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Normalizing the cyclic modules of Connes*

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§1. Introduction

1.1. Summary. This paper deals with simplicial modules X which, in each dimension $n \geq 0$, have an extra degeneracy map $s_{n+1}: X_n \to X_{n+1}$ (satisfying the usual identities, except that, in general, $d_0 s_{n+1} \neq s_n d_0 : X_n \to X_n$). We call them *duplicial modules*, because omission of the initial face maps $d_0 : X_{n+1} \to X_n$ $(n \geq 0)$ leaves a cosimplicial module (with the degeneracy maps as coface maps and the remaining face maps as codegeneracy maps). Our key observation (in §3) then is, that one can *normalize* duplicial modules, just like simplicial and cosimplicial ones, and that *the resulting normalization functor is an equivalence between the category of duplicial modules and the category of "duchain complexes"*, i.e. diagrams of modules of the form

$$U_0 \underset{\partial}{\overset{\delta}{\rightleftarrows}} U_1 \underset{\partial}{\overset{\delta}{\rightleftarrows}} U_2 \underset{\partial}{\overset{\delta}{\leftrightarrows}} \cdots$$

in which $\partial^2 = 0$ and $\delta^2 = 0$, but in which the ∂'s and the δ's are otherwise independent.

1.2. Motivation and application. In our investigations of the cyclic objects of Connes [2] we noted that *a cyclic module X is just a duplicial module which satisfies the cyclic identities $(d_0 s_{n+1})^{n+1} = id : X_n \to X_n$ $(n \geq 0)$*. This suggests that a study of duplicial modules could be of use for a better understanding of cyclic modules. Indeed, the normalization result mentioned above (1.1) immediately implies that *the category of cyclic modules is equivalent to a full subcategory of duchain complexes*, and our results (in §6) on the natural self maps of duplicial modules yield a rather simple *characterization of this subcategory in terms of polynomial identities in $\partial\delta$ and $\delta\partial$.*

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1.3. Main results and organization of the paper

(*i*) *The duplicial indexing category* $\mathbf{K}^{op}$. This is a small category with the property: if R is a ring with $1 \neq 0$ and $\mathbf{R}$ denotes the category of (left). R-modules, then *a duplicial R-module is just a factor* $\mathbf{K}^{op} \to \mathbf{R}$. In §2 we first define $\mathbf{K}^{op}$ directly, as an extension of the simplicial indexing category $\boldsymbol{\Delta}^{op}$ and then show, that $\mathbf{K}^{op}$ can be constructed as an *amalgamation of the simplicial indexing category* $\boldsymbol{\Delta}^{op}$ *and the cosimplicial indexing category* $\boldsymbol{\Delta}$, by identifying, in each dimension, the degeneracy operators (in $\boldsymbol{\Delta}^{op}$) with all but one of the coface operators (in $\boldsymbol{\Delta}$) and the codegeneracy operators (in $\boldsymbol{\Delta}$) with all but one of the face operators (in $\boldsymbol{\Delta}^{op}$). We also note that *the cyclic indexing category* $\boldsymbol{\Lambda}^{op}$ *of Connes* [2] *can be obtained from* $\mathbf{K}^{op}$ *by the addition of certain "cyclic" relations.*

(*ii*) *The normalization.* Next (in §3) we observe that one can normalize duplicial modules, just like simplicial and cosimplicial ones, and that *this normalization induces an equivalence between the category* $\mathbf{R}^{\mathbf{K}^{op}}$ *of duplicial R-modules and the category* $\mathbf{R}(\partial, \delta)$ *of* (*see* 1.1) *duchain complexes over R.*

(*iii*) *Homotopy theories of duplicial modules and duchain complexes.* If one defines weak equivalences between duplicial modules as maps which induce isomorphisms on the homotopy groups of the underlying (see 1.1) simplicial modules as well as on the cohomotopy groups of the underlying (see 1.1) cosimplicial modules, then (§4) *the resulting homotopy theory is equivalent to a more familiar homotopy theory of the maps $A' \to A$ of differential graded modules, for which A' has trivial homology in positive dimensions and A has trivial homology in negative dimensions.* Of course (ii), a similar result holds for duchain complexes.

(*iv*) *An Eilenberg–Zilber theorem.* An application (in §5) is an Eilenberg–Zilber theorem for duplicial modules: the normalization of the (dimensionwise) tensor product of two duplicial modules is, as a duchain complex, naturally weakly equivalent to the tensor product of their normalizations.

(*v*) *The ring of natural self maps.* The normalization (ii) induces an isomorphism between the ring $\operatorname{End} \mathbf{R}^{\mathbf{K}^{op}}$ of the natural self maps of duplicial R-modules and the ring $\operatorname{End} \mathbf{R}(\partial, \delta)$ of the natural self maps of duchain complexes over R. In §6 we show that these rings are isomorphic to *the ring ER of sequences of polynomials in one variable with coefficients in the center of R and with the same constant term,* and we then use this result to give (as promised in 1.2) a simple *characterization of the full subcategory of* $\mathbf{R}(\partial, \delta)$ *which* (*under the normalization functor*) *is equivalent to the category of cyclic R-modules.*

(*vi*) *Homotopy theories.* In the appendix (§7) we make clear what exactly we mean by *the homotopy theory* of a category with respect to a subcategory (of weak equivalences), and when two such homotopy theories will be called *equivalent.*

§2. The duplicial indexing category $\mathbf{K}^{op}$.

After defining the *duplicial indexing category* $\mathbf{K}^{op}$ as an extension of the simplicial indexing category Δ^{op}, we show that $\mathbf{K}^{op}$ admits the *self dual presentation in terms of Δ^{op} and Δ* mentioned in 1.3(i). We also note that *the cyclic indexing category Λ^{op} of Connes* [2] *can be obtained from $\mathbf{K}^{op}$ by the addition of certain "cyclic" relations,* which are closely related to a curious natural transformation from the identity functor of $\mathbf{K}^{op}$ to itself.

Recall that the *simplicial indexing category Δ^{op}* is the category with objects $\mathbf{0}, \mathbf{1}, \mathbf{2}, \ldots$ and generating maps

$$d_i : \mathbf{n} \to \mathbf{n-1} \qquad 0 \le i \le n, \qquad n > 0$$

$$s_i : \mathbf{n} \to \mathbf{n+1} \qquad 0 \le i \le n$$

subject to the usual relations [7, p. 1], and that dually the *cosimplicial indexing category Δ* is the category with objects $\mathbf{0}, \mathbf{1}, \mathbf{2}, \ldots$ and generating maps

$$d^i : \mathbf{n-1} \to \mathbf{n} \qquad 0 \le i \le n, \qquad n > 0$$

$$s^i : \mathbf{n+1} \to \mathbf{n} \qquad 0 \le i \le \mathbf{n}$$

subject to the dual relations. Then one can define as follows:

2.1. The duplicial indexing category $\mathbf{K}^{op}$. This is the category with objects $\mathbf{0}, \mathbf{1}, \mathbf{2}, \ldots$ and generating maps

$$d_i : \mathbf{n} \to \mathbf{n-1} \qquad 0 \le i \le n$$

$$s_i : \mathbf{n} \to \mathbf{n+1} \qquad 0 \le i \le n+1$$

subject to the relations

$$d_i d_j = d_{j-1} d_i \quad \text{and} \quad s_j s_i = s_i s_{j-1} \qquad 0 < j-1$$

$$\begin{aligned} d_i s_j &= s_{j-1} d_i : \mathbf{n} \to \mathbf{n} && 0 < j-1 \le n \\ &= id && -1 \le j-i \le 0 \\ &= s_j d_{i-1} && j-i < -1 \end{aligned}$$

Note that $d_0 s_{n+1} \ne s_n d_0 : \mathbf{n} \to \mathbf{n}$.

Clearly *the obvious functor* $j:\mathbf{\Delta}^{\mathrm{op}} \to \mathbf{K}^{\mathrm{op}}$ *is* 1-1 *and so is dually the functor* $k:\mathbf{\Delta} \to \mathbf{K}^{\mathrm{op}}$ *given by*

$$(d^i \cdot \mathbf{n-1} \to \mathbf{n}) \to (s_{n-1}:\mathbf{n-1} \to \mathbf{n})$$

$$(s^i:\mathbf{n} \to \mathbf{n-1}) \to (d_{n-1}:\mathbf{n} \to \mathbf{n-1})$$

There is also an *isomorphism* $m:\mathbf{K}^{\mathrm{op}} \approx \mathbf{K}$ (the opposite of $\mathbf{K}^{\mathrm{op}}$) given by

$$(d_i:\mathbf{n} \to \mathbf{n-1}) \to (s_{n-i}:\mathbf{n-1} \to \mathbf{n})$$

$$(s_i:\mathbf{n-1} \to \mathbf{n}) \to (d_{n-i}:\mathbf{n} \to \mathbf{n-1})$$

which has the property that $mj = k^{\mathrm{op}}$ *and* $mk = j^{\mathrm{op}}$.

2.2. *Remark.* One often identifies the category $\mathbf{\Delta}$ with the category of the weakly monotone functions between the finite ordered sets of integers $(0, \ldots, n)$. In a similar manner *the category* $\mathbf{K}$ *can be identified with a category* $\mathbf{P}$ *of weakly monotone and "periodic" functions between copies of the ordered set of the non-negative integers* $N = (0, 1, 2, \ldots)$. More precisely:

Let P be the category whose objects consist of one copy p_n of N for each integer $n \geq 0$ and which has as maps $p_n \to p_{n'}$ the weakly monotone functions f which are *periodic*, i.e. $f(j+n+1) = f(j) + n' + 1$ for all $j \in N$. One then readily verifies that $\mathbf{P}$ is indeed isomorphic to $\mathbf{K}$ (the opposite of $\mathbf{K}^{\mathrm{op}}$); the opposite of the map $d_i:\mathbf{n} \to \mathbf{n-1} \in \mathbf{K}^{\mathrm{op}}$ corresponds to the function $p_{n-1} \to p_n$ given by $j \to j$ for $j < i$ and $j \to j+1$ for $j \geq i$, and the opposite of the map $s_i:\mathbf{n-1} \to \mathbf{n} \in \mathbf{K}^{\mathrm{op}}$ corresponds to the function $p_n \to p_{n-1}$ given by $j \to j$ for $j \leq i$ and $j \to j-1$ for $j > i$. This implies that the opposite of the map $d_0 s_{n+1}:\mathbf{n} \to \mathbf{n}$ corresponds to the function $p_n \to p_n$ given by $j \to j+1$ for all j.

It is also not difficult to see that the copy of $\mathbf{\Delta}$ (resp. $\mathbf{\Delta}^{\mathrm{op}}$) contained in $\mathbf{K}$ corresponds to the subcategory of $\mathbf{P}$ which consists of the functions $f:p_n \to p_{n'}$ such that $f(n) \leq n'$ (resp. $f(0) = 0$).

An immediate consequence of this remark is the existence of

2.3. A curious natural transformation. *The functor* v *which assigns to every object* $\mathbf{n} \in \mathbf{K}^{\mathrm{op}}$ *the map*

$$v\mathbf{n} = (d_0 s_{n+1})^{n+1}:\mathbf{n} \to \mathbf{n} \in \mathbf{K}^{\mathrm{op}}$$

is a natural transformation from the identity of $\mathbf{K}^{\mathrm{op}}$ *to itself.*

Moreover one has

2.4. PROPOSITION. *The cyclic indexing category* Λ^{op} [6] *can be obtained from* $\mathbf{K}^{op}$ *by adding the "cyclic" relations*

$$v\mathbf{n} = (d_0 s_{n+1})^{n+1} = id : \mathbf{n} \to \mathbf{n} \qquad n \geq 0$$

Proof. This is a straightforward calculation in which one takes $t_{n+1} = (d_0 s_{n+1})^n = (d_0 s_{n+1})^{-1} : \mathbf{n} \to \mathbf{n}$, or equivalently $s_n = t_{n+1}^{-1} s_0 : \mathbf{n-1} \to \mathbf{n}$.

§3. The normalization

Let $\mathbf{R}$ be the category of (left) modules over a ring R with $1 \neq 0$, let $\mathbf{R}^{\mathbf{K}^{op}}$ denote the category of duplicial R-modules (the objects are the functors $\mathbf{K}^{op} \to \mathbf{R}$ and the maps are the natural transformations between them) and let $\mathbf{R}(\partial, \delta)$ be the category of duchain complexes over R (1.1). The key result of this paper then is that (3.3) *one can normalize duplicial* R*-modules*, just like simplicial and cosimplicial ones, and that (3.5) *the resulting functor* $N : \mathbf{R}^{\mathbf{K}^{op}} \to \mathbf{R}(\partial, \delta)$ *is an equivalence of categories.*

We start with a brief review of the relevant facts in the simplicial and cosimplicial cases [7, §22].

3.1. The simplicial case. Let $\mathbf{R}(\partial)$ denote the category of chain complexes over R, i.e. diagrams in $\mathbf{R}$

$$U_0 \xleftarrow{\partial} U_1 \xleftarrow{\partial} U_2 \xleftarrow{\partial} \cdots \quad \text{with} \quad \partial^2 = 0$$

and consider, for every simplicial R-modules X, its *total complex* $TX \in \mathbf{R}(\partial)$ given by $T_n X = X_n$ $(n \geq 0)$ and

$$\partial x = \sum_{i=0}^{n} (-1)^i d_i x \qquad x \in X_n$$

as well as two subcomplex of TX, the *normalized complex* NX and the *bulk complex* BX, given by

$$N_n X = X_n \cap \ker d_1 \cap \cdots \cap \ker d_n \qquad n \geq 0$$

$$B_n X = X_n \cap (\operatorname{im} s_0 \cup \cdots \cup \operatorname{im} s_{n-1}) \qquad n \geq 0.$$

Conversely, for $U \in \mathbf{R}(\partial)$, its *denormalization* $N'U$ is the simplicial R-module which, in dimension n, consists of the direct sum

$$N'_n U = U_n \oplus \left(\sum_I s_{i_k} \cdots s_{i_1} U_{n-k} \right)$$

where each $s_{i_k} \cdots s_{i_1} U_{n-k}$ denotes a copy of U_{n-k} and the direct sum is taken over the set I of all non-empty sequences of integers $(i_k, \ldots, i_1)$ such that $n > i_k > \cdots > i_1 \geq 0$; the degeneracy operators are the obvious ones and the face operators are determined by the requirement that $d_0 x = \partial x$ and $d_i x = 0$ $(0 \leq i \leq n)$ for all $x \in U_n \subset N'_n U$. Then,

(i) *BX has trivial homology in all dimensions,*

(ii) *$TX = BX \oplus NX$, and*

(iii) *the functor $N : \mathbf{R}^{\Delta^{\mathrm{op}}} \to \mathbf{R}(\partial)$ is an equivalence of categories with as inverse the functor $N' : \mathbf{R}(\partial) \to \mathbf{R}^{\Delta^{\mathrm{op}}}$*, i.e. the compositions $N'N$ and NN' are naturally equivalent to the identity functors of $\mathbf{R}^{\Delta^{\mathrm{op}}}$ and $\mathbf{R}(\partial)$ respectively.

Dually one has

3.2. The cosimplicial case. Let $\mathbf{R}(\delta)$ denote the category of cochain complexes over R, i.e. diagrams in $\mathbf{R}$

$$U_0 \xrightarrow{\delta} U_1 \xrightarrow{\delta} U_2 \xrightarrow{\delta} \cdots \quad \text{with} \quad \delta^2 = 0$$

and consider, for every cosimplicial R-module X, its *total complex* $TX \in \mathbf{R}(\delta)$ given by $T_n X = X_n$ $(n \geq 0)$ and

$$\delta x = \sum_{i=0}^{n+1} (-1)^i d^i x \qquad x \in X_n,$$

as well as two subcomplexes of TX, the *normalized complex* NX and the *bulk complex* BX, *given by*

$$N_n X = X_n \cap \ker s^0 \cap \cdots \cap \ker s^{n-1} \qquad n \geq 0$$

$$B_n X = X_n \cap (\operatorname{im} d^1 \cap \cdots \cap \operatorname{im} d^n) \qquad n \geq 0.$$

Conversely, for $U \in \mathbf{R}(\delta)$, its *denormalization* $N'U$ is the cosimplicial R-module which, in dimension n, consists of the direct sum

$$N'U = U_n \oplus \left(\sum_J d^{i_k} \cdots d^{i_1} U_{n-k} \right)$$

where each $d^{j_k}\cdots d^{j_1}U_{n-k}$ denotes a copy of U_{n-k} and the direct sum is taken over the set J of all non-empty sequences of integers $(j_k,\ldots,j_l)$ such that $n\ge j_k>\cdots>j_l>0$; the coface operators d^i $(i>0)$ are the obvious ones and d^0 and the codegeneracy operators are determined by the requirement that, for all $x\in U_n\subset N'_nU$, one has $s^ix=0$ $(0\le i\le n)$ and

$$d^0x=\delta x-\sum_{i=1}^{n+1}(-1)^id^ix.$$

Then again:

(i) *BX has trivial cohomology in all dimensions,*

(ii) *$TX=BX\oplus NX$, and*

(iii) *the functor $N:\mathbf{R}^{\Delta}\to\mathbf{R}(\delta)$ is an equivalence of categories with as inverse the functor $N':\mathbf{R}(\delta)\to\mathbf{R}^{\Delta}$.*

Combining 3.1 and 3.2 one finally gets

3.3. The duplicial case. For a duplicial R-module X, one can consider its *total complex* $TX\in\mathbf{R}(\partial,\delta)$ (1.2(ii)) given by $T_nX=X_n$ $(n\ge0)$ and

$$\partial x=\sum_{i=0}^{n}(-1)^id_ix\quad\text{and}\quad\delta x=\sum_{i=0}^{n+1}(-1)^is_{n+1-i}x\qquad x\in X_n$$

as well as two subcomplexes of TX, the *normalized complex* NX and the bulk complex BX, given by

$$N_nX=X_n\cap\ker d_1\cap\cdots\cap\ker d_n\qquad n\ge0$$

$$B_nX=X_n\cap(\operatorname{im}s_0\cup\cdots\cup\operatorname{im}s_{n-1})\qquad n\ge0.$$

Conversely, for $U\in\mathbf{R}(\partial,\delta)$, its *denormalization* $N'U$ is the duplicial R-module which, in dimension n, consists of the direct sum

$$N'_nU=U_n\oplus\Big(\sum_I s_{i_k}\cdots s_{i_1}U_{n-k}\Big)$$

where each $s_{i_k}\cdots s_{i_1}U_{n-k}$ denotes a copy of U_{n-k} and I is as in 3.1; all but the last degeneracy operator are the obvious ones, while the last degeneracy operator and the face operators are determined by the requirement that, for all $x\in U_n\subset N'_nU$, $d_0x=\partial x$, $d_ix=0$ $(0\le i\le n)$ and

$$s_{n+1}x=\delta x-\sum_{i=1}^{n+1}(-1)^is_{n+1-i}x.$$

A lengthy but straightforward calculation now yields:

3.4. THEOREM. *Let* $X \in \mathbf{R}^{\mathbf{K}^{op}}$. *Then*
(i) $H_*(BX) = H^*(BX) = 0$, *and*
(ii) $TX = BX \oplus NX$.

3.5. THEOREM. *The functor* $N : \mathbf{R}^{\mathbf{K}^{op}} \to \mathbf{R}(\partial, \delta)$ *is an equivalence of categories with as inverse the functor* $N' : \mathbf{R}(\partial, \delta) \to \mathbf{R}^{\mathbf{K}^{op}}$.

3.6. *Remark.* Another way to verify 3.5 is to note that there is a push out diagram of categories

$$\begin{array}{ccc} \boldsymbol{\Delta} \cap \boldsymbol{\Delta}^{op} & \longrightarrow & \boldsymbol{\Delta}^{op} \\ \downarrow & & \downarrow \\ \boldsymbol{\Delta} & \longrightarrow & \mathbf{K}^{op} \end{array}$$

where $\boldsymbol{\Delta} \cap \boldsymbol{\Delta}^{op}$ denotes the subcategory of $\mathbf{K}^{op}$ generated by the maps $d_i : \mathbf{n} \to \mathbf{n-1}$ $(1 \le i \le n)$ and $s_i : \mathbf{n} \to \mathbf{n+1}$ $(0 \le i \le n)$. The appropriate normalization functor then yields an equivalence of categories $\mathbf{R}^{\boldsymbol{\Delta} \cap \boldsymbol{\Delta}^{op}} \to$ (**graded** R-**modules**) and therefore, to give an object of $\mathbf{R}^{\mathbf{K}^{op}}$ amounts, after normalization, to giving a chain complex A over R and a cochain complex B over R such that A and B agree as graded R-modules.

§4. Homotopy theories of duplicial modules and duchain complexes

We show that *the homotopy theory of duplicial modules*, which takes into account both the underlying simplicial and cosimplicial structures, *is equivalent to the homotopy theory of certain maps of differential graded modules* (1.3(iii)). Of course a similar result holds for duchain complexes. What we mean by "homotopy theories" and when two such homotopy theories will be called "equivalent" will be made precise in the appendix (§7).

We start with some preliminaries (4.1 and 4.2).

4.1. Differential graded modules

(i) We denote by $\mathbf{dgR}$ the category of differential graded (left) R-modules (in which the differentials are of degree -1) and by $\mathbf{dgR}_-$ and $\mathbf{dgR}_+ \subset \mathbf{dgR}$ the full subcategories spanned by the objects with trivial homology in positive and negative dimensions respectively.

(ii) A map $A \to B \in \mathbf{dgR}$ is called a *weak equivalence* if it induces isomorphisms $H_iA \approx H_iB$ on the homology groups. The category $\mathbf{dgR}$ then admits a *closed model category structure* in the sense of Quillen [3, §3] with these weak equivalences and with as fibrations the maps which are onto.

(iii) Two maps $f, g : A \to B \in \mathbf{dgR}$ are called *chain homotopic* if there exists a *chain homotopy* $D : f \sim g$, i.e. a sequence of maps $D_i : A_i \to B_{i+1} \in \mathbf{R}$ such that $\partial D_i + D_{i-1}\partial = f_i - g_i$ for all i. Clearly *chain homotopic maps are homotopic*, i.e. (7.7), they have the same image in the localization of $\mathbf{dgR}$ with respect to the weak equivalences.

4.2. Maps of differential graded modules

(i) We denote by $(\mathbf{dgR}, \mathbf{dgR})$ the category of maps in $\mathbf{dgR}$, (i.e. the category which has as objects the maps $a : A' \to A \in \mathbf{dgR}$ and as maps $(a : A' \to A) \to (b : B' \to B)$ the pairs of maps $f' : A' \to B'$, $f : A \to B \in \mathbf{dgR}$ such that $bf' = fa$) and by $(\mathbf{dgR}_-, \mathbf{dgR}_+) \subset (\mathbf{dgR}, \mathbf{dgR})$ the full subcategory spanned by the maps $A' \to A \in \mathbf{dgR}$ with $A' \in \mathbf{dgR}_-$ and $A \in \mathbf{dgR}_+$.

(ii) A map $(f', f) : (a : A' \to A) \to (b : B' \to B) \in (\mathbf{dgR}, \mathbf{dgR})$ will be called a *weak equivalence* if $f' : A' \to B'$ and $f : A \to B$ are weak equivalences in $\mathbf{dgR}$. The category $(\mathbf{dgR}, \mathbf{dgR})$ then admits a *closed model category structure* with these weak equivalences and with as fibrations the maps (f', f) for which both f' and f are fibrations in $\mathbf{dgR}$.

(iii) Two maps (f', f), $(g', g) : (a : A' \to A) \to (b : B' \to B)$ in $(\mathbf{dgR}, \mathbf{dgR})$ are called *chain homotopic* if there exists a *chain homotopy* $(D', D, E) : (f', f) \sim (g', g)$, i.e. sequences of maps $D'_i : A'_i \to B'_{i+1}$, $D_i : A_I \to B_{i+1}$ and $E_i : A'_i \to B_{i+2} \in \mathbf{R}$ such that $\partial D'_i + D'_{i-1}\partial = f'_i - g'_i$, $\partial D_i + D_{i-1}\partial = f_i - g_i$ and $\partial E_i - E_{i-1}\partial = b_{i+1}D'_i - D_i a_i$ for all i. Again one readily verifies that *chain homotopic maps in* $(\mathbf{dgR}, \mathbf{dgR})$ *are homotopic* (7.7).

We also have to define

4.3. Weak equivalences in $\mathbf{R}^{\mathbf{K}^{op}}$ and in $\mathbf{R}(\partial, \delta)$. A map $X \to Y \in \mathbf{R}^{\mathbf{K}^{op}}$ will be called a *weak equivalence* if it induces isomorphisms $\pi_i j^* X \approx \pi_i j^* Y$ $(i > 0)$ on the homotopy groups of the underlying simplicial modules as well as isomorphisms $\pi^i k^* X \approx \pi^i k^* Y$ $(i \geq 0)$ on the cohomotopy groups [1, Ch. X, §7] of the underlying cosimplicial modules. Similarly a map $U \to V \in \mathbf{R}(\partial, \delta)$ will be called a *weak equivalence* if it induces isomorphisms $H_iU \approx H_iV$ and $H^iU \approx H^iV$ $(i \geq 0)$ on the homology and cohomology groups.

Theorems 3.4 and 3.5 now immediately imply

4.4. THEOREM. *The functors* $T, N : \mathbf{R}^{\mathbf{K}^{op}} \to \mathbf{R}(\partial, \delta)$ *of* 3.3 *preserve weak equivalences and induce equivalences between the homotopy theory of* $\mathbf{R}^{\mathbf{K}^{op}}$ *and the* *homotopy theory of* $\mathbf{R}(\partial, \delta)$ (with respect to the weak equivalence of 4.3, of course).

The main result of this section now relates those homotopy theories to that of $(\mathbf{dgR}_-, \mathbf{dgR}_+)$. If $s : \mathbf{R}(\partial, \delta) \to (\mathbf{dgR}_-, \mathbf{dgR}_+)$ denotes the *splitting functor* which sends an object $U \in \mathbf{R}(\partial, \delta)$ to the map $K' \to K \in \mathbf{dgR}$ with $K'_{-n} = K_n = U_n$ for $n \geq 0$ and $K'_{-n} = K_n = 0$ for $n < 0$, then one has

4.5. THEOREM. *The splitting functor* $s : \mathbf{R}(\partial, \delta) \to (\mathbf{dgR}_-, \mathbf{dgR}_+)$ *preserves weak equivalences and induces an equivalence between the homotopy theories of* $\mathbf{R}(\partial, \delta)$ *and* $(\mathbf{dgR}_-, \mathbf{dgR}_+)$ (with respect to the weak equivalences of 4.3 and 4.2 respectively).

As *a map* $U \to V \in \mathbf{R}(\partial, \delta)$ *is a weak equivalence iff the induced map* $sU \to sV \in (\mathbf{dgR}_-, \mathbf{dgR}_+)$ *is so*, an immediate consequence is the following result on

4.6. Duchain homotopies. Call two maps $f, g : U \to V \in \mathbf{R}(\partial, \delta)$ *duchain homotopic* if there exists a *duchain homotopy* $(D', D, E) : f \sim g$, i.e. maps $D'_i : U_i \to V_{i-1}$ $(i > 0)$, $D_i : U_i \to V_{i+1}$ $(i \geq 0)$ and $E_i : U_i \to V_{2-i}$ $(0 \leq i \leq 2)$ in $\mathbf{R}$ such that $\delta D'_i + D'_{i+1}\delta = \partial D_i + D_{i-1}\partial = f_i - g_i$ for $i > 0$, $D'_i\delta = 0$, $\partial D_0 = 0$, $\partial E_1 - E_2\delta = D'_1$ and $\partial E_0 - E_1\delta = 0$. Then *duchain homotopic maps are homotopic* (7.7).

Another consequence of 4.4 and 4.5, which can also easily be obtained directly is the

4.7. COROLLARY ON ONE-SIDED HOMOTOPY THEORIES. *The homotopy theory of* $\mathbf{R}^{\mathbf{K}^{op}}$ *with respect to the maps which induce isomorphisms on the homotopy* (resp. *cohomotopy*) *groups of the underlying simplicial* (resp. *cosimplicial*) *modules is equivalent to the homotopy theory of the category* $\mathbf{dgR}_+$ (resp. $\mathbf{dgR}_-$) (with weak equivalences as in 4.1).

Proof of 4.5. Let $\mathbf{C} \subset (\mathbf{dgR}_-, \mathbf{dgR}_+)$ be the full subcategory spanned by the maps $K' \to K \in \mathbf{dgR}$ such that $K'_0 \to K_0$ is 1-1 and $K'_i = K_{-i} = 0$ for $i > 0$. The restriction $s' : \mathbf{R}(\partial, \delta) \to \mathbf{C}$ of s then has a left adjoint $r : \mathbf{C} \to \mathbf{R}(\partial, \delta)$ with the property that, for every object $(K' \to K) \in \mathbf{C}$, $H_* r(K' \to K)$ is naturally isomorphic to $H_* K$, while $H^* r(K' \to K)$ is naturally isomorphic to $H_* K'$. As r preserves push outs and $K' \to K$ fits into a push out diagram

$$\begin{array}{ccc} (O \to M) & \longrightarrow & (O \to K) \\ \downarrow & & \downarrow \\ (K' \to M) & \longrightarrow & (K' \to K) \end{array}$$

where 0 denotes the trivial object and M is given by $M_0 = K'_0$ and $M_i = 0$ for $i \neq 0$, one only has to verify this property for the other three corners, which is straightforward. Using this property one now readily shows that r and s' both preserve weak equivalences and that both adjunction maps are weak equivalences.

On the other hand, given an object $(A' \to A) \in (\mathbf{dgR}_-, \mathbf{dgR}_+)$, one can, in a functorial manner, construct a commutative diagram in **dgR**

$$\begin{tikzcd} A' \arrow[r] \arrow[d] & C' \arrow[d] & E' \arrow[l] \arrow[d] & G' \arrow[l, "\approx"'] \arrow[r] \arrow[d] & K' \arrow[d] \\ A \arrow[r, "\approx"] & C & E \arrow[l] & G \arrow[l] \arrow[r] & K \end{tikzcd}$$

in which

(i) the horizontal maps are weak equivalences,
(ii) the maps $A \to C$ and $G' \to E'$ are isomorphisms,
(iii) the map $C' \to C$ is onto,
(iv) the second square is a pull back and $E_i = 0$ for $i < 0$,
(v) $G_i = 0$ for $i < 0$ and the map $G'_i \to G_i$ is 1-1 for $i \geq 0$, and
(vi) $K'_0 = 0$ for $i > 0$,

and hence $(K' \to K) \in \mathbf{C}$.

The theorem now follows by combining the above results with 4.2, 7.5 and 7.6.

We end with the construction of

4.8. A closed model category structure for $\mathbf{R}(\partial, \delta)$. *The category* $\mathbf{R}(\partial, \delta)$ *admits a closed model category structure in which the weak equivalences are as in* 4.3 *and in which a map is a fibration iff it is onto in dimensions* >0.

4.9. *Remark.* The normalization functor N (3.3) of course induces a corresponding *closed model category structure on* $\mathbf{R}^{\mathbf{K}^{op}}$ with weak equivalences as in 4.3.

Proof of 4.8. Let 0 denote the trivial object and, for every integer $n \geq 0$, let D^n stand for the (free) object with one generator x_n the dimension n, and let S^n_∂ (resp. S^n_δ) be the object with one generator y_n (resp. y'_n) in dimension n and one relation $\partial y_n = 0$ (resp. $\delta y'_n = 0$). Then one readily verifies that a map in $\mathbf{R}(\partial, \delta)$ is a fibration (resp. a trivial fibration) iff it has the right lifting property with respect to the maps $0 \to D^n$ with $n > 0$ (resp. the maps $S^{n-1}_\partial \to D^n$ given by $y_{n-1} \to \partial x_n$ $(n > 0)$, the maps $S^{n+1}_\delta \to D^n$ given by $y'_{n+1} \to \delta x_n$ $(n \geq 0)$ and the map $0 \to S^0_\partial = D^0$). The rest of the proof then is as in [3, 3.1].

§5. An Eilenberg–Zilber theorem

An application is the following Eilenberg-Zilber theorem for duplicial modules.

5.1. THEOREM. *Let* $X, Y \in \mathbf{R}^{\mathbf{K}^{op}}$. *Then the duchain complexes* $N(X\otimes Y)$ *and* $NX\otimes NY$ (where N is as in 3.3 and $\otimes$ denotes the dimensionwise tensor product in $\mathbf{R}^{\mathbf{K}^{op}}$ and the obvious graded tensor product in $\mathbf{R}(\partial, \delta)$) *are naturally weakly equivalent* (in the sense of 4.3).

To prove this we first recall simplicial and cosimplicial versions of the Eilenberg–Zilber theorem [7, p. 129].

5.2. The simplicial case. Given two simplicial R-modules X and Y, let $X\otimes Y$ be their dimensionwise tensor product (which is often denoted by $X\times Y$) and consider the maps

$$T(X\otimes Y)\xrightarrow{f} TX\otimes TY \quad \text{and} \quad TX\otimes TY\xrightarrow{g} T(X\otimes Y)\in \mathbf{R}(\partial)$$

given by the formulas

$$f(x_n\otimes y_n)=\sum_{i=0}^{n}(d_{i+1}\cdots d_nx_n)\otimes(d_0\cdots d_{i-1}y_n) \qquad x_n\in X_n, \qquad y_n\in Y_n.$$

$$g(x_p\otimes y_q)=\sum_{(a,b)}(-1)^{e(a)}(s_{b_q}\cdots s_{b_1}x_p)\otimes(s_{a_p}\cdots s_{a_1}y_q) \qquad x_p\in X_p, \qquad y_q\in Y_q,$$

where $e(a)=\sum_{i=1}^{p}(a_i+1-i)$ and the sum $\sum_{(a,b)}$ is taken over all (p, q)-shuffles (a, b). Then *the compositions gf and fg are the identity in dimension* 0 *and there are natural chain homotopies* $D: gf\sim id$ *and* $\bar{D}: fg\sim id$ *for which* D_0 *and* $\bar{D}_0$ *are the zero maps.*

Dually one has

5.3. The cosimplicial case. Given two cosimplicial R-modules X and Y, let $X\otimes Y$ denote their dimensionwise tensor product and consider the maps

$$T(X\otimes Y)\xrightarrow{f} TX\otimes TY \quad \text{and} \quad TX\otimes TY\xrightarrow{g} T(X\otimes Y)\in \mathbf{R}(\delta)$$

given by the formulas (see 5.2)

$$f(x_n\otimes y_n)=\sum_{p+q=n}\sum_{(a,b)}(-1)^{e(a)}(s^{b_1}\cdots s^{b_q}x_n)\otimes(s^{a_1}\cdots s^{a_p}y_n)$$

$$g(x_p\otimes y_q)=(d^n\cdots d^{p+1}x_p)\otimes(d^{p-1}\cdots d^0y_q).$$

Then *the compositions* gf *and* fg *are the identity in dimension* 0 *and there are natural cochain homotopies* $D': gf \sim id$ *and* $\bar{D}': fg \sim id$ *for which* D'_1 *and* $\bar{D}'_1$ *are the zero maps.*

In view of 4.2 and 4.5, combination of 5.2 and 5.3 yields

5.4. A duplicial version. Given $X, Y \in \mathbf{R}^{\mathbf{K}^{op}}$, consider the maps

$$sT(X\otimes Y) \xrightarrow{f} sTX\otimes sTY \quad \text{and} \quad sTX\otimes sTY \xrightarrow{g} sT(X\otimes Y) \in (\mathbf{dgR}_-, \mathbf{dgR}_+)$$

which, in dimensions ≥ 0, are given by the formulas (5.2)

$$f(x_n\otimes y_n) = \sum_{i=0}^{n} (d_{i+1}\cdots d_n x_n)\otimes(d_0\cdots d_{i-1}y_n)$$

$$g(x_p\otimes y_q) = \sum_{(a,b)} (-1)^{e(a)}(s_{b_q}\cdots s_{b_1}x_p)\otimes(s_{a_p}\cdots s_{a_1}y_q)$$

and, in dimensions ≤ 0, by (5.3)

$$f(x_n\otimes y_n) = \sum_{p+q=n}\sum_{(a,b)} (-1)^{e(a)}(d_{n-b_q}\cdots d_{n-b_1}x_n)\otimes(d_{n-a_p}\cdots d_{n-a_1}y_n)$$

$$g(x_p\otimes y_q) = (s_{q-1}\cdots s_0x_p)\otimes(s_n\cdots s_{q+1}y_q).$$

Then *the compositions* gf *and* fg *are the identity in dimension* 0 *and there are natural chain homotopies* $(D', D, E): gf \sim id$ *and* $(\bar{D}', \bar{D}, \bar{E}): fg \sim id$ *for which* D'_1, $\bar{D}'_1$, D_0, $\bar{D}_0$, *the* E_i *and the* $\bar{E}_i$ *are the zero maps.*

It remains to give a

5.5. Proof of 5.1. Using 3.4, 3.5, 4.2 and 4.5, Theorem 5.1 now follows readily from 5.4 and the fact that $sTX\otimes sTY \approx s(TX\otimes TY)$.

§6. The ring of natural self maps

As another application of the normalization of §3 we

(i) show that the ring End $\mathbf{R}^{\mathbf{K}^{op}}$ of the natural self maps of duplicial R-modules is isomorphic to *the ring* ER *of sequences* $(f_0, f_1, f_2, \ldots)$ *of polynomials in one variable with coefficients in the center of* R *and with the same constant coefficient* (in which the multiplication is termwise multiplication of the polynomials),

(ii) compute the element of ER which corresponds to the natural self map induced by the curious natural transformation of 2.2, and

(iii) use this to *characterize the full subcategory of* $\mathbf{R}(\partial, \delta)$ *which (under the normalization functor) is equivalent to the category of the cyclic* R*-modules* of Connes [2].

First we observe that Theorem 3.5 immediately implies

6.1. PROPOSITION. *The normalization functor* $N:\mathbf{R}^{\mathbf{K}^{op}} \to \mathbf{R}(\partial, \delta)$ *induces an isomorphism between the ring* $\operatorname{End}\mathbf{R}^{\mathbf{K}^{op}}$ *of the natural self maps of duplicial* R*-modules and the ring* $\operatorname{End}\mathbf{R}(\partial, \delta)$ *of the natural self maps of duchain complexes over* R.

Next we consider

6.2. The ring ER. It is convenient to consider ER as the ring of sequences $(f_{-1}, f_0, f_1, \ldots)$ of polynomials in one variable z with coefficients in the center of R such that

(i) the f_i $(i \geq -1)$ have the same constant term, and

(ii) f_{-1} consists of the constant term only.

A simple calculation now yields that, given an element $f = (f_{-1}, f_0, f_1, \ldots) \in ER$, one can construct an element $\varphi f \in \operatorname{End}\mathbf{R}(\partial, \delta)$ which, to an object $U \in \mathbf{R}(\partial, \delta)$, assigns the self map $\varphi f: U \to U \in \mathbf{R}(\partial, \delta)$ given by the formula

$$(\varphi f)x = f_n(\partial\delta)x + (f_{n-1} - f_{-1})(\partial\delta)x \qquad x \in U_n, \qquad n \geq 0.$$

Moreover

6.3. THEOREM. *The function* $\varphi: ER \to \operatorname{End}\mathbf{R}(\partial, \delta)$ *is an isomorphism of rings.*

Proof. Let $e \in \operatorname{End}\mathbf{R}(\partial, \delta)$ have components e_n $(n \geq 0)$. By checking the action of e on the free duchain complex with one generator in dimension n, one sees that

$$e_n = f_n(\partial\delta) + g_n(\delta\partial)$$

for some uniquely determined polynomials f_n and g_n with coefficients in the center of R and with g_n having zero constant term. It then follows easily from the fact that $e_n\partial = \partial e_{n+1}$ and $e_{n+1}\delta = \delta e_n$ $(n \geq 0)$ that

(i) the f_n $(n \geq 0)$ have the same constant term which we denote by f_{-1}, and

(ii) $g_n = f_n - f_{-1}$ $(n \geq 0)$,

i.e. $e = \varphi(f_{-1}, f_0, f_1, \ldots)$.

Furthermore

6.4. THEOREM. *Let* $f=(f_{-1}, f_0, f_1, \ldots)\in ER$ *and let* $U\in \mathbf{R}(\partial, \delta)$. *Then the map* $\varphi f: U\to U\in \mathbf{R}(\partial,\delta)$ *is naturally duchain homotopic* (4.6) *to "multiplication by* f_{-1}".

Proof. For every integer $n\geq 0$, let $\bar f_n$ be defined by $z\bar f_n = f_n - f_{-1}$. The desired duchain homotopy (D', D, E) then is given by $D'_{n+1}=\partial \bar f_n(\delta\partial)=\bar f_n(\partial\delta)\partial$ and $D_n=\bar f_n(\delta\partial)\delta=\delta\bar f_n(\partial\delta)$ for $n\geq 0$, $E_0=E=0$ and $E_1=\bar f_0(\delta\partial)$.

Next we prove

6.5. PROPOSITION. *Let* $X\in \mathbf{R}^{\mathbf{K}^{\mathrm{op}}}$ *and let* $u: X\to X\in\mathbf{R}^{\mathbf{K}^{\mathrm{op}}}$ *be the natural self map given by* (see 2.3)

$$ux=(d_0 s_{n+1})^{n+1}x \qquad x\in X_n, \qquad n\geq 0.$$

Then $Nu=\varphi(f_{-1}, f_0, f_1, \ldots): NX\to NX\in \mathbf{R}(\partial,\delta)$, *where*

$$f_n=(1+(-1)^n z)^{n+1} \qquad \textit{for all } n\geq -1.$$

In view of 2.4 and 3.5 this implies

6.6. COROLLARY. *The normalization functor of* 3.3 *induces an equivalence between the category of cyclic* R*-modules* (i.e. functors $\Lambda^{\mathrm{op}}\to\mathbf{R}$ (2.4) and natural transformations between them) *and the full subcategory of* $\mathbf{R}(\partial,\delta)$ *spanned by the objects* $U\in\mathbf{R}(\partial,\delta)$ *such that, for every integer* $n\geq 0$,

$$f_n(\partial\delta)+(f_{n-1}-f_{-1})(\delta\partial)=id: U_n\to U_n\in\mathbf{R}$$

where f_n *is as in* 6.5, *or equivalently*

$$(1+(-1)^n\partial\delta)^{n+1}(1+(-1)^{n-1}\delta\partial)^n=id: U_n\to U_n\in\mathbf{R}$$

Proof of 6.5. If $x\in N_nX$, i.e. $d_ix=0$ for $i>0$ and $d_0x=\partial x$, then (3.3) $s_{n+1}x=\delta x+s_nx-\cdots+(-1)^n s_0x$ and hence $d_0s_{n+1}x=\partial\delta x+s_{n-1}\partial x-\cdots+(-1)^n x$, i.e. $d_0s_{n+1}x=(\partial\delta+(-1)^n)x$ modulo terms involving ∂x. This, together with 6.3, now readily implies that, for all $n\geq -1$

$$f_n=(z+(-1)^n)^{n+1}=(1+(-1)^n z)^{n+1}.$$

We end with

6.7. REMARK. If X is a cyclic R-module, then [6, II.2] *the normalized version of the commuting differentials b and B of Connes is* (up to a possible sign) *given by*

$$b=\partial: N_{n+1}X \to N_nX \quad \textit{and} \quad B=\delta\bar{f}_n(\partial\delta): N_nX \to N_{n+1}X,$$

where the polynomial $\bar{f}_n$ $(n \geq 0)$ is determined by $1+z\bar{f}_n=(1+(-1)^n z)^{n+1}$. Moreover, one can show [5] that *the resulting functor from "cyclic R-modules" to "duchain complexes over R with commuting differentials" is an equivalence of homotopy theories.*

§7. Appendix on homotopy theories

In this appendix we hope to make clear (by reformulating some of the results of [4])

(i) what we mean by *the homotopy theory* of a category $\mathbf{C}$ with respect to an admissible subcategory $\mathbf{W}$ (the maps of which are usualty called weak equivalences),

(ii) when two such homotopy theories will be called *equivalent*, and

(iii) when we say that *a functor induces an equivalence of homotopy theories.*

7.1. Simplicial categories. By a *simplicial category* we will mean something slightly different from usual. We assume, as is often done, that a simplicial category have *the same objects in each dimension.* However, we do not require that the "simplicial hom-sets" be small, but only that they be *homotopically small* in the sense explained below (7.2). *A simplicial category for which the simplicial hom-sets are discrete then is just an ordinary category.*

7.2. Homotopically small simplicial sets. A (not necessarily small) simplicial set X will be called *homotopically small* if $\pi_n(X; v)$ is small, for every vertex $v \in X$ and every integer $n \geq 0$. This is clearly equivalent to requiring that X contain a small simplicial set U with the property that, for every small simplicial set $V \subset X$ containing U, there is a small simplicial set $W \subset X$ containing V, such that the inclusion $U \to W$ is a weak homotopy equivalence. Clearly the homotopy type of such a U is unique and it thus makes sense to talk of the *homotopy type* of a homotopically small simplicial set and of *weak homotopy equivalences* between homotopically small simplicial sets. One can therefore define as follows

7.3. Weak equivalences between simplicial categories. A *weak equivalence* $S:\mathbf{C}\to\mathbf{D}$ between two simplicial categories is a functor which

(i) induces an equivalence $\pi_0\mathbf{C}\approx\pi_0\mathbf{D}$ between the "categories of components", and

(ii) induces, for every two objects $X, Y\in\mathbf{C}$, a weak homotopy equivalence $\mathbf{C}(X, Y)\approx\mathbf{D}(SX, SY)$ between the simplicial hom-sets.

Similarly two simplicial categories will be called *weakly equivalent* if they can be connected by a finite string of simplicial categories and weak equivalences between them (in alternating directions).

Next we consider the key construction of

7.4. The simplicial localization of a category with respect to an admissible subcategory. Let $\mathbf{C}$ be a category, let $\mathbf{W}\subset\mathbf{C}$ be a subcategory and consider, for every two objects $X, Y\in\mathbf{C}$, the (not necessarily small) *simplicial set of the reduced hammocks between X and Y*, which has as k-simplices the commutative diagrams in $\mathbf{C}$ of the form

$$\begin{tikzcd}
 & C_{0,1} \arrow[r, dash] \arrow[d] & C_{0,2} \arrow[r, dash] \arrow[d] & \cdots \arrow[r, dash] & C_{0,n-1} \arrow[d] \arrow[ddr, dash] & \\
 & C_{1,1} \arrow[r, dash] \arrow[d] & C_{1,2} \arrow[r, dash] \arrow[d] & \cdots \arrow[r, dash] & C_{1,n-1} \arrow[d] \arrow[dr, dash] & \\
X \arrow[uur, dash] \arrow[ur, dash] \arrow[ddr, dash] & \vdots \arrow[d] & \vdots \arrow[d] & & \vdots \arrow[d] & Y \\
 & & & & & \\
 & C_{k,1} \arrow[r, dash] & C_{k,2} \arrow[r, dash] & \cdots \arrow[r, dash] & C_{k,n-1} \arrow[uur, dash] &
\end{tikzcd}$$

in which

(i) n, the length of the hammock, is any integer ≥ 0,

(ii) all vertical maps are in $\mathbf{W}$,

(iii) in each column (of horizontal maps) all maps go in the same direction; if they go to the left, then they are in $\mathbf{W}$,

(iv) the maps in adjacent columns go in different directions, and

(v) no column contains only identity maps,

and in which the faces and degeneracies are defined in the obvious manner, i.e. the i-face is obtained by omitting the i-row and the i-degeneracy by repeating the i-row; if the resulting hammock is not reduced (i.e. does not satisfy (iv) and (v)), then it can easily be made so by repeatedly (iv)′ composing two adjacent columns whenever their maps go in the same direction, and

(v)′ omitting any column which contains only identity maps.

One then calls $\mathbf{W}$ an *admissible* subcategory of $\mathbf{C}$ if $\mathbf{W}$ contains all the objects of $\mathbf{C}$ and if, for every two objects $X, Y \in \mathbf{C}$, the simplicial set of the reduced hammocks between X and Y is homotopically small. Moreover, in this case, one defines the *simplicial localization* of $\mathbf{C}$ *with respect to* $\mathbf{W}$ as the simplicial category $L(\mathbf{C}, \mathbf{W})$ (or short $L\mathbf{C}$) which has the same objects as $\mathbf{C}$ and in which, for every two objects $X, Y \in \mathbf{C}$, the simplicial hom-set $L\mathbf{C}(X, Y)$ is the (homotopically small) simplicial set of the reduced hammocks between X and Y, with the obvious (see above) composition.

Note that, *for every two objects*, $X, Y \in \mathbf{C}$, *the components of* $L\mathbf{C}(X, Y)$ *are in* 1-1 *correspondence with the maps* $X \to Y \in \mathbf{C}[W^{-1}]$, *i.e.* $\pi_0 L\mathbf{C} = \mathbf{C}[W^{-1}]$, where $\mathbf{C}[W^{-1}]$ denotes the (ordinary) localization of $\mathbf{C}$ with respect to $\mathbf{W}$ (i.e. the category obtained from $\mathbf{C}$ by "formally inverting" the maps of $\mathbf{W}$).

7.5. EXAMPLE. Let $\mathbf{C}_1$ and $\mathbf{C}_2$ be categories, let $\mathbf{W}_1 \subset \mathbf{C}_1$ and $\mathbf{W}_2 \subset \mathbf{C}_2$ be subcategories containing all the objects and call the maps of $\mathbf{W}_1$ and $\mathbf{W}_2$ weak equivalences. Furthermore let $S: \mathbf{C}_1 \to \mathbf{C}_2$ and $T: \mathbf{C}_2 \to \mathbf{C}_1$ be functors such that

(i) S and T preserve weak equivalences, and

(ii) the compositions TS and ST are naturally weakly equivalent to the identity functors of $\mathbf{C}_1$ and $\mathbf{C}_2$ respectively.

Then it is easy to see that $\mathbf{W}_1$ *is admissible iff* W_2 *is so* and that, in that case, *the functors* S *and* T *induce weak equivalences* $LS: L\mathbf{C}_1 \to L\mathbf{C}_2$ and $LT: L\mathbf{C}_2 \to L\mathbf{C}_1$.

7.6. EXAMPLES. Let $\mathbf{C}$ be a closed model category in the sense of Quillen [3, §3]. Then one readily verifies that

(i) *the subcategory* $\mathbf{W} \subset \mathbf{C}$ *of the weak equivalences is admissible*, and

(ii) *if* $\mathbf{C}' \subset \mathbf{C}$ *is a full subcategory such that every weak equivalence is either in* $\mathbf{C}'$ *or has neither its domain nor its range in* $\mathbf{C}'$, *then* $\mathbf{W} \cap \mathbf{C}'$ *is an admissible subcategory of* $\mathbf{C}'$.

Now we are ready for our definition of

7.7. HOMOTOPY THEORIES. By *the homotopy theory of a category* $\mathbf{C}$ *with respect to an admissible subcategory* $\mathbf{W}$, we just mean the simplicial localization $L\mathbf{C}$. The maps of $\mathbf{W}$ are then called *weak equivalences*. The category of components $\pi_0 \mathbf{LC} = \mathbf{C}[\mathbf{W}^{-1}]$ is usually referred to as *the homotopy category* of $\mathbf{C}$ with respect to $\mathbf{W}$ and maps in $\mathbf{C}$ are called *homotopic* if they have the same image in this homotopy category. (The results of [4] show that, in the model category case, this notion of homotopy theory is a natural enrichment of Quillen's original notion).

7.8. Equivalent homotopy theories. Let $\mathbf{W}_1 \subset \mathbf{C}_1$ and $\mathbf{W}_2 \subset \mathbf{C}_2$ be admissible subcategories. Then we call the resulting homotopy theories $L\mathbf{C}_1$ and $L\mathbf{C}_2$ *equivalent* if they are weakly equivalent (7.3) as simplicial categories, and we say that *a functor* $S : \mathbf{C}_1 \to \mathbf{C}_2$ *induces an equivalence of homotopy theories* if

(i) S preserves weak equivalences, and

(ii) the induced functor $L\mathbf{C}_1 \to L\mathbf{C}_2$ is a weak equivalence.

7.9. REMARK. It is sometimes useful to extend definition 7.7 somewhat and to define "*a*" *homotopy theory of a category* $\mathbf{C}$ *with respect to an admissible subcategory* $\mathbf{W}$ as any simplicial category which is weakly equivalent to $L\mathbf{C}$.

BIBLIOGRAPHY

[1] A. K. Bousfield and D. M. Kan, *Homotopy limits, completions and localizations*, Lect. Notes in Math. 304 (1972), Springer.
[2] A. Connes, *Cohomologie cyclique et foncteurs* Ext^n, Comptes Rendus Acad. Sc. Paris *296* (1983), 953–958.
[3] W. G. Dwyer, M. J. Hopkins and D. M. Kan, *The homotopy theory of cyclic sets*, Trans. AMS (to appear).
[4] W. G. Dwyer and D. M. Kan, *Function complexes in homotopical algebra*, Topology *19* (1980), 427–440.
[5] W. G. Dwyer and D. M. Kan, *Three homotopy theories for cyclic modules*, (to appear).
[6] T. C. Goodwillie, *Cyclic homology, derivations and the free loop space*, preprint.
[7] J. P. May, *Simplicial objects in algebraic topology*, (1967) Van Nostrand.

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