac{11}{42}) \\ \frac{11}{40} + (\frac{9}{20})t & (\frac{11}{42} \leq t < \frac{1}{2}). \end{cases} \quad (82)$$

*Remark 6.* Most authors who studied the  $S_k$  restricted themselves to the case  $|k| \leq 1$  ("to avoid unnecessary complications" according to Cramér [4]). Estimates of

$$F_k(x), \int_1^x F_k(t) dt \quad \text{and} \quad \int_1^x F_k^2(t) dt$$

for the case  $|k| > 1$  are apparently unavailable in the literature. With the help of the existing proofs [3] of such estimates for  $|k| \leq 1$ , together with Ramanujan's estimate [21] of  $F_k(x) - x^k F_{-1}(x)$  for  $0 < k < \infty$ , extending the domain of validity of (43) to  $|k| > 1$  is only a matter of tedious and unoriginal calculation. We now observe that

$$\lim_{k \rightarrow \infty} \frac{8}{3} \left( 1 - \frac{\zeta(2k)}{4\zeta(2+2k)} \right) = 2; \quad (83)$$

with (69), this shows that the constant in (43) is in some sense best possible.

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