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The rational homotopy of Thom spaces and the smoothing of homology classes

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1. Introduction

Let $G \overset{j}{\hookrightarrow} SO(n)$ be a closed connected subgroup and let V^m be a closed oriented manifold. A homology class $z \in H_{m-n}(V^m; Z)$ is said to be *G-smoothable* if $z = [W^{m-n}]$, the submanifold W having G as structure group for its normal bundle. In his famous paper [13] Thom showed, among other things, that the G-smoothability problem is of a homotopy theoretic nature. One first has to construct the universal Thom space MG, by taking the Thom space of the bundle γ^G over BG, which is the pull-back of the universal oriented n-plane bundle γ^n over $BSO(n)$. If V is a finite complex, a cohomology class $u \in H^n(V; Z)$ is said to be *G-realizable* if $u = g^*(u_G)$ for some map $V \xrightarrow{g} MG$, u_G being the universal Thom class. If u is the Poincaré dual of z, then Thom's result reads: z is G-smoothable if and only if u is G-realizable.

The problem of deciding the G-smoothability (G-realizability) is in general a difficult one. There are very few general results in this direction, the oldest and perhaps the most important also belonging to Thom:

THEOREM ([13], Théorème II.25). *If $G = SO(n)$ then, for any u, some nonzero multiple of u is G-realizable.*

The question of G-realizability up to a nonzero factor turns out to be a rational homotopy problem. The answer is strongly influenced by the nature of the universal Euler class $e_G \in H^n(BG; Q)$, $e_G = e(\gamma^G)$. The main result of this paper is:

1.1. THEOREM. (i) *If $e_G = 0$ then: some nonzero multiple of u is G-realizable if and only if u^2 is a torsion element.*

(ii) *If $e_G \neq 0$ then: some nonzero multiple of any u is G-realizable if and only if e_G is not decomposable in H^*BG.*

(iii) *If V is a finite connected complex such that $H^i(V; Q) = 0$ for $i > 2n+3$ then, for any $G \hookrightarrow SO(n)$ with $e_G \neq 0$, some nonzero multiple of any u is G-realizable.*

COROLLARY. *If G is one of the classical groups $U(r)$, $Sp(r)$, $r \geq 1$, or $SU(r)$, $r \geq 2$, with standard embeddings, then some nonzero multiple of any u is G-realizable.*

Proof. By simply checking the conditions on e_G given in (ii) above (see e.g. [4]). The first two examples are implicit in [13] (see the remarks following Théorème II.25). For the dependence on the embedding, see the beginning of Section 4, which is devoted to the proof of Theorem 1.1 and ends with a discussion of the general solution of the G-realizability problem up to a nonzero factor.

The rational homotopy approach goes as follows: first "tensor" with Q the given problem. For the moment, this means nothing but just making a new definition: let us say that $v \in H^n(V; Q)$ is *G-realizable* if there is a map $V \xrightarrow{f} MG_0$ such that $f^*(u_G)_0 = v$, where the subscript denotes rationalization ([12], [8]). One can then try to solve this problem by using the purely algebraic technique of the minimal models of Sullivan [12].

Supposing that one is able to construct in this way a rational solution $V_0 \xrightarrow{f} MG_0$, there is still one more thing to do, namely to find $V \xrightarrow{g} MG$ such that $g_0 = f$. This is the general delocalization problem: given a map between localized nilpotent spaces $V_0 \xrightarrow{f} M_0$, is there $V \xrightarrow{g} M$ such that $g_0 = f$?

Section 3 is devoted to this problem. Roughly speaking, the main results (Proposition 3.1 and the remark following it) assert that, whenever V and M are finite complexes and one of them is a 1-connected *formal* space, there is a delocalization, for any f. As a typical application we offer the following:

1.2. PROPOSITION. *If $u \in H^n(V; Z)$, where V is a finite complex then: some Z^*-multiple of u is G-realizable if and only if some Q^*-multiple of u_0 is G-realizable.*

Both Sec. 4 and the above result depend on the analysis of the homotopy type of MG_0, which is carried out in §2. The starting point is to observe a very simple but very useful fact, namely that the universal Thom spaces are formal. A *formal space* is defined by the property that its Q-type is entirely determined by the cohomology algebra ([12]). Using formality arguments, we give in Theorem 2.6 a concrete convenient description of the Q-types of universal Thom spaces.

The rest of Sec. 2 contains results of independent interest, derived along the lines of the central idea of this paper, which says that the universal Thom constructions possess strong formality properties facilitating precise rational homotopy computations. For example, Proposition 2.7 and Corollary 2.8 give the Whitehead product structure of the rational homotopy Lie algebra of MG in terms of H^*BG and e_G, thus answering a question raised in [2].

More applications are contained in [10]. I owe many thanks to the referee for his valuable suggestions which led to many improvements of this paper.

2. Formality properties of universal Thom spaces

The study of the Q-type of universal Thom spaces was started in [2], where the cases of $MO(r)$, $MSO(r)$ and $MU(r)$ were considered. In this section we shall extend the results of [2] to the case $G \xrightarrow{j} SO(n)$ closed connected arbitrary, using a new point of view, namely the formality.

2.1. DEFINITION. S is a *formal space* if, denoting by $(\mathcal{M}, d)$ its minimal model, there exists a differential graded algebra map $(\mathcal{M}, d) \xrightarrow{\rho} (H^*\mathcal{M}, 0)$ such that $\rho^* = id$.

2.2. LEMMA. *Let $\mathcal{M}$ be a formal minimal algebra and $u \in Z^n\mathcal{M}$ such that $[u] \in H^n\mathcal{M}$ is not a zero divisor.*

Consider: $\mathcal{M} \xrightarrow{p} \mathcal{M} \otimes_d \Lambda_{n-1}(y) = \mathcal{N}$, where the right hand extension of $\mathcal{M}$ is constructed by setting $dy = u$. Then p^* is onto, and $\mathcal{N}$ is again formal.

Proof. The first assertion is immediate, by looking at the algebraic Serre spectral sequence of p. Since $\mathcal{M}$ is formal, there is a d.g.a. map $\mathcal{M} \xrightarrow{f_M} H^*\mathcal{M}$ such that $f_M^* = id$. Construct a d.g.a. map $\mathcal{N} \xrightarrow{f_N} H^*\mathcal{N}$ by $f_{N|\mathcal{M}} = p^* f_M$ and $f_N(y) = 0$. Since p^* is onto, $f_N^* = id$.

2.3. LEMMA. *Let $G \hookrightarrow O(n)$ be a closed subgroup. If G is connected or if $G = O(n)$ then the Thom space MG is formal.*

Proof. MG appears as a cofibre in the sequence $SG \xrightarrow{p} BG \xrightarrow{k} MG$, where SG is the associated sphere bundle of γ^G. We are going to use the fact that the cofibre of a formal map is a formal space [3]. A *formal map* is defined by the property that its minimal model, say $\mathcal{M} \xrightarrow{f} \mathcal{M}'$, satisfies an algebraic homotopy commutativity condition of the form $\rho' f \simeq f^* \rho$, where ρ and ρ' are formalization maps as in Definition 2.1. Since H^*BG is a polynomial algebra we only have to check that SG is a formal space.

If $G = O(n)$, SG has the homotopy type of $BO(n-1)$ and we are done. If G is connected, then $S^{n-1} \hookrightarrow SG \to BG$ is a rational fibration in the sense of [5], hence we may write a nilpotent model of SG which fits into an algebraic fibre sequence of the form

$$(H^*BG, 0) \hookrightarrow H^*BG \otimes_d \Lambda(z_{n-1}) \xrightarrow{\text{proj}} (\Lambda(z_{n-1}), \bar{d})$$

when n is even, or of the form

$$(H^*BG, 0) \hookrightarrow H^*BG \otimes_d \Lambda(x_{n-1}, y_{2n-3}) \xrightarrow{\text{proj}} (\Lambda(x_{n-1}, y_{2n-3}), \bar{d})$$

when n is odd, where the d.g.a.'s with induced differential $\bar{d}$ are the minimal models of the appropriate spheres.The differential structure of the extensions is given by $dz = e_G$, $dx = 0$ and $dy = x^2 + sx + t$, for some $s, t \in H^*BG$. If n is odd or n is even and e_G is nonzero, Lemma 2.2 is available. In the remaining case, the formality of SG is obvious.

It will often happen that our discussion splits into two rather contrasting cases, according to the nature of the Euler class. This can be seen as follows: if $e_G = 0$ the Gysin sequence of the fibration $S^{n-1} \hookrightarrow SG \to BG$ shows that p^* is injective and identifies coker $p^* = \sum^{n-1} H^*BG$; the Barratt–Puppe sequence of the cofibration $SG \to BG \to MG$ gives then $H^+MG = \sum^n H^*BG$, with trivial multiplication. If $e_G \neq 0$, again by the Gysin sequence p^* is onto and $\ker p^* = e_G \cdot H^*BG$; the exact sequence of the cofibration identifies $H^+MG = e_G \cdot H^*BG$, with the multiplication induced from H^*BG.

The above discussion also shows that H^*MG is determined by H^*BG and by e_G. By the previous lemma, this gives all the information about MG_0, at least theoretically. We are going to be more precise about this point in what follows. To start with, we shall describe H^*MG by generators and relations, in the case e_G is nonzero.

Choose a graded complement, X, for $e_G \cdot H^*BG$ in H^*BG, with homogenous well-ordered indexed basis x_a and write $|a|$ for $\deg x_a$. Let $x_0 = 1 \in H^0BG$. Then a minimal set of generators for H^*MG is given by

$$z_a = e_G \cdot x_a. \tag{2.4}$$

They form a basis for a graded space $Z_0 = \sum^n X (\sum^n x = e_G x,\ x \in X)$. The relations among them are described as follows: write

$$x_a x_b = \sum_{k \geq 0} e_G^k y_k, \qquad y_k \in X.$$

Then a complete set of relations is given by

$$r_{ab} = z_a z_b - \sum_{k \geq 0} z_0^{k+1} \cdot \sum{}^n y_k = 0, \qquad b \geq a > 0. \tag{2.5}$$

In particular they all have degrees greater than or equal to $2n+4$.

2.6. THEOREM. *If $e_G=0$ then MG has the Q-type of a wedge of spheres. If $e_G\neq 0$ then MG fibres over $K(Z,n)$, the fibre having the Q-type of a wedge of spheres.*

Proof. The first assertion is clear. Suppose then that $e_G\neq 0$ and consider the fibration given by the universal Thom class $F\overset{p}{\hookrightarrow} MG\xrightarrow{u_G}K(Z,n)$ and the localized fibration $K(Q,n-1)\hookrightarrow F_0\xrightarrow{p_0}MG_0$. Since $H^*(MG;Q)$ may be identified with a subalgebra of $H^*(BG;Q)$ in such a way that the rational Thom class is identified with e_G, we may use Lemma 2.2 to deduce that F is a formal space whose cohomology algebra has trivial multiplication. Consequently, F_0 is again a wedge of spheres. (More precisely, $F_0\simeq(\bigvee_{a>0}S^{n+|a|})_0$.)

The rest of this section is not needed for the proof of Theorem 1.1. The results below show that the rational homotopy theory of universal Thom spaces is a *direct* consequence of the cohomological picture, which is a striking formality property, worth to be included here.

The proofs are slightly more technical; they are based on the bigraded models introduced in [6], which are the most appropriate ones when dealing with formality. We shall recall briefly, following [6], the construction of the *bigraded model* $(\Lambda Z,d)$ of a connected graded algebra H. The generators are bigraded by $Z=\oplus_{\substack{n>0\\ p\geq 0}}Z_p^n$, the differential is bihomogenous of upper degree $+1$ and lower degree -1. The graded space Z_0 is isomorphic to a minimal system of homogenous algebra generators for H and $\sum Z_1$ is isomorphic to a minimal system of relations. The modelling map $(\Lambda Z,d)\xrightarrow{f}(H,0)$ is defined in the obvious way on Z_0 and sends Z_+ to zero, and $H(\Lambda Z)=H_0(\Lambda Z)=H$. If S is a formal 1-connected space, it inherits by Sullivan duality a new grading $\pi_n(S)\otimes Q=\oplus_{p\geq 0}\pi_n^p(S)$, with π_*^p dual to Z_p^*.

2.7. PROPOSITION. (i) *$\pi(MG)$ is generated as a Lie algebra by π^0.*
(ii) *For any $p\geq 0$, $\pi^p(MG)=ad(\pi^0)^p(\pi^0)$.*

Proof. For any space the Whitehead bracket is dual to the quadratic part of the differential of its minimal model [12]. Due to the homogeneity properties of the differential of the bigraded model there is an inclusion valid for any formal space, namely $ad(\pi^0)^p(\pi^0)\subset\pi^p$. Therefore (ii) is an immediate consequence of (i).

If S is a formal 1-connected space a similar degree argument provides an inclusion $[\pi(S),\pi(S)]\subset\pi^+(S)$. Moreover, one has a dual Hurewicz exact sequence [6] $0\to H^+(\Lambda Z)\cdot H^+(\Lambda Z)\to H^+(\Lambda Z)\xrightarrow{h^*}Z_0\to 0$ which gives by Sullivan duality an exact Hurewicz sequence:

$$0 \to \pi^+(S) \to \pi(S) \xrightarrow{h} PH_*(S) \to 0.$$

Putting these facts together, it is readily seen that the property stated in (i) is equivalent with the equality $[\pi, \pi] = \ker h$ (which does not hold in general even for formal spaces).

If $e_G = 0$, MG is rationally a wedge of spheres and, by the theorem of Hilton [7], we are done. If $e_G \neq 0$, recall the fibration $F \overset{p}{\hookrightarrow} MG \xrightarrow{u_G} K(Z, n)$ in Theorem 2.6 and the fact that p induces an injection in rational homology (by Lemma 2.2), and apply the Hilton theorem again.

2.8. COROLLARY. *Suppose that* e_G *is nonzero. Then*

(i) $\pi(MG) = \pi_n^0 \oplus \{$*free Lie algebra generated by* $\pi_{>n}^0\}$, *where* $\pi_n^0 = \pi_n$, *which is generated by the Hurewicz dual of* u_G, *appears as a subalgebra, the Lie algebra extension is nontrivial unless* rank $G = 1$, *and the full bracket structure may be explicitly written down in general using the relations* (2.5).

(ii) $\pi^p = ad(\pi_{>n}^0)^p(\pi_{>n}^0)$, *for any* $p > 0$.

Proof. The decomposition in (i) comes from the exact homotopy sequence of the fibration $F \overset{p}{\hookrightarrow} MG \xrightarrow{u_G} K(Z, n)$, namely $\pi_*(MG) = \pi_n(MG) \oplus p_{\#}\pi_*(F) = \pi_n(MG) \oplus \pi_{>n}(MG)$. The same argument used in Proposition 2.7 shows that $\pi_{>n}^+ = [\pi_{>n}, \pi_{>n}]$. The free Lie algebra $p_{\#}\pi_*(F)$ being thus generated by the graded subspace $\pi_{>n}^0$ which has the right dimensions in each (lower) degree (see the proof of Theorem 2.6), the assertion on freeness follows again by Hilton's theorem. The equality $\pi_n = \pi_n^0$ is a consequence of the Hurewicz theorem.

The Whitehead product $\pi^0 \Lambda \pi^0 \to \pi^1$ is dual to the quadratic part of the differential $Z_1 \to Z_0 \Lambda Z_0$, which in turn is obtained by simply taking the quadratic part of the relations (2.5). To be a little bit more precise, denote by $\{z_a^*\}$ the basis of π^0 which is dual to the basis $\{z_a\}$ of Z_0 given by (2.4) and notice that z_0^* generates π_n^0, while $\{z_a^*\}_{a>0}$ generate $\pi_{>n}^0$. It is straightforward to see that $[z_0^*, z_0^*] = 0$ and that $[z_a^*, z_b^*]$ is Sullivan dual to the element of Z_1 corresponding to the relation r_{ab} (2.5), for any $b \geq a > 0$. It is now easy to find the coefficients in

$$[z_0^*, z_c^*] = \sum_{b \geq a > 0} C_{ab}^c [z_a^*, z_b^*], \quad \text{for any } c > 0.$$

In particular, if the decomposition in (i) is a Lie product one then it follows from the construction of the relations (2.5) that $x_a x_b \in e_G \cdot H^*BG$, for any $a, b > 0$. We infer that $H^+BG \cdot H^+BG \subset e_G \cdot H^*BG$, which forces rank $G = 1$, H^*BG being a polynomial algebra.

The equality (ii) follows exactly as in the previous proposition, with $\pi_{>n}(MG)$ replacing $\pi(MG)$.

3. Delocalization

Let K be a finite simply connected complex and consider the following subset of the self 0-equivalences of K

$$E = \{K \xrightarrow{f} K \mid f^*(x) = t^{|x|} \cdot x, \text{ any } x \in H^*(K; Z), \text{ for some } t \in Z^*\}.$$

We are going to exploit in this section the fact that the formality of K is equivalent to the existence, for any $t \in Z^*$, of an $f \in E$ which acts on $H^*(K; Z)$ by $f^*(x) = s^{|x|} \cdot x$, for some nonzero multiple s of t ([12], [11]).

3.1. PROPOSITION. *Let V be a finite complex and K be a finite simply connected complex with localization map l. If K is formal then, given any map $V \xrightarrow{h} K_0$, there is an $f \in E$ and $V \xrightarrow{H} K$ such that $f_0 h = lH$.*

Proof. K being formal, the same argument used in [9] (with f restricted to lie in E) shows that for any integers n, s there is an $f \in E$ such that for $i \leqslant n$ $\pi_i(f)$ kills the s-torsion of $\pi_i(K)$ and Im $\pi_i(f) \subset s \cdot \pi_i(K)$.

Now let $F \to K$ be the homotopy fibre of $K \xrightarrow{l} K_0$. Any $f \in E$ localizes to f_0, and induces $f_F : F \to F$. The remark above implies (via the long exact homotopy sequence) that for any finite subgroup $S \subset \pi_*(F)$ there exists $f \in E$ such that $\pi_*(f_F)$ annihilates S.

In order to construct f and H satisfying the desired conclusion we proceed by induction, putting $h_m = h\,|_{V^m}$ and supposing that $H_m : V^m \to K$ and $f_m \in E$ have been constructed so that $(f_m)_0 h_m = lH_m$. The obstruction to extending H_m to $H_{m+1}: V^{m+1} \to K$ so that $(f_m)_0 h_{m+1} = lH_{m+1}$ lies in $H^{m+1}(V^{m+1}, V^m; \pi_m(F))$. Since V is finite there is, in fact, a finite subgroup $S \subset \pi_m(F)$ such that the obstruction lies in the image of $H^{m+1}(V^{m+1}, V^m; S) \to H^{m+1}(V^{m+1}, V^m; \pi_m(F))$.

Choose $f' \in E$ such that $\pi_m(f'_F)$ annihilates S. Then $f'H_m$ extends to H_{m+1} such that $lH_{m+1} = (f')_0 (f_m)_0 h_{m+1}$. Put $f_{m+1} = f'f_m$. Because V is finite this proves the proposition.

3.2. *Remark.* A similar (and even simpler) argument shows that, for a formal

1-connected finite complex K and a 1-connected space X with localization map l, for any map $h:K\to X_0$ there is an $f\in E$ and $H:K\to X$ such that $hf=lH$ (see also [12], [9]).

3.3. *Proof of Proposition* 1.2. Suppose we have a map $V\xrightarrow{g} MG_0$ such that $g^*(u_G)_0=q\cdot u_0$, for some $q\in Q^*$. Since V is finite we may suppose g maps into $(MG^N)_0$ for some skeleton MG^N of MG. Since MG is formal this is a formal ([12]) finite complex. We may apply the previous Proposition and obtain $f:MG^N\to MG^N$ such that $H^nf=k^n\cdot id$, for some $k\in Z^*$, and $H:V\to MG^N$ such that $lH=f_0g$. It follows that H^*u_G is a Q^*-multiple of u in $H^n(V;Q)$, hence $r\cdot H^*u_G=s\cdot u$ in $H^n(V;Z)$, for a suitable choice of $r,s\in Z^*$. Use the formality again and choose $f'\in E$, $MG^N\xrightarrow{f'}MG^N$ such that $(f')^*u_G=rt\cdot u_G$; set then $g'=f'H:V\to MG$ and conclude that $st\cdot u$ is G-realizable.

There is one more application of Proposition 3.1:

3.4. COROLLARY. *If G is a compact connected Lie group let us choose elements $c_i\in H^*(BG;Z)$ which freely generate the algebra $H^*(BG;Q)$. Given any collection of classes $a_i\in H^{|c_i|}(V;Z)$, where V is a finite complex, there exist $t\in Z^*$ and a map $\xi:V\to BG$ such that*

$$\xi^*c_i=t^{|a_i|}\cdot a_i,\quad \textit{for any } i.$$

Proof. Define $h:V\to BG_0$ by $h^*(c_i)_0=(a_i)_0$, any i, and pick a large number N such that $h:V\to(BG^N)_0$. By Proposition 3.1 one obtains $f:BG^N\to BG^N$ such that $f^*c_i=s^{|c_i|}\cdot c_i$, any i, for some $s\in Z^*$, and $H:V\to BG^N$ such that $lH=f_0h$. As in the previous proof one may choose a sufficiently large $k\in Z^*$ such that $k\cdot H^*c_i=ks^{|a_i|}\cdot a_i$ in $H^*(V;Z)$, for any i. The formality of BG^N produces a map $f':BG^N\to BG^N$ and a Z^*-multiple of k, say r, such that $(f')^*c_i=r^{|c_i|}\cdot c_i$, any i. We may then take $t=rs$ and $\xi=f'H$.

4. *G*-realizability up to a nonzero factor

4.1. Let $G\overset{j}{\hookrightarrow}SO(n)$ be an embedding of a compact connected Lie group and let V^m be a closed oriented manifold. We ought to point out from the beginning that all the previous constructions, definitions and notations coming from this situation (MG, G-smoothability, G-realizability, $e_G,\ldots$) are in fact depending on the embedding j and not only on the isomorphism class of G. The G-abbreviation is only a notational simplification and should not be misleading.

As a very simple example we may take $G = SO(2)$, embedded in two ways in $SO(4)$:

$$SO(2) \underset{(\mathrm{id},\, \mathrm{id})}{\overset{(\mathrm{id},\, 1)}{\rightrightarrows}} SO(2)\times SO(2) \hookrightarrow SO(4).$$

Denote by j_1 the embedding onto the first factor of the maximal torus and by j_2 the diagonal embedding. For $z \in H_{m-4}(V^m; Z)$ they give rise to two distinct notions of G-smoothability, namely z is G-smoothable if and only if

$$z = [W^{m-4}], \quad \text{with} \quad \nu(W) = \zeta \oplus \theta \qquad (\text{via } j_1)$$

$$z = [W^{m-4}], \quad \text{with} \quad \nu(W) = \zeta \oplus \zeta \qquad (\text{via } j_2)$$

where ζ is a complex line bundle and θ is the trivial complex line bundle. Indeed we may take $V = P^4C$ and $u = a^2$ as the Poincaré dual of z, where $a \in H^2(P^4C; Z)$ is the canonical generator. Since u is dual to P^2C, it is G-realizable via j_2. On the other hand, since u^2 is a generator of $H^8(P^4C; Z)$, Theorem 1.1(i) shows that u is not G-realizable via j_1. The point is that the Euler classes constructed via the two embeddings are different, namely $e_G = 0$ via j_1, but $e_G = c_1^2$ via j_2. We mention that we have no example (in connection with Theorem 1.1(ii)) where the property of indecomposability of the Euler class really depends on j.

4.2. *Proof of Theorem* 1.1

(i) If $e_G = 0$ then Theorem 2.6 gives a decomposition $(MG)_0 = S_0^n VX$ with S_0^n carrying $(u_G)_0$. On the other hand, it is a classical fact that classes $a \in H^n(V; Q)$ satisfying $a^2 = 0$ all arise from maps $V \to S_0^n$. Now apply Proposition 1.2.

(ii) Suppose that e_G is not a decomposable of H^*BG. This implies that there is a graded algebra map $H^*(BG; Q) \to H^*(K(Z, n); Q)$ sending e_G to $(a_n)_0$, $a_n \in H^n(K(Z, n); Z)$ being the canonical generator. Since we know that $H^+(MG; Q) = e_G \cdot H^*(BG; Q) \subset H^*(BG; Q)$ as a subalgebra, the formality of MG insures the existence of a map $r : K(Z, n) \to MG_0$ with the property that $r^*(u_G)_0 = (a_n)_0$.

Passing to a large skeleton, we may delocalize it via Proposition 1.2, obtaining a map $s : K(Z, n)^N \to MG$ such that $s^*(u_G) = q \cdot a_n$, for some $q \in Z^*$.

Finally, given $u \in H^n(V^m; Z)$, represent it by a map $\bar{u} : V \to K(Z, n)^N$ and set $g = s\bar{u}$. Note that we have $g^*(u_G) = q \cdot u$, where the factor q depends only on m and on the embedding $G \overset{j}{\hookrightarrow} SO(n)$.

Suppose now that $e_G \neq 0$ but $e_G \in H^+BG \cdot H^+BG$. We shall construct a closed manifold V and a class $u \in H^n(V; Z)$ with the property that no nonzero multiple of u is G-realizable.

Our hypothesis implies that e_G may be written in the form $e_G = e_2 + \cdots e_{t+1}$, where each e_i is a linear combination of monomials containing exactly i generators of the polynomial algebra H^*BG(= free graded algebra on $\{c_j\}$). Choose integers k and l satisfying

$$2k \geqslant t+2, \qquad 4l+2 > (t+2)n \tag{*}$$

and construct a graded algebra $H = \Lambda_n(x)/(x^{2k+1}) \otimes \Lambda_{4l+2}(z)/(z^2)$ keeping in mind that, since e_G is nonzero, n must be even. Notice that H is 1-connected, satisfies Poincaré duality and has top dimension $\neq 0 \pmod 4$. Taking the formal Q-space having it as cohomology algebra, there is no obstruction to rational surgery on it ([12], [1]). Therefore there exists a closed manifold V such that $H^*(V; Q) = H$. Choose $u \in H^n(V; Z)$ such that u_0 is nonzero in $H^n(V; Q)$.

In order to finish the proof, we are going to show that any graded algebra map $f: H^*MG \to H^*$ sends $(u_G)_0$ to zero.

We have a graded algebra map $\Lambda(\Sigma^n H^*BG) \xrightarrow{p} H^*MG$ given by $p(\Sigma^n a) = u_G \cdot a$, for any $a \in H^*BG$. Putting $g = fp$, it is immediate to see that

$$g\left(\Sigma^n a_1\right) \cdots g\left(\Sigma^n a_{r+1}\right) = g\left(\Sigma^n 1\right)^r \cdot g\left(\Sigma^n a_1 \cdots a_{r+1}\right)$$

for any $a_1, \ldots, a_{r+1} \in H^*BG$. Suppose that $g(\Sigma^n 1) = q \cdot x$, with $q \neq 0$.

We first remark that, due to the inequalities (*), the elements $\{x^j \mid 0 \leqslant j \leqslant t+2\}$ represent a basis for $H^{\leqslant (t+2)n}$. If the monomial $c_1 \cdots c_{r+1}$ appears in e_G then necessarily $1 \leqslant r \leqslant t$ and $0 < |c_i| < n$, for any i. These imply that

$$q^r \cdot g\left(\Sigma^n c_1 \cdots c_{r+1}\right) \cdot x^r = 0, \quad \text{hence } g\left(\Sigma^n e_G\right) = 0.$$

Since $g(\Sigma^n e_G) = g(\Sigma^n 1)^2$ this gives $x^2 = 0$, a contradiction.

(iii) Suppose $e_G \neq 0$. As observed in Section 2, $H^*(MG; Q)$ has relations only in degrees $\geqslant 2n+4$. On the other hand, the assumption on V implies that it has the same minimal model as a d.g.a. (A, d_A) which is zero in degrees $> 2n+3$. Thus given $u_0 \in H^n(V; Q)$ we may find a d.g.a. map $(H^*MG, 0) \to (A, d_A)$ sending $(u_G)_0$ to a representative of u_0. Since MG is formal this yields a map $g: V \to MG_0$ such that $g^*(u_G)_0 = u_0$. Now apply Proposition 1.2.

A solution of the G-realizability problem up to a nonzero factor may be obtained in general as follows:

4.3. PROPOSITION. *Suppose $e_G \neq 0$ and let V be a finite connected complex with minimal model $\mathcal{M}_V$. One may construct an affine algebraic variety over Q, $A \subset \mathcal{M}_V^n \times X$, the Q-vector space X and the variety A depending on the embedding j and on $\mathcal{M}_V$, such that:*

(i) $pr_1(A) \subset Z^n\mathcal{M}_V$

(ii) *given $u \in H^n(V; Z)$, some Z^*-multiple of u is G-realizable if and only if some Q^*-multiple of u_0 lies in $[pr_1(A)] \subset H^n\mathcal{M}_V = H^n(V; Q)$.*

Proof. Denote by $\mathcal{M}$ the minimal model of MG and consider the map $[\mathcal{M}, \mathcal{M}_V] \xrightarrow{ev} H^n\mathcal{M}_V$, which sends the homotopy class of the d.g.a. map g to $g^*[U]$. Here we have written $\mathcal{M} = (\Lambda Z, d)$ observing that Z^n is generated by an element U such that $dU = 0$ and $[U] = (u_G)_0$, by the Hurewicz theorem. Proposition 1.2 may be restated as follows: some Z^*-multiple of u is G-realizable if and only if some Q^*-multiple of u_0 lies in the image of ev.

We therefore have to construct A and show that $\operatorname{Im} ev = [pr_1A]$. Pick $m > n$ such that $H^i(V; Q) = 0$ for $i > m$. Standard algebraic obstruction theory shows that there is a bijection induced by restriction $[\Lambda Z, \mathcal{M}_V] \to [\Lambda Z^{\leq m}, \mathcal{M}_V]$ so we may consider a second evaluation map $\operatorname{Hom}_{\text{d.g.a.}}(\Lambda Z^{\leq m}, \mathcal{M}_V) \xrightarrow{ev} H^n\mathcal{M}_V$ defined in the same way and having the same image as the previous one.

The graded algebra maps between $\Lambda Z^{\leq m}$ and $\mathcal{M}_V$ are identified with the vector space $\prod_{i=n}^m \operatorname{Hom}(Z^i, \mathcal{M}_V^i) = \mathcal{M}_V^n \times X$. The d.g.a. maps are those determined by the algebraic conditions imposed by the property of commuting with the differentials. We shall thus take $A = \operatorname{Hom}_{\text{d.g.a.}}(\Lambda Z^{\leq m}, \mathcal{M}_V)$ and simply remark that $pr_1(A) \subset Z^n\mathcal{M}_V$ and $[pr_1A] = \operatorname{Im} ev$.

4.4. *Remarks.* The proof of Theorem 1.1(ii) shows in fact that whenever $e_G \neq 0$ but $e_G \in H^+BG \cdot H^+BG$ there exists a class $v \in H^n(V; Q)$ for which the G-realizability problem cannot be solved even at the cohomological level, that is there is no algebra map $H^*MG \to H^*V$ sending $(u_G)_0$ to some Q^*-multiple of v.

On the other hand, by exploiting more carefully the structure of the bigraded model of the formal space MG([6]), it can be shown that if V is a finite connected complex such that $H^i(V; Q) = 0$ for $i > 3n+4$ then, for any $G \hookrightarrow SO(n)$ with $e_G \neq 0$ and for any $u \in H^n(V; Z)$, some nonzero multiple of u is G-realizable if and only if there is an algebra map $h: H^*MG \to H^*V$ such that $h(u_G)_0 = q \cdot u_0$, with $q \in Q^*$.

The example below shows that outside this range the existence of a cohomological solution does not in general imply the G-realizability, thus indicating the complexity of the problem, as reflected in the fact that in general A depends on more than $H^*(V; Q)$.

4.5. EXAMPLE. We shall start with $G = SO(2)\times SO(2)$ cannonically embedded in $SO(4)$ and we shall construct a 1-connected complex V of dimension 17 and an algebra map $h: H^*MG \to H^*V$ such that, setting $h(u_G)_0 = v$, no Q^*-multiple of v is G-realizable.

H^*BG is freely generated by two elements of degree 2, say e and f, and $e_G = ef$. Writing as in §2

$$H^*BG = e_G \cdot H^*BG \oplus Q \cdot 1 \oplus \operatorname{span}\{e^k \mid k \geq 1\} \oplus \operatorname{span}\{f^l \mid l \geq 1\}$$

use (2.4) to conclude that H^*MG is generated by the elements: $U = ef$, $x_k = e^{k+1}f$, $k \geq 1$, and $y_l = ef^{l+1}$, $l \geq 1$. The relations (2.5) among them are of the following types:

$$\begin{array}{lll} (I_{jk}) & x_jx_k - Ux_{j+k}, & k \geq j \geq 1 \\ (II_{jk}) & y_jy_k - Uy_{j+k}, & k \geq j \geq 1 \\ (III_{kl}) & x_ky_l - U^{l+1}x_{k-l}, & k > l \geq 1 \\ (IV_{kl}) & x_ly_k - U^{l+1}y_{k-l}, & k > l \geq 1 \\ (V_k) & x_ky_k - U^{k+2}, & k \geq 1. \end{array}$$

With this information at hand, it is not difficult to construct a 16-stage minimal model $\rho: (\Lambda Z^{\leq 16}, d) \to (H^*MG, 0)$ which, due to the formality of MG, may be extended to a minimal model of MG, $(\Lambda Z, d)$ (see also [6]). Explicitly, $Z^{\leq 16}$ has a basis consisting of elements labeled $U, x_1, y_1, \ldots, x_6, y_6, z_1, \ldots, z_{14}, v_1, v_2$, on which ρ acts by sending the first of them to the corresponding generators of H^*MG, and the rest of them to zero. The differential structure is defined by setting $dU = 0$, $dx_i = 0$, $dy_i = 0$, for $1 \leq i \leq 6$, the elements dz_j correspond to the relations (I)–(V) with degrees ≤ 16 as follows: $dz_1 = V_1$, $dz_2 = I_{11}$, $dz_3 = II_{11}$, $dz_4 = I_{21}$, $dz_5 = II_{21}$, $dz_6 = III_{21}$, $dz_7 = IV_{21}$, $dz_8 = V_2$, $dz_9 = I_{31}$, $dz_{10} = II_{31}$, $dz_{11} = III_{31}$, $dz_{12} = IV_{31}$, $dz_{13} = I_{22}$, $dz_{14} = II_{22}$; finally:

$$dv_1 = Uz_6 + y_1z_2 - x_1z_1$$
$$dv_2 = Uz_7 - y_1z_1 + x_1z_3.$$

Note that $[dv_1]$ and $[dv_2]$ form a basis of $H^{17}(\Lambda Z^{\leq 15}, d)$. Construct a 16-stage minimal algebra

$$\mathcal{M}_{16} = (\Lambda Z^{\leq 15}, d) \otimes (\Lambda(x_6, y_6), 0)$$

and extend it to a minimal $\mathcal{M}$ by defining inductively:

$$\mathcal{M}_p = \mathcal{M}_{p-1} \otimes_{id} \Lambda_p(H^{p+1}\mathcal{M}_{p-1}), \quad \text{for } p > 16$$

in order to have $H^i\mathcal{M} = 0$ for $i > 17$. Take then V to be a 1-connected finite complex with the property that $\mathcal{M}_V = \mathcal{M}$.

Associating to each generator of H^*MG of degree ≤ 16 the corresponding cohomology class of $\mathcal{M}$ gives rise to an algebra map $h: H^*MG \to H^*V$, due to the fact that all relations of degree ≤ 16 hold by construction in $H^*\mathcal{M}$.

Suppose now that v is G-realizable up to a nonzero factor, which implies that there exists a d.g.a. map $F: (\Lambda Z^{\leq 16}, d) \to \mathcal{M}_{16}$ such that $F(U) = k \cdot U$, with $k \in Q^*$.

Consider the d.g.a. involution $(\Lambda Z^{\leq 16}, d) \xrightarrow{s} (\Lambda Z^{\leq 16}, d)$ defined by: $U \leftrightarrow U$, $x_i \leftrightarrow y_i$ $(1 \leq i \leq 6)$, $z_1 \leftrightarrow z_1$, $z_2 \leftrightarrow z_3$, $z_4 \leftrightarrow z_5$, $z_6 \leftrightarrow z_7$, $z_8 \leftrightarrow z_8$, $z_9 \leftrightarrow z_{10}$, $z_{11} \leftrightarrow z_{12}$, $z_{13} \leftrightarrow z_{14}$, $v_1 \leftrightarrow v_2$. Writing the conditions of commutation with the differentials for F, one finds out that there exist constants a and b with $ab = k^3$ and such that the following hold

$$G(U) = kU, \quad G(x_1) = ax_1, \quad G(y_1) = by_1, \quad G(z_1) = k^3 z_1,$$
$$G(z_2) = a^2 z_2, \quad G(z_3) = b^2 z_3, \quad G(z_6) = k^2 a z_6, \quad G(z_7) = k^2 b z_7$$

either for $G = F$ or for $G = Fs$.

Restrict F and s to 15-stage models. Since $F^*[dv_i] = 0$ in $H^{17}\mathcal{M}_{15}$ $(i = 1, 2)$, we infer that in both cases we must have $G^*[dv_i] = 0$ in $H^{17}\mathcal{M}_{15}$ $(i = 1, 2)$. This would imply that $[dv_i] = 0$ in H^{17} $(\Lambda Z^{\leq 15}, d)$, a contradiction.

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