Zeitschrift: Commentarii Mathematici Helvetici
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 60 (1985)

Artikel: The Smale invariant of a knot.
Autor: Hughes, John F. / Melvin, Paul M.
DOI: https://doi.org/10.5169/seals-46334

Nutzungsbedingungen
Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Mehr erfahren

Conditions d'utilisation
L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. En savoir plus

Terms of use
The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. Find out more

Download PDF: 08.02.2026

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

Comment. Math. Helvetici 60 (1985) 615–627 0010–2571/85/040615–13$01.50 + 0.20/0
© 1985 Birkhäuser Verlag, Basel

The Smale invariant of a knot

JOHN F. HUGHES and PAUL M. MELVIN

Smale [S2] associates to each immersion $f: S^n \looparrowright \mathbb{R}^k$ an element $s(f)$ in $\pi_n V_n(\mathbb{R}^k)$, where $V_n(\mathbb{R}^k)$ is the Stiefel manifold of n-frames in $\mathbb{R}^k$. The map s is an isomorphism from the set of regular homotopy classes of immersions of S^n in $\mathbb{R}^k$ to the set $\pi_n V_n(\mathbb{R}^k)$. Smale [S2, p. 329, questions (2) and (3)] asks for a characterization of those elements $s(f)$ where f is an *embedding*. Kervaire [K3] solves this problem for $k \geqq \frac{3}{2}n + 1$ and then, together with Milnor [K3] [KM], for $k = n + 1$. (In all of these cases, $s(f) = 0$ when f is an embedding.) Haefliger [Ha, 4.7] gives a homotopy theoretic solution for all $k \geqq n + 3$, which, however, does not lend itself to simple computations. We solve the problem for the case $k = n + 2$ in this paper, including an explicit means for computing the Smale invariant (Corollary 2).

If $n = 1$, then there is only one regular homotopy class, and it is represented by an embedding. The case $n = 2$ is solved by Smale [S1], who shows that regular homotopy classes correspond to elements of the set $\pi_2 V_2(\mathbb{R}^4) = \mathbb{Z}$, and the integer associated with a self-transverse immersion is the algebraic number of double points of the immersion. Thus there is only one immersion represented by an embedding, and its Smale invariant $s(f)$ is zero.

For $n > 2$, the group $\pi_n V_n(\mathbb{R}^{n+2})$ can be identified with the group $\pi_n SO(n+2)$. We call the image of $s(f)$ under this identification $i(f)$. The main result of this paper may be stated as follows:

THEOREM. *Let $f: S^n \looparrowright \mathbb{R}^{n+2}$ be an immersion. Then f is regularly homotopic to an embedding if and only if $J(i(f)) = 0$.*

Here J denotes the Hopf–Whitehead J homomorphism from $\pi_n SO(n+2)$ to $\pi_{2n+2} S^{n+2}$.

The proof consists of identifying $i(f)$ geometrically in a more convenient form than Smale's original definition, understanding the J homomorphism geometrically, and then combining these when f is an embedding to see that $J(i(f)) = 0$. The proof of the converse is by construction, using examples provided by Brieskorn [Br].

Using known properties of the J homomorphism, it follows that there exist non-trivial embeddings $S^n \hookrightarrow \mathbb{R}^{n+2}$ (i.e. embeddings not regularly homotopic to the standard inclusion) if and only if $n \equiv 3 \pmod 4$ (Corollary 1). This answers negatively the question raised in Kervaire [K3, §5]: "Is the Smale invariant of an embedding $f: S^n \hookrightarrow \mathbb{R}^k$ with $k \leqq n+3$ always zero?" Ironically, a proof that $J(i(f)) = 0$ (properly intepreted) is implicit in [K3], which together with the results of [MK] might have indicated where to look for counterexamples.

1. Preliminaries

$\mathbb{R}^n$ denotes coordinate n-space, which we consider naturally embedded in $\mathbb{R}^{n+1}$ as the points with last coordinate zero. B^n denotes the closed unit ball in $\mathbb{R}^n$, and S^n the boundary of B^{n+1}.

$V_n(\mathbb{R}^k)$ denotes the Stiefel manifold of n-frames in $\mathbb{R}^k$, which we identify with the space of injective linear maps from $\mathbb{R}^n$ to $\mathbb{R}^k$ (associating the frame $v_1, \ldots, v_n$ with the linear map sending e_i to v_i). Similarly we identify $GL(k)$, the set of $k \times k$ invertible matrices, with the space $\mathrm{Aut}(\mathbb{R}^k)$ of linear automorphisms of $\mathbb{R}^k$. $GL_+(k)$ denotes the matrices of positive determinant in $GL(k)$, or equivalently the orientation preserving maps in $\mathrm{Aut}(\mathbb{R}^k)$. $SO(k)$ denotes the orthogonal matrices of determinant one in $GL(k)$, identified with the rotations of $\mathbb{R}^k$.

Throughout this paper, all manifolds and maps are smooth. If M is a manifold, then τ_M denotes the tangent bundle of M, and ε^k denotes the trivial bundle over M with fiber $\mathbb{R}^k$.

$\mathrm{Imm}(S^n, \mathbb{R}^k)$ denotes the set of all regular homotopy classes of immersions $f: S^n \looparrowright \mathbb{R}^k$. We often do not distinguish between an immersion and its regular homotopy class; thus we may write $f \in \mathrm{Imm}(S^n, \mathbb{R}^k)$.

$\mathrm{Emb}(S^n, \mathbb{R}^k)$ denotes the subset of $\mathrm{Imm}(S^n, \mathbb{R}^k)$ consisting of all regular homotopy classes containing an embedding.

DEFINITION 1. Let $f: S^n \looparrowright \mathbb{R}^{n+2}$ be an immersion. We define two invariants,

$$i(f) \in \pi_n SO(n+2)$$

and the *Smale invariant*

$$s(f) \in \pi_n V_n(\mathbb{R}^{n+2}),$$

as follows:

The invariant $i(f)$:

Extend f to an orientation preserving immersion

$$F: N(S^n) \looparrowright \mathbb{R}^{n+2}$$

where $N(S^n)$ is a neighborhood of the standard S^n in $\mathbb{R}^{n+2}$. Then dF, the differential of F, maps $N(S^n)$ into $GL_+(n+2)$. Define $i(f)$ to be the homotopy class of the map

$$S^n \to SO(n+2): x \mapsto GS \circ dF_x,$$

where GS: $GL_+(n+2) \to SO(n+2)$ is the Gram–Schmidt map. It is not hard to see that if $n > 1$, then $i(f)$ is independent of the choice of F, and in fact depends only on the regular homotopy class of f. Thus there is a well defined map

$$i: \mathrm{Imm}\,(S^n, \mathbb{R}^{n+2}) \to \pi_n SO(n+2)$$

for $n > 1$.

The Smale invariant $s(f)$:

Consider S^n as lying in $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$, and write points in $\mathbb{R}^{n+1}$ as pairs (v, t), where $v \in \mathbb{R}^n$ and $t \in \mathbb{R}$. The northern and southern hemispheres of S^n are then

$$N = \{(v, t) \in S^n : t \geqq 0\}$$

$$S = \{(v, t) \in S^n : t \leqq 0\}.$$

If $x = (v, t)$, write $\bar{x}$ for $(v, -t)$. Stereographic projection from the south pole, $sp = (0, -1)$, to a plane tangent to the north pole, $np = (0, 1)$, is given by the formula

$$p: S^n - \{sp\} \to \mathbb{R}^n : (v, t) \mapsto \frac{2}{1+t} v.$$

Let $q: \mathbb{R}^n \to \mathbb{R}^{n+2}$ be the inverse of p, followed by the natural inclusion of S^n in $\mathbb{R}^{n+2}$.

Now alter the immersion f by a regular homotopy so that the restriction of f to the southern hemisphere S agrees with the standard inclusion of S into $\mathbb{R}^{n+2}$. Define $s(f)$ to be the homotopy class of the map

$$S^n \to V_n(\mathbb{R}^{n+2}) : x \mapsto \begin{cases} d(f \circ q)_{p(x)} & x \in N \\ dq_{p(\bar{x})} & x \in S \end{cases}$$

(compare Smale [S2]). It turns out that $s(f)$ is independent of the choice of regular homotopy used to alter f, and so there is a well-defined map

$$s : \mathrm{Imm}\,(S^n, \mathbb{R}^{n+2}) \to \pi_n V_n(\mathbb{R}^{n+2}).$$

Smale [S2] shows that s is a bijection. In fact, using the operation of oriented connected sum on $\mathrm{Imm}\,(S^n, \mathbb{R}^{n+2})$, s is an isomorphism of groups (see Kervaire [K2], Hughes [Hu]).

DEFINITION 2. Let $j : \mathbb{R}^n \to \mathbb{R}^{n+2}$ denote the standard inclusion. Define $\phi : SO(n+2) \to V_n(\mathbb{R}^{n+2})$ by sending h to $h \circ j$. (Here we are thinking of elements of $SO(n+2)$ and $V_n(\mathbb{R}^{n+2})$ as linear maps. On the matrix level, ϕ is simply "drop the last two columns of the matrix".) Observe that ϕ induces an isomorphism

$$\phi_* : \pi_n SO(n+2) \to \pi_n V_n(\mathbb{R}^{n+2})$$

for $n > 2$. (To see this, consider the commutative diagram

$$\begin{tikzcd} GL_+(n+2) \arrow[dr, "\psi"] & \\ SO(n+2) \arrow[u, phantom, "\cup"] \arrow[r, "\phi"'] & V_n(\mathbb{R}^{n+2}) \end{tikzcd}$$

where $\psi(g) = g \circ j$. ψ is a fibration with a fiber which is homotopy equivalent to $GL_+(2)$, which is in turn homotopy equivalent to S^1. Hence ψ induces an isomorphism on π_n for $n > 2$. The inclusion $SO(n+2) \subset GL_+(n+2)$ is a homotopy equivalence, so induces an isomorphism on π_n for every n.)

Combining definitions 1 and 2 we have a diagram

$$\begin{tikzcd} & \pi_n V_n(\mathbb{R}^{n+2}) & \\ \mathrm{Imm}\,(S^n, \mathbb{R}^{n+2}) \arrow[ur, "s"] \arrow[rr, "i"'] & & \pi_n(SO(n+2) \arrow[ul, "\phi_*"'] \end{tikzcd}$$

(for $n > 1$) with s and ϕ_* isomorphisms (for $n > 2$).

PROPOSITION. *$s = \phi_* \circ i$. Thus i is an isomorphism for $n > 2$.*

Proof. Let $f \in \mathrm{Imm}\,(S^n, \mathbb{R}^{n+2})$. As in definition 1, we may assume that f agrees with the standard inclusion on the southern hemisphere S. It is easy to arrange that F (in the definition of $i(f)$) is the identity in a neighborhood of S in $\mathbb{R}^{n+2}$.

Notice that since $f=F$ on image(q), we may write $d(fq)_v = d(Fq)_v$ for v in $\mathbb{R}^n$. Thus $s(f)$ is represented by the map

$$S^n \to V_n(\mathbb{R}^{n+2}) : x \mapsto \begin{cases} d(fq)_{p(x)} = dF_x dq_{p(x)} & x \in N \\ dq_{p(\bar{x})} & x \in S \end{cases}$$

(applying the chain rule). This map can be altered by the homotopy

$$(x, t) \mapsto \begin{cases} dF_x dq_{(1-t)p(x)} & x \in N \\ dq_{(1-t)p(\bar{x})} & x \in S \end{cases}$$

resulting in

$$s(f) = \left[x \mapsto \begin{cases} dF_x dq_0 & x \in N \\ dq_0 & x \in S \end{cases} \right] = [x \mapsto dF_x \circ j]$$

where j denotes the inclusion of $\mathbb{R}^n$ into $\mathbb{R}^{n+2}$. The last equality follows because $dq_0 = j$, as is easily verified. But the map $x \mapsto dF_x \circ j$ is homotopic to $x \mapsto GS \circ dF_x \circ j$, which by definition represents $\phi_*(i(f))$, proving the proposition.

DEFINITION 3. Suppose that M is a manifold, and P and Q are codimension zero submanifolds with $P \cap Q$ a submanifold and $P \cup Q = M$. Given a map $f: P \cap Q \to GL(k)$, denote by

$$\beta(P, Q, f)$$

the $\mathbb{R}^k$-bundle whose total space is $(P \times \mathbb{R}^k) \cup (Q \times \mathbb{R}^k)/\sim$, where $\sim$ is the equivalence relation identifying $(x, v) \in P \times \mathbb{R}^k$ with $(x, f(x)v) \in Q \times \mathbb{R}^k$, for all x in $P \cap Q$. The projection map for this bundle sends (x, v) to x.

It follows from the homotopy axiom for vector bundles that if f and g are homotopic maps from $P \cap Q$ to $GL(k)$, then $\beta(P, Q, f)$ and $\beta(P, Q, g)$ are isomorphic bundles. Also, if $\tilde{f}$ is defined by $\tilde{f}(x) = f(x)^{-1}$, then $\beta(P, Q, f) \cong \beta(Q, P, \tilde{f})$ and $\beta(P, Q, f) \oplus \beta(P, Q, \tilde{f}) \cong \varepsilon^{2k}$.

If an orientable bundle ξ over M is trivial away from a point (almost parallelizable), then there is an isomorphism $\xi \cong \beta(P, Q, f)$ with $Q = B^{n+1}$, $P \cap Q = S^n = \partial B^{n+1}$, and $f: S^n \to SO(k)$. The class $[f] \in \pi_n SO(k)$ is called the *obstruction to framing* ξ.

2. The main theorem

From the previous section, there is a commutative diagram:

$$\begin{tikzcd}
 & \pi_n V_n(\mathbb{R}^{n+2}) & \\
\mathrm{Emb}\,(S^n, \mathbb{R}^{n+2}) \subset \mathrm{Imm}\,(S^n, \mathbb{R}^{n+2}) \arrow[ur, "s"] \arrow[r, "i"'] & \pi_n SO(n+2) \arrow[u, "\phi_*"'] \arrow[r, "J"'] & \pi_{2n+2}(S^{n+2}).
\end{tikzcd}$$

Our main result is:

THEOREM. $s(\mathrm{Emb}\,(S^n, \mathbb{R}^{n+2})) = \phi_*(\ker\,(J))$.

Proof. It suffices to show $i(\mathrm{Emb}\,(S^n, \mathbb{R}^{n+2})) = \ker\,(J)$. The proof is in two steps.

STEP 1. *If* $f \in \mathrm{Emb}\,(S^n, \mathbb{R}^{n+2})$, *then* $J(i(f)) = 0$.

Extend f to an embedding $f: M_0 \hookrightarrow \mathbb{R}^{n+2}$ of some compact oriented $(n+1)$-manifold M_0 with $\partial M_0 = S^n$. (M_0 is called a Seifert surface for f.) Consider the closed, smooth manifold

$$M = M_0 \cup B^{n+1}$$

the union being along $\partial B^{n+1} = S^n = \partial M_0$. We will show that $i(f)$ is an obstruction to framing the stable normal bundle of M. Step 1 then follows from Lemma 1 of [MK]. The details follow:

A suitable neighborhood U of B^{n+1} in M can be identified with $\mathbb{R}^{n+1}$. Thus we view $\mathbb{R}^{n+2} = \mathbb{R}^{n+1} \times \mathbb{R} = U \times \mathbb{R} \subset M \times \mathbb{R}$. The standard orientation on $\mathbb{R}^{n+2}$ induces an orientation on $M \times \mathbb{R}$. Within $M \times \mathbb{R}$, we identify M with $M \times \{0\}$. Set $V = M - \{0\}$ (here 0 denotes the origin of $\mathbb{R}^{n+1}$ = center of B^{n+1}).

Now further extend f to an orientation preserving embedding

$$F: V \times \mathbb{R} \hookrightarrow \mathbb{R}^{n+2}$$

(see Figure 1).

Let

$$g = dF \mid S^n : S^b \to GL_+(n+2).$$

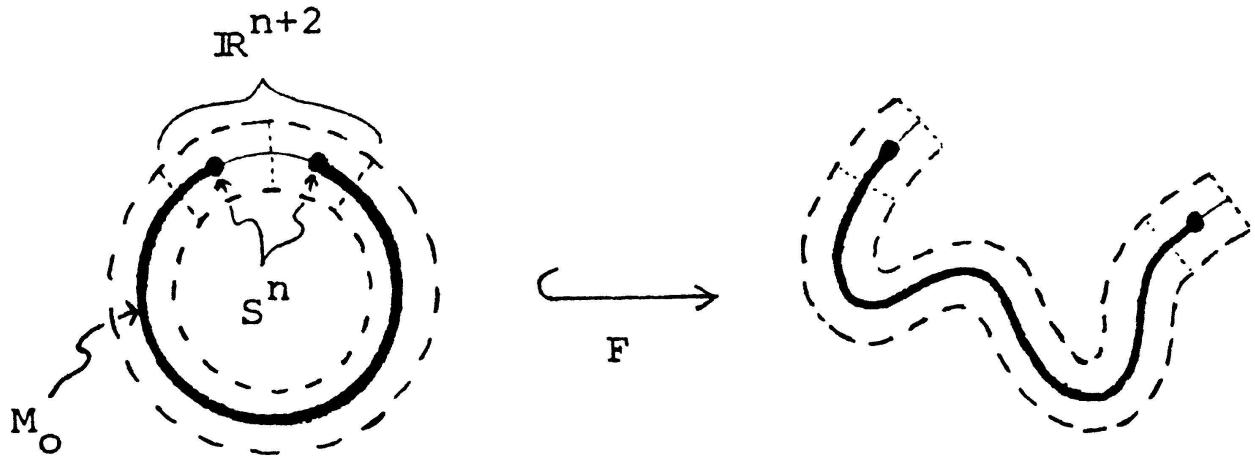

Figure 1

Then

$$\tau_{M\times\mathbb{R}} \mid M \cong \beta(B^{n+1}, M_0, g). \tag{1}$$

An explicit isomorphism between the bundles is given by assigning to the tangent vector v to $M\times\mathbb{R}$ at the point x in $M = M\times\{0\}$, either

$$(x, v) \in B^{n+1}\times\mathbb{R}^{n+2}$$

if $x \in B^{n+1}$ (here we think of $B^{n+1}\subset\mathbb{R}^{n+1}$, so $v\in\mathbb{R}^{n+2}$), or

$$(x, dF_x(v)) \in M_0\times\mathbb{R}^{n+2}$$

if $x\in M_0$. This is well-defined, for if $x\in M_0\cap B^{n+1} = S^n$, then $(x, v)\in B^{n+1}\times\mathbb{R}^{n+2}$ is identified with $(x, g(x)v) = (x, dF_x(v)) \in M_0\times\mathbb{R}^{n+2}$, by the definition of $\beta(B^{n+1}, M_0, g)$ and g.

Let $h = GS\circ g$, so that

$$h : S^n \to SO(n+2).$$

Note that $[h] = i(f)$, by definition. Furthermore, g and h are homotopic maps (in $GL_+(n+2)$), so we have

$$\begin{aligned}\tau_M \oplus \epsilon^1 &\cong \tau_{M\times\mathbb{R}|M} \cong \beta(B^{n+1}, M_0, g) \quad \text{(by (1))}\\ &\cong \beta(B^{n+1}, M_0, h).\end{aligned}$$

Now using the Whitney embedding theorem, embed M in S^{2n+3} and let ν be the normal $(n+2)$-plane bundle of the embedding. Then $(\tau_M \oplus \varepsilon^1) \oplus \nu \cong \varepsilon^{2n+4} \cong \beta(B^{n+1}, M_0, h) \oplus \beta(B^{n+1}, M_0, \tilde{h})$ (where $\tilde{h}(x) = h(x)^{-1}$), and so

$$\nu \cong \beta(B^{n+1}, M_0, \tilde{h}) \cong \beta(M_0, B^{n+1}, h)$$

(both bundles are stable normal bundles of M). Since $[h] = i(f)$,

$$i(f) \text{ is the obstruction to framing } \nu. \tag{2}$$

By Lemma 1 of [MK], it follows that $J(i(f)) = 0$.

Remark. For the reader's convenience, here are the details of a proof of the lemma cited above:

Embed M in S^{2n+3} so that it is perpendicular to the equatorial S^{2n+2}, intersecting it in the standard $S^n \subset S^{2n+2}$, with M_0 lying in the northern hemisphere of S^{2n+3} (see Figure 2). This may be accomplished by taking a height function for S^{2n+3}, making it transverse to the embedding of M, and then identifying a minimum point. An isotopy taking a neighborhood of this minimum onto the southern hemisphere alters the embedding to one satisfiying the conditions above.

Consider the normal framing $\mathbb{F}$ on $S^n = M \cap S^{2n+2}$ in S^{2n+2}, given by assigning to a point $x \in S^n$ the frame $\begin{pmatrix} 0 \\ h(x) \end{pmatrix} \in V_{n+2}(\mathbb{R}^{2n+3})$. The Thom–Pontrjagin construction applied to this framed submanifold of S^{2n+2} gives an element of $\pi_{2n+2}S^{n+2}$

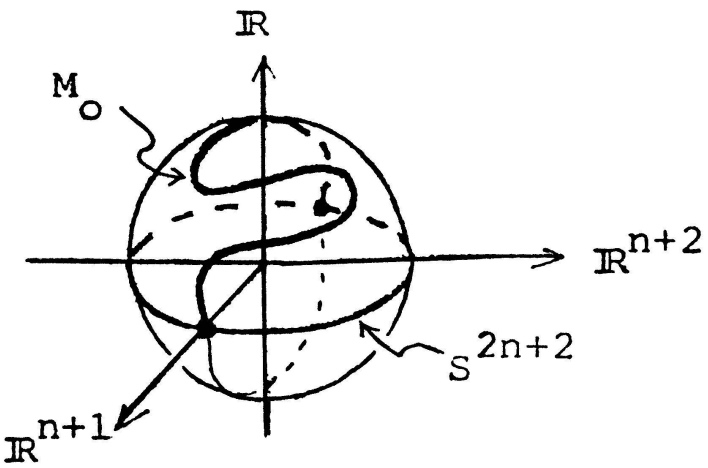

Figure 2

which can be identified with $J([h])$: Both elements are represented by the map

$$\begin{array}{l} S^{2n+2} \subset \mathbb{R}^{2n+3} = \mathbb{R}^{n+1} \times \mathbb{R}^{n+2} \\ \quad\downarrow \\ S^{n+2} \subset \mathbb{R}^{n+3} = \mathbb{R}^{n+2} \times \mathbb{R} \end{array}$$

sending the minimal geodesic arc joining $x \in S^n \times \{0\}$ with $y \in \{0\} \times S^{n+1}$ to the minimal geodesic arc joining the south pole with the north pole of S^{n+2} and passing through $h(x)y \in S^{n+1} \times \{0\}$. (Explicitly, $J([h])$ maps $(u, v) \in S^{2n+2}$ to $(0, 1) \in S^{n+2}$ if $u = 0$, and to $(2\|u\| h(u/\|u\|)v, \|v\|^2 - \|u\|^2)$ otherwise.) Compare Kervaire [K1, 1.8].

Finally observe that the framing $\mathbb{F}$ on S^n extends over M_0: $\beta(M_0, B^{n+1}, h)$ is abstractly isomorphic to the normal bundle ν of M in S^{2n+3}. We may choose an isomorphism over B^{n+1} which is standard over $S^n = \partial B^{n+1}$ (i.e. maps the standard frame on $\mathbb{R}^{n+2}$ to the standard frame on $\{0\} \times \mathbb{R}^{n+2} \subset \mathbb{R}^{n+1} \times \mathbb{R}^{n+2} = \mathbb{R}^{2n+3}$), and extend this to an isomorphism Ψ over the rest of M. But on $S^n = \partial M_0$, the standard frame on $\mathbb{R}^{n+2}$ maps to $\mathbb{F}$ under Ψ. Hence the image under Ψ of the standard frame on $\mathbb{R}^{n+2}$ over M_0 provides an extension of $\mathbb{F}$.

Now because the framing extends over M_0, the Thom–Pontrjagin construction yields 0 in $\pi_{2n+2}(S^{n+2})$, hence so must J.

STEP 2. *If $J(x) = 0$, then there exists $f \in \mathrm{Emb}\,(S^n, \mathbb{R}^{n+2})$ with $i(f) = x$.*

Bott [Bo] computes

$$\pi_n SO(n+2) = \begin{cases} \mathbb{Z} & \text{if } n \equiv 3 \text{ or } 7 \pmod 8 \\ \mathbb{Z}_2 & \text{if } n \equiv 0 \text{ or } 1 \pmod 8 \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, by the work of Adams, $J: \pi_n SO(n+2) \to \pi_{2n+2} S^{n+2}$ is injective for $n \equiv 0$ or $1 \pmod 8$ (see Switzer [S, p. 487]). Thus there is nothing to prove except in the case $n \equiv 3$ or $7 \pmod 8$, i.e. $n \equiv 3 \pmod 4$.

So let $n = 4m - 1$. Write j_m for the order of the image of $J: \pi_{4m-1} SO(4m+1) \to \pi_{8m} S^{4m+1}$. Identifying $\pi_{4m-1} SO(4m+1)$ with $\mathbb{Z}$, it suffices to produce an embedding $f: S^{4m-1} \to \mathbb{R}^{4m+1}$ with $i(f) = \pm j_m$.

First consider the collection of all closed, oriented, almost-parallelizable $4m$-manifolds M. The associated signatures $\sigma(M)$ form a subgroup of $\mathbb{Z}$; let $\sigma_m > 0$ denote the generator. Similarly let $p_m > 0$ denote the generator of the group of all top Pontrjagin numbers $p_m(M)$. Observe that if $\sigma(M) = \sigma_m$, then by

the Hirzebruch Index Theorem, $p_m(M) = p_m$. Also, it is known that $\sigma_m \equiv 0 \pmod 8$ (see [KM, p. 531]).

Case 1: $m > 1$. Let f be the inclusion of the Brieskorn homotopy $(4m-1)$-sphere $\sum(2, \ldots, 2, 3, 6(\sigma_m/8) - 1)$ into $\mathbb{R}^{4m+1} = S^{4m+1} - \{\text{point}\}$, bounding the Milnor fiber $M_0 \subset S^{4m+1}$ [Br]. Brieskorn computes

$$\sigma(M_0) = \pm\sigma_m,$$

so by Kervaire–Milnor [KM, 7.5] and the h-cobordism Theorem [S3], ∂M_0 is diffeomorphic to S^{4m-1}. Capping off M_0 with a $4m$-ball to get a closed, almost-parallelizable $4m$-manifold M, we have $\sigma(M) = \pm\sigma_m$, and so

$$p_m(M) = \pm p_m.$$

Case 2: $m = 1$. Let M be the Kummer surface (see, for example Milnor [M]), and let M_0 be the complement of an open ball in M. Note that

$$p_1(M) = p_1 = 48.$$

It is known that M_0 can be constructed from the 4-ball by attaching 2-handles with even framings [Hr][AK] from which it follows easily that there is an embedding $M_0 \hookrightarrow \mathbb{R}^5$ (cf. Ruberman [R]). Let f be the restriction of this embedding to $\partial M_0 = S^3$.

Now in either case we have an embedding $f: S^{4m-1} \hookrightarrow \mathbb{R}^{4m+1}$ whose image bounds a submanifold M_0, with

$$p_m(M) = \pm p_m,$$

where M is M_0 capped off with a $4m$-ball. By Theorems 1 and 2 in Milnor–Kervaire [MK]

$$p_m = \pm a_m(2m-1)!\, j_m,$$

where a_m is defined to be 1 for m even and 2 for m odd. Also, by Lemma 2 in [MK]

$$p_m(M) = \pm a_m(2m-1)!\, o, \tag{3}$$

where o is the obstruction to framing the stable normal bundle ν of M. Thus

$$o = \pm j_m.$$

But by (2) in Step 1,

$$i(f) = o. \tag{4}$$

Hence

$$i(f) = \pm j_m,$$

and so f is the desired embedding.

This completes the proof of the Theorem.

Since $J: \pi_n SO(n+2) \to \pi_{2n+2} S^{n+2}$ is a monomorphism if $n \not\equiv 3 \pmod 4$ (as noted above), $\pi_{2n+2} S^{n+2}$ is finite, and $\pi_n SO(n+2) = \mathbb{Z}$ if $n \equiv 3 \pmod 4$, we deduce:

COROLLARY 1. $\mathrm{Emb}(S^n, \mathbb{R}^{n+2})$ *is isomorphic to* $\mathbb{Z}$ *if* $n \equiv 3$ (*mod* 4) *and to* 0 *otherwise.*

In fact in the case $n \equiv 3 \pmod 4$ (say $n = 4m-1$), one may identify explicitly the subgroup $\mathrm{Emb}(S^n, \mathbb{R}^{n+2}) = j_m \mathbb{Z}$ of $\mathrm{Imm}(S^n, \mathbb{R}^{n+2}) = \mathbb{Z}$ using the following formula for j_m:

$$v_2(j_m) = v_2(m) + 3$$

$$v_p(j_m) = \begin{cases} v_p(m) + 1 & \text{if } m \equiv 0 \left(\mathrm{mod}\, \dfrac{p-1}{2}\right) \\ 0 & \text{otherwise} \end{cases}$$

(for p an odd prime)

where $v_p(k)$ denotes the exponent of the prime p in the prime decomposition of k. This formula follows from Lemma 3 in [MK] and the Adams conjecture (compare Switzer [S, pp. 479, 488]). The first few values of j_m are $j_1 = 24$, $j_2 = 240$, $j_3 = 504$, and $j_4 = 480$.

One may also give a formula relating the invariant $i(f)$ (for an embedding $f: S^n \hookrightarrow \mathbb{R}^{n+2}$) to the signature of a Seifert surface for f:

COROLLARY 2. *If* $f: S^n \hookrightarrow \mathbb{R}^{n+2}$ *is an embedding,* $n = 4m-1$, *and* M_0 *is an oriented* $4m$*-manifold in* $\mathbb{R}^{n+2}$ *with* $\partial M_0 = f(S^n)$, *then identifying* $\mathrm{Imm}(S^n, \mathbb{R}^{n+2})$

with $\mathbb{Z}$ *we have*

$$i(f) = \pm \frac{m}{2^{2m-1}(2^{2m-1}-1)B_m a_m} \sigma(M_0)$$

where B_m *is the* m*-th Bernoulli number and* a_m *is* 1 *or* 2 *depending upon whether* m *is even or odd.*

Proof. Let M denote M_0 capped off with a $4m$-ball ($\sigma(M) = \sigma(M_0)$). By (3) and (4) of the proof of the theorem

$$i(f) = \pm \frac{1}{a_m^{\cdot}(2m-1)!} p_m(M).$$

The Hirzebruch Index Theorem (see [MK, p. 457]) gives

$$p_m(M) = \frac{(2m)!}{2^{2m}(2^{2m-1}-1)B_m} \sigma(M),$$

as M is almost parallelizable, and the Corollary follows.

For example, if $m = 1$, then $i(f) = \pm\frac{3}{2}\sigma(M_0)$.

Remark. Our viewpoint also sheds light on the case of embeddings $S^n \hookrightarrow \mathbb{R}^k$ for $k > n+2$: If $\mathrm{Emb}_F(S^n, \mathbb{R}^k)$ denotes the set of regular homotopy classes containing embeddings which bound framed submanifolds of $\mathbb{R}^k$, then one has by an analogous argument to the proof of the theorem

$$s(\mathrm{Emb}_F(S^n, \mathbb{R}^k)) = \phi_*(\ker(J))$$

where

$$\phi_* : \pi_n SO(k) \to \pi_n V_n(\mathbb{R}^k)$$

is the natural map. (Note that ϕ_* is generally not an isomorphism.) As a consequence, for example, one has

$$s(\mathrm{Emb}_F(S^3, \mathbb{R}^6)) = 0$$

(in fact $\mathrm{Emb}(S^3, \mathbb{R}^6) = 0$ by [S2]), and

$$s(\mathrm{Emb}_F(S^7, \mathbb{R}^{10})) = 720\mathbb{Z} \oplus \{0\} \subset \mathbb{Z} \oplus \mathbb{Z}_4 = \pi_7 V_7(\mathbb{R}^{10}).$$

QUESTIONS. (1) Is $\mathrm{Emb}_F(S^n, \mathbb{R}^{n+3}) = \mathrm{Emb}(S^n, \mathbb{R}^{n+3})$? (2) For a given n, what is the largest value of k for which $\mathrm{Emb}(S^n, \mathbb{R}^k) \neq 0$?

Added in proof: Sylvain Cappell has informed us that our theorem can be deduced from an unpublished version of his paper with J. Shaneson, "Singularities and immersions", Ann. of Math. *105* (1977), 539–552.

REFERENCES

[AK] S. Akbulut and R. Kirby, *Branched covers of surfaces in 4-manifolds*, Math. Ann. *252* (1980), 111–133.

[Bo] R. Bott, *The stable homotopy of the classical groups*, Proc. Nat. Acad. Sci. U.S.A. *43* (1957), 933–935.

[Br] E. Brieskorn, *Beispiele zur Differentialtopologie von Singularitäten*, Invent. Math. *2* (1966), 1–14.

[Ha] A. Haefliger, *Differential embeddings of S^n in S^{n+q} for $q>2$*, Ann. of Math. *83* (1966), 402–436.

[Hr] J. Harer, *On handlebody structures for hypersurfaces in $\mathbb{C}^3$ and $\mathbb{C}P^3$*, Math. Ann. *238* (1978), 51–58.

[Hu] J. F. Hughes, *Invariants of bordism and regular homotopy of low dimensional immersions*, Ph.D. Thesis, Berkeley (1982).

[K1] M. A. Kervaire, *An interpretation of G. Whitehead's generalization of H. Hopf's invariant*, Ann. of Math. *69* (1959), 345–365.

[K2] ——, *Sur le fibré normal à une sphère immergée dans un espace euclidien*, Comment. Math. Helv. *33* (1959), 121–131.

[K3] ——, *Sur l'invariant de Smale d'un plongement*, Comment. Math. Helv. *34* (1960), 127–139.

[KM] M. A. Kervaire and J. W. Milnor, *Groups of homotopy spheres I*, Ann. of Math. *77* (1963), 504–537.

[M] J. Milnor, *On simply connected 4-manifolds*, Symp. Intern. de Top. Alg., Mexico (1958), 122–128.

[MK] J. W. Milnor and M. A. Kervaire, *Bernoulli numbers, homotopy groups, and a theorem of Rohlin*, Proc. Int. Congress of Math., Edinburgh (1958), 454–458.

[R] D. Ruberman, *Imbedding four-manifolds and slicing links*, Math. Proc. Cam. Phil. Soc. *91* (1982), 107–110.

[S1] S. Smale, *A classification of immersions of the two-sphere*, Trans. Amer. Math. Soc. *90* (1958), 281–290.

[S2] ——, *The classification of immersions of spheres in euclidean spaces*, Ann. of Math. *69* (1959), 327–344.

[S3] ——, *On the structure of manifolds*, Amer. J. Math. *84* (1962), 387–399.

[S] R. M. Switzer, *Algebraic Topology–Homotopy and Homology*, Springer-Verlag, 1975.

Department of Mathematics
Bryn Mawr College
Bryn Mawr, PA 19010

Received September 4, 1984