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The Smale invariant of a knot

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Smale [S2] associates to each immersion $f: S^n \looparrowright \mathbb{R}^k$ an element $s(f)$ in $\pi_n V_n(\mathbb{R}^k)$, where $V_n(\mathbb{R}^k)$ is the Stiefel manifold of n-frames in $\mathbb{R}^k$. The map s is an isomorphism from the set of regular homotopy classes of immersions of S^n in $\mathbb{R}^k$ to the set $\pi_n V_n(\mathbb{R}^k)$. Smale [S2, p. 329, questions (2) and (3)] asks for a characterization of those elements $s(f)$ where f is an *embedding*. Kervaire [K3] solves this problem for $k \geqq \frac{3}{2}n+1$ and then, together with Milnor [K3] [KM], for $k=n+1$. (In all of these cases, $s(f)=0$ when f is an embedding.) Haefliger [Ha, 4.7] gives a homotopy theoretic solution for all $k \geqq n+3$, which, however, does not lend itself to simple computations. We solve the problem for the case $k=n+2$ in this paper, including an explicit means for computing the Smale invariant (Corollary 2).

If $n=1$, then there is only one regular homotopy class, and it is represented by an embedding. The case $n=2$ is solved by Smale [S1], who shows that regular homotopy classes correspond to elements of the set $\pi_2 V_2(\mathbb{R}^4)=\mathbb{Z}$, and the integer associated with a self-transverse immersion is the algebraic number of double points of the immersion. Thus there is only one immersion represented by an embedding, and its Smale invariant $s(f)$ is zero.

For $n>2$, the group $\pi_n V_n(\mathbb{R}^{n+2})$ can be identified with the group $\pi_n SO(n+2)$. We call the image of $s(f)$ under this identification $i(f)$. The main result of this paper may be stated as follows:

THEOREM. *Let $f: S^n \looparrowright \mathbb{R}^{n+2}$ be an immersion. Then f is regularly homotopic to an embedding if and only if $J(i(f))=0$.*

Here J denotes the Hopf–Whitehead J homomorphism from $\pi_n SO(n+2)$ to $\pi_{2n+2} S^{n+2}$.

The proof consists of identifying $i(f)$ geometrically in a more convenient form than Smale's original definition, understanding the J homomorphism geometrically, and then combining these when f is an embedding to see that $J(i(f))=0$. The proof of the converse is by construction, using examples provided by Brieskorn [Br].

Using known properties of the J homomorphism, it follows that there exist non-trivial embeddings $S^n \hookrightarrow \mathbb{R}^{n+2}$ (i.e. embeddings not regularly homotopic to the standard inclusion) if and only if $n \equiv 3 \pmod 4$ (Corollary 1). This answers negatively the question raised in Kervaire [K3, §5]: "Is the Smale invariant of an embedding $f: S^n \hookrightarrow \mathbb{R}^k$ with $k \leqq n+3$ always zero?" Ironically, a proof that $J(i(f)) = 0$ (properly intepreted) is implicit in [K3], which together with the results of [MK] might have indicated where to look for counterexamples.

1. Preliminaries

$\mathbb{R}^n$ denotes coordinate n-space, which we consider naturally embedded in $\mathbb{R}^{n+1}$ as the points with last coordinate zero. B^n denotes the closed unit ball in $\mathbb{R}^n$, and S^n the boundary of B^{n+1}.

$V_n(\mathbb{R}^k)$ denotes the Stiefel manifold of n-frames in $\mathbb{R}^k$, which we identify with the space of injective linear maps from $\mathbb{R}^n$ to $\mathbb{R}^k$ (associating the frame $v_1, \ldots, v_n$ with the linear map sending e_i to v_i). Similarly we identify $GL(k)$, the set of $k \times k$ invertible matrices, with the space $\mathrm{Aut}(\mathbb{R}^k)$ of linear automorphisms of $\mathbb{R}^k$. $GL_+(k)$ denotes the matrices of positive determinant in $GL(k)$, or equivalently the orientation preserving maps in $\mathrm{Aut}(\mathbb{R}^k)$. $SO(k)$ denotes the orthogonal matrices of determinant one in $GL(k)$, identified with the rotations of $\mathbb{R}^k$.

Throughout this paper, all manifolds and maps are smooth. If M is a manifold, then τ_M denotes the tangent bundle of M, and ε^k denotes the trivial bundle over M with fiber $\mathbb{R}^k$.

$\mathrm{Imm}(S^n, \mathbb{R}^k)$ denotes the set of all regular homotopy classes of immersions $f: S^n \looparrowright \mathbb{R}^k$. We often do not distinguish between an immersion and its regular homotopy class; thus we may write $f \in \mathrm{Imm}(S^n, \mathbb{R}^k)$.

$\mathrm{Emb}(S^n, \mathbb{R}^k)$ denotes the subset of $\mathrm{Imm}(S^n, \mathbb{R}^k)$ consisting of all regular homotopy classes containing an embedding.

DEFINITION 1. Let $f: S^n \looparrowright \mathbb{R}^{n+2}$ be an immersion. We define two invariants,

$$i(f) \in \pi_n SO(n+2)$$

and the *Smale invariant*

$$s(f) \in \pi_n V_n(\mathbb{R}^{n+2}),$$

as follows:

The invariant $i(f)$:

Extend f to an orientation preserving immersion

$$F: N(S^n) \looparrowright \mathbb{R}^{n+2}$$

where $N(S^n)$ is a neighborhood of the standard S^n in $\mathbb{R}^{n+2}$. Then dF, the differential of F, maps $N(S^n)$ into $GL_+(n+2)$. Define $i(f)$ to be the homotopy class of the map

$$S^n \to SO(n+2): x \mapsto GS \circ dF_x,$$

where $GS: GL_+(n+2) \to SO(n+2)$ is the Gram–Schmidt map. It is not hard to see that if $n > 1$, then $i(f)$ is independent of the choice of F, and in fact depends only on the regular homotopy class of f. Thus there is a well defined map

$$i: \mathrm{Imm}\,(S^n, \mathbb{R}^{n+2}) \to \pi_n SO(n+2)$$

for $n > 1$.

The Smale invariant $s(f)$:

Consider S^n as lying in $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$, and write points in $\mathbb{R}^{n+1}$ as pairs (v, t), where $v \in \mathbb{R}^n$ and $t \in \mathbb{R}$. The northern and southern hemispheres of S^n are then

$$N = \{(v, t) \in S^n : t \geqq 0\}$$
$$S = \{(v, t) \in S^n : t \leqq 0\}.$$

If $x = (v, t)$, write $\bar{x}$ for $(v, -t)$. Stereographic projection from the south pole, $sp = (0, -1)$, to a plane tangent to the north pole, $np = (0, 1)$, is given by the formula

$$p: S^n - \{sp\} \to \mathbb{R}^n : (v, t) \mapsto \frac{2}{1+t} v.$$

Let $q: \mathbb{R}^n \to \mathbb{R}^{n+2}$ be the inverse of p, followed by the natural inclusion of S^n in $\mathbb{R}^{n+2}$.

Now alter the immersion f by a regular homotopy so that the restriction of f to the southern hemisphere S agrees with the standard inclusion of S into $\mathbb{R}^{n+2}$. Define $s(f)$ to be the homotopy class of the map

$$S^n \to V_n(\mathbb{R}^{n+2}) : x \mapsto \begin{cases} d(f \circ q)_{p(x)} & x \in N \\ dq_{p(\bar{x})} & x \in S \end{cases}$$

(compare Smale [S2]). It turns out that $s(f)$ is independent of the choice of regular homotopy used to alter f, and so there is a well-defined map

$$s : \mathrm{Imm}(S^n, \mathbb{R}^{n+2}) \to \pi_n V_n(\mathbb{R}^{n+2}).$$

Smale [S2] shows that s is a bijection. In fact, using the operation of oriented connected sum on $\mathrm{Imm}(S^n, \mathbb{R}^{n+2})$, s is an isomorphism of groups (see Kervaire [K2], Hughes [Hu]).

DEFINITION 2. Let $j : \mathbb{R}^n \to \mathbb{R}^{n+2}$ denote the standard inclusion. Define $\phi : SO(n+2) \to V_n(\mathbb{R}^{n+2})$ by sending h to $h \circ j$. (Here we are thinking of elements of $SO(n+2)$ and $V_n(\mathbb{R}^{n+2})$ as linear maps. On the matrix level, ϕ is simply "drop the last two columns of the matrix".) Observe that ϕ induces an isomorphism

$$\phi_* : \pi_n SO(n+2) \to \pi_n V_n(\mathbb{R}^{n+2})$$

for $n > 2$. (To see this, consider the commutative diagram

$$\begin{tikzcd} GL_+(n+2) \arrow[dr, "\psi"] & \\ SO(n+2) \arrow[u, phantom, "\cup"] \arrow[r, "\phi"'] & V_n(\mathbb{R}^{n+2}) \end{tikzcd}$$

where $\psi(g) = g \circ j$. ψ is a fibration with a fiber which is homotopy equivalent to $GL_+(2)$, which is in turn homotopy equivalent to S^1. Hence ψ induces an isomorphism on π_n for $n > 2$. The inclusion $SO(n+2) \subset GL_+(n+2)$ is a homotopy equivalence, so induces an isomorphism on π_n for every n.)

Combining definitions 1 and 2 we have a diagram

$$\begin{tikzcd} & \pi_n V_n(\mathbb{R}^{n+2}) & \\ \mathrm{Imm}(S^n, \mathbb{R}^{n+2}) \arrow[ur, "s"] \arrow[rr, "i"'] & & \pi_n(SO(n+2)) \arrow[ul, "\phi_*"'] \end{tikzcd}$$

(for $n > 1$) with s and ϕ_* isomorphisms (for $n > 2$).

PROPOSITION. $s = \phi_* \circ i$. *Thus i is an isomorphism for $n > 2$.*

Proof. Let $f \in \mathrm{Imm}(S^n, \mathbb{R}^{n+2})$. As in definition 1, we may assume that f agrees with the standard inclusion on the southern hemisphere S. It is easy to arrange that F (in the definition of $i(f)$) is the identity in a neighborhood of S in $\mathbb{R}^{n+2}$.

Notice that since $f=F$ on image(q), we may write $d(fq)_v=d(Fq)_v$ for v in $\mathbb{R}^n$. Thus $s(f)$ is represented by the map

$$S^n \to V_n(\mathbb{R}^{n+2}): x \mapsto \begin{cases} d(fq)_{p(x)} = dF_x dq_{p(x)} & x \in N \\ dq_{p(\bar{x})} & x \in S \end{cases}$$

(applying the chain rule). This map can be altered by the homotopy

$$(x, t) \mapsto \begin{cases} dF_x dq_{(1-t)p(x)} & x \in N \\ dq_{(1-t)p(\bar{x})} & x \in S \end{cases}$$

resulting in

$$s(f) = \left[x \mapsto \begin{cases} dF_x dq_0 & x \in N \\ dq_0 & x \in S \end{cases} \right] = [x \mapsto dF_x \circ j]$$

where j denotes the inclusion of $\mathbb{R}^n$ into $\mathbb{R}^{n+2}$. The last equality follows because $dq_0 = j$, as is easily verified. But the map $x \mapsto dF_x \circ j$ is homotopic to $x \mapsto GS \circ dF_x \circ j$, which by definition represents $\phi_*(i(f))$, proving the proposition.

DEFINITION 3. Suppose that M is a manifold, and P and Q are codimension zero submanifolds with $P \cap Q$ a submanifold and $P \cup Q = M$. Given a map $f: P \cap Q \to GL(k)$, denote by

$$\beta(P, Q, f)$$

the $\mathbb{R}^k$-bundle whose total space is $(P \times \mathbb{R}^k) \cup (Q \times \mathbb{R}^k)/\sim$, where $\sim$ is the equivalence relation identifying $(x, v) \in P \times \mathbb{R}^k$ with $(x, f(x)v) \in Q \times \mathbb{R}^k$, for all x in $P \cap Q$. The projection map for this bundle sends (x, v) to x.

It follows from the homotopy axiom for vector bundles that if f and g are homotopic maps from $P \cap Q$ to $GL(k)$, then $\beta(P, Q, f)$ and $\beta(P, Q, g)$ are isomorphic bundles. Also, if $\bar{f}$ is defined by $\bar{f}(x) = f(x)^{-1}$, then $\beta(P, Q, f) \cong \beta(Q, P, \bar{f})$ and $\beta(P, Q, f) \oplus \beta(P, Q, \bar{f}) \cong \varepsilon^{2k}$.

If an orientable bundle ξ over M is trivial away from a point (almost parallelizable), then there is an isomorphism $\xi \cong \beta(P, Q, f)$ with $Q = B^{n+1}$, $P \cap Q = S^n = \partial B^{n+1}$, and $f: S^n \to SO(k)$. The class $[f] \in \pi_n SO(k)$ is called the *obstruction to framing* ξ.

2. The main theorem

From the previous section, there is a commutative diagram:

$$
\begin{tikzcd}
 & & \pi_n V_n(\mathbb{R}^{n+2}) & \\
\mathrm{Emb}\,(S^n, \mathbb{R}^{n+2}) \subset & \mathrm{Imm}\,(S^n, \mathbb{R}^{n+2}) \arrow[ur, "s"] \arrow[r, "i"'] & \pi_n SO(n+2) \arrow[u, "\phi_*"'] \arrow[r, "J"'] & \pi_{2n+2}(S^{n+2}).
\end{tikzcd}
$$

Our main result is:

THEOREM. $s(\mathrm{Emb}\,(S^n, \mathbb{R}^{n+2})) = \phi_*(\ker\,(J))$.

Proof. It suffices to show $i(\mathrm{Emb}\,(S^n, \mathbb{R}^{n+2})) = \ker\,(J)$. The proof is in two steps.

STEP 1. *If $f \in \mathrm{Emb}\,(S^n, \mathbb{R}^{n+2})$, then $J(i(f)) = 0$.*

Extend f to an embedding $f: M_0 \hookrightarrow \mathbb{R}^{n+2}$ of some compact oriented $(n+1)$-manifold M_0 with $\partial M_0 = S^n$. (M_0 is called a Seifert surface for f.) Consider the closed, smooth manifold

$$M = M_0 \cup B^{n+1}$$

the union being along $\partial B^{n+1} = S^n = \partial M_0$. We will show that $i(f)$ is an obstruction to framing the stable normal bundle of M. Step 1 then follows from Lemma 1 of [MK]. The details follow:

A suitable neighborhood U of B^{n+1} in M can be identified with $\mathbb{R}^{n+1}$. Thus we view $\mathbb{R}^{n+2} = \mathbb{R}^{n+1} \times \mathbb{R} = U \times \mathbb{R} \subset M \times \mathbb{R}$. The standard orientation on $\mathbb{R}^{n+2}$ induces an orientation on $M \times \mathbb{R}$. Within $M \times \mathbb{R}$, we identify M with $M \times \{0\}$. Set $V = M - \{0\}$ (here 0 denotes the origin of $\mathbb{R}^{n+1}$ = center of B^{n+1}).

Now further extend f to an orientation preserving embedding

$$F: V \times \mathbb{R} \hookrightarrow \mathbb{R}^{n+2}$$

(see Figure 1).

Let

$$g = dF \mid S^n : S^b \to GL_+(n+2).$$

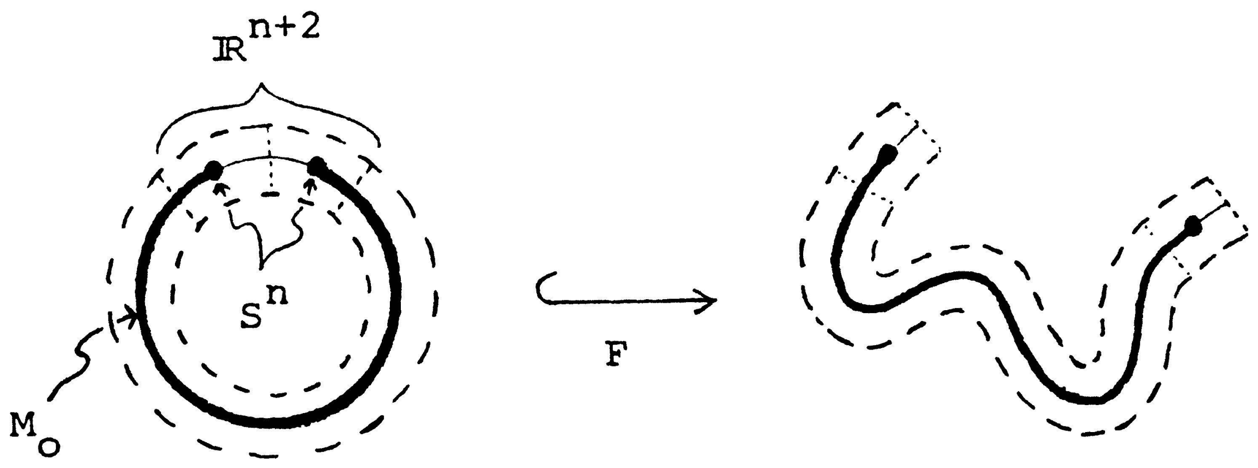

Figure 1

Then

$$\tau_{M\times\mathbb{R}}\,|\,M \cong \beta(B^{n+1}, M_0, g). \tag{1}$$

An explicit isomorphism between the bundles is given by assigning to the tangent vector v to $M\times\mathbb{R}$ at the point x in $M = M\times\{0\}$, either

$$(x, v) \in B^{n+1}\times\mathbb{R}^{n+2}$$

if $x \in B^{n+1}$ (here we think of $B^{n+1}\subset\mathbb{R}^{n+1}$, so $v\in\mathbb{R}^{n+2}$), or

$$(x, dF_x(v)) \in M_0\times\mathbb{R}^{n+2}$$

if $x \in M_0$. This is well-defined, for if $x\in M_0\cap B^{n+1} = S^n$, then $(x, v)\in B^{n+1}\times\mathbb{R}^{n+2}$ is identified with $(x, g(x)v) = (x, dF_x(v)) \in M_0\times\mathbb{R}^{n+2}$, by the definition of $\beta(B^{n+1}, M_0, g)$ and g.

Let $h = GS\circ g$, so that

$$h : S^n \to SO(n+2).$$

Note that $[h] = i(f)$, by definition. Furthermore, g and h are homotopic maps (in $GL_+(n+2)$), so we have

$$\begin{aligned}\tau_M \oplus \epsilon^1 \cong \tau_{M\times\mathbb{R}|M} &\cong \beta(B^{n+1}, M_0, g) \quad \text{(by (1))}\\ &\cong \beta(B^{n+1}, M_0, h).\end{aligned}$$

Now using the Whitney embedding theorem, embed M in S^{2n+3} and let ν be the normal $(n+2)$-plane bundle of the embedding. Then $(\tau_M \oplus \varepsilon^1) \oplus \nu \cong \varepsilon^{2n+4} \cong \beta(B^{n+1}, M_0, h) \oplus \beta(B^{n+1}, M_0, \tilde{h})$ (where $\tilde{h}(x) = h(x)^{-1}$), and so

$$\nu \cong \beta(B^{n+1}, M_0, \tilde{h}) \cong \beta(M_0, B^{n+1}, h)$$

(both bundles are stable normal bundles of M). Since $[h] = i(f)$,

$$i(f) \text{ is the obstruction to framing } \nu. \tag{2}$$

By Lemma 1 of [MK], it follows that $J(i(f)) = 0$.

Remark. For the reader's convenience, here are the details of a proof of the lemma cited above:

Embed M in S^{2n+3} so that it is perpendicular to the equatorial S^{2n+2}, intersecting it in the standard $S^n \subset S^{2n+2}$, with M_0 lying in the northern hemisphere of S^{2n+3} (see Figure 2). This may be accomplished by taking a height function for S^{2n+3}, making it transverse to the embedding of M, and then identifying a minimum point. An isotopy taking a neighborhood of this minimum onto the southern hemisphere alters the embedding to one satisfiying the conditions above.

Consider the normal framing $\mathbb{F}$ on $S^n = M \cap S^{2n+2}$ in S^{2n+2}, given by assigning to a point $x \in S^n$ the frame $\begin{pmatrix} 0 \\ h(x) \end{pmatrix} \in V_{n+2}(\mathbb{R}^{2n+3})$. The Thom–Pontrjagin construction applied to this framed submanifold of S^{2n+2} gives an element of $\pi_{2n+2}S^{n+2}$

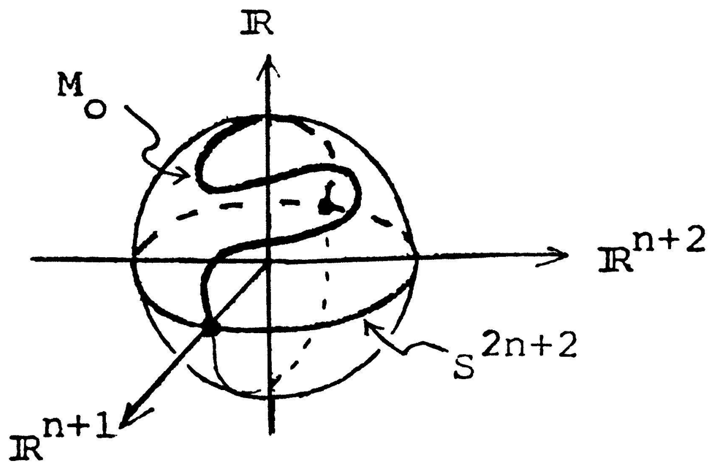

Figure 2

which can be identified with $J([h])$: Both elements are represented by the map

$$\begin{array}{c} S^{2n+2} \subset \mathbb{R}^{2n+3} = \mathbb{R}^{n+1} \times \mathbb{R}^{n+2} \\ \downarrow \\ S^{n+2} \subset \mathbb{R}^{n+3} = \mathbb{R}^{n+2} \times \mathbb{R} \end{array}$$

sending the minimal geodesic arc joining $x \in S^n \times \{0\}$ with $y \in \{0\} \times S^{n+1}$ to the minimal geodesic arc joining the south pole with the north pole of S^{n+2} and passing through $h(x)y \in S^{n+1} \times \{0\}$. (Explicitly, $J([h])$ maps $(u, v) \in S^{2n+2}$ to $(0, 1) \in S^{n+2}$ if $u = 0$, and to $(2\|u\|\, h(u/\|u\|)v, \|v\|^2 - \|u\|^2)$ otherwise.) Compare Kervaire [K1, 1.8].

Finally observe that the framing $\mathbb{F}$ on S^n extends over M_0: $\beta(M_0, B^{n+1}, h)$ is abstractly isomorphic to the normal bundle ν of M in S^{2n+3}. We may choose an isomorphism over B^{n+1} which is standard over $S^n = \partial B^{n+1}$ (i.e. maps the standard frame on $\mathbb{R}^{n+2}$ to the standard frame on $\{0\} \times \mathbb{R}^{n+2} \subset \mathbb{R}^{n+1} \times \mathbb{R}^{n+2} = \mathbb{R}^{2n+3}$), and extend this to an isomorphism Ψ over the rest of M. But on $S^n = \partial M_0$, the standard frame on $\mathbb{R}^{n+2}$ maps to $\mathbb{F}$ under Ψ. Hence the image under Ψ of the standard frame on $\mathbb{R}^{n+2}$ over M_0 provides an extension of $\mathbb{F}$.

Now because the framing extends over M_0, the Thom–Pontrjagin construction yields 0 in $\pi_{2n+2}(S^{n+2})$, hence so must J.

STEP 2. *If $J(x) = 0$, then there exists $f \in \mathrm{Emb}\,(S^n, \mathbb{R}^{n+2})$ with $i(f) = x$.*

Bott [Bo] computes

$$\pi_n SO(n+2) = \begin{cases} \mathbb{Z} & \text{if } n \equiv 3 \text{ or } 7 \pmod 8 \\ \mathbb{Z}_2 & \text{if } n \equiv 0 \text{ or } 1 \pmod 8 \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, by the work of Adams, $J: \pi_n SO(n+2) \to \pi_{2n+2} S^{n+2}$ is injective for $n \equiv 0$ or $1 \pmod 8$ (see Switzer [S, p. 487]). Thus there is nothing to prove except in the case $n \equiv 3$ or $7 \pmod 8$, i.e. $n \equiv 3 \pmod 4$.

So let $n = 4m - 1$. Write j_m for the order of the image of $J: \pi_{4m-1} SO(4m+1) \to \pi_{8m} S^{4m+1}$. Identifying $\pi_{4m-1} SO(4m+1)$ with $\mathbb{Z}$, it suffices to produce an embedding $f: S^{4m-1} \to \mathbb{R}^{4m+1}$ with $i(f) = \pm j_m$.

First consider the collection of all closed, oriented, almost-parallelizable $4m$-manifolds M. The associated signatures $\sigma(M)$ form a subgroup of $\mathbb{Z}$; let $\sigma_m > 0$ denote the generator. Similarly let $p_m > 0$ denote the generator of the group of all top Pontrjagin numbers $p_m(M)$. Observe that if $\sigma(M) = \sigma_m$, then by the Hirzebruch Index Theorem, $p_m(M) = p_m$. Also, it is known that $\sigma_m \equiv 0 \pmod 8$ (see [KM, p. 531]).

Case 1: $m > 1$. Let f be the inclusion of the Brieskorn homotopy $(4m-1)$-sphere $\sum(2, \ldots, 2, 3, 6(\sigma_m/8) - 1)$ into $\mathbb{R}^{4m+1} = S^{4m+1} - \{\text{point}\}$, bounding the Milnor fiber $M_0 \subset S^{4m+1}$ [Br]. Brieskorn computes

$$\sigma(M_0) = \pm\sigma_m,$$

so by Kervaire–Milnor [KM, 7.5] and the h-cobordism Theorem [S3], ∂M_0 is diffeomorphic to S^{4m-1}. Capping off M_0 with a $4m$-ball to get a closed, almost-parallelizable $4m$-manifold M, we have $\sigma(M) = \pm\sigma_m$, and so

$$p_m(M) = \pm p_m.$$

Case 2: $m = 1$. Let M be the Kummer surface (see, for example Milnor [M]), and let M_0 be the complement of an open ball in M. Note that

$$p_1(M) = p_1 = 48.$$

It is known that M_0 can be constructed from the 4-ball by attaching 2-handles with even framings [Hr][AK] from which it follows easily that there is an embedding $M_0 \hookrightarrow \mathbb{R}^5$ (cf. Ruberman [R]). Let f be the restriction of this embedding to $\partial M_0 = S^3$.

Now in either case we have an embedding $f: S^{4m-1} \hookrightarrow \mathbb{R}^{4m+1}$ whose image bounds a submanifold M_0, with

$$p_m(M) = \pm p_m,$$

where M is M_0 capped off with a $4m$-ball. By Theorems 1 and 2 in Milnor–Kervaire [MK]

$$p_m = \pm a_m(2m-1)!\, j_m,$$

where a_m is defined to be 1 for m even and 2 for m odd. Also, by Lemma 2 in [MK]

$$p_m(M) = \pm a_m(2m-1)!\, o, \tag{3}$$

where o is the obstruction to framing the stable normal bundle ν of M. Thus

$$o = \pm j_m.$$

But by (2) in Step 1,

$$i(f) = o. \tag{4}$$

Hence

$$i(f) = \pm j_m,$$

and so f is the desired embedding.

This completes the proof of the Theorem.

Since $J: \pi_n SO(n+2) \to \pi_{2n+2} S^{n+2}$ is a monomorphism if $n \not\equiv 3 \pmod 4$ (as noted above), $\pi_{2n+2} S^{n+2}$ is finite, and $\pi_n SO(n+2) = \mathbb{Z}$ if $n \equiv 3 \pmod 4$, we deduce:

COROLLARY 1. $\operatorname{Emb}(S^n, \mathbb{R}^{n+2})$ *is isomorphic to* $\mathbb{Z}$ *if* $n \equiv 3$ (*mod* 4) *and to* 0 *otherwise.*

In fact in the case $n \equiv 3 \pmod 4$ (say $n = 4m-1$), one may identify explicitly the subgroup $\operatorname{Emb}(S^n, \mathbb{R}^{n+2}) = j_m \mathbb{Z}$ of $\operatorname{Imm}(S^n, \mathbb{R}^{n+2}) = \mathbb{Z}$ using the following formula for j_m:

$$v_2(j_m) = v_2(m) + 3$$

$$v_p(j_m) = \begin{cases} v_p(m) + 1 & \text{if } m \equiv 0 \left(\operatorname{mod} \dfrac{p-1}{2}\right) \\ 0 & \text{otherwise} \end{cases}$$

(for p an odd prime)

where $v_p(k)$ denotes the exponent of the prime p in the prime decomposition of k. This formula follows from Lemma 3 in [MK] and the Adams conjecture (compare Switzer [S, pp. 479, 488]). The first few values of j_m are $j_1 = 24$, $j_2 = 240$, $j_3 = 504$, and $j_4 = 480$.

One may also give a formula relating the invariant $i(f)$ (for an embedding $f: S^n \hookrightarrow \mathbb{R}^{n+2}$) to the signature of a Seifert surface for f:

COROLLARY 2. *If* $f: S^n \hookrightarrow \mathbb{R}^{n+2}$ *is an embedding,* $n = 4m-1$, *and* M_0 *is an oriented* $4m$*-manifold in* $\mathbb{R}^{n+2}$ *with* $\partial M_0 = f(S^n)$, *then identifying* $\operatorname{Imm}(S^n, \mathbb{R}^{n+2})$

with $\mathbb{Z}$ we have

$$i(f)=\pm\frac{m}{2^{2m-1}(2^{2m-1}-1)B_m a_m}\sigma(M_0)$$

where B_m is the m-th Bernoulli number and a_m is 1 *or* 2 *depending upon whether m is even or odd.*

Proof. Let M denote M_0 capped off with a $4m$-ball ($\sigma(M)=\sigma(M_0)$). By (3) and (4) of the proof of the theorem

$$i(f)=\pm\frac{1}{a_m(2m-1)!}p_m(M).$$

The Hirzebruch Index Theorem (see [MK, p. 457]) gives

$$p_m(M)=\frac{(2m)!}{2^{2m}(2^{2m-1}-1)B_m}\sigma(M),$$

as M is almost parallelizable, and the Corollary follows.

For example, if $m=1$, then $i(f)=\pm\frac{3}{2}\sigma(M_0)$.

Remark. Our viewpoint also sheds light on the case of embeddings $S^n \hookrightarrow \mathbb{R}^k$ for $k>n+2$: If $\mathrm{Emb}_F(S^n,\mathbb{R}^k)$ denotes the set of regular homotopy classes containing embeddings which bound framed submanifolds of $\mathbb{R}^k$, then one has by an analogous argument to the proof of the theorem

$$s(\mathrm{Emb}_F(S^n,\mathbb{R}^k))=\phi_*(\ker(J))$$

where

$$\phi_*:\pi_n SO(k)\to\pi_n V_n(\mathbb{R}^k)$$

is the natural map. (Note that ϕ_* is generally not an isomorphism.) As a consequence, for example, one has

$$s(\mathrm{Emb}_F(S^3,\mathbb{R}^6))=0$$

(in fact $\mathrm{Emb}(S^3,\mathbb{R}^6)=0$ by [S2]), and

$$s(\mathrm{Emb}_F(S^7,\mathbb{R}^{10}))=720\mathbb{Z}\oplus\{0\}\subset\mathbb{Z}\oplus\mathbb{Z}_4=\pi_7 V_7(\mathbb{R}^{10}).$$

QUESTIONS. (1) Is $\mathrm{Emb}_F(S^n, \mathbb{R}^{n+3}) = \mathrm{Emb}(S^n, \mathbb{R}^{n+3})$? (2) For a given n, what is the largest value of k for which $\mathrm{Emb}(S^n, \mathbb{R}^k) \neq 0$?

Added in proof: Sylvain Cappell has informed us that our theorem can be deduced from an unpublished version of his paper with J. Shaneson, "Singularities and immersions", Ann. of Math. *105* (1977), 539–552.

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