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The Smale invariant of a knot

JOHN F. HUGHES and PAUL M. MELVIN

Smale [S2] associates to each immersion $f: S^n \looparrowright \mathbb{R}^k$ an element $s(f)$ in $\pi_n V_n(\mathbb{R}^k)$, where $V_n(\mathbb{R}^k)$ is the Stiefel manifold of n -frames in \mathbb{R}^k . The map s is an isomorphism from the set of regular homotopy classes of immersions of S^n in \mathbb{R}^k to the set $\pi_n V_n(\mathbb{R}^k)$. Smale [S2, p. 329, questions (2) and (3)] asks for a characterization of those elements $s(f)$ where f is an *embedding*. Kervaire [K3] solves this problem for $k \geq \frac{3}{2}n + 1$ and then, together with Milnor [K3] [KM], for $k = n + 1$. (In all of these cases, $s(f) = 0$ when f is an embedding.) Haefliger [Ha, 4.7] gives a homotopy theoretic solution for all $k \geq n + 3$, which, however, does not lend itself to simple computations. We solve the problem for the case $k = n + 2$ in this paper, including an explicit means for computing the Smale invariant (Corollary 2).

If $n = 1$, then there is only one regular homotopy class, and it is represented by an embedding. The case $n = 2$ is solved by Smale [S1], who shows that regular homotopy classes correspond to elements of the set $\pi_2 V_2(\mathbb{R}^4) = \mathbb{Z}$, and the integer associated with a self-transverse immersion is the algebraic number of double points of the immersion. Thus there is only one immersion represented by an embedding, and its Smale invariant $s(f)$ is zero.

For $n > 2$, the group $\pi_n V_n(\mathbb{R}^{n+2})$ can be identified with the group $\pi_n SO(n + 2)$. We call the image of $s(f)$ under this identification $i(f)$. The main result of this paper may be stated as follows:

THEOREM. *Let $f: S^n \looparrowright \mathbb{R}^{n+2}$ be an immersion. Then f is regularly homotopic to an embedding if and only if $J(i(f)) = 0$.*

Here J denotes the Hopf–Whitehead J homomorphism from $\pi_n SO(n + 2)$ to $\pi_{2n+2} S^{n+2}$.

The proof consists of identifying $i(f)$ geometrically in a more convenient form than Smale's original definition, understanding the J homomorphism geometrically, and then combining these when f is an embedding to see that $J(i(f)) = 0$. The proof of the converse is by construction, using examples provided by Brieskorn [Br].

Using known properties of the J homomorphism, it follows that there exist non-trivial embeddings $S^n \hookrightarrow \mathbb{R}^{n+2}$ (i.e. embeddings not regularly homotopic to the standard inclusion) if and only if $n \equiv 3 \pmod{4}$ (Corollary 1). This answers negatively the question raised in Kervaire [K3, §5]: “Is the Smale invariant of an embedding $f: S^n \hookrightarrow \mathbb{R}^k$ with $k \leq n+3$ always zero?” Ironically, a proof that $J(i(f)) = 0$ (properly interpreted) is implicit in [K3], which together with the results of [MK] might have indicated where to look for counterexamples.

1. Preliminaries

\mathbb{R}^n denotes coordinate n -space, which we consider naturally embedded in \mathbb{R}^{n+1} as the points with last coordinate zero. B^n denotes the closed unit ball in \mathbb{R}^n , and S^n the boundary of B^{n+1} .

$V_n(\mathbb{R}^k)$ denotes the Stiefel manifold of n -frames in \mathbb{R}^k , which we identify with the space of injective linear maps from \mathbb{R}^n to \mathbb{R}^k (associating the frame v_1, \dots, v_n with the linear map sending e_i to v_i). Similarly we identify $GL(k)$, the set of $k \times k$ invertible matrices, with the space $\text{Aut}(\mathbb{R}^k)$ of linear automorphisms of \mathbb{R}^k . $GL_+(k)$ denotes the matrices of positive determinant in $GL(k)$, or equivalently the orientation preserving maps in $\text{Aut}(\mathbb{R}^k)$. $SO(k)$ denotes the orthogonal matrices of determinant one in $GL(k)$, identified with the rotations of \mathbb{R}^k .

Throughout this paper, all manifolds and maps are smooth. If M is a manifold, then τ_M denotes the tangent bundle of M , and ε^k denotes the trivial bundle over M with fiber \mathbb{R}^k .

$\text{Imm}(S^n, \mathbb{R}^k)$ denotes the set of all regular homotopy classes of immersions $f: S^n \hookrightarrow \mathbb{R}^k$. We often do not distinguish between an immersion and its regular homotopy class; thus we may write $f \in \text{Imm}(S^n, \mathbb{R}^k)$.

$\text{Emb}(S^n, \mathbb{R}^k)$ denotes the subset of $\text{Imm}(S^n, \mathbb{R}^k)$ consisting of all regular homotopy classes containing an embedding.

DEFINITION 1. Let $f: S^n \hookrightarrow \mathbb{R}^{n+2}$ be an immersion. We define two invariants,

$$i(f) \in \pi_n SO(n+2)$$

and the *Smale invariant*

$$s(f) \in \pi_n V_n(\mathbb{R}^{n+2}),$$

as follows:

The invariant $i(f)$:

Extend f to an orientation preserving immersion

$$F: N(S^n) \hookrightarrow \mathbb{R}^{n+2}$$

where $N(S^n)$ is a neighborhood of the standard S^n in \mathbb{R}^{n+2} . Then dF , the differential of F , maps $N(S^n)$ into $GL_+(n+2)$. Define $i(f)$ to be the homotopy class of the map

$$S^n \rightarrow SO(n+2): x \mapsto GS \circ dF_x,$$

where $GS: GL_+(n+2) \rightarrow SO(n+2)$ is the Gram–Schmidt map. It is not hard to see that if $n > 1$, then $i(f)$ is independent of the choice of F , and in fact depends only on the regular homotopy class of f . Thus there is a well defined map

$$i: \text{Imm}(S^n, \mathbb{R}^{n+2}) \rightarrow \pi_n SO(n+2)$$

for $n > 1$.

The Smale invariant $s(f)$:

Consider S^n as lying in $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$, and write points in \mathbb{R}^{n+1} as pairs (v, t) , where $v \in \mathbb{R}^n$ and $t \in \mathbb{R}$. The northern and southern hemispheres of S^n are then

$$N = \{(v, t) \in S^n : t \geq 0\}$$

$$S = \{(v, t) \in S^n : t \leq 0\}.$$

If $x = (v, t)$, write \bar{x} for $(v, -t)$. Stereographic projection from the south pole, $sp = (0, -1)$, to a plane tangent to the north pole, $np = (0, 1)$, is given by the formula

$$p: S^n - \{sp\} \rightarrow \mathbb{R}^n : (v, t) \mapsto \frac{2}{1+t} v.$$

Let $q: \mathbb{R}^n \rightarrow \mathbb{R}^{n+2}$ be the inverse of p , followed by the natural inclusion of S^n in \mathbb{R}^{n+2} .

Now alter the immersion f by a regular homotopy so that the restriction of f to the southern hemisphere S agrees with the standard inclusion of S into \mathbb{R}^{n+2} . Define $s(f)$ to be the homotopy class of the map

$$S^n \rightarrow V_n(\mathbb{R}^{n+2}): x \mapsto \begin{cases} d(f \circ q)_{p(x)} & x \in N \\ dq_{p(\bar{x})} & x \in S \end{cases}$$

(compare Smale [S2]). It turns out that $s(f)$ is independent of the choice of regular homotopy used to alter f , and so there is a well-defined map

$$s : \text{Imm}(S^n, \mathbb{R}^{n+2}) \rightarrow \pi_n V_n(\mathbb{R}^{n+2}).$$

Smale [S2] shows that s is a bijection. In fact, using the operation of oriented connected sum on $\text{Imm}(S^n, \mathbb{R}^{n+2})$, s is an isomorphism of groups (see Kervaire [K2], Hughes [Hu]).

DEFINITION 2. Let $j : \mathbb{R}^n \rightarrow \mathbb{R}^{n+2}$ denote the standard inclusion. Define $\phi : SO(n+2) \rightarrow V_n(\mathbb{R}^{n+2})$ by sending h to $h \circ j$. (Here we are thinking of elements of $SO(n+2)$ and $V_n(\mathbb{R}^{n+2})$ as linear maps. On the matrix level, ϕ is simply “drop the last two columns of the matrix”.) Observe that ϕ induces an isomorphism

$$\phi_* : \pi_n SO(n+2) \rightarrow \pi_n V_n(\mathbb{R}^{n+2})$$

for $n > 2$. (To see this, consider the commutative diagram

$$\begin{array}{ccc} GL_+(n+2) & & \\ \cup & \searrow \psi & \\ SO(n+2) & \xrightarrow{\phi} & V_n(\mathbb{R}^{n+2}) \end{array}$$

where $\psi(g) = g \circ j$. ψ is a fibration with a fiber which is homotopy equivalent to $GL_+(2)$, which is in turn homotopy equivalent to S^1 . Hence ψ induces an isomorphism on π_n for $n > 2$. The inclusion $SO(n+2) \subset GL_+(n+2)$ is a homotopy equivalence, so induces an isomorphism on π_n for every n .)

Combining definitions 1 and 2 we have a diagram

$$\begin{array}{ccc} & \pi_n V_n(\mathbb{R}^{n+2}) & \\ & \nearrow s & \nwarrow \phi_* \\ \text{Imm}(S^n, \mathbb{R}^{n+2}) & \xrightarrow{i} & \pi_n(SO(n+2)) \end{array}$$

(for $n > 1$) with s and ϕ_* isomorphisms (for $n > 2$).

PROPOSITION. $s = \phi_* \circ i$. Thus i is an isomorphism for $n > 2$.

Proof. Let $f \in \text{Imm}(S^n, \mathbb{R}^{n+2})$. As in definition 1, we may assume that f agrees with the standard inclusion on the southern hemisphere S . It is easy to arrange that F (in the definition of $i(f)$) is the identity in a neighborhood of S in \mathbb{R}^{n+2} .

Notice that since $f = F$ on $\text{image}(q)$, we may write $d(fq)_v = d(Fq)_v$ for v in \mathbb{R}^n . Thus $s(f)$ is represented by the map

$$S^n \rightarrow V_n(\mathbb{R}^{n+2}) : x \mapsto \begin{cases} d(fq)_{p(x)} = dF_x dq_{p(x)} & x \in N \\ dq_{p(\bar{x})} & x \in S \end{cases}$$

(applying the chain rule). This map can be altered by the homotopy

$$(x, t) \mapsto \begin{cases} dF_x dq_{(1-t)p(x)} & x \in N \\ dq_{(1-t)p(\bar{x})} & x \in S \end{cases}$$

resulting in

$$s(f) = \left[x \mapsto \begin{cases} dF_x dq_0 & x \in N \\ dq_0 & x \in S \end{cases} \right] = [x \mapsto dF_x \circ j]$$

where j denotes the inclusion of \mathbb{R}^n into \mathbb{R}^{n+2} . The last equality follows because $dq_0 = j$, as is easily verified. But the map $x \mapsto dF_x \circ j$ is homotopic to $x \mapsto GS \circ dF_x \circ j$, which by definition represents $\phi_*(i(f))$, proving the proposition.

DEFINITION 3. Suppose that M is a manifold, and P and Q are codimension zero submanifolds with $P \cap Q$ a submanifold and $P \cup Q = M$. Given a map $f : P \cap Q \rightarrow GL(k)$, denote by

$$\beta(P, Q, f)$$

the \mathbb{R}^k -bundle whose total space is $(P \times \mathbb{R}^k) \cup (Q \times \mathbb{R}^k) / \sim$, where \sim is the equivalence relation identifying $(x, v) \in P \times \mathbb{R}^k$ with $(x, f(x)v) \in Q \times \mathbb{R}^k$, for all x in $P \cap Q$. The projection map for this bundle sends (x, v) to x .

It follows from the homotopy axiom for vector bundles that if f and g are homotopic maps from $P \cap Q$ to $GL(k)$, then $\beta(P, Q, f)$ and $\beta(P, Q, g)$ are isomorphic bundles. Also, if \tilde{f} is defined by $\tilde{f}(x) = f(x)^{-1}$, then $\beta(P, Q, f) \cong \beta(Q, P, \tilde{f})$ and $\beta(P, Q, f) \oplus \beta(P, Q, \tilde{f}) \cong \varepsilon^{2k}$.

If an orientable bundle ξ over M is trivial away from a point (almost parallelizable), then there is an isomorphism $\xi \cong \beta(P, Q, f)$ with $Q = B^{n+1}$, $P \cap Q = S^n = \partial B^{n+1}$, and $f : S^n \rightarrow SO(k)$. The class $[f] \in \pi_n SO(k)$ is called the *obstruction to framing* ξ .

2. The main theorem

From the previous section, there is a commutative diagram:

$$\begin{array}{ccccc}
 & & \pi_n V_n(\mathbb{R}^{n+2}) & & \\
 & \nearrow s & & \nwarrow \phi_* & \\
 \text{Emb}(S^n, \mathbb{R}^{n+2}) \subset \text{Imm}(S^n, \mathbb{R}^{n+2}) & \xrightarrow{i} & \pi_n SO(n+2) & \xrightarrow{J} & \pi_{2n+2}(S^{n+2}).
 \end{array}$$

Our main result is:

THEOREM. $s(\text{Emb}(S^n, \mathbb{R}^{n+2})) = \phi_*(\ker(J))$.

Proof. It suffices to show $i(\text{Emb}(S^n, \mathbb{R}^{n+2})) = \ker(J)$. The proof is in two steps.

STEP 1. *If $f \in \text{Emb}(S^n, \mathbb{R}^{n+2})$, then $J(i(f)) = 0$.*

Extend f to an embedding $f: M_0 \hookrightarrow \mathbb{R}^{n+2}$ of some compact oriented $(n+1)$ -manifold M_0 with $\partial M_0 = S^n$. (M_0 is called a Seifert surface for f .) Consider the closed, smooth manifold

$$M = M_0 \cup B^{n+1}$$

the union being along $\partial B^{n+1} = S^n = \partial M_0$. We will show that $i(f)$ is an obstruction to framing the stable normal bundle of M . Step 1 then follows from Lemma 1 of [MK]. The details follow:

A suitable neighborhood U of B^{n+1} in M can be identified with \mathbb{R}^{n+1} . Thus we view $\mathbb{R}^{n+2} = \mathbb{R}^{n+1} \times \mathbb{R} = U \times \mathbb{R} \subset M \times \mathbb{R}$. The standard orientation on \mathbb{R}^{n+2} induces an orientation on $M \times \mathbb{R}$. Within $M \times \mathbb{R}$, we identify M with $M \times \{0\}$. Set $V = M - \{0\}$ (here 0 denotes the origin of \mathbb{R}^{n+1} = center of B^{n+1}).

Now further extend f to an orientation preserving embedding

$$F: V \times \mathbb{R} \hookrightarrow \mathbb{R}^{n+2}$$

(see Figure 1).

Let

$$g = dF|_{S^n}: S^n \rightarrow GL_+(n+2).$$

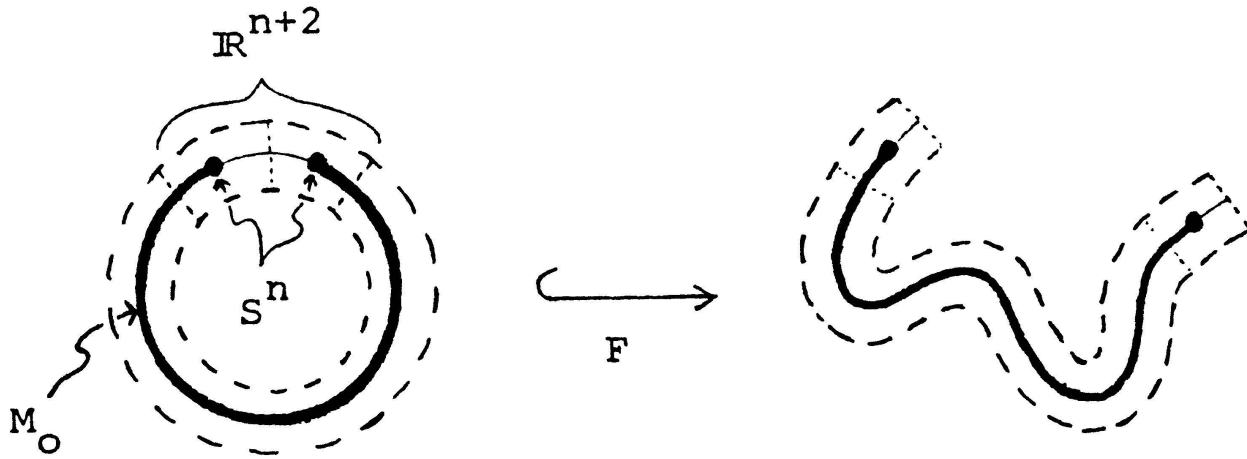


Figure 1

Then

$$\tau_{M \times \mathbb{R}} | M \cong \beta(B^{n+1}, M_0, g). \tag{1}$$

An explicit isomorphism between the bundles is given by assigning to the tangent vector v to $M \times \mathbb{R}$ at the point x in $M = M \times \{0\}$, either

$$(x, v) \in B^{n+1} \times \mathbb{R}^{n+2}$$

if $x \in B^{n+1}$ (here we think of $B^{n+1} \subset \mathbb{R}^{n+1}$, so $v \in \mathbb{R}^{n+2}$), or

$$(x, dF_x(v)) \in M_0 \times \mathbb{R}^{n+2}$$

if $x \in M_0$. This is well-defined, for if $x \in M_0 \cap B^{n+1} = S^n$, then $(x, v) \in B^{n+1} \times \mathbb{R}^{n+2}$ is identified with $(x, g(x)v) = (x, dF_x(v)) \in M_0 \times \mathbb{R}^{n+2}$, by the definition of $\beta(B^{n+1}, M_0, g)$ and g .

Let $h = GS \circ g$, so that

$$h : S^n \rightarrow SO(n+2).$$

Note that $[h] = i(f)$, by definition. Furthermore, g and h are homotopic maps (in $GL_+(n+2)$), so we have

$$\begin{aligned} \tau_M \oplus \epsilon^1 &\cong \tau_{M \times \mathbb{R}} | M \cong \beta(B^{n+1}, M_0, g) \quad (\text{by (1)}) \\ &\cong \beta(B^{n+1}, M_0, h). \end{aligned}$$

Now using the Whitney embedding theorem, embed M in S^{2n+3} and let ν be the normal $(n+2)$ -plane bundle of the embedding. Then $(\tau_M \oplus \epsilon^1) \oplus \nu \cong \epsilon^{2n+4} \cong \beta(B^{n+1}, M_0, h) \oplus \beta(B^{n+1}, M_0, \tilde{h})$ (where $\tilde{h}(x) = h(x)^{-1}$), and so

$$\nu \cong \beta(B^{n+1}, M_0, \tilde{h}) \cong \beta(M_0, B^{n+1}, h)$$

(both bundles are stable normal bundles of M). Since $[h] = i(f)$,

$$i(f) \text{ is the obstruction to framing } \nu. \tag{2}$$

By Lemma 1 of [MK], it follows that $J(i(f)) = 0$.

Remark. For the reader's convenience, here are the details of a proof of the lemma cited above:

Embed M in S^{2n+3} so that it is perpendicular to the equatorial S^{2n+2} , intersecting it in the standard $S^n \subset S^{2n+2}$, with M_0 lying in the northern hemisphere of S^{2n+3} (see Figure 2). This may be accomplished by taking a height function for S^{2n+3} , making it transverse to the embedding of M , and then identifying a minimum point. An isotopy taking a neighborhood of this minimum onto the southern hemisphere alters the embedding to one satisfying the conditions above.

Consider the normal framing \mathbb{F} on $S^n = M \cap S^{2n+2}$ in S^{2n+2} , given by assigning to a point $x \in S^n$ the frame $\begin{pmatrix} 0 \\ h(x) \end{pmatrix} \in V_{n+2}(\mathbb{R}^{2n+3})$. The Thom–Pontrjagin construction applied to this framed submanifold of S^{2n+2} gives an element of $\pi_{2n+2}S^{n+2}$

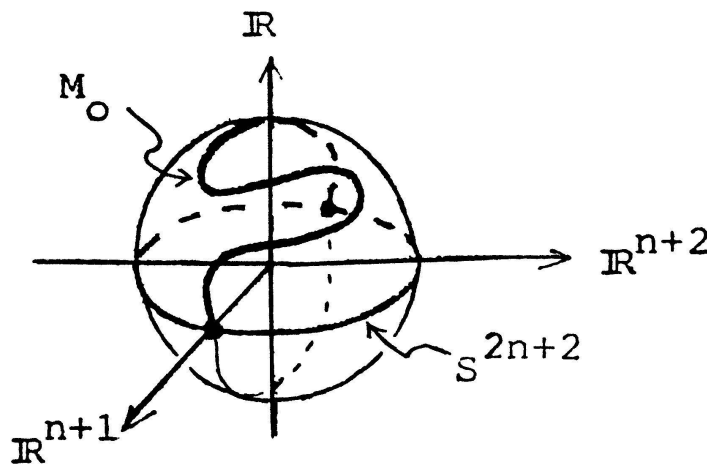


Figure 2

which can be identified with $J([h])$: Both elements are represented by the map

$$\begin{array}{c} S^{2n+2} \subset \mathbb{R}^{2n+3} = \mathbb{R}^{n+1} \times \mathbb{R}^{n+2} \\ \downarrow \\ S^{n+2} \subset \mathbb{R}^{n+3} = \mathbb{R}^{n+2} \times \mathbb{R} \end{array}$$

sending the minimal geodesic arc joining $x \in S^n \times \{0\}$ with $y \in \{0\} \times S^{n+1}$ to the minimal geodesic arc joining the south pole with the north pole of S^{n+2} and passing through $h(x)y \in S^{n+1} \times \{0\}$. (Explicitly, $J([h])$ maps $(u, v) \in S^{2n+2}$ to $(0, 1) \in S^{n+2}$ if $u = 0$, and to $(2 \|u\| h(u/\|u\|)v, \|v\|^2 - \|u\|^2)$ otherwise.) Compare Kervaire [K1, 1.8].

Finally observe that the framing \mathbb{F} on S^n extends over $M_0: \beta(M_0, B^{n+1}, h)$ is abstractly isomorphic to the normal bundle ν of M in S^{2n+3} . We may choose an isomorphism over B^{n+1} which is standard over $S^n = \partial B^{n+1}$ (i.e. maps the standard frame on \mathbb{R}^{n+2} to the standard frame on $\{0\} \times \mathbb{R}^{n+2} \subset \mathbb{R}^{n+1} \times \mathbb{R}^{n+2} = \mathbb{R}^{2n+3}$), and extend this to an isomorphism Ψ over the rest of M . But on $S^n = \partial M_0$, the standard frame on \mathbb{R}^{n+2} maps to \mathbb{F} under Ψ . Hence the image under Ψ of the standard frame on \mathbb{R}^{n+2} over M_0 provides an extension of \mathbb{F} .

Now because the framing extends over M_0 , the Thom–Pontrjagin construction yields 0 in $\pi_{2n+2}(S^{n+2})$, hence so must J .

STEP 2. *If $J(x) = 0$, then there exists $f \in \text{Emb}(S^n, \mathbb{R}^{n+2})$ with $i(f) = x$.*

Bott [Bo] computes

$$\pi_n SO(n+2) = \begin{cases} \mathbb{Z} & \text{if } n \equiv 3 \text{ or } 7 \pmod{8} \\ \mathbb{Z}_2 & \text{if } n \equiv 0 \text{ or } 1 \pmod{8} \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, by the work of Adams, $J: \pi_n SO(n+2) \rightarrow \pi_{2n+2} S^{n+2}$ is injective for $n \equiv 0$ or $1 \pmod{8}$ (see Switzer [S, p. 487]). Thus there is nothing to prove except in the case $n \equiv 3$ or $7 \pmod{8}$, i.e. $n \equiv 3 \pmod{4}$.

So let $n = 4m - 1$. Write j_m for the order of the image of $J: \pi_{4m-1} SO(4m+1) \rightarrow \pi_{8m} S^{4m+1}$. Identifying $\pi_{4m-1} SO(4m+1)$ with \mathbb{Z} , it suffices to produce an embedding $f: S^{4m-1} \rightarrow \mathbb{R}^{4m+1}$ with $i(f) = \pm j_m$.

First consider the collection of all closed, oriented, almost-parallelizable $4m$ -manifolds M . The associated signatures $\sigma(M)$ form a subgroup of \mathbb{Z} ; let $\sigma_m > 0$ denote the generator. Similarly let $p_m > 0$ denote the generator of the group of all top Pontrjagin numbers $p_m(M)$. Observe that if $\sigma(M) = \sigma_m$, then by

the Hirzebruch Index Theorem, $p_m(M) = p_m$. Also, it is known that $\sigma_m \equiv 0 \pmod{8}$ (see [KM, p. 531]).

Case 1: $m > 1$. Let f be the inclusion of the Brieskorn homotopy $(4m - 1)$ -sphere $\Sigma(2, \dots, 2, 3, 6(\sigma_m/8) - 1)$ into $\mathbb{R}^{4m+1} = S^{4m+1} - \{\text{point}\}$, bounding the Milnor fiber $M_0 \subset S^{4m+1}$ [Br]. Brieskorn computes

$$\sigma(M_0) = \pm\sigma_m,$$

so by Kervaire–Milnor [KM, 7.5] and the h -cobordism Theorem [S3], ∂M_0 is diffeomorphic to S^{4m-1} . Capping off M_0 with a $4m$ -ball to get a closed, almost-parallelizable $4m$ -manifold M , we have $\sigma(M) = \pm\sigma_m$, and so

$$p_m(M) = \pm p_m.$$

Case 2: $m = 1$. Let M be the Kummer surface (see, for example Milnor [M]), and let M_0 be the complement of an open ball in M . Note that

$$p_1(M) = p_1 = 48.$$

It is known that M_0 can be constructed from the 4-ball by attaching 2-handles with even framings [Hr][AK] from which it follows easily that there is an embedding $M_0 \hookrightarrow \mathbb{R}^5$ (cf. Ruberman [R]). Let f be the restriction of this embedding to $\partial M_0 = S^3$.

Now in either case we have an embedding $f: S^{4m-1} \hookrightarrow \mathbb{R}^{4m+1}$ whose image bounds a submanifold M_0 , with

$$p_m(M) = \pm p_m,$$

where M is M_0 capped off with a $4m$ -ball. By Theorems 1 and 2 in Milnor–Kervaire [MK]

$$p_m = \pm a_m (2m - 1)! j_m,$$

where a_m is defined to be 1 for m even and 2 for m odd. Also, by Lemma 2 in [MK]

$$p_m(M) = \pm a_m (2m - 1)! o, \tag{3}$$

where o is the obstruction to framing the stable normal bundle ν of M . Thus

$$o = \pm j_m.$$

But by (2) in Step 1,

$$i(f) = o. \tag{4}$$

Hence

$$i(f) = \pm j_m,$$

and so f is the desired embedding.

This completes the proof of the Theorem.

Since $J: \pi_n SO(n+2) \rightarrow \pi_{2n+2} S^{n+2}$ is a monomorphism if $n \not\equiv 3 \pmod{4}$ (as noted above), $\pi_{2n+2} S^{n+2}$ is finite, and $\pi_n SO(n+2) = \mathbb{Z}$ if $n \equiv 3 \pmod{4}$, we deduce:

COROLLARY 1. *Emb (S^n, \mathbb{R}^{n+2}) is isomorphic to \mathbb{Z} if $n \equiv 3 \pmod{4}$ and to 0 otherwise.*

In fact in the case $n \equiv 3 \pmod{4}$ (say $n = 4m - 1$), one may identify explicitly the subgroup $\text{Emb}(S^n, \mathbb{R}^{n+2}) = j_m \mathbb{Z}$ of $\text{Imm}(S^n, \mathbb{R}^{n+2}) = \mathbb{Z}$ using the following formula for j_m :

$$v_2(j_m) = v_2(m) + 3$$

$$v_p(j_m) = \begin{cases} v_p(m) + 1 & \text{if } m \equiv 0 \pmod{\frac{p-1}{2}} \\ 0 & \text{otherwise} \end{cases}$$

(for p an odd prime)

where $v_p(k)$ denotes the exponent of the prime p in the prime decomposition of k . This formula follows from Lemma 3 in [MK] and the Adams conjecture (compare Switzer [S, pp. 479, 488]). The first few values of j_m are $j_1 = 24$, $j_2 = 240$, $j_3 = 504$, and $j_4 = 480$.

One may also give a formula relating the invariant $i(f)$ (for an embedding $f: S^n \hookrightarrow \mathbb{R}^{n+2}$) to the signature of a Seifert surface for f :

COROLLARY 2. *If $f: S^n \hookrightarrow \mathbb{R}^{n+2}$ is an embedding, $n = 4m - 1$, and M_0 is an oriented $4m$ -manifold in \mathbb{R}^{n+2} with $\partial M_0 = f(S^n)$, then identifying $\text{Imm}(S^n, \mathbb{R}^{n+2})$*

with \mathbb{Z} we have

$$i(f) = \pm \frac{m}{2^{2m-1}(2^{2m-1}-1)B_m a_m} \sigma(M_0)$$

where B_m is the m -th Bernoulli number and a_m is 1 or 2 depending upon whether m is even or odd.

Proof. Let M denote M_0 capped off with a $4m$ -ball ($\sigma(M) = \sigma(M_0)$). By (3) and (4) of the proof of the theorem

$$i(f) = \pm \frac{1}{a_m (2m-1)!} p_m(M).$$

The Hirzebruch Index Theorem (see [MK, p. 457]) gives

$$p_m(M) = \frac{(2m)!}{2^{2m}(2^{2m-1}-1)B_m} \sigma(M),$$

as M is almost parallelizable, and the Corollary follows.

For example, if $m = 1$, then $i(f) = \pm \frac{3}{2} \sigma(M_0)$.

Remark. Our viewpoint also sheds light on the case of embeddings $S^n \hookrightarrow \mathbb{R}^k$ for $k > n + 2$: If $\text{Emb}_F(S^n, \mathbb{R}^k)$ denotes the set of regular homotopy classes containing embeddings which bound framed submanifolds of \mathbb{R}^k , then one has by an analogous argument to the proof of the theorem

$$s(\text{Emb}_F(S^n, \mathbb{R}^k)) = \phi_*(\ker(J))$$

where

$$\phi_*: \pi_n SO(k) \rightarrow \pi_n V_n(\mathbb{R}^k)$$

is the natural map. (Note that ϕ_* is generally not an isomorphism.) As a consequence, for example, one has

$$s(\text{Emb}_F(S^3, \mathbb{R}^6)) = 0$$

(in fact $\text{Emb}(S^3, \mathbb{R}^6) = 0$ by [S2]), and

$$s(\text{Emb}_F(S^7, \mathbb{R}^{10})) = 720\mathbb{Z} \oplus \{0\} \subset \mathbb{Z} \oplus \mathbb{Z}_4 = \pi_7 V_7(\mathbb{R}^{10}).$$

QUESTIONS. (1) Is $\text{Emb}_F(S^n, \mathbb{R}^{n+3}) = \text{Emb}(S^n, \mathbb{R}^{n+3})$? (2) For a given n , what is the largest value of k for which $\text{Emb}(S^n, \mathbb{R}^k) \neq 0$?

Added in proof: Sylvain Cappell has informed us that our theorem can be deduced from an unpublished version of his paper with J. Shaneson, "Singularities and immersions", *Ann. of Math.* 105 (1977), 539–552.

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