

**Zeitschrift:** Commentarii Mathematici Helvetici  
**Herausgeber:** Schweizerische Mathematische Gesellschaft  
**Band:** 60 (1985)

**Artikel:** Earthquakes are analytic.  
**Autor:** Kerckhoff, Steven P.  
**DOI:** <https://doi.org/10.5169/seals-46297>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 20.08.2025

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

## Earthquakes are analytic

STEVEN P. KERCKHOFF\*

Various approaches to the study of Teichmüller space tend to emphasize different properties which are natural in a given context. For example, its complex analytic structure is natural when considering it as the space ( $T_g$ ) of all marked Riemann surfaces of genus  $g$  (up to equivalence) or as a subset of quasi-Fuchsian groups. It is well-known that the complex analytic structure of  $T_g$  is quite inhomogeneous. In particular, the only biholomorphic self-mappings come from the properly discontinuous action of the (Teichmüller) modular group ([7]).

On the other hand,  $T_g$  (via the uniformization theorem) is also the space of hyperbolic structures on a surface of genus  $g$ . From this point of view,  $T_g$  is naturally a *real* analytic manifold, its structure coming from the isomorphism between  $PSL(2, \mathbb{R})$  and the group of isometries of two-dimensional hyperbolic space. In contrast to the complex analytic case there are many real analytic maps of  $T_g$  to itself. It is reasonable, therefore, to further restrict oneself to maps which arise from geometric deformations of the hyperbolic structure, or to those which preserve some geometric quantity on the surfaces themselves.

The maps discussed in this paper are closely related to geodesic length functions (generalized from the length of closed geodesics to the length of geodesic laminations) in that they preserve the hypersurface level sets of these functions. They are the time 1 maps of a  $6g-6$  dimensional family of flows, no two of which agree at any point (see Proposition 2.6 at the end of this paper).

The flows are parametrized by the space  $\mathcal{ML}$  of geodesic laminations  $\mu \in \mathcal{ML}$  and are denoted by  $\mathcal{F}_\mu$ . The integral curves of these flows are the earthquake deformations of hyperbolic structures which generalize the classical Fenchel-Nielsen twist deformations. That these older deformations are real analytic is well-known; since they are “dense” in the set of earthquake flows, we can think of the general  $\mathcal{F}_\mu$  as a limit of these twist flows. (Indeed, that is how they are usually defined.) The primary purpose of this paper is to show that this limiting process is geometrically and analytically well-controlled.

---

\* During preparation of this work the author was supported in part by NSF Grant MCS 82-03806.

**THEOREM 1.** *The earthquake flows  $\mathcal{F}_\mu$  on  $T_g$  are real analytic for every geodesic lamination  $\mu$ .*

The approach taken here is not quite as direct as the preceding discussion suggests, since it doesn't distinguish between the classical twist case and the general case. It is, however, often useful to keep the limiting process in mind.

The proof of Theorem 1 is fairly straightforward, combining an elementary normal families argument with known facts about both the real analytic structure of  $T_g$  and about the behavior of the geodesic length function under earthquake deformations.

As a corollary we find that the length,  $l_\mu$ , of a geodesic lamination,  $\mu$ , is equally smooth. In particular (Corollary 2.2), for  $\mu$  a fixed lamination, it is real analytic as a function of  $T_g$ . As  $\mu$  varies the  $l_\mu$  vary continuously in the  $C^\infty$ -topology for functions on compact subsets of  $T_g$ .

Since much of the background material is hard to reference, we have given an expository account of it in Section I. Further discussion of geodesic laminations and earthquakes may be found in [8] and [4]. The proof of Theorem 1 is contained in Section II.

## I.A.

A hyperbolic surface,  $M$ , is a surface of genus  $g$ ,  $g \geq 2$ , with a metric of constant curvature-1. It is isometric to a surface of the form  $\mathbb{H}^2/\Gamma$  where  $\mathbb{H}^2$  is two-dimensional hyperbolic space and  $\Gamma$  is a discrete subgroup of isometries isomorphic to  $\pi_1 M$ .  $M$  determines  $\Gamma$  up to conjugacy in the group of isometries of  $\mathbb{H}^2$ , which we identify with  $PSL(2, \mathbb{R})$ . Let  $\Sigma$  be a fixed topological surface of genus  $g$ . The Teichmüller space of genus  $g$  ( $T_g$ ) is the space of marked hyperbolic surfaces; i.e., hyperbolic surfaces with a fixed isomorphism of  $\pi_1 \Sigma$  to  $\Gamma$  where two surfaces are thought to be equivalent if there is an isometry between them respecting this isomorphism. Equivalently,  $T_g$  is the subset of discrete representations of  $\pi_1 \Sigma$  into  $PSL(2, \mathbb{R})$  up to conjugacy. It is known to be diffeomorphic to an open cell of dimension  $6g - 6$ . The space of Fuchsian groups  $\Gamma$  together with an isomorphism from  $\pi_1 \Sigma$  to  $\Gamma$  will be denoted by  $R_g$ ; it is diffeomorphic to  $T_g \times PSL(2, \mathbb{R})$  and will be called the representation space of genus  $g$ .

For computational purposes, the upper half-space of  $\mathbb{C}$  serves as a convenient model for  $\mathbb{H}^2$ , but the point at infinity has a less (artificially) distinguished character if we identify the upper half-space with the upper hemisphere of the Riemann sphere  $\hat{\mathbb{C}}$  ( $=\mathbb{C} \cup \infty$ ). The extended real axis  $\mathbb{R} \cup \infty$  will be denoted by  $\hat{\mathbb{R}}$ : it is preserved by isometries of  $\mathbb{H}^2$ . However, since it will be necessary to consider

homeomorphism of  $\hat{\mathbb{C}}$  to itself which do not preserve  $\hat{\mathbb{R}}$ , it is useful to consider  $\hat{\mathbb{R}}$  as a circle bounding the upper hemisphere, thus emphasizing that its topological character is unchanged under homeomorphism.

Any isometry of  $\mathbb{H}^2$  extends continuously to its boundary, denoted by  $S_\infty^1$ , and called the circle at infinity. (This is the unit circle in the Poincaré disk model,  $\hat{\mathbb{R}}$  in the upper half space model.) Since  $M$  is a closed, non-singular surface, all of the elements  $\gamma$  of  $\Gamma$  are hyperbolic; i.e.,  $\gamma$  acting on the closure of  $H^2$  has exactly two fixed points, both on  $S_\infty^1$ , one attracting and one repelling. Pairs of points on  $S_\infty^1$  are in 1–1 correspondence to geodesics in  $H^2$ ; the geodesic corresponding to the fixed points of  $\gamma \in \Gamma$  projects to the unique geodesic in  $M$  in the free homotopy class of  $\gamma \in \pi_1 \Sigma$  (under the isomorphism of  $\Gamma$  with  $\pi_1 \Sigma$ ).

Since there is a given isomorphism between any two  $\Gamma, \Gamma' \in R_g$ , there is a canonical 1–1 correspondence between elements in  $\Gamma$  and those in  $\Gamma'$  which induces a like correspondence between closed geodesics on the quotient surfaces  $M$  and  $M'$ . In other words, we can talk about *the* geodesic corresponding to the conjugacy class of  $\gamma \in \pi_1 \Sigma$  on every  $M \in T_g$ . Similarly, since two geodesics in  $H^2$  intersect at most once, different points of intersection between two closed geodesics in  $M$  correspond to intersections between distinct lifts of the geodesics to  $H^2$ . Thus the correspondence between endpoints of  $S_\infty^1$  via the isomorphism between  $\Gamma$  and  $\Gamma'$  induces an identification between points of intersection of geodesics on  $M$  and  $M'$ .

Because fixed points of  $\Gamma$  and  $\Gamma'$  are both dense in  $S_\infty^1$ , there is a unique homeomorphism of the circles at infinity for  $\Gamma$  and  $\Gamma'$  extending the correspondence between fixed points. It follows that the identification between geodesics and their intersections on  $M, M' \in T_g$  carries over to infinite, non-closed geodesics as well. Nielsen showed that every lift to  $H^2$  of any homotopy equivalence between  $M$  and  $M'$  (respecting the isomorphisms to  $\pi_1 \Sigma$  as usual) extends continuously to a homeomorphism on  $S_\infty^1$ , depending only on  $M, M'$  and the choice of lift. These extensions are precisely the maps given by extending continuously the isomorphism between  $\Gamma$  and  $\Gamma'$ . (Different choices of  $\Gamma$  and  $\Gamma'$  with quotients  $M$  and  $M'$  amount to different lifts.)

$R_g$  inherits a real analytic structure as a subset of the set of representations of  $\pi_1 \Sigma$  into the real analytic Lie group  $PSL(2, \mathbb{R})$ .  $T_g$  similarly inherits an analytic structure as a quotient space of  $R_g$ . If  $\Gamma \in R_g$  and an element is represented by a matrix  $A \in \Gamma$  (well-defined up to multiplication by  $-I$ ) then it is an elementary fact that the geodesic representing  $\gamma$  in  $\mathbb{H}^2/\Gamma = M$  has length  $l_\gamma(M)$  where  $\cosh l_\gamma(M) = \frac{1}{2} |\text{tr } A|$ . In particular,  $l_\gamma$  is a real analytic function on  $T_g$  and  $R_g$ . In fact, the lengths of finitely many closed curves completely determine the hyperbolic structure on  $M$ ; locally  $6g - 6$  lengths serve as co-ordinates. (See e.g., [2], [3].) Whenever analyticity on  $T_g$  or  $R_g$  is discussed in this paper, it is with respect



to this analytic structure; lengths of closed curves will generally serve as convenient co-ordinates. It should be noted that any two sets of “length co-ordinates” are analytic functions of each other since they are both determined by the traces of finite products of a fixed generating set for  $\pi_1\Sigma$ .

Finally, we need to know how  $R_g$  sits in the space of all representations of  $\pi_1\Sigma$  into  $PSL(2, \mathbb{C})$ , in particular, in the subset  $\mathbb{C}R_g$  of quasi-Fuchsian groups. A quasi-Fuchsian group is a quasi-conformal deformation of a Fuchsian group. By this we mean that  $\tilde{\Gamma} \subset PSL(2, \mathbb{C})$  satisfies  $\tilde{\Gamma} = f\Gamma f^{-1}$  where  $\Gamma$  is Fuchsian and  $f$  and  $f^{-1}$  are quasi-conformal maps of the Riemann sphere  $\hat{\mathbb{C}}$  to itself. These groups act properly discontinuously on two connected, simply-connected domains  $\Omega_i$ ,  $i = 0, 1$  in  $\mathbb{C}$ , and have as limit set  $\Lambda$  a topological circle, which separates the  $\Omega_i$  and which is the image under  $f$  of the circle limit set of the Fuchsian group.

As in the Fuchsian case,  $\tilde{\Gamma} \subset \mathbb{C}R_g$  is assumed to possess an isomorphism to  $\pi_1\Sigma$  so that  $f$  is uniquely determined on  $S_\infty^1$  and the limit sets for different  $\Gamma$ 's are canonically identified. (Fixed points of group elements in  $\tilde{\Gamma}$  are still dense in  $\Lambda$ .) Moreover, the Riemann surfaces  $S_i$  defined by  $\Omega_i/\tilde{\Gamma}$  ( $\tilde{\Gamma}$  acts conformally on  $\hat{\mathbb{C}}$ ) define points in  $T_g$ , and this ordered pair of points determines  $\tilde{\Gamma}$  up to conjugacy in  $PSL(2, \mathbb{C})$ . Thus,  $\mathbb{C}R_g \approx T_g \times T_g \times PSL(2, \mathbb{C})$  (although as a complex manifold it is probably best to write it as  $T_g \times \bar{T}_g \times PSL(2, \mathbb{C})$  if  $T_g$  is given its usual complex structure.) The subset of groups conjugate to a Fuchsian group are characterized by the property that  $S_0$  and  $S_1$  are mirror image surfaces, or equivalently, that  $\Lambda$  is a geometric circle.  $R_g \subset \mathbb{C}R_g$  is the subset where  $\Lambda$  is the circle  $\mathbb{R} \subset \hat{\mathbb{C}}$ .

Although  $T_g$  has a complex structure, it is not natural in our context; in particular, the functions to be considered in Section 2 are not complex analytic. When extended to  $\mathbb{C}R_g$ , however, they *are* complex analytic which greatly simplifies convergence questions. The main relationship between  $\mathbb{C}R_g$  and  $R_g$  which we need in this paper is the following:

**PROPOSITION 1.1.**  *$R_g$  is a real analytic submanifold of  $\mathbb{C}R_g$ . The induced structure is the analytic structure determined by the geodesic lengths of closed curves.*

This proposition is well-known and there are numerous possible proofs. The proof below is included for completeness and follows Bers' proof in [1] that  $\mathbb{C}R_g$  is a  $6g - 3$  complex dimensional manifold.

*Proof.* Let  $\Gamma$  be a Fuchsian group and let  $a_i, b_i$  be the standard generators for  $\pi_1\Sigma$  so that  $\prod_{i=1}^g [a_i, b_i] = 1$  is the single defining relation. If  $A_i, B_i$  are matrices representing  $a_i$  and  $b_i$  respectively (choose  $2g - 1$  signs arbitrarily), then by conjugation assume that

$$A_g = \begin{pmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{pmatrix} \quad B_g = \begin{pmatrix} \gamma & \sigma \\ \sigma & \delta \end{pmatrix} \quad \sigma \neq 0, \gamma\delta - \sigma^2 = 1. \quad (1)$$

Then, if  $\prod_{i=1}^{g-1} [A_i, B_i] = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$ , the equation

$$B_g A_g B_g^{-1} A_g^{-1} = \begin{pmatrix} 1 + \sigma^2(1 - \rho^{-2}) & \sigma\gamma(1 - \rho^2) \\ \sigma\gamma(1 - \rho^2) & 1 + \sigma^2(1 - \rho^2) \end{pmatrix} = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \quad (2)$$

is satisfied. Similarly, the groups  $\tilde{\Gamma}$  whose matrices,  $\tilde{A}_i, \tilde{B}_i$ , near those of  $\Gamma$  and satisfying (1) and (2) (with all entries replaced by nearby entries) determine a neighborhood of  $\Gamma$  in the submanifold of  $\mathbb{C}R_g$  normalized by (1). It is not hard to see that  $\tilde{A}_g, \tilde{B}_g$  are uniquely determined (in  $PSL(2, \mathbb{C})$ ) by (2) for arbitrary  $\tilde{x}, \tilde{y}, \tilde{z}, \tilde{w}$  so that the matrices  $\tilde{A}_i, \tilde{B}_i, i = 1, 2, \dots, g-1$ , serve as local co-ordinates (i.e., choose three entries from each matrix) for groups in  $\mathbb{C}R_g$  normalized by (1). The groups which are Fuchsian have matrices with real entries. Conversely, if  $x, y, z, w$  are real, then, by (2),  $\tilde{\rho}, \tilde{\delta}, \tilde{\gamma}$  and  $\tilde{\sigma}$  are either real or pure imaginary, and if either  $\tilde{\delta}$  or  $\tilde{\gamma}$  fails to be real, then so does  $\tilde{\sigma}$ . Since  $\tilde{\sigma}, \tilde{\rho} \neq 0$  by hypothesis, it follows that all solutions near  $\Gamma$  for which  $\tilde{A}_i, \tilde{B}_i, i = 1, 2, \dots, g-1$ , are real have real entries. Thus all  $2g$  generators are real iff the first  $2g-2$  are; hence in these local co-ordinates, the Fuchsian groups are precisely those with all real co-ordinates. A group is in  $R_g \subset \mathbb{C}R_g$  iff it is conjugate by an element in  $PSL(2, \mathbb{R})$  (which preserves  $\hat{\mathbb{R}}$ ) to one with real entries satisfying (1). Thus the proposition follows.  $\square$

## I.B.

A geodesic lamination  $\mathcal{L}$  is a closed subset of a hyperbolic surface which is a union of simple geodesics and which satisfies the following local condition: There are open sets  $U_i$  covering  $\mathcal{L}$  together with diffeomorphisms  $\phi_i$  from  $U_i$  to  $\mathbb{R}^2$  such that  $\phi_i(U_i \cap \mathcal{L}) = (0, 1) \times B_i$  where  $B_i$  is a closed subset of  $\mathbb{R}$ . On the overlaps  $U_i \cap U_j$ ,  $\phi_i \circ \phi_j^{-1}$  is of the form  $(x, y) \rightarrow (f(x, y), (g(y)))$ ; i.e., it preserves horizontal arcs. In other words,  $\mathcal{L}$  is a “partial foliation” of  $M$ . We also assume that  $\mathcal{L}$  has a transverse Borel measure  $\mu$  invariant under translation along  $\mathcal{L}$  whose support is all of  $\mathcal{L}$ . We will drop the notational distinction between  $\mathcal{L}$  and its measure and denote both by “ $\mu$ ”, “ $\nu$ ”, etc. The space of all such laminations  $\mathcal{ML}$  on  $M$  is homeomorphic to  $\mathbb{R}^{6g-6}$ . (For a discussion of the topology on the space of geodesic laminations see [4] or [8].) If we throw out the “zero” lamination and identify two laminations which are equal under multiplication of the transverse measure by a scalar, we get the space  $\mathcal{PL}$  of projective classes of laminations which is homeomorphic to  $S^{6g-7}$ .

Given  $\mu$  we can define  $\int_A d\mu$  for any suitable transverse arc  $A$  by integrating the transverse measure over  $A$ . The *intersection number*  $i(\gamma, \mu)$  of  $\mu$  with any

simple closed curve  $\gamma$  on  $M$  is  $\inf_{\gamma'} \int_{\gamma'} d\mu$  where the  $\gamma'$  run all curves isotopic to  $\gamma$ . Similarly  $i(A, \mu)$ ,  $A$ , a transverse arc, is  $\inf_{A'} \int_{A'} d\mu$  where  $A'$  runs over all arcs isotopic to  $A$  with endpoints fixed. In both cases the infimum is realized by the unique geodesic in the corresponding isotopy class.

The simplest example of a geodesic lamination is a simple closed geodesic  $\phi$  with  $r$  times the counting measure as its transverse measure. Then  $i(\gamma, \mu) = r i(\gamma, \phi)$  where  $i(\gamma, \phi)$  is the minimum number of intersections under isotopy of  $\gamma$  between  $\gamma$  and  $\phi$ . The function  $i(\gamma, \cdot)$  is continuous on  $\mathcal{ML}$ ; one way to define the topology on  $\mathcal{ML}$  is to embed it as a subset of function space  $\mathbb{R}^S$ , where  $S$  is the set of isotopy classes of non-trivial, simple closed curves.

If  $\gamma$  is a closed geodesic on  $M$ , we can define the *total cosine*  $\cos(\gamma, \mu) = \int_{\gamma} \cos \theta d\mu$  of  $\gamma$  with  $\mu$  where  $\theta$  is the angle of intersection of  $\gamma$  with  $\mu$  (measured counterclockwise from  $\gamma$  to  $\mu$ ). The integral exists because the simplicity of  $\mu$  uniformly bounds the local variation of  $\theta$ . (See [4] for a more detailed discussion.) If  $\mu = (\phi, r) \in S \times \mathbb{R}_+$  then  $\cos(\gamma, \mu) = r \sum \cos \theta$ , where the sum is over the intersections of the geodesics  $\phi$  and  $\gamma$ .

Although weighted simple closed curves are very simple examples of geodesic laminations, Thurston [8] shows that  $S \times \mathbb{R}_+$  is dense in  $\mathcal{ML}$  and  $S$  is dense in  $\mathcal{PL}$ . This allows one to extend many operations and concepts from simple closed curves to general geodesic laminations. The deformations defined below are one such example.

If a lamination is lifted to  $\mathbb{H}^2$  each (infinite) geodesic converges to a point on  $S_{\infty}^1$  in each direction. Conversely, the pairs of points on  $S_{\infty}^1$  determine the geodesic. Therefore, the map between the circles at infinity for two surfaces  $M$  and  $M'$  discussed in IA allows a canonical identification between the laminations on  $M$  and those on  $M'$ . Simplicity of leaves is invariant under this equivariant map since it is equivalent to nonlinking of the endpoints of a leaf and all of its lifts. We will implicitly make this identification by talking about a lamination  $\mu$  on all  $M \in T_g$  simultaneously.

Given any hyperbolic surface  $M$  and simple closed geodesic  $\gamma$  on  $M$  we can define a new hyperbolic structure  $M_t$  by cutting along  $\gamma$  and glueing it back with a left twist of distance  $t$ . To determine a well-defined point in  $T_g$ , we must keep track of homotopy classes of curves. This is done by identifying the homotopy class of a closed curve  $\phi$  on  $M$  with the homotopy class of the curve  $\phi'$  on  $M_t$  determined by following the image of  $\phi$  in  $M_t$  until it hits  $\gamma$  (assuming it does), going along  $\gamma$  to the left distance  $t$ , continuing along the image of  $\phi$  and so on.

This cutting and glueing operation will be called the *time  $t$  twist along  $\gamma$*  (often called a Fenchel-Nielsen twist). As  $t$  varies, the surfaces  $M_t$  define a path in  $T_g$  denoted by  $\mathcal{E}_{\gamma}(t)$  ( $M$  will always be implicit) and called the *time  $t$  twist deformation along  $\gamma$* . The time  $t$  twist along  $\gamma$  can be generalized to a time  $t$  twist deformation

determined by  $(\gamma, r) \in S \times \mathbb{R}_+ \subset \mathcal{ML}$  by taking it to be the time  $tr$  twist along  $\gamma$ . Since  $S \times \mathbb{R}_+$  is dense in  $\mathcal{ML}$  we make the following:

**DEFINITION.** For any  $M \in T_g$ ,  $\mu \in \mathcal{ML}$ , the *time  $t$  earthquake deformation*,  $\mathcal{E}_\mu(t)$ , determined by  $\mu$  is the limit in  $T_g$  (for each  $t$ ) of the time  $t$  twist deformations of  $M$  determined by  $(\gamma_i, r_i) \in S \times \mathbb{R}_+$  where  $(\gamma_i, r_i) \rightarrow \mu$  in  $\mathcal{ML}$ .

The following result is proved in [4]:

**PROPOSITION 1.2.** *The limits  $\mathcal{E}_{(\gamma_i, r_i)}(t)$ ,  $(\gamma_i, r_i) \rightarrow \mu$  are independent of the approximating sequences so that  $\mathcal{E}_\mu(t)$  is well-defined.  $\mathcal{E}_\mu(t)$  is a  $C^1$  curve in  $T_g$  for all  $\mu \in \mathcal{ML}$ .*

It follows from the work of Wolpert [9] that  $\mathcal{E}_\mu(t)$  is  $C^2$ . We will show in this paper that the curves are analytic; in fact, they are the integral curves of an analytic flow defined on  $T_g$ . The geodesic lengths of closed curves provide analytic co-ordinates for  $T_g$  so the first step is to see what the derivatives of the length function,  $l_\phi$ , of a fixed closed geodesic,  $\phi$ , are along  $\mathcal{E}_\mu(t)$ . This is contained in

**PROPOSITION 1.3 ([4]).**  *$dl_\phi/dt = \int_\phi \cos \theta d\mu$  along the earthquake path  $\mathcal{E}_\mu(t)$ .*

The goal of Section II is to study how  $\cos \theta$  and hence  $dl_\phi/dt$  varies as a function of  $M \in T_g$ .

## II.

With the background material established in the previous section, we proceed here to the proof of the main theorem, which is restated below. As previously discussed, we can identify a fixed geodesic lamination  $\mu \in \mathcal{ML}$  on every hyperbolic surface  $M \in T_g$  simultaneously. This allows us to identify, for each  $t \in \mathbb{R}$ , the time  $t$  earthquake deformation of  $M$  determined by  $\mu$  for every  $M \in T_g$ . Thus, for any fixed  $\mu \in \mathcal{ML}$  a flow  $\mathcal{F}_\mu$  is defined on  $T_g$ . Although the earthquake maps on the surfaces are complicated and not generally  $C^1$ , the flows are very smooth.

**THEOREM 1.** *The earthquake flows  $\mathcal{F}_\mu$  are real analytic for every  $\mu \in \mathcal{ML}$ .*

**COROLLARY 2.1.** *The geodesic length function,  $l_\phi$ ,  $\phi$  any closed curve, is analytic along every earthquake path  $\mathcal{E}_\mu(t)$ .*

The length  $l_\nu$  of a geodesic lamination  $\nu \in \mathcal{ML}$  is defined as the total mass on the surface  $M$  of the measure which is the product of Lebesgue measure along the leaves of  $\nu$  and the measure  $\nu$  transverse to the leaves. Equivalently,  $l_\nu$  is the limit of  $r_i l_{\gamma_i}$  where  $(\gamma_i, r_i) \in S \times \mathbb{R}_+ \subset \mathcal{ML}$  converges to  $\nu$  in  $\mathcal{ML}$ . (This equivalence is proved during the proof of Corollary 2.2.)

**COROLLARY 2.2.** *The length  $l_\nu$  of the lamination  $\nu \in \mathcal{ML}$  is analytic along  $\mathcal{E}_\mu(t)$  for all  $\mu \in \mathcal{ML}$ . It is analytic on all of  $T_g$  and constant along  $\mathcal{E}_\nu(t)$ . Hence  $\mathcal{F}_\nu$  preserves  $l_\nu$ . As  $\mu$  varies the  $l_\mu$  vary continuously in the  $C^\infty$ -topology for functions on compact subsets of  $T_g$ .*

The proof of Corollary 2.2 is at the end of this section. Corollary 2.1 follows immediately from Theorem 1.

We will show that the vector fields on Teichmüller space which generate the earthquake flows are real analytic. Since the lengths of finitely many closed curves provide local (analytic) co-ordinates, it suffices to show that the first derivative of the geodesic length function,  $l_\phi$ , of any closed curve  $\phi$  in the direction of the flow is an analytic function of the point in  $T_g$ . By Proposition 1.3, this derivative at  $M \in T_g$  in the direction of  $\mathcal{F}_\mu$  equals the total cosine,  $\int_\phi \cos \theta d\mu$ , of  $\mu$  with  $\phi$ , where  $\theta$  is the angle on  $M$  from  $\phi$  to  $\mu$  at every point of intersection between  $\phi$  and  $\mu$ . Our first goal, therefore, is to understand how  $\theta$  varies with  $M$  for each such intersection.

First, notice that points of intersection between  $\phi$  and  $\mu$  on two distinct surfaces  $M$  and  $M'$  are in a canonical 1–1 correspondence. This correspondence is induced by the maps on the circles at infinity for  $M$  and  $M'$  respectively as discussed in Section IA. The angle of intersection between the two geodesics can be computed in terms of the cross-ratio of their endpoints.

**DEFINITION.** The *cross-ratio*  $\chi(a, b, c, d)$  of four points in  $\hat{C}$  is equal to

$$(a, b, c, d) = \frac{(a-c)(b-d)}{(a-d)(b-c)}$$

The cross-ratio is invariant under linear fractional transformations and calculation shows that if  $a_i, b_i, i = 1, 2$  lie on  $\hat{\mathbb{R}}$  then

$$\chi(a_1, b_1, b_2, a_2) = \cos^2 \frac{\theta}{2} = \frac{\cos \theta + 1}{2} \quad (1)$$

where  $(a_1, b_1, a_2, b_2)$  are the endpoints of two geodesics  $l_i, i = 1, 2$  arranged

counterclockwise around  $\hat{\mathbb{R}}$  and  $\theta$  is the angle of intersection between the geodesics. The angle  $\theta$  is clearly an analytic function of the endpoints so we want to show that these vary analytically on  $R_g$ .

When an endpoint,  $x$ , is a fixed point for an element  $\gamma \in \Gamma_0$  analyticity is clear, for we can write  $\gamma$  as a finite product of fixed generators of  $\Gamma_0$  which, by definition, vary analytically as functions of  $R_g$ . Since  $\gamma$  varies analytically so does  $x$ . Fixed points of group elements are dense in  $S_\infty^1$  so the general case will follow if the functions  $f_\Gamma(x_i): R_g \rightarrow S^1$ , ( $x_i$  fixed points of  $\gamma_i$ ,  $x_i \rightarrow x$ ), converge nicely. Since this situation is most simply analyzed, via normal families, when the maps are holomorphic, we allow deformations within the complex Lie group  $PSL(2, \mathbb{C})$ . This is the reason for the discussion of quasi-Fuchsian groups in Section I.A.

**PROPOSITION 2.3.** *Fix  $\Gamma_0 \in R_g$  and denote by  $f_\Gamma: S_\infty^1 \rightarrow \Lambda \subset \hat{\mathbb{C}}$  the map from the circle at infinity of  $\Gamma_0$  to the limit set  $\Lambda$  of  $\Gamma \in \mathbb{C}R_g$ . Then, for any  $x \in S_\infty^1$ , the map  $\phi(\Gamma) = f_\Gamma(x)$  from  $\mathbb{C}R_g$  to  $\hat{\mathbb{C}}$  is complex analytic.*

*Proof.* In the case  $x$  is a fixed point of some element  $\gamma \in \Gamma_0$ , the proposition follows as before from the fact that  $\gamma$  is finite product of generators which vary analytically. For a general  $x \in S_\infty^1$  let  $x_i \rightarrow x$ ,  $x_i$  fixed points of  $\gamma_i \in \Gamma_0$ . Let  $\phi_i(\Gamma) = f_{\gamma_i}(x_i)$  and  $\phi(\Gamma) = f_\Gamma(x)$ .

Since  $f_\Gamma$  is continuous for each  $\Gamma$  and  $x_i \rightarrow x$ , then  $f_\Gamma(x_i) \rightarrow f_\Gamma(x)$  for each  $\Gamma$  so  $\phi_i \rightarrow \phi$  pointwise. Furthermore, for  $\Gamma$  restricted to a compact set of  $\mathbb{C}R_g$ ,  $f_\Gamma$  is the restriction to  $S_\infty^1 = \hat{\mathbb{R}}$  of a family of  $K$ -quasiconformal mappings of the Riemann sphere to itself. A sequence of  $K$ -quasiconformal mappings converging to another  $K$ -quasiconformal mapping converges uniformly on compact sets ([6]). Therefore, for every  $\Gamma \in \mathbb{C}R_g$ , there is an open set containing  $\Gamma$  for which either  $1/|\phi_i|$  or  $|\phi_i|$  is bounded (depending on whether or not  $\phi(\Gamma) = \infty$ ) for  $i > N$ , some  $N$ . In other words, the  $\phi_i$  are locally bounded. Since they are all complex analytic, they form a normal family, and the limit  $\phi(\Gamma) = f_\Gamma(x)$  is analytic.  $\square$

The function  $\cos \theta$  can be extended to arbitrary collections  $(a_1, b_1, a_2, b_2)$  by formula (1). It will be a complex analytic function of the endpoints  $\{a_i, b_i\}$ . We can then extend the function  $\int_\phi \cos \theta d\mu$  to a neighborhood of  $R_g$  in  $\mathbb{C}R_g$ . The integral exists and is approximated uniformly on compact sets of  $\mathbb{C}R_g$  by its Riemann sums. To see this, note that it is true on  $R_g$  because the leaves of  $\mu$  do not cross. (This was discussed in I.B.) Furthermore, the maps  $f_\Gamma$  from  $S_\infty^1$  to  $\Lambda$  are equicontinuous on compact subsets of  $\mathbb{C}R_g$  by the proof of Proposition 2.3 so the local variation of  $\cos \theta$  is still uniformly bounded. Thus the integral is approximated uniformly by its Riemann sums as claimed.

We can now prove Theorem 1.



*Proof of Theorem 1.* By Proposition 2.3  $\cos \theta$  is complex analytic on  $\mathbb{C}R_g$ , and, by the discussion above, the integral  $\int_\phi \cos \theta d\mu$  is approximated uniformly on compact subsets of  $\mathbb{C}R_g$  by its Riemann sums so that it is also complex analytic. The real part of the integral is analytic and since  $\cos \theta$  is real on the real analytic submanifold  $R_g$ ,  $\int_\phi \cos \theta d\mu$  is real analytic on  $R_g$ .

The real analytic structure of  $T_g$  is determined by the lengths  $l_\phi$  of finitely many simple closed geodesics  $\phi$  and the derivative of  $l_\phi$  along the flow  $\mathcal{F}_\mu$  is  $\int_\phi \cos \theta d\mu$  by Proposition 1.3. Thus  $l_\phi$  itself is analytic along  $\mathcal{F}_\mu$  and  $\mathcal{F}_\mu$  is an analytic flow.  $\square$

Before proving Corollary 2.2, we digress for a brief discussion of the length  $l_\nu$  of a geodesic lamination  $\nu$  and its derivative along  $\mathcal{F}_\mu$ .

**DEFINITION.** The *length*,  $l_\nu(M)$ ,  $M \in T_g$ , of a geodesic lamination  $\nu$  is the integral over  $M$  of the product measure  $d\nu \times dl$  where  $dl$  is the length measure along the leaves of  $\nu$ .

**LEMMA 2.4.** *Given any sequence  $c_i\phi_i$  of weighted simple closed geodesics converging in  $\mathcal{ML}$  to  $\nu$ , and any  $M \in T_g$ ,  $c_i l_{\phi_i}(M)$  converges to  $l_\nu(M)$ . The convergence is uniform on compact subsets of  $T_g$ ; hence  $l_\nu$  is continuous on  $T_g$ . In fact  $l_\nu(M)$  is continuous with respect to the pair  $(M, \nu)$ .*

*Proof.* If we denote by  $d\phi_i$  the counting measure on  $\phi_i$  then  $l_{\phi_i}(M) = \int_M d\phi_i \times dl$ . Cover the support of  $\nu$  with finitely many quadrilaterals with the following properties:

- i) Two opposite (“horizontal”) sides are disjoint from  $\nu$ .
- ii) The remaining two (“vertical”) sides are transverse to  $\nu$  and each leaf of  $\nu$  crosses from one side to the other.

It suffices to prove the lemma on a single quadrilateral  $Q$ .

By definition (see [4]), convergence of  $c_i\phi_i$  to  $\nu$  implies that, on any finite set of transverse geodesic arcs,  $A_j$ , endpoints disjoint from  $\nu$ , the intersection numbers and total cosines of  $c_i\phi_i$  with the  $A_j$  converge to those of  $\nu$  with the  $A_j$ . Moreover, if we let the  $A_j$  vary continuously with the hyperbolic structure, the intersection number and total cosine are continuous as functions on  $T_g \times \mathcal{ML}$  so the convergence is uniform on compact subsets of  $T_g$ .

To see that the intersection number is continuous on the product, note that because the leaves of all laminations and the arcs  $A_j$  move continuously with the hyperbolic structure,  $M$ , we can assume, by restricting to a small neighborhood of a given structure  $M_0$ , that only geodesics in a given neighborhood of the endpoints of the  $A_j$  move across the endpoints as we vary  $M$ . Since the measure



with respect to  $\nu$  of some neighborhood of the endpoints is zero, the measure of this neighborhood with respect to other laminations can be made arbitrarily small on  $M_0$  by restricting to small neighborhoods of  $\nu$  in  $\mathcal{ML}$ . Thus the change in the intersection number with the  $A_j$  is uniformly small on this neighborhood of  $\nu$  as we vary over the chosen neighborhood of  $M_0$ . Continuity follows. The same argument shows that the angles of intersection and total cosines are also continuous as functions on the product.

In particular the above discussion applies to any finite subdivision of one of the horizontal arcs  $A$  of  $Q$  if  $Q$ ,  $A$ , and its subdivision vary continuously with the hyperbolic structure. Restricting to a single sub-arc, we see that the variation of the angle of intersection for the leaves of a lamination is bounded, independent of the lamination, depending only on the length of the sub-arc  $A_j$ . This is because the leaves of a lamination do not cross. It follows that the shortest and longest pieces of leaves in  $Q$  going from a fixed  $A_j$  to the opposite side of  $Q$  are universally close, depending only on the shape and size of  $Q$ , and going to zero as the length of the  $A_j$  go to zero. Since the shape and size of  $Q$  and the length of  $A_j$  vary continuously with the hyperbolic structure, the estimate for the difference between the shortest and longest pieces of leaves can be made uniform on a compact subset of  $T_g$ , independent of the lamination.

It follows that the integrals defining the length, restricted to  $Q$ , can be uniformly approximated by their Riemann sums, i.e., for any  $\varepsilon > 0$  there is a subdivision  $A_j$  of  $A$  such that

$$\left| \int_Q d\phi_i \times dl - \sum i(A_j, \phi_i) l_j^{(i)} \right| < \varepsilon i(A, \phi_i)$$

Similarly,

$$\left| \int_Q d\nu \times dl - \sum i(A_j, \nu) l_j \right| < \varepsilon i(A, \nu)$$

where  $l_j$  ( $l_j^{(i)}$ ) is the length of any arc of  $\nu(c_i\phi_i)$  going from  $A_j$  to the opposite side of  $Q$ . The estimates are uniform on compact subsets of  $T_g$  and the only restriction on the weighted curves  $c_i\phi_i$  is that they are close enough to  $\nu$  so that the total length of the arcs crossing  $A$  but hitting the top or the bottom of  $Q$  is small. (These arcs are counted in the integral but not in the sum.)

Finally, since for any finite collection of  $A_j$ 's and a given compact subset of  $T_g$ , the intersection numbers and angles of intersection (hence the  $l_j^{(i)}$  also) converge uniformly, we can always find an integer  $N$  such that for the collection of arcs needed for the first estimates we have

$$\left| \sum i(A_j, \nu) l_j - \sum c_i i(A_j, \phi_i) l_j^{(i)} \right| < \varepsilon, \quad i > N$$

Since  $\varepsilon$  is arbitrary the lemma follows from the three estimates and the triangle inequality.  $\square$

Given any two geodesic laminations,  $\mu, \nu$ , we can define the product measure  $d\mu \times d\nu$  on  $M$ . First, consider any quadrilateral  $Q$  whose “horizontal” sides,  $A_1, A_2$ , are parts of leaves of  $\nu$  and whose “vertical” sides  $B_1, B_2$  are parts of leaves of  $\mu$ . Furthermore, we require that  $i(A_1, \mu) = i(A_2, \mu)$  and  $i(B_1, \nu) = i(B_2, \nu)$ ; i.e., no leaf of either  $\mu$  or  $\nu$  hits the same side twice. Then, by definition,  $d\mu \times d\nu(Q)$  equals the product  $i(B_1, \nu)i(A_1, \mu)$ . The measure of an arbitrary Borel set is defined in the usual way. The measure is defined to be zero at any point of tangency of  $\mu$  and  $\nu$ ; i.e., on any common leaf.

When  $\mu$  and  $\nu$  are weighted simple closed geodesics, the derivative of  $l_\nu$  with respect to twisting along  $\mu$  is easily seen to be the weighted sum of  $\cos \theta$  at the finite number of intersections of  $\mu$  and  $\nu$ . (See, e.g., [4] Lemma 3.2.) This is just  $\int_M \cos \theta d\mu \times d\nu$  when  $d\mu$  and  $d\nu$  are both atomic. Proposition 1.3 covers the case when only  $d\nu$  is atomic. For the general case we have the following:

**PROPOSITION 2.5.** *The function  $l_\nu$  is  $C^1$  along the earthquake path  $\mathcal{E}_\mu(t)$  with derivative  $\int_M \cos \theta d\mu \times d\nu$ , where  $\theta$  is the angle (measured counterclockwise) from  $\nu$  to  $\mu$  at each point of intersection of  $\mu$  and  $\nu$ .*

*Proof.* Take any sequence  $c_i \phi_i$  of weighted simple closed curves converging to  $\nu$  in  $\mathcal{ML}$ . Then we will show that

$$c_i \int_M \cos \theta d\mu \times d\phi_i \rightarrow \int_M \cos \theta d\mu \times d\nu \quad (2)$$

uniformly on compact sets of  $T_g$ . Thus

$$c_i l_{\phi_i} \rightarrow l_\nu \quad \text{and} \quad c_i \frac{dl_{\phi_i}}{dt} \rightarrow \int_M \cos \theta d\mu \times d\nu$$

along  $\mathcal{E}_\mu(t)$ , uniformly for  $t \leq T$ . The proposition will then follow.

The proof of (2) is essentially the same as that of Lemma 2.4. Cover the support of  $d\mu \times d\nu$  by finitely many geodesic quadrilaterals of the type described above with vertical sides in  $\mu$  and horizontal sides in  $\nu$ . Restrict, without loss of generality, to a single such quadrilateral  $Q$  which we further subdivide into similar quadrilaterals  $Q_j$  with sides of length less than  $\delta$ . To apply the discussion from the proof of Lemma 2.4 perturb the  $Q_j$  slightly so that the vertical sides are disjoint from  $\mu$ , the horizontal sides from  $\nu$  and assume that they move continuously with the hyperbolic structure,  $M$ . For  $\delta$  sufficiently small,  $\cos \theta$  will vary less than any

given  $\varepsilon$  on every  $Q_j$ , so

$$\left| \int_Q \cos \theta d\mu \times d\nu - \sum \cos \theta(x_j) \int_{Q_j} d\mu \times d\nu \right| < \varepsilon \int_Q d\mu \times d\nu$$

for any choice of  $x_j$  in  $Q_j$ . Similarly,

$$\left| c_j \int_{Q_j} \cos \theta d\mu \times d\phi_i - \sum \cos \theta(y_j) c_j \int_{Q_j} d\mu \times d\phi_i \right| < \varepsilon c_j \int_{Q_j} d\mu \times d\phi_i$$

for any choice of  $y_j$  in  $Q_j$  in the intersection of  $\mu$  with  $\phi_i$ .

These estimates depend only on  $\delta$ ; hence they are uniform in  $M$ . As in the proof of Lemma 2.4, for a fixed subdivision  $Q_j$  and compact region of  $T_g$ , we can find an  $N$  such that for  $i > N$  the Riemann sums are approximated uniformly by those of the  $c_i \phi_i$ . From the estimates above and the triangle inequality the integrals are similarly estimated. But, by choosing  $\delta$  sufficiently small and  $N$  sufficiently large, this holds for any  $\varepsilon$  and the proposition follows.  $\square$

Now the proof of Corollary 2.2 is straightforward.

*Proof of Corollary 2.2.* From Proposition 2.5, the derivative of  $l_\nu$  along  $\mathcal{E}_\mu(t)$  is:

$$\frac{dl_\nu}{dt} = \int_M \cos \theta d\mu \times d\nu$$

where  $\theta$  is the angle (measured counterclockwise) from  $\nu$  to  $\mu$  at each intersection of  $\mu$  and  $\nu$ . Proposition 2.3 implies that  $\cos \theta$  varies analytically over  $\mathbb{C}R_g$  at each intersection of  $\mu$  and  $\nu$ . As in the proof of Theorem 1, this implies that  $l_\nu$  varies analytically along  $\mathcal{E}_\mu(t)$ . When  $\mu = \nu$  the first derivative is identically zero so  $l_\nu$  is constant. To see that  $l_\nu$  is analytic on all of  $T_g$  note that all of its directional derivatives are analytic either by Proposition 2.6 below or by the fact that tangents to classical twist flows span the tangent space at every point (see [10]).

That the functions  $l_{\mu_i}$  converge to  $l_\mu$  in the  $C^\infty$ -topology (on compacts) as  $\mu_i$  converges to  $\mu$  similarly follows from  $l_{\mu_i}(M)$  converging to  $l_\mu(M)$  and the fact that when complexified, the derivatives of  $l_{\mu_i}$  along classical twist paths converge uniformly to those of  $l_\mu$  on compact neighborhoods of  $R_g$  in  $\mathbb{C}R_g$ .  $\square$

Although it is not necessary in the proof of Corollary 2.2, it seems worthwhile to point out (Proposition 2.6) that every tangent vector in  $T_g$  is tangent to a

unique earthquake path. The main point is the following theorem which is proved in [5] where it is of more central importance:

**THEOREM.** *If two geodesic laminations  $\mu$  and  $\nu$  on  $M \in T_g$  have the same total cosine with every closed geodesic on  $M$ , then  $\mu = \nu$ .*

**PROPOSITION 2.6.** *Every tangent vector in  $T_g$  is tangent to a unique earthquake path in  $T_g$ .*

*Proof.* Since the space of geodesic laminations and the tangent space at any point  $M$  in  $T_g$  are homeomorphic to  $6g-6$  dimensional balls, it suffices, by invariance of domain, to show that the map associating a lamination  $\mu$  with the tangent to the integral curve (earthquake path) of  $\mathcal{F}_\mu$  through  $M$  is continuous, proper, and 1-1. From Proposition 1.3 continuity and properness are immediate. Similarly, from this proposition, it follows that if  $\mathcal{F}_\mu$  and  $\mathcal{F}_\nu$  are tangent at  $M$ , then  $\int_\phi \cos \theta d\mu = \int_\phi \cos \tilde{\theta} d\nu$  for every closed geodesic  $\phi$ . In other words, all the total cosines are equal, so, by the Theorem above, this implies that  $\mu = \nu$  and the map is 1-1.  $\square$

## BIBLIOGRAPHY

- [1] BERS, L. *On boundaries of Teichmuller space and on Kleinian groups, I*, Ann. of Math. 91 (1970), 570-600.
- [2] FATHI, A., et al., *Travaux de Thurston Sur les Surfaces*, Asterisque (Orsay Seminaire), 1979.
- [3] HARVEY, W. (ed.), *Discrete Groups and Automorphic Functions*, Academic Press (London), 1977.
- [4] KERCKHOFF, S., "The Nielsen realization problem," Ann. of Math., 117 (1983), 235-265.
- [5] KERCKHOFF, S., "Lines of minima in Teichmuller space," to appear.
- [6] LEHTO, O. and VIRTANEN, K., *Quasiconformal Mappings in the Plane*, Springer-Verlag (New York), 1973.
- [7] ROYDEN, H. "Automorphism and isometries of Teichmuller space," *Advances in the Theory of Riemann Surfaces*, Ahlfors, L. (ed.), Ann. of Math. Studies #66, (Princeton), 1971.
- [8] THURSTON, W., *The Geometry and Topology of 3-Manifolds*, notes, Princeton University.
- [9] WOLPERT, S., *On the symplectic geometry of deformations of a hyperbolic surface*, Ann. of Math., 117 (1983), 207-234.
- [10] —, *The Fenchel-Nielsen twist deformation*, Ann. of Math. 115 (1982), 501-528.

*The Institute of Advanced Study*  
*School of Mathematics*  
*Princeton, New Jersey 08540*  
*USA*

Received September 23, 1983