Zeitschrift: Commentarii Mathematici Helvetici

Herausgeber: Schweizerische Mathematische Gesellschaft

Band: 60 (1985)

Artikel: More Denjoy minimal sets for area preserving diffeomorphisms.

Autor: Mather, John N.

DOI: https://doi.org/10.5169/seals-46330

Comment. Math. Helvetici 60 (1985) 508–557
0010-2571/85/040508-50$01.50 + 0.20/0

More Denjoy minimal sets for area preserving diffeomorphisms

JOHN N. MATHER[1]

Abstract. For an area preserving, monotone twist diffeomorphism and an irrational number ω, we prove that if there is no invariant circle of angular rotation number ω, then there are uncountably many Denjoy minimal sets of angular rotation number ω. For each pair of positive integers n and R we prove that the space (with the vague topology) of Denjoy minimal sets of angular rotation number ω and intrinsic rotation number $(\omega + R)/n$ (mod. 1) contains a disk of dimension $n-1$.

Contents

[1] Partially supported by NSF contract #MCS82-01604.

§1. Introduction

This paper continues the study of monotone twist area preserving diffeomorphisms of the annulus which we have pursued in [16–20]. In [16], we proved the existence of quasi-periodic orbits of all irrational frequencies ω (cf. §5). This result was obtained independently by Aubry and Le Daeron [4] and had been found numerically and non-rigorously earlier by Aubry [cf. 3]. The quasi-periodic orbits of frequency ω in [16] or [4] lie on a Denjoy minimal set (or a circle), of angular rotation number ω, when ω is irrational.

In the case that there is an invariant circle of angular rotation number ω, where ω is irrational, its unique minimal set Σ_ω is either the whole circle or a Denjoy minimal set of angular rotation number ω. Moreover, in this case, it is easy to see that there are no other Denjoy minimal sets of angular rotation number ω (cf. §4). It is natural to ask whether this uniqueness result holds in other cases, i.e. whether the minimal set of angular rotation number ω, constructed in [16], is unique. In [19], we gave examples where this uniqueness result does not hold, but we left open the question as to whether there exist examples with no invariant circle of rotation number ω (with ω irrational), but uniqueness for the Denjoy minimal set of rotation number ω.

In this paper, we will answer this question by showing that when ω is irrational and there is no circle of angular rotation number ω, there are uncountably many distinct Denjoy minimal sets of angular rotation number ω.

More precisely, under these conditions the set of Denjoy minimal sets of angular rotation number ω contains a topological disk of arbitrary high dimension, if it is provided with an appropriate topology. The appropriate topology is what we call the *vague topology* on Denjoy minimal sets. This is defined as follows. Each Denjoy minimal set carries a unique invariant measure and is the support of that measure. The vague topology on measures, i.e. the weak topology defined by continuous functions of compact support, induces a topology on Denjoy minimal sets, which we continue to call the vague topology. In §3, we define the angular and intrinsic rotation numbers of a Denjoy minimal set of a monotone twist diffeomorphism. Our principal result is stated in §15 and is proved in §§17–25. It has the consequence (Theorem 29 and the remarks following it) and if ω is an irrational number, n is a positive integer, and R is any integer, then the space of Denjoy minimal sets of angular rotation number ω and intrinsic rotation number $(\omega+R)/n$ (mod. 1) contains a disk of dimension $n-1$.

I announced this result in talks I gave at the Institute for Advanced Study and City University of New York in May, 1984 and at the Berkeley Math. Sciences Research Institute and ETH, Zürich (Forschungsinstitut für Mathematik) in June, 1984.

The method of proof in this paper combines elements of both the method of [16] and the method of [4]. We recall the method of [16]. We defined a functional F_ω (called Percival's Lagrangian) on a space Y of mappings of $\mathbb{R}$ into itself. In a suitable topology, Y is compact and F_ω is continuous; therefore F_ω takes a maximum value. Let ϕ_ω be a point of Y where F_ω takes its maximum value. In the case ω is rational, we showed how to construct a periodic orbit, using ϕ_ω. The existence of such a periodic orbit follows from a famous theorem of Birkhoff [6]. However, our proof also showed that this periodic orbit was ordered, which apparently was not known prior to the work of Aubry and the author. In the case ω is irrational, we showed how to associate a Denjoy minimal set Σ_ω of rotation number ω to ϕ_ω.

Percival introduced his Lagrangian in [22] and [23]. However, the domain of definition of this Lagrangian was not specified by Percival, and the proof in [16] depended crucially on choosing the right domain of definition for F_ω.

In this paper, we will use the opposite sign convention from that in [16], in order to agree with the sign convention used by the physicists (e.g. in [22], [23], and [4]). Thus, we replace F_ω with $-F_\omega$ and seek a ϕ_ω which minimizes Percival's Langrangian.

In §§6–12, we tie up some loose ends from previous papers on this subject and show that certain results of [16] are equivalent to the corresponding results of [4]. That this is the case is not at all surprising, but requires proof. The principal result here, Theorem 6, is that ϕ_ω is a minimum not only over Y, but also over a larger space Y_n^*, where no order condition is imposed on the elements of the space. Theorem 6 is stated in §6 and proved in §§7–10. We prove Theorem 6 because the method of proof is used in the proof of the principal result, Theorem 15, of this paper. The proof of Theorem 6 is based on the ideas of Aubry and Le Daeron [4].

The existence of new Denjoy minimal sets (Proposition 16 and the discussion following it) follows easily from Theorem 15, which in turn depends on a result which was discovered independently by Katok and the author, on the one hand (cf. [18]), and by Aubry, Le Daeron, and André, on the other (cf. [5]). In the terminology of [18], there is an invariant, homotopically non-trivial circle of rotation number ω if and only if $\Delta W_\omega = 0$. Under the hypothesis of Theorem 15 that there is no such invariant circle, we therefore have $\Delta W_\omega > 0$.

In the suggestive terminology of Aubry, the condition that $\Delta W_\omega > 0$ means that there is a barrier, called Peierl's energy barrier, which prevents a minimal energy configurations from sliding freely along the line. Minimal energy configurations are defined and discussed in §§11–13 and Peierls's energy barrier is defined in §14.

Using the positivity of Peierls's energy barrier, one can find $\phi \in Y_n^*$ which minimizes Percival's Lagrangian subject to constraints defined in terms of Peierls's energy barrier. One considers configurations where some of the atoms are constrained to lie on the opposite side of the barrier from where they would be if the configuration were a minimal energy configuration. The existence of such relative local minima is the content of Theorem 15. See also §24, where we interpret Theorem 15 in terms of configurations.

The method of constructing new Denjoy minimal sets in this paper is related to the method that we used in [16]. There, as here, Denjoy minimal sets were constructed by minimizing Percival's Lagrangian subject to constraints. In order to obtain invariant set in this way, one must show that the minimizing element satisfies Percival's Euler–Lagrange equation (in the terminology of [16]).

The main difficulty consists of showing that the minimizing element does not bump up against the constraints. In this paper, the relevant constraints are inequalities 2) and 3) of §15; the fact that the minimizing element does not bump up against the constraints means that these inequalities are strict for the minimizing element, i.e. the inequalities 2′) and 3′) of Theorem 15 hold.

The method of [16] does not appear to be well adapted to proving that the minimizing element does not bump up against the constraints, under the hypotheses considered in this paper. Instead, I use a method which relies heavily on ideas from the paper of Aubry and Le Daeron [4], in order to prove this.

§2. Monotone twist diffeomorphisms (Definitions)

This paper is the study of certain properties of area preserving, monotone twist diffeomorphisms of an infinite cylinder $(\mathbb{R}/\mathbb{Z}) \times \mathbb{R}$. We will consider one such diffeomorphism $\bar{f}$ which will be fixed throughout this paper. For notational reasons, it is more convenient to pass to the universal cover $\mathbb{R}^2$ of the cylinder and discuss an appropriate lift f of the given twist diffeomorphism.

The conditions which we impose of f in this paper are the following: First, we require that f be the lift to $\mathbb{R}^2$ of a C^1 diffeomorphism of the infinite cylinder, so $fT = Tf$, where T is the unit horizontal translation, i.e. $T(x, y) = (x+1, y)$. Second, we require that the form $y'\,dx' - y\,dx$ on $(\mathbb{R}/\mathbb{Z}) \times \mathbb{R}$ be exact, where $f(x, y) = (x', y')$. In particular, $dy' \wedge dx' = dy \wedge dx$, so f is area preserving and orientation preserving, and the *flux* $\int y'\,dx' - y\,dx$ vanishes, where the integral is taken over any curve going around the cylinder. Third, we require that f preserve each end of the cylinder. Fourth, we require that f satisfy a positive monotone twist condition, i.e. $\partial x'/\partial y > 0$, everywhere. Fifth, we require that $\bar{f}$ twist the cylinder infinitely at either end. This means that for fixed x, we have $x' \to +\infty$ as $y \to +\infty$ and $x' \to -\infty$ as $y \to -\infty$.

The exact hypotheses which we have just imposed on f are chosen for reasons of technical convenience. It seems very likely that the results we prove here can be generalized slightly. Thus, it seems to be enough to assume that f is a positively tilted homeomorphism rather than a monotone twist diffeomorphism. The hypothesis that f twists infinitely at both ends seems unnecessary, as does our assumption that the domain (and range) of f is an infinite cylinder rather than an arbitrary annulus. A method for removing the last two hypotheses has been found by R. Douady [10, Chapt. 2, III.2]: one compactifies the domain B of the generating function h (defined below) and extends h to a semi-continuous function on $\bar{B}$. One then has involved discussions concerning semi-continuous functions. In our case, $B = \mathbb{R}^2$ and no extension is necessary. Not only does this avoid a rather technical discussion of semi-continuous functions, but it also avoids the necessity of always having to specify the domains where the functions are defined. In the Fall semester of 1983, I explained the Aubry–Le Daeron theory in my class under the more general hypotheses which I have described above; it seems plausible that the theory of this paper should also work under these more general hypotheses, but J. Bellisard has pointed out some difficulties which I have been unable to overcome.

The Generating Function. The above hypotheses imply that there exists a C^2 function $h : \mathbb{R}^2 \to \mathbb{R}$ such that for $(x, y, x, y') \in \mathbb{R}^4$, we have $f(x, y) = (x', y')$ if and only if $y = -h_1(x, x')$ and $y' = h_2(x, x')$, where h_1 and h_2 denote the first partial derivatives with respect to x and x'. The periodicity condition $fT = Tf$ implies $h(x+1, x'+1) - h(x, x')$ is a constant; the hypothesis that the flux vanishes implies that $h(x+1, x'+1) = h(x, x')$.

In classical mechanics, h is known as the generating function for f. Its construction is a special case of classical work of various nineteenth century mathematicians, such as Lagrange, Hamilton and Jacobi. (Cf. Arnold and Avez [2] or Abraham and Marsden [1].) In the special situation we consider here, its construction is carried out in [16]. The point is that graph $f \subset \mathbb{R}^4 = (\{x, y, x', y'\})$ projects diffeomorphically onto the (x, x') plane; the restriction of $y'\,dx' - y\,dx$ to graph f is exact by hypothesis; hence, $y'\,dx' - y\,dx = dh$, where $h = h(x, x')$ is the generating function.

Note that in defining the generating function, we have adopted the opposite sign convention from that of our previous papers [16–19]. The sign convention used here agrees with that in [4], [22], and [23].

§3. Denjoy minimal sets

In topological dynamics, a *minimal set* is a pair (X, ψ), where X is a compact topological space and ψ is a homeomorphism of X whose every orbit is dense. A minimal set is called a *Denjoy minimal set* if it admits an embedding into, but not onto, the circle such that ψ extends to an orientation preserving homeomorphism of the circle with irrational rotation number. If $\bar{g}$ is an orientation preserving homeomorphism of the circle $\mathbb{R}/\mathbb{Z}$ and g is a lifting of $\bar{g}$ to $\mathbb{R}$, then the *rotation number of* $\bar{g}$ is defined to be $\lim_{n\to\pm\infty} (g^n(x) - x)/n \pmod{1}$. Note that this limit always exists and is independent of x. (See, e.g., Herman [12, II.2.3].) If (X, ψ) is a Denjoy minimal set, then X is a Cantor set. If X is a Cantor set, then necessary and sufficient conditions for (X, ψ) to be a Denjoy minimal set are that X admit a cyclic order which the homeomorphism ψ preserves and ψ has no periodic points. In this case, X admits only one other cyclic order preserved by ψ, namely, the reverse cyclic order. For any cyclic order preserving embedding of X in the circle, ψ extends to a homeomorphism of the circle. The resulting rotation number is independent of the embedding or the extension; we will call it the *intrinsic rotation number* of the pair (X, ψ).

Note, however, that this intrinsic rotation number depends on the cyclic order. If α is the rotation number for one cyclic order, then $1-\alpha$ is the rotation number for the reverse cyclic order. Moreover, it is defined only modulo 1.

A subset Σ of $(\mathbb{R}/\mathbb{Z}) \times \mathbb{R}$ will be said to be a *Denjoy minimal set for the monotone twist diffeomorphism* $\bar{f}$ if it is invariant for $\bar{f}$, i.e. $\bar{f}\Sigma = \Sigma$, and $(\Sigma, \bar{f} \mid \Sigma)$ is a Denjoy minimal set. The main purpose of this paper is to construct examples of such sets, but in this section, we wish to discuss properties of Denjoy minimal sets for $\bar{f}$.

Let pr_1 be the projection of $\mathbb{R}/\mathbb{Z} \times \mathbb{R}$ on its first factor. Let Σ be a Denjoy minimal set for $\bar{f}$, let $\bar{x} \in \Sigma$, and let $x \in \mathbb{R}^2$ be an element which projects to $\bar{x}$ under the covering mapping $\mathbb{R}^2 \to (\mathbb{R}/\mathbb{Z}) \times \mathbb{R}$. Then $\lim_{n\to\pm\infty} (pr_1 f^n(x) - pr_1(x))/n$ exists and is independent for x. This may be seen as follows: Let $u(x) = pr_1 f(x) - pr_1(x)$, for $x \in \Sigma$. Then $pr_1 f^n(x) - pr_1(x) = \sum_{i=0}^{n-1} u(f^i x)$. Since Σ is a Denjoy minimal set, it is uniquely ergodic. (See, e.g., Herman [12, II.8.5].) Consequently, $\lim_{n \to \pm\infty} \sum_{i=0}^{n-1} u(f^i x)/n$ exists and is independent of x, as claimed. (See, e.g., Herman [12, II.8.4].)

The number $\lim_{n \to \pm\infty} (pr_1 f^n(x) - pr_1(x))/n$ will be called the *angular rotation number* of Σ. Unlike the intrinsic rotation number, it does not depend on the choice of cyclic ordering of Σ. However, it does depend on the choice of the lift f of $\bar{f}$. If f is replaced by fT^r, then the angular rotation number is increased by r. If the angular rotation number were defined only modulo 1, then it would be independent of this lift. However, we wish to consider it as a real number. Since we have made the convention that f is fixed throughout this paper, it is a well defined real number for any Denjoy minimal set Σ for $\bar{f}$.

§4. The case when an invariant circle exists

For an $\bar{f}$ invariant circle in the cylinder $(\mathbb{R}/\mathbb{Z}) \times \mathbb{R}$, the two rotation numbers can be defined, just as for a Denjoy minimal set. In this case, however, the situation is very simple. If the circle is null homotopic in the cylinder, then the angular rotation number is an integer. If, on the other hand, the invariant circle goes around the cylinder, then the intrinsic rotation number is congruent (mod. 1) to the angular rotation number, if the invariant circle is assigned the cyclic order which makes its projection on $\mathbb{R}/\mathbb{Z}$ a degree 1 (rather than -1) mapping.

Suppose that Γ is an $\bar{f}$-invariant circle which goes around the annulus. According to a theorem of G. D. Birkhoff [6, §3], [13, Chapter I and appendix by Fathi], there is a Lipschitz function $u: \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ such that $\Gamma = \text{graph } u$.

PROPOSITION 4. *Let Σ be a Denjoy minimal set for $\bar{f}$, whose angular rotation number ω is the same as that of Γ. If ω is irrational, then $\Sigma \subset \Gamma$.*

Proof. Since Σ is a minimal set, it is contained in a closed invariant set if it meets that set. Applying this remark to Γ, to the part of $(\mathbb{R}/\mathbb{Z}) \times \mathbb{R}$ which lies on or above Γ, and to the part of $(\mathbb{R}/\mathbb{Z}) \times \mathbb{R}$ which lies on or below Γ, we obtain that Σ lies entirely above Γ, lies in Γ, or lies entirely below Γ.

Suppose Σ lies entirely above Γ. Consider a point in Σ and express it in coordinates as $(\bar{x}, y)$ where $\bar{x} \in \mathbb{R}/\mathbb{Z}$, $y \in \mathbb{R}$. Let x be a point in $\mathbb{R}$ which projects onto $\bar{x} \in \mathbb{R}/\mathbb{Z}$. By the monotone twist condition, $pr_1 f(x, y) - pr_1 f(x, u(\bar{x})) > 0$, since $y > u(x)$, in view of the fact that Σ lies entirely above Γ. Since Σ is compact, there exists $\varepsilon > 0$ such that $pr_1 f(x, y) > pr_1 f(x, u(\bar{x})) + \varepsilon$, for any $(\bar{x}, y) \in \Sigma$ and any $x \in \mathbb{R}$

which projects onto $\bar{x}$. From this inequality, it follows by a known argument (see, e.g. Herman [12, III 4.1]) that the angular rotation number of Σ is greater than that of Γ. (For this argument to work, we need that the angular rotation number of Γ is irrational.) But this contradicts the hypothesis that Σ and Γ have the same rotation number. This contradiction shows that Σ cannot lie above Γ.

Since the argument of [12, III 4.1] does not apply directly to our situation, we give a version of it applicable in our situation: Let $g:\mathbb{R}\to\mathbb{R}$ be defined by $g(t)=pr_1 f(t, u(\bar{t}))$, for $t\in\mathbb{R}$ and $\bar{t}$ its image in $\mathbb{R}/\mathbb{Z}$. We have $g(t+1)=g(t)+1$ and the rotation number of g = the rotation number of $\Gamma=\omega$. Let $\bar{g}:\mathbb{R}/\mathbb{Z}\to\mathbb{R}/\mathbb{Z}$ be the induced homeomorphism of the circle. Since ω is irrational, $\bar{g}$ has a unique minimal set Π, which is either a Cantor set or the whole circle.

Choose $t\in\mathbb{R}$ whose projection $\bar{t}$ in $\mathbb{R}/\mathbb{Z}$ is in Π, but is not the endpoint of a complementary interval of Π. Since Π is a minimal set for $\bar{g}$, there exist integers p, q, with q positive, such that $t+p-\varepsilon/2<g^q(t)<t+p$.

Consider the interval $(\bar{t}, \bar{t}+\varepsilon/2)\subset\mathbb{R}/\mathbb{Z}$. Since $\bar{t}\in\Pi$ and is not the endpoint of a complementary interval of Π, there is an open subinterval $U\subset(\bar{t}, \bar{t}+\varepsilon/2)$, both of whose endpoints lie in Π, and $U=h^{-1}h(U)$, where $h:\mathbb{R}/\mathbb{Z}\to\mathbb{R}/\mathbb{Z}$ is a semi-conjugacy of $\bar{g}$ with the rotation R_ω of $\mathbb{R}/\mathbb{Z}$, i.e. $h\bar{g}=R_\omega h$. (See [12, II.7.1], for the existence of such a semi-conjugacy; note that is is necessarily weakly cyclic order preserving.) Since ω is irrational, there exists $N>0$ such that $\bigcup_{i=0}^{N} R_\omega^{-i}(h(U))=\mathbb{R}/\mathbb{Z}$; then $\bigcup_{i=0}^{N}\bar{g}^{-i}U=\mathbb{R}/\mathbb{Z}$. In other words, for every $\bar{x}\in\mathbb{R}/\mathbb{Z}$, there exists a positive integer $n\leq N$ such that $\bar{g}^n(\bar{x})\in(\bar{t}, \bar{t}+\varepsilon/2)$. This implies that for every $x\in\mathbb{R}$, there exists a positive integer $n=n(x)\leq N$ and an integer $m=m(x)$ such that $t+m<g^n(x)<t+m+\varepsilon/2$.

Consider $(x, y)\in\Sigma$ and $x\in\mathbb{R}$ whose projection on $\mathbb{R}/\mathbb{Z}$ is $\bar{x}$. We have shown above that $pr_1 f(x, y)>pr_1 f(x, u(\bar{x}))+\varepsilon=g(x)+\varepsilon$. Since g is order preserving, we get, by induction on n, that $pr_1 f^n(x, y)>g(pr_1 f^{n-1}(x, y))+\varepsilon>g(g^{n-1}(x)+\varepsilon)+\varepsilon>g^n(x)+\varepsilon$. Let $n_1=n(x)$, $m_1=m(x)$. By definition of $n(x)$ and $m(x)$, we have $t+m_1<g^{n_1}(x)<t+m_1+\varepsilon/2$. Moreover, p and q were chosen so that $t+p-\varepsilon/2<g^q(t)<t+p$; hence

$$
\begin{aligned}
pr_1 f^{n_1+q}(x, y)&>g^{n_1+q}(x)+\varepsilon>g^q(t)+m_1+\varepsilon\\
&>t+p+m_1+\varepsilon/2>g^{n_1}(x)+p.
\end{aligned}
$$

Let $n_2=n(pr_1 f^{n_1+q}(x, y))$. Repeating the argument just given, we get

$$
\begin{aligned}
pr_1 f^{n_1+n_2+2q}(x, y)&>g^{n_2}(pr_1 f^{n_1+q}(x, y))+p\geq g^{n_2}(g^{n_1}(x)+p)+p\\
&=g^{n_1+n_2}(x)+2p.
\end{aligned}
$$

Continuing in this way, we define, by induction, $n_{k+1}=n(pr_1 f^{n_1+\cdots+n_k+kq}(x, y))$ and obtain, by the same argument,

$$pr_1 f^{n_1+\cdots+n_k+kq}(x, y) > g^{n_1+\cdots+n_k}(x) + kp.$$

Then

$$pr_1 f^{n_1+\cdots+n_k+kq}(x, y) - x > g^{n_1+\cdots+n_k}(x) - x + kp > (n_1+\cdots+n_k)\omega - 1 + kp,$$

since the rotation number of g is ω. Since $p/q > \omega$ and $0 \leqslant n_i \leqslant N$, we obtain

$$\frac{pr_1 f^{n_1+\cdots+n_k+kq}(x, y) - x}{n_1+\cdots+n_k+kq} > \frac{N\omega + p - k^{-1}}{N+q} > \omega + \delta$$

for k large enough, where δ is a small positive number which depends only on N, ω, p, and q.

Since $(x, y) \in \Sigma$, we thus obtain a contradiction to the hypothesis that the angular rotation number of Σ is ω. This shows that Σ cannot be entirely above Γ. A similar argument shows that if Σ is entirely below Γ, then its angular rotation number is less than ω, so we obtain a contradiction in this case, too. We have previously shown that the only remaining possibility is $\Sigma \subset \Gamma$. $\square$

§5. Percival's Lagrangian (definition)

Slightly modifying the notation of [16], we let Y denote the set of all weakly ordering preserving mappings $\phi : \mathbb{R} \to \mathbb{R}$ such that $\phi(t+1) = \phi(t) + 1$ and ϕ is continuous from the left. For $\phi \in Y$ and $\omega \in \mathbb{R}$, we set

$$F_\omega(\phi) = \int_a^{a+1} h(\phi(t), \phi(t+\omega))\, dt.$$

This is what we have called *Percival's Lagrangian* in [16] and elsewhere. Note that because of our change in sign convention, F_ω has the opposite sign from the corresponding function in [16–19]. The function F_ω is independent of a, because $h(x+1, x'+1) = h(x, x')$, cf. [16, 3.2].

In [16], we considered a mapping of a bounded annulus; here, an infinite cylinder. Consequently, we must make certain slight modifications to apply the results of [16] to our present situation. Next, we explain how to modify [16].

As in [16], we define, for $\phi \in Y$, $\phi(t-) = \lim_{s \uparrow t} \phi(s)$, $\phi(t+) = \lim_{s \downarrow t} \phi(s)$, graph $\phi = \{(t, x) \in \mathbb{R}^2 : \phi(t-) \leqslant x \leqslant \phi(t+)\}$. We provide Y with the metric coming from the Hausdorff metric on graphs, i.e. for ϕ, $\psi \in Y$, we set

$$d(\phi, \psi) = \max\left\{\sup_{\xi} \inf_{\eta} |\xi - \eta|, \sup_{\eta} \inf_{\xi} |\xi - \eta|\right\},$$

where ξ ranges over graph ϕ, η ranges over graph ψ, and $| \ |$ denotes the Euclidean norm on $\mathbb{R}^2$.

For $a \in \mathbb{R}$, we let $T_a(t) = t + a$. From the formula $h(x+1, x'+1) = h(x, x')$, the definition of F_ω, and $\phi(t+1) = \phi(t) + 1$, it follows easily that F_ω is *translation invariant*, i.e. $F_\omega(\phi T_a) = F_\omega(\phi)$, for all $a \in \mathbb{R}$.

Let $\mathbb{R}$ act on Y by $(a, \phi) \to \phi T_a$, and let $X = Y/\mathbb{R}$ denote the set of orbits of this action. For $\phi \in Y$, we let $[\phi] \in X$ denote the orbit ϕ; we set $d'([\phi], [\psi]) = \inf_{a \in \mathbb{R}} d(\phi, \psi T_a)$. It is easily seen that d' is a metric on X, because $\phi \to \phi T_a$ is an isometry of Y, for each $a \in \mathbb{R}$. Since F_ω is translation invariant, it induces a function on X which we continue to denote by the same symbol.

The function $F_\omega: Y \to \mathbb{R}$ is continuous with respect to the metric d. This may be shown by a slight modification of the proof in [16, §6]. The only necessary change is in the definition of M. Note that in [16], B was the domain of h; here $\mathbb{R}^2$ is the domain of h. If we replace B by $\mathbb{R}^2$ in the definition of M (using $g = -\partial h/\partial x$ and $g' = \partial h/\partial x'$), we get $M = \infty$, which won't do. Instead, we set

$$M = \sup_{|x - x'| \leq 2} \max\{1, |h_1(x, x')|, |h_2(x, x)|\},$$

which is finite, since $h(x+1, x'+1) = h(x, x')$ and h is C^1. We still have

$$|F_\omega\phi - F_\omega\psi| \leq M \int_0^1 (|\phi(t) - \psi(t)| + |\psi(t+\omega) - \phi(t+\omega)|)\, dt,$$

when $d(\phi, \psi) < \delta$, where δ is defined as in [16, §6], since $d(\phi, \psi) < \delta$ implies that $|\phi(t) - \psi(t)| < 2$, for all $t \in \mathbb{R}$, as we showed in [16, §6]. The rest of the proof in [16, §6] works without change. We obtain that $F_\omega: Y \to \mathbb{R}$ is continuous, with respect to the metric d.

The topology on X associated to the metric d' is clearly the quotient topology of the topology on Y associated to the metric d. Consequently, $F_\omega: X \to \mathbb{R}$ is continuous with respect to the metric d'.

The space (X, d') is compact. For, let $Y_0 = \{\phi \in Y : \phi(t) \geq 0$ if $t \geq 0$ and $\phi(t) \leq 0$ if $t < 0\}$. The argument of [16, §5] shows that Y_0 is compact. For, the mapping $\phi \to \text{graph}\, \phi \cap [0, 1]$ embeds (Y, d) isometrically as a closed subset in the space of closed subsets of $[0, 1]^2$ with the Hausdorff metric. Since it is well known, and easy to prove, that the latter space is compact, it follows that (Y_0, d) is compact.

But the projection $Y_0 \to X$ is continuous and surjective, so it follows that (X, d') is compact.

Since F_ω is a continuous function on the compact space X, it takes a minimum value. Consequently, $F_\omega : Y \to \mathbb{R}$ also takes a minimum value. From now on, we will let $\phi_\omega \in Y$ be an element where F_ω takes its minimum value. We choose one such element once and for all. It will be fixed throughout the discussion. From the fact that F_ω takes its minimum value at ϕ_ω, it may be shown that ϕ_ω satisfies the corresponding Euler–Lagrange equation, which may be written formally as $\delta \int_a^{a+1} h(\phi(t), \phi(t+\omega))\, dt = 0$, or

$$h_2(\phi(t-\omega), \phi(t)) + h_1(\phi(t), \phi(t+\omega)) = 0,$$

for all $t \in \mathbb{R}$. The reasoning of [16, §10] (with Y_ω replaced by Y) shows that this equation is satisfied for $\phi = \phi_\omega$. (The sign we are using here is opposite that in [16].)

Because we require the elements of Y to be order preserving, it is not an immediate consequence of the fact that ϕ_ω minimizes F_ω over Y that the Euler–Lagrange equation is satisfied; indeed, the Euler–Lagrange equation is not satisfied at a point ϕ of Y where F_ω takes its maximum value. On the other hand, it is an immediate consequence of Theorem 6 of the next section that the Euler–Lagrange equation is satisfied by ϕ_ω, since no order condition is imposed on elements of Y_n^*, and consequently there is no difficulty seeing that there are enough test curves. Thus, Theorem 6 provides an alternative proof of the Euler–Lagrange equation for ϕ_ω. Note that F_ω does not take a maximum value on Y_n^*.

Set $\eta_\omega(t) = -h_1(\phi_\omega(t), \phi_\omega(t+\omega))$. From the definition of the generating function, it follows that $f(\phi_\omega(t), \eta_\omega(t)) = (\phi_\omega(t+\omega), \eta_\omega(t+\omega))$. We set

$$M_\omega = \overline{\{(\phi_\omega(t), \eta_\omega(t)) : t \in \mathbb{R}\}}$$

According to what we have just proved, M_ω is invariant for f. In the case ω is irrational, the projection of M_ω on the cylinder is the Denjoy minimal set Σ_ω whose existence we proved in [16].

§6. Application of the Aubry–Le Daeron method: statement of the theorem

Let n be a positive integer. Let Y_n^* denote the set of measurable and locally bounded mappings $\phi : \mathbb{R} \to \mathbb{R}$ such that $\phi(t+n) = \phi(t) + n$. For $\phi \in Y_n$ and $\omega \in \mathbb{R}$, we set

$$F_\omega(\phi)=n^{-1}\int_a^{a+n} h(\phi(t),\phi(t+\omega))\,dt.$$

This independent of the choice of $a\in\mathbb{R}$. We continue to call F_ω *Percival's Lagrangian*. Clearly, $Y\subset Y_n^*$ and we have extended the previously defined F_ω. We continue to let ϕ_ω denote an element of Y which minimizes F_ω over Y. In §§7–10, we will prove:

THEOREM 6. *ϕ_ω minimizes F_ω over Y_n^*, for every positive integer n. Moreover, if ω is irrational and ϕ is any element of Y_n^* which minimizes F_ω over Y_n^*, then there exists $a\in\mathbb{R}$ such that $\phi=\phi_\omega T_a$, almost everywhere, where $T_a(t)=t+a$.*

The proof depends on ideas which are due to Aubry and Le Daeron [4]. There are three main steps in the proof that ϕ_ω minimizes F_ω over Y_n^*.

First, we will show in §7 that if F_ω takes a minimum value over Y_n^*, it takes its minimum value at ϕ_ω (Lemma 7.3). Second, we will show that in §8 that F_ω takes a minimum value over Y_n^* when ω is rational, say $\omega=p/q$ in lowest terms. The trick is to replace the original problem of minimizing $F_{p/q}$ over Y_n^* with an equivalent problem of minimizing another function W over a finite dimensional space $\mathscr{X}_{pqn}/T$, which has the property that for any $a\in\mathbb{R}$, $\{W\leq a\}$ is compact. (Lemma 8.1). This is possible because ω is rational. In view of the result of §7, it will follow that $F_{p/q}$ takes its minimum value over Y_n^* at $\phi_{p/q}$. Thus, the first statement of Theorem 6 will be proved in §8 for the case when ω is rational.

Third, we consider in §9 the possibility that F_ω does not take a minimum value in Y_n^*. In this case, for any $\phi\in Y_n^*$, there exists $\phi'\in Y_n^*$ such that $F_\omega(\phi')<F_\omega(\phi)$. By the result of §8, $F_{p/q}(\phi_{p/q})\leq F_{p/q}(\phi')$. By suitable approximation lemmas, we show in §9 that $|F_{p/q}(\phi')-F_\omega(\phi')|$ and $|F_{p/q}(\phi_{p/q})-F_\omega(\phi_{p/q})|$ are arbitrarily small when p/q is close enough to ω. Thus $F_\omega(\phi_{p/q})<F_\omega(\phi)$, if p/q is close enough to ω. Since $\phi_{p/q}\in Y$ and ϕ_ω minimizes F_ω over Y, we then obtain $F_\omega(\phi_\omega)<F_\omega(\phi)$, so ϕ_ω minimizes F_ω over Y_n^*, proving the first statement in Theorem 6.

The last statement of Theorem 6 will be proved in §10.

§7. Application of the Aubry–Le Daeron method: the existence of a minimizing element implies ϕ_ω minimizes

We need an inequality which (in slightly different form) is due to Aubry and Le Daeron [4]. Given $\phi,\phi'\in Y_n^*$, we define the latatice operations, as usual, by

$$\phi\vee\phi'(t)=\max(\phi(t),\phi'(t)),\qquad \phi\wedge\phi'(t)=\min(\phi(t),\phi'(t)).$$

It is easily seen that if ϕ, $\phi' \in Y_n^*$, then so are $\phi \vee \phi'$ and $\phi \wedge \phi'$. The following result is a slight variant of earlier results of [4]. Its proof follows the method of [4]. Bangert has pointed out a remarkable connection of this method with the ideas of Morse [21] and Hedlund [11]. Here is our formulation of the Aubry–Le Daeron result:

LEMMA 7.1. *If ϕ, $\phi' \in Y_n^*$, then $F_\omega(\phi \vee \phi') + F_\omega(\phi \wedge \phi') \leq F_\omega(\phi) + F_\omega(\phi')$.*

Proof. Write $h_{12}(x, x')$ for the mixed second partial derivative $\partial^2 h(x, x')/\partial x\, \partial x'$. Clearly,

$$h(x \wedge x', x'' \wedge x''') + h(x \vee x', x'' \vee x''') - h(x, x'') - h(x', x''')$$
$$= \int_{x''\wedge x'''}^{x''\vee x'''} d\xi' \int_{x\wedge x'}^{x \vee x'} d\xi h_{12}(\xi, \xi'),$$

when either of the conditions 1) $x < x'$ and $x'' > x'''$ or 2) $x > x'$ and $x'' < x'''$ is satisfied. Otherwise, the left side of this equation vanishes. Consequently,

$$F_\omega(\phi \vee \phi') + F_\omega(\phi \wedge \phi') - F_\omega(\phi) - F_\omega(\phi') = n^{-1} \int_E dt \iint_{D(t)} h_{12}(x, x')\, dx\, dx',$$

where $E = \{t \in [0, n] : (\phi(t) - \phi'(t))(\phi(t+\omega) - \phi'(t+\omega)) < 0\}$ and $D(t) = \{(x, x') \in \mathbb{R}^2 : \phi(t) \wedge \phi'(t) \leq x \leq \phi(t) \vee \phi'(t)$ and $\phi(t+\omega) \wedge \phi'(t+\omega) \leq x' \leq \phi(t+\omega) \vee \phi'(t+\omega)\}$. Since $h_{12} < 0$, we obtain the desired inequality. □

LEMMA 7.2. *Let $\phi \in Y_n^*$ and suppose ϕ minimizes F_ω over Y_n^*. Let $\psi(t) =$ ess. $\inf_{s \geq t} \phi(s)$. Then $F_\omega(\phi) = F_\omega(\psi)$.*

Proof. From the translation invariance of F_ω, we have $F_\omega(\phi T_a) = F_\omega(\phi)$, for any $a \in \mathbb{R}$. From Lemma 7.1 and the assumption that ϕ minimizes F_ω over Y_n^*, we then obtain

$$F_\omega(\phi \wedge \phi T_a) = F_\omega(\phi \vee \phi T_a) = F_\omega(\phi) = F_\omega(\phi T_a).$$

Let $a(1), \ldots a(m), \ldots$ be an enumeration of the positive rational numbers. Repeating the argument we have just given m times, we obtain $F_\omega(\psi_m) = F_\omega(\psi)$, where $\psi_m = \phi \wedge \phi_{a(1)} \wedge \cdots \wedge \phi_{a(m)}$. Since ϕ is locally bounded and $\phi(t+n) = \phi(t) + n$, we have that $\psi_m(t)$ is bounded below by $\inf\{\phi(s) : t \leq s \leq t+n\}$, for all m. We have $\phi \geq \psi_1 \geq \cdots \geq \psi_m \geq \cdots$ and there is, for each finite interval $[a, b]$, a number C, such that $\psi_m \mid [a, b] \geq C$ for all m, namely $C = \inf\{\phi(s): a \leq s \leq a+n\}$. Consequently, $\psi_\infty(t) = \lim_{m\to\infty} \psi_m(t)$ exists, for all $t \in \mathbb{R}$.

Moreover, $\{|h(\psi_m(t), \psi_m(t+\omega))| : m$ is a positive integer, $t \in [0, n]\}$ is bounded, so $F_\omega(\psi_\infty) = \lim_{m\to\infty} F_\omega(\psi_m) = F_\omega(\phi)$ by the dominated convergence theorem.

In order to complete the proof, it will be enough to show that $\psi = \psi_\infty$, almost everywhere. First, we have that $\psi \leq \psi_\infty$, almost everywhere. Let $t_0 \in \mathbb{R}$, $a > 0$. Let $\varepsilon = a/2$. By definition of the essential infinum, $\phi(t+a) \geq \text{ess.}\inf\{\phi(s): t_0 + a - \varepsilon < s < t_0 + a + \varepsilon\}$, for almost all $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$. For any such t, we have $\phi(t+a) \geq \psi(t)$. Since t_0 is arbitrary, we have $\phi(t+a) \geq \psi(t)$, for almost all $t \in \mathbb{R}$. Since $\psi_\infty(t) = \inf\{\phi(t+a): a$ is a positive rational number$\}$, it follows that $\psi_\infty(t) \geq \psi(t)$, for almost all $t \in \mathbb{R}$.

In order to show that $\psi_\infty \leq \psi$, almost everywhere, we first show that ψ_∞ is order preserving except on a set of zero measure. For a positive rational number a, we have $\psi_\infty T_a \geq \psi_\infty$, by definition of ψ_∞. Since $\psi_\infty \in \mathscr{L}^\infty_{\text{loc}}(\mathbb{R})$, i.e. is measurable and bounded on bounded sets, we have that $\psi_\infty T_a \to \psi_\infty T_b$ in $\mathscr{L}^1_{\text{loc}}(\mathbb{R})$ as $a \to b$, i.e. $\int_I |\psi_\infty T_a - \psi_\infty T_b| \to 0$ as $a \to b$, for any finite interval I. Since $\psi_\infty T_a \geq \psi_\infty$, for positive rationals, it follows that if $b > 0$, then $\psi_\infty T_b \geq \psi_\infty$, almost everywhere, by this convergence result. In other words, for each $b > 0$, the set of $t \in \mathbb{R}$ for which $\psi_\infty(t) > \psi_\infty(t+b)$ has zero measure; it follows by Fubini's theorem that the set of $(t, b) \in \mathbb{R}^2$, $b > 0$ such that $\psi_\infty(t) > \psi_\infty(t+b)$ has zero planar measure. This implies that $\{(t, s) \in \mathbb{R}^2 : (t-s)(\psi_\infty(t) - \psi_\infty(s)) < 0\}$ has zero measure. By Fubini's theorem, there is a set $E \subset \mathbb{R}$ of zero measure, such that if $t \notin E$, then $\{s \in \mathbb{R} : (t-s) \times (\psi_\infty(t) - \psi_\infty(s)) < 0\}$ has zero measure. Suppose $t, s \notin E$, $t < s$. Then, for almost all u satisfying $t < u < s$ we have $\psi_\infty(t) \leq \psi_\infty(u)$ and $\psi_\infty(u) \leq \psi_\infty(s)$. Consequently $\psi_\infty(t) \leq \psi_\infty(s)$. We have proved that if s, $t \notin E$, $s < t$, then $\psi_\infty(s) \leq \psi_\infty(t)$.

For $t \notin E$, we therefore have $\psi_\infty(t) \leq \text{ess.}\inf_{s \geq t} \psi_\infty(s) \leq \text{ess.}\inf_{s \geq t} \phi(s) = \psi(t)$. Since E has zero measure, we have proved that $\psi_\infty \leq \psi$ almost everywhere.

Since $\psi_\infty = \psi$, almost everywhere, we have $F_\psi(\psi_\infty) = F_\psi(\psi) = F_\omega(\phi)$. □

LEMMA 7.3. *If F_ω takes a minimum value over Y^*_n, then it takes its minimum value at ϕ_ω.*

Proof. Suppose F_ω takes its minimum value at $\phi \in Y^*_n$. Let $\psi(t) = \text{ess.}\inf_{s \geq t} \phi(s)$. Then $F_\omega(\psi) = F_\omega(\phi)$, by Lemma 7.2. We have that ψ is ordered and $\psi(t+n) = \psi(t) + n$, so $\psi \wedge \psi T_1 \wedge \cdots \wedge \psi T_{n-1} \in Y$. Since ψ minimizes F_ω over Y^*_n, Lemma 7.1 and the translation invariance of F_ω imply $F_\omega(\psi \wedge \psi T) = F_\omega(\psi)$. By induction, $F_\omega(\psi \wedge \psi T_1 \wedge \cdots \wedge \psi T_{n-1}) = F_\omega(\psi)$. Since $\psi \wedge \cdots \wedge \psi T_{n-1} \in Y$ and minimizes F_ω over Y^*_n, and since ϕ_ω minimizes F_ω over Y, we obtain that ϕ_ω minimizes F_ω over Y^*_n.

§8. Application of the Aubry–Le Daeron method: rational rotation number

We follow the Aubry–Le Daeron terminology [4] and call any bi-infinite sequence $x=(\ldots, x_i, \ldots)$ of real numbers a *configuration.* Given $\phi \in Y_n^*$, $\omega \in \mathbb{R}$, and $t \in \mathbb{R}$, we let $x = x_{\phi\omega t}$ denote the configuration defined by $x_i = \phi(t+\omega i)$.

Suppose ω is rational. Let $\omega = p/q$ in lowest terms, with $q>0$. The configuration $x = x_{\phi\omega t}$ obviously satisfies $x_{i+qn/r} = x_i + pn/r$. Let $\mathscr{X}_{pqn}$ denote the set of all configurations which satisfy this condition. Given $x \in \mathscr{X}_{pqn}$, we define

$$W(x) = rn^{-1} \sum_{i=0}^{nr^{-1}q-1} h(x_i, x_{i+1}),$$

where r is the gratest common divisor of n and p.

As i runs from 0 to $nr^{-1}q-1$, the congruence class of ωi (mod. n) takes each value in $q^{-1}r\mathbb{Z}/n\mathbb{Z}$ exactly once. For, multiplication by qr^{-1} defines an isomorphism $q^{-1}r\mathbb{Z}/n\mathbb{Z} \approx \mathbb{Z}/qr^{-1}n\mathbb{Z}$ and $qr^{-1}\omega = pr^{-1}$ is an invertible element in this ring. Consequently,

$$W(x_{\phi\omega t}) = rn^{-1} \sum_{i=0}^{nr^{-1}q-1} h(\phi(t+q^{-1}ir), \phi(t+q^{-1}(ir+p)),$$

for $\phi \in Y_n^*$ and $t \in \mathbb{R}$. This implies

$$F_\omega(\phi) = r^{-1} \int_a^{a+rq^{-1}} W(x_{\phi\omega t})\, dt.$$

Given $x \in \mathscr{X}_{pqn}$, there exists a unique $\phi \in Y_n^*$ such that $x_{\phi\omega t} = x$, for $a \leq t < a + rq^{-1}$.

It follows that if x minimizes W over $\mathscr{X}_{pqn}$, then ϕ minimizes F_ω over Y_n^*, where ϕ is the unique element in Y_n^* such that $x = x_{\phi\omega t}$ for $a \leq t < a + rq^{-1}$. Therefore, in order to prove that there exists $\phi \in Y_n^*$ which minimizes F_ω over Y_n^*, it is enough to prove that there exists $x \in \mathscr{X}_{pqn}$ which minimizes W over $\mathscr{X}_{pqn}$. We may prove the existence of such an x by means of the following simple topological argument.

A point $x = (\ldots, x_i, \ldots)$ in $\mathscr{X}_{pqn}$ is determined by $(x_1, \ldots, x_{qn/r})$. Thus, we may identify $\mathscr{X}_{pqn}$ with $\mathbb{R}^{qn/r}$. We use this identification to topologize $\mathscr{X}_{pqn}$. We let $T: \mathscr{X}_{pqn} \to \mathscr{X}_{pqn}$ be defined by $(Tx)_i = x_i + 1$. Clearly $WT = W$, so W induces a function, which we continue to denote by W, on the quotient space $\mathscr{X}_{pqn}/T$.

LEMMA 8.1. *$W: \mathscr{X}_{pqn}/T \to \mathbb{R}$ is proper; in fact, for any $a \in \mathbb{R}$, $\{W \leq a\}$ is compact.*

Proof. Set $f(x, y) = (x', y')$. Since f commutes with $(x, y) \mapsto (x+1, y)$, we have that $|x - x'|$ is defined on the quotient space $(\mathbb{R}/\mathbb{Z}) \times \mathbb{R}$. Since $(\mathbb{R}/\mathbb{Z}) + [-1, 1]$ is compact, it follows that $|x - x'|$ is bounded on $\{|y| \leq 1\}$. Likewise, $|x - x'|$ is bounded on $\{|y'| \leq 1\}$. Let C be the maximum of $|x - x'|$ over $\{|y| \leq 1\} \cup \{|y'| \leq 1\}$. Then $|y| \geq 1$ and $|y'| \geq 1$ on $\{|x - x'| \geq C\}$. Since f satisfies a positive twist condition, we have $y \geq 1$ and $y' \geq 1$ when $x' - x \geq C$ and $y \leq -1$, $y' \leq -1$, when $x' - x \leq -C$. Since $y = -h_1(x, x')$, $y' = h_2(x, x')$, we obtain

$$h_1(x, x') \leq -1, h_2(x, x') \geq 1, \quad \text{if} \quad x' - x \geq C,$$

$$h_1(x, x') \geq 1, h_2(x, x') \leq -1, \quad \text{if} \quad x' - x \leq -C.$$

Since $h(x+1, x'+1) = h(x, x')$ and h is continuous, there exists a constant A such that $h(x, x') \geq A$ on $|x' - x| \leq C$. Combining this with the inequalities above, we obtain

$$h(x, x') \geq A - C + |x - x'|,$$

everywhere. Hence

$$W(x) \geq q(A - C) + \sum_{i=0}^{nqr^{-1}-1} rn^{-1} |x_i - x_{i+1}|,$$

for $x = (\ldots, x_i \ldots) \in \mathscr{X}_{pqn}$ and it follows immediately that W is proper on $\mathscr{X}_{pqn}/T$. □

It follows immediately from Lemma 8.1 that W takes its minimum value on $\mathscr{X}_{pqn}$. Let $x \in \mathscr{X}_{pqn}$ minimize W. Let ϕ be the unique element of Y_n^* such that $x = x_{\phi\omega t}$ for $0 \leq t < rq^{-1}$. By the remarks preceding Lemma 8.1, we see that ϕ minimizes F_ω over Y_n^*. By Lemma 7.3, we obtain that ϕ_ω minimizes F_ω over Y_n^*. Recalling that ω was an arbitrary rational number throughout this section, we obtain:

LEMMA 8.2. *When ω is rational, ϕ_ω minimizes F_ω over Y_n^*.* □

§9. Approximation by rationals

Having established in Lemma 8.2 that ϕ_ω minimizes F_ω over Y_n^* when ω is rational, we will now prove it when ω any real number, by means of an approximation argument. This argument will depend on two facts: first, for

$\phi \in Y_n^*$, we have that $\omega \mapsto F_\omega(\phi)$ is continuous. Second, for $\phi \in Y$, this continuity is uniform in ϕ. These facts are stated and proved in the next two lemmas.

LEMMA 9.1. *For any $\phi \in Y_n^*$, the mapping $\omega \mapsto F_\omega(\phi)$ is continuous.*

Proof. Let $\omega \in \mathbb{R}$ and let $\varepsilon > 0$. Let $C = \sup |\phi| \mid [0, n] \cup [\omega - 1, \omega + n + 1]$. We have $C < \infty$, since $\phi \in Y_n^*$. Let $B = \sup |h| \mid [-C, C]^2$. Since ϕ is measurable, there exists a compact set $X \subset [\omega - 1, \omega + n + 1]$ such that the Lebesque measure of $[\omega - 1, \omega + n + 1] \backslash X$ is less than $n\varepsilon/5B$ and $\phi \mid X$ is continuous. Let $\delta_1 > 0$ be such that if $|x|$, $|x'|$, $|x''| \leq C$ and $|x' - x''| < \delta_1$, then $|h(x, x') - h(x, x'')| < \varepsilon/5$. Let $1 > \delta > 0$ be such that if t, $t' \in X$ and $|t - t'| < \delta$, then $|\phi(t) - \phi(t')| < \delta_1$.

Suppose $\omega' \in \mathbb{R}$ and $|\omega' - \omega| < \delta$. Let $t \in [0, n]$. If $t + \omega$ and $t + \omega'$ are both members of X, we have $|\phi(t + \omega) - \phi(t + \omega')| < \delta_1$ and consequently, $|h(\phi(t), \phi(t + \omega)) - h(\phi(t), \phi(t + \omega'))| < \varepsilon/5$.

Let E be the set of $t \in [0, n]$ such that at least one of $t + \omega$ or $t + \omega'$ is not in X. Clearly, the Lebesque measure of E is less than $2n\varepsilon/5B$, and $|h(\phi(t), \phi(t + \omega)) - h(\phi(t), \phi(t + \omega'))| \leq |h(\phi(t), \phi(t + \omega))| + |h(\phi(t), \phi(t + \omega'))| \leq 2B$, for all $t \in [0, n]$. Consequently, setting $E' = [0, n] \backslash E$, we have

$$|F_\omega \phi - F_{\omega'} \phi| \leq \left[\int_E + \int_{E'}\right] |h(\phi(t), \phi(t + \omega)) - h(\phi(t), \phi(t + \omega'))|\, dt$$

$$< [(2n\varepsilon/5B)(2B) + n\varepsilon/5] = \varepsilon. \quad \square$$

LEMMA 9.2. *For any compact set K of real numbers, there exists a constant C such that if $\phi \in Y$ and ω, $\omega' \in K$, then $|F_\omega(\phi) - F_{\omega'}(\phi)| \leq C\,|\omega' - \omega|$.*

Proof. Let N be an integer such that $|\omega| \leq N$ and $|\omega'| \leq N$, if $\omega, \omega' \in K$. Since $\phi \in Y$, we have $|\phi(t) - \phi(t + \omega)| \leq N$ and $|\phi(t) - \phi(t + \omega')| \leq N$, if $\omega, \omega' \in K$. Let $C = \sup \{|h_2(x, x')| : |x' - x| \leq N\}$. We have $C < \infty$, since h is C^1 and $h(x + 1, x' + 1) = h(x, x')$. Then

$$|F_\omega(\phi) - F_{\omega'}(\phi)| = \left| \int_0^1 (h(\phi(t), \phi(t + \omega)) - h(\phi(t), (t + \omega')))\, dt \right|$$

$$\leq C \int_0^1 |\phi(t + \omega) - \phi(t + \omega')|\, dt = C\,|\omega - \omega'|,$$

if $\omega, \omega' \in K$. The inequality is a consequence of the mean value theorem, the definition of C, and the fact that $|\phi(t) - \phi(t + \omega)|$ and $|\phi(t) - \phi(t + \omega')|$ are both less than N.

From the periodicity $\phi(t+1)=\phi(t)+1$, we get

$$\int_0^1 \phi(t+\omega)\,dt=\int_\omega^{\omega+1}\phi(t)\,dt=\int_\omega^{[\omega]+1}\phi(t)\,dt+\int_{[\omega]+1}^{\omega+1}\phi(t)\,dt$$

$$=\int_{\omega-[\omega]}^{1}(\phi(t)+[\omega])\,dt+\int_0^{\omega-[\omega]}(\phi(t)+[\omega]+1)\,dt=\int_0^1\phi(t)\,dt+\omega.$$

Since ϕ is weakly order preserving, the sign of $\phi(t+\omega)-\phi(t+\omega')$ never changes, and we obtain the last equation. □

Proof that ϕ_ω minimizes F_ω over Y_n^*. In §5, we showed (by a slight modification of the argument in [16]) that there exists an element ϕ_ω in Y which minimizes F_ω over Y. Now we will finish the proof that ϕ_ω minimizes F_ω over the larger space Y_n^*, which is the first assertion in Theorem 6.

Let $\phi\in Y_n^*$. We wish to prove that $F_\omega(\phi_\omega)\le F_\omega(\phi)$. It is enough to prove that there exists $\phi^*\in Y$ such that $F_\omega(\phi^*)\le F_\omega(\phi)$, since ϕ_ω minimizes F_ω over Y.

By Lemma 7.3, if ϕ minimizes F_ω over Y_n^*, then $F_\omega(\phi)=F_\omega(\phi_\omega)$ and we are done. Therefore, we may assume that ϕ does not minimize F_ω over Y_n^*. Then there exists $\phi'\in Y_n^*$ such that $F_\omega(\phi')<F_\omega(\phi)$. Let $\varepsilon=F_\omega(\phi)-F_\omega(\phi')$. Let K be a compact subset of $\mathbb{R}$ which contains ω in its interior. Let C be the constant given by Lemma 9.2. Let p/q be a rational number in K such that $C\,|\omega-p/q|<\varepsilon/2$ and $|F_\omega(\phi')-F_{p/q}(\phi')|<\varepsilon/2$. Such a number exists by Lemma 9.1. By Lemma 8.2, $F_{p/q}(\phi_{p/q})\le F_{p/q}(\phi')$. By Lemma 9.2, $|F_\omega(\phi_{p/q})-F_{p/q}(\phi_{p/q})|\le C\,|\omega-p/q|<\varepsilon/2$, since, by definition, $\phi_{p/q}\in Y$. Hence,

$$F_\omega(\phi_{p/q})<F_{p/q}(\phi_{p/q})+\varepsilon/2\le F_{p/q}(\phi')+\varepsilon/2<F_\omega(\phi')+\varepsilon=F_\omega(\phi).$$

We may therefore take $\phi^*=\phi_{p/q}$. □

§10. Uniqueness up to translation of the minimizing element

In §9, we finished the proof of the first assertion in Theorem 6. In this section, we will prove the remaining assertion in Theorem 6.

Let A denote the set of $a\in\mathbb{R}$ such that $\{t\in\mathbb{R}:\phi(t)>\phi_\omega T_a(t)\}$ has positive measure. Let B denote the set of $a\in\mathbb{R}$ such that $\{t\in\mathbb{R}:\phi(t)<\phi_\omega T_a(t)\}$ has positive measure. If $a\in\mathbb{R}\backslash(A\cup B)$, then $\phi=\phi_\omega T_a$, almost everywhere. Consequently, to prove the last assertion in Theorem 6, it is enough to show that $A\cup B\ne\mathbb{R}$. We will suppose that $A\cup B=\mathbb{R}$ and obtain a contradiction.

By the order preserving property of ϕ_ω, we have that if $b<a$ and $a\in A$ then $b\in A$, and that if $b<a$ and $b\in B$ then $a\in B$. Moreover, since ϕ_ω is order

preserving, it is continuous at all but at most countably many points. Consequently, A and B are open. Therefore, the hypothesis that $A \cup B = \mathbb{R}$ implies that $A \cap B$ is an open interval.

Let $E_a = \{t \in [0, n] : (\phi(t) - \phi_\omega T_a(t))(\phi(t+\omega) - \phi_\omega T_a(t+\omega)) < 0\}$. Taking $\phi' = \phi_\omega T_a$, we have that E_a is the set E which appears in the proof of Lemma 7.1. It is clear from the proof of Lemma 7.1 that the inequality in Lemma 7.1 is strict if and only if E has positive measure. By hypothesis, ϕ minimizes F_ω over Y_n^*. By the first assertion in Theorem 6 and the translation invariance of F_ω, $\phi' = \phi_\omega T_a$ minimizes F_ω over Y_n^*. Since both ϕ and ϕ' minimize F_ω over Y_n^*, the inequality in Lemma 7.1 cannot be strict. Consequently, $E_a = E$ has zero measure, whatever the real number a is.

Let $F_a = \{t \in [0, n] : \phi(t) = \phi_\omega T_a(t)\}$, $F = \{(t, a) : t \in F_a\}$. Clearly, F is a measurable subset of $\mathbb{R}^2$. The mapping ϕ_ω is strictly order preserving according to Addendum 2 to the Theorem in [16]. (Strictly speaking, we cannot apply the results of [16], since the conditions we have imposed on f in this paper are slightly different from the conditions which we have imposed in [16], as we have pointed out in §5. However, the proof of Addendum 2 in [16, §12] applies in the context of this paper, without change.) Since ϕ_ω is strictly increasing, for each $t \in \mathbb{R}$ there is at most one $a \in \mathbb{R}$ such that $\phi(t) = \phi_\omega T_a(t)$. By Fubini's theorem, it follows that F has vanishing planar measure; a second application of Fubini's theorem shows that F_a has zero linear measure, for almost all $a \in \mathbb{R}$.

Since $A \cap B$ is an open interval, it follows that we may choose $a \in A \cap B$ such that F_a has zero measure. Let $G_a = \{t : \phi(t) > \phi_\omega T_a(t)\}$, $H_a = \{t : \phi(t) < \phi_\omega T_a(t)\}$. The translation $t \mapsto t + \omega$ maps $G_a \backslash E_a$ into $G_a \cup F_a$ and $H_a \backslash E_a$ into $H_a \cup F_a$. Moreover, the image of $G_a \cup F_a$ under this translation contains $G_a \backslash E_a$ and the image of $H_a \cup F_a$ contains $H_a \backslash E_a$. Since E_a and F_a have zero measure, it follows that G_a and H_a are invariant (mod. sets of zero measure) under $t \mapsto t + \omega$. By the periodicity property $\phi(t+n) = \phi(t) + n$ of elements of Y_n^*, G_a and H_a are invariant under $t \mapsto t + n$.

Since ω is irrational and n is an integer, the group of translations generated by ω and n is ergodic, i.e. every set invariant (mod. sets of zero measure) under both these translations has zero measure or full measure. But this contradicts the facts that G_a and H_a both have positive measure (since $a \in A \cap B$), both are invariant (mod. sets of zero measure) under both these translations, and $G_a \cap H_a = \varnothing$ (by definition). □

§11. The Aubry–Le Daeron notion of ground-state configurations

When ω is irrational, the set M_ω which we defined near the end of §5 (and in [16]) corresponds to the set of ground-state configurations of mean atomic distance ω, in the terminology of Aubry and Le Daeron [16]. The purpose of this section is threefold: first, to explain the Aubry–Le Daeron notion of stationary configurations and of ground state configurations of mean atomic distance ω; second, to explain how stationary configurations correspond to orbits; and third, to show that a stationary configuration corresponds to an orbit in M_ω if and only if it is a ground-state configuration of mean atomic distance ω.

These results should be obvious for anyone who has mastered the theory of Aubry and Le Daeron [16]. We have included them for the convenience of the reader and for the sake of completeness. The third point above does not seem to follow easily from the theory developed by the author in [16]; for this, the theory of Aubry and Le Daeron seems superior. In this section, we will deduce this third point from Theorem 6 and results of Aubry and Le Daeron [16].

In the Aubry–Le Daeron terminology, a bi-infinite sequence $x = (\dots, x_i, \dots)$ of real numbers is a *configuration of atoms*, the x_i's representing the atoms. Given a configuration $x = (\dots, x_i, \dots)$ and integers $m < n$, we set $W_{mn}(x) = \sum_{i=m}^{n-1} h(x_i, x_{i+1})$. Following the Aubry–Le Daeron terminology, we say that a configuration is a *minimal energy configuration* if for any pair of integers $m < n$ and any configuration x' such that $x'_m = x_m$ and $x'_n = x_n$, we have $W_{mn}(x) \leq W_{mn}(x')$. It is an immediate consequence of the Fundamental Lemma of [4] that if x is a minimal energy configuration and x' is any *other* configuration with $x'_m = x_m$ and $x'_n = x_n$ then we actually have *strict* inequality; $W_{mn}(x) < W_{mn}(x')$. A minimal energy configuration is clearly a stationary configuration, in the sense that $h_2(x_{i-1}, x_i) + h_1(x_i, x_{i+1}) = 0$, where h_1 and h_2 denote the first partial derivations of h, with respect to the first and second variables, respectively. If we set $y_i = -h_1(x_i, x_{i+1})$, then $f(x_i, y_i) = (x_{i+1}, y_{i+1})$, if x is a stationary configuration; in this way, we obtain a one-one correspondence between stationary configurations and orbits of f. Thus, the minimal energy configurations correspond to a class of orbits, which we call *minimal energy orbits*.

PROPOSITION 11.1. *Let ω be irrational. Every orbit in M_ω is a minimal energy orbit.*

Proof. Given $\phi \in Y$ and $t \in \mathbb{R}$, we define configurations $x_{\phi\omega t+}$ and $x_{\phi\omega t-}$ by $(x_{\phi\omega t+})_i = \phi(t + \omega i +)$, $x_{\phi\omega t-} = \phi(t + \omega i -)$, for $i \in \mathbb{Z}$. In view of the definition of M_ω (end of §5), the order preserving property of ϕ_ω, and the left-continuity of ϕ_ω, an orbit of f is in M_ω if and only if there exists $t \in \mathbb{R}$ such that the corresponding stationary configuration is either $x_{\phi\omega t+}$ or $x_{\phi\omega t-}$, with $\phi = \phi_\omega$. Thus, it is enough to prove that these configurations are minimal energy configurations.

Suppose, for example, that $x_{\phi\omega t+}$ (where $\phi = \phi_\omega$) is not a minimal energy configuration, so there exists a configuration x' and integers $m < n$ such that

$x'_m = (x_{\phi\omega t+})_m$, $x'_n = (x_{\phi\omega t+})_n$, and $W_{mn}(x') < W_{mn}(x_{\phi\omega t+})$. Let $\delta > 0$ and set

$$\phi'(s) = x'_i, \quad \text{if } m \leq i \leq n,\ t+\omega i \leq s \leq t+\omega i+\delta,$$

$$\phi'(s+1) = \phi'(s), \quad \text{for all } s, \text{ and}$$

$$\phi'(t) = \phi_\omega(t), \quad \text{whenever } \phi'(t) \text{ is not defined by the previous two conditions.}$$

For $\delta > 0$ small enough, there is no contradiction between the first two conditions, since ω is irrational. Consequently, ϕ' is defined. Clearly, $\phi' \in Y_n^*$. Moreover,

$$F_\omega(\phi_\omega) - F_\omega(\phi') = \int_t^{t+\delta} (A(s) + B(s) + C(s))\, ds,$$

where

$$A(s) = W_{mn}(x_{\phi\omega s}) - W_{mn}(x')$$

$$B(s) = h(\phi_\omega(s+\omega(m-1)), \phi_\omega(s+\omega m)) - h(\phi_\omega(s+\omega(m-1), \phi_\omega(t+\omega m+)))$$

$$C(s) = h(\phi_\omega(s+\omega n), \phi_\omega(s+\omega(n+1)) - h(\phi_\omega(t+\omega n+), \phi_\omega(s+\omega(n+1)))).$$

Clearly, $A(s) \to W_{mn}(x_{\phi\omega t+}) - W_{mn}(x') > 0$, $B(s) \to 0$, and $C(s) \to 0$, as $s \downarrow t$. Consequently, $F_\omega(\phi_\omega) - F_\omega(\phi') > 0$, for $\delta > 0$ small enough. But this contradicts the fact that ϕ_ω minimizes F_ω over Y_1^* (Theorem 6).

This contradiction shows that $x_{\phi\omega t+}$ is a minimal energy configuration. The proof that $x_{\phi\omega t-}$ is a minimal energy configuration is similar.

Aubry and Le Daeron have shown [4, Theorem 3] that if $x = (\ldots, x_i \ldots)$ is a minimal energy configuration, then

$$l = \lim_{|i-j| \to \infty} \frac{x_i - x_j}{|i-j|}$$

exists. In their terminology, l is called the *mean atomic distance* of the configuration x. It is the same as the angular rotation number of the corresponding orbit. (The conditions that Aubry and Le Daeron impose on f are slightly different from the conditions which we impose, but their proof works without essential change under our hypothesis. This comment applies also to the other results of Aubry and Le Daeron which we quote in this and the next section.)

Aubry and Le Daeron have also shown [4, Theorem 4] that if ω is irrational, then the set of minimal energy configurations of mean atomic distance ω is totally ordered, viz. if x and x' are minimal energy configurations of mean atomic distance ω, then one of the following holds: $x_i < x'_i$, for all $i \in \mathbb{Z}$; $x_i = x'_i$, for all $i \in \mathbb{Z}$; or $x_i > x'_i$, for all $i \in \mathbb{Z}$.

Let $\mathrm{Min}_\omega \subset \mathbb{R}^2$ be the union of all minimal energy orbits of angular rotation number ω. The statement that the set of all minimal energy configurations of mean atomic distance ω is ordered is equivalent to the statement that Min_ω is f-monotone in the sense of [18], i.e. if pr_1 denotes the projection of $\mathbb{R}^2$ on its first factor, then $pr_1 : \mathrm{Min}_\omega \to \mathbb{R}$ is injective and f preserves the order on Min_ω induced from the order on $\mathbb{R}$.

From the definition of minimal energy orbit, it follows easily that the union of all minimal energy orbits is a closed subset of $\mathbb{R}^2$. If ω is irrational, the fact that Min_ω is f-monotone is easily seen to imply that Min_ω is closed. Moreover, Min_ω is bounded in the vertical direction, i.e. there exist constants $A < B$ such that $\mathrm{Min}_\omega \subset \mathbb{R} \times [A, B]$.

For, if we write $f(x, y) = (x', y')$, we have that $x' \to \pm\infty$, as $y \to \pm\infty$, for fixed x, by the fifth condition imposed on f in §2. It is an easy consequence of the positive monotone twist condition $\partial x'/\partial y > 0$ (i.e., the fourth condition in §2) and the periodicity condition $fT = Tf$ (i.e. the first condition in §2) that this convergence is uniform in the sense that $x' - x \to \pm\infty$ as $y \to \pm\infty$, uniformly in x. Consequently, we may choose A such that $x' < x + [\omega]$ when $y \leq A$ and B such that $x' > x + [\omega] + 1$ when $y \geq B$, where $[\omega]$ denotes the greatest integer $\leq \omega$. Taking into account the hypothesis that ω is irrational and therefore is not an integer, the fact that Min_ω if f monotone, the fact that $fT = Tf$, and the fact that every orbit in Min_ω has angular rotation number ω, we obtain that $[\omega] \leq x' - x \leq [\omega] + 1$ if $(x, y) \in \mathrm{Min}_\omega$ and $(x', y') = f(x, y)$. Consequently, $\mathrm{Min}_\omega \subset \mathbb{R} \times [A, B]$, as asserted.

Since Min_ω is closed, bounded in the vertical direction, and T-invariant, it follows that $\mathrm{Min}_\omega/T \subset (\mathbb{R}/\mathbb{Z}) \times \mathbb{R}$ is compact. Let pr_1 denote the projection of $(\mathbb{R}/\mathbb{Z}) \times \mathbb{R}$ on its first factor. Since Min_ω is f-monotone, it follows that $pr_1 : \mathrm{Min}_\omega/T \to \mathbb{R}/\mathbb{Z}$ is injective and $\bar{f}$ preserves the cyclic order on Min_ω/T induced from that on $\mathbb{R}/\mathbb{Z}$. The rotation number of $\bar{f} : \mathrm{Min}_\omega/T \to \mathrm{Min}_\omega/T$ defined with respect to this cyclic order is obviously ω.

In summary, $\bar{f} : \mathrm{Min}_\omega/T \to \mathrm{Min}_\omega/T$ is a homeomorphism of a cyclically ordered, compact metric space, of rotation number ω. Because ω is irrational, it follows that the set of recurrent orbits in Min_ω/T is the unique minimal set of Min_ω/T. The proof of this is the same as in the case of an orientation preserving homeomorphism of the circle with irrational rotation number. See, for example, Herman [12, II.7].

By Proposition 11.1, $M_\omega \subset \mathrm{Min}_\omega$. Because $\Sigma_\omega = M_\omega/T \subset \mathrm{Min}_\omega/T$ is a Denjoy minimal set, it is the unique minimal set of Min_ω/T, which, as we have seen, is the set of recurrent points of Min_ω/T. We have proved:

PROPOSITION 11.2. *Σ_ω is the set of $\bar{f}$-recurrent points in* Min_ω/T. $\square$

Our purpose in going through this lengthy but elementary discussion is to show the relation between the Aubry–Le Daeron theory [4] and our theory [16]. (See also Chenciner [9].) Aubry and Le Daeron call a minimal energy configuration a *ground-state configuration* if the corresponding orbit in $(\mathbb{R}/\mathbb{Z})\times\mathbb{R}$ is recurrent. We may restate Proposition 11.2 in the Aubry–Le Daeron terminology, as follows:

PROPOSITION 11.3. *An f-orbit is in M_ω if and only if the corresponding stationary configuration is a ground-state configuration of atomic mean distance ω.* $\square$

§12. The Aubry–Le Daeron notion of minimal energy configurations

Throughout this section, we suppose that ω is irrational. We recall the definition of M'_ω from [18] and show that an f-orbit is in M'_ω if and only if the corresponding stationary configuration is a minimal energy configuration in the sense defined in the previous sedtion.

We continue to let ϕ_ω be an element of Y where F_ω takes its minimum value. For $t\in\mathbb{R}$, we let $\mathscr{X}_{\omega t}$ denote the set of states $x=(\dots,x_i,\dots)$ such that $\phi_\omega(t+\omega i-)\leq x_i\leq\phi_\omega(t+\omega i+)$. Thus $\mathscr{X}_{\omega t}=\prod_{i=-\infty}^{\infty}[\phi_\omega(t+\omega i-),\phi_\omega(t+\omega i+)]$. We provide $\mathscr{X}_{\omega t}$ with the product topology. When t is a point of continuity of ϕ_ω, we have that $\mathscr{X}_{\omega t}$ is one point; otherwise, $\mathscr{X}_{\omega t}$ is the Hilbert cube. For $x\in\mathscr{X}_{\omega t}$, we set

$$G_{\omega t}(x)=\sum_{i=-\infty}^{\infty} h(x_i,x_{i+1})-h(x_i^-,x_{i+1}^-),$$

where $x_i^\pm=\phi_\omega(t+\omega i\pm)$. We have $\sum_{i=-\infty}^{\infty} x_i^+-x_i^-\leq 1$, since the intervals (x_i^-,x_i^+) are distinct holes in the Cantor set $pr_1\Sigma_\omega\subset\mathbb{R}/\mathbb{Z}$ (where pr_1 denotes the projection of $\mathbb{R}/\mathbb{Z}\times\mathbb{R}$ on its first factor; cf. [18, §13]). Consequently, the sum above is absolutely convergent and $G_{\omega t}$ is a continuous function on $\mathscr{X}_{\omega t}$. According to [18, Lemma 6.2], $G_{\omega t}(x^-)=G_{\omega t}(x^+)=0$ and $G_{\omega t}\geq 0$, everywhere on $\mathscr{X}_{\omega t}$. (Note that the sign convention in [18] is opposite that which we are using in this paper. What we called G_ω there is $G_{\omega t}$, x^0 is x^-, and x^1 is x^+.)

As in [18, §9], we define $M_{\omega t}$ to be the set of all (x_i,y_i), where x ranges over

$G_{\omega t}^{-1}(0)$, i ranges over all integers, and $y_i = -h_1(x_i, x_{i+1})$. We let

$$M'_\omega = M_\omega \cup \bigcup_t (M'_{\omega t}),$$

where t ranges over $\mathbb{R}$. This is the definition given in [18, §9]. This is related to the Aubry–De Daeron notion of minimal energy configurations by:

PROPOSITION 12. *If ω is irrational, then* $\mathrm{Min}_\omega = M'_\omega$.

In other words, an f-orbit is in M'_ω if and only if the corresponding stationary configuration is a minimal energy configuration of atomic mean distance ω. Note that M'_ω is f-invariant by [18, Proposition 9.1].

Proof. First, consider an orbit in $M'_{\omega t}$ and let $x = (\ldots, x_i \ldots)$ be the corresponding stationary configuration. If the orbit is in M_ω, then the corresponding stationary configuration is a minimal energy configuration of atomic mean distance ω, by Proposition 11.3. Otherwise, by definition of M'_ω, there is a point t of discontinuity of ϕ'_ω such that $x \in \mathscr{X}_{\omega t}$ and $G_{\omega t}(x) = 0$.

It follows that x is a minimal energy configuration. For, otherwise, there is a second configuration x' and integers m and n with $x'_i = x_i$ for $i \leq m$ or $i \geq n$ and $W_{mn}(x') < W_{mn}(x)$. Consider integers $m' < m$ and $n' > n$. Let $x_i^- = \phi_\omega(t + \omega i -)$. Let $x_i^* = x_i^-$, for $i \leq m'$ and $i \geq n'$, and $x_i^* = x'_i$, for $m' < i < n'$. Then

$$\begin{aligned} W_{m'n'}(x^-) - W_{m'n'}(x^*) &= \sum_{i=-\infty}^{\infty} (h(x_i^-, x_{i+1}^-) - h(x_i^*, x_{i+1}^*)) \\ &= -G_{\omega t}(x) + \sum_{i=-\infty}^{\infty} (h(x_i, x_{i+1}) - h(x_i^*, x_{i+1}^*)) \\ &= A + B + C + D, \end{aligned}$$

where

$$A = \sum_{i=-\infty}^{m'-1} + \sum_{i=n'}^{\infty} (h(x_i, x_{i+1}) - h(x_i^-, x_{i+1}^-)),$$

$$B = h(x_{m'}, x_{m'+1}) - h(x_{m'}^-, x_{m'+1}),$$

$$C = h(x_{n'-1}, x_{n'}) - h(x_{n'-1}, x_{n'}^-),$$

$$D = W_{mn}(x) - W_{mn}(x').$$

The second equality above follows from the definition of $G_{\omega t}(x)$. The third follows from the fact that $G_{\omega t}(x) = 0$, the definition of x^*, and the fact that $x_i = x'_i$, for

$i \leqslant m$ or $i \geqslant n$. We have $D > 0$ by the choice of x'. We may arrange for $|A|$, $|B|$, and $|C|$ to be as small as we like by taking $-m'$ and n' large enough. For, $\sum_{i=-\infty}^{\infty}(x_i^+ - x_i^-) \leqslant 1$, as we showed in §11, and $x_i^- \leqslant x_i \leqslant x_i^+$, since $x \in \mathscr{X}_{\omega t}$. The fact that $|A|$, $|B|$, and $|C|$ may be taken to be arbitrarily small therefore follows from the mean value theorem, since h is C^1, $h(x+1, x'+1) = h(x, x')$, and $[\omega] \leqslant x_{i+1}^- - x_i^+ \leqslant x_{i+1}^+ - x_i^- \leqslant [\omega] + 1$. Since $|A|$, $|B|$, and $|C|$ may be taken as small as we like and $D > 0$, we therefore obtain $W_{m'n'}(x^-) - W_{m'n'}(x^*) > 0$, when $-m'$ and n' are large enough. But x^- is a minimal energy configuration by Proposition 11.3, since it corresponds to an orbit in M_ω. But $W_{m'n'}(x^-) - W_{m'n'}(x^*) > 0$ contradicts this fact. This contradiction shows that x is a minimal energy configuration. Since x^- and x^+ have atomic mean distance ω, so does x.

We have shown that every orbit in M'_ω corresponds to a minimal energy configuration of atomic mean distance ω, i.e. $M'_\omega \subset \mathrm{Min}_\omega$. It remains only to prove that this inclusion is an equality.

As we have pointed out in §11, the Aubry–Le Daeron theory implies that Min_ω is f-monotone. Therefore, if x is a configuration in corresponding to an orbit in Min_ω, but not in M_ω, then there exists a point t of discontinuity of ϕ_ω such that $x \in \mathscr{X}_{\omega t}$. It remains only to prove that $G_{\omega t}(x) = 0$.

Suppose, to the contrary, that $G_{\omega t}(x) > 0$. Let $x_i^{\mp} = \phi(t + \omega i \pm)$. Choose $m < n$ and let $x'_i = x_i$, for $i \leqslant m$ or $i \geqslant n$ and $x'_i = x_i^-$ for $m < i < n$. We have

$$W_{mn}(x) - W_{mn}(x') = G_{\omega t}(x) + A + B + C,$$

where

$$A = \sum_{i=-\infty}^{m-1} + \sum_{i=n}^{\infty} (h(x_i^-, x_{i+1}^-) - h(x_i, x_{i+1}))$$

$$B = h(x_m^-, x_{m+1}^-) - h(x_m, x_{m+1}^-)$$

$$C = h(x_{n-1}^-, x_n^-) - h(x_{n-1}, x_n^-).$$

Just as before, we many show that $|A|$, $|B|$, and $|C|$ may be taken to be arbitrarily small, by taking $-m$ and n to be sufficiently large. Since $G_{\omega t}(x) > 0$, we then obtain a contradiction to the hypothesis that x is a minimal energy configuration.

This contradiction shows that $G_{\omega t}(x) = 0$ and the orbit corresponding to x is in M'_ω. Thus, $\mathrm{Min}_\omega = M'_\omega$. □

§13. The theory of Aubry and Le Daeron for rational ω

In this section, we state without proof some results of Aubry and Le Daeron, in order to complete the discussion of the relation of their results to our results.

We refer to their paper for proofs. We will not need the results of this section later in this paper.

Let $\omega = p/q$ be a rational number expressed in lowest terms, $q>0$. Let x be a minimal energy configuration of atomic mean distance p/q. Then one of the following three possibilities holds: a) $x_{i+q} = x_i + p$, for all $i \in \mathbb{Z}$, b) $x_{i+q} > x_i + p$, for all $i \in \mathbb{Z}$, or c) $x_{i+q} < x_i + p$, for all $i \in \mathbb{Z}$. Assuming that one of these three possibilities holds, then the necessary and sufficient condition for x to be a ground-state configuration is that a) holds. Slightly modifying the terminology of Aubry and Le Daeron, we will call x an *advancing* minimal energy configuration when b) holds and a retreating minimal energy configuration when c) holds.

The statement that one of a), b), or c) holds is a restatement of part of [4, Theorem 5]. To see this, we introduce the notion of a *translate* of a configuration. If x and x' are configurations, one will be said to be the *translate* of the other if there exist integers j and k such that $x'_i = x_{i+j} + k$, for all integers i. It is clear that if x is a minimal energy configuration of atomic mean distance ω, then the set of its translates is totally ordered (in the sense of §11) if and only if one of a), b), or c) hold. According to [4, Theorem 5], a minimal energy configuration of mean atomic distance p/q is either a ground-state configuration, an "advanced elementary discommensuration" (in the terminology of [4]), or a "delayed elementary discommensuration" (cf. [4, formula (41)]). It follows from the definitions given in [4] that the translate of a minimal energy configuration of any one of these types is one of the same. Moreover, according to [4, Theorem 5] the set of all minimal energy configurations of mean atomic distance p/q and of one of these three types is totally ordered. In particular, the set of all translates of a given minimal energy configuration of mean atomic distance p/q is totally ordered, so one of a), b) or c) must hold.

It follows from the definitions given in [4] that, of these three possibilities, only a) can hold when x is a ground-state configuration; only b) can hold when x is an "advanced elementary discommensuration", and only c) can hold when x is a "delayed elementary discommensuration." Thus, what Aubry and Le Daeron call an "advanced (resp. delayed) elementary discommensuration of atomic mean distance p/q" is what we call an "advancing (resp. retreating) minimal energy configuration of atomic mean distance p/q."

Other results in [4, Theorem 5] are: First, the set of all minimal energy configurations of atomic mean distance p/q satisfying a) or b) is totally ordered, as is the set of all minimal energy configurations of atomic mean distance p/q satisfying a) or c). Second, for any minimal energy configuration x of atomic mean distance p/q satisfying b) (resp. c)), there are ground-state configurations x^-, x^+ of atomic mean distance p/q satisfying $x_i^- < x_i < x_i^+$, for all $i \in \mathbb{Z}$, $x_i^+ - x_i \to 0$ as $i \to +\infty$, (resp. as $i \to -\infty$) and $x_i - x_i^- \to 0$, as $i \to -\infty$ (resp. as $i \to +\infty$). Third, there always are ground-state configurations of atomic mean distance p/q and their union is a closed subset of $\mathbb{R}$. Fourth, suppose that x^- and x^+ are successive ground-state configurations of atomic mean distance p/q in the sense that $x^- < x^+$ and there are no ground-state configurations x of atomic mean distance p/q satisfying $x^- < x < x^+$. Then there are both advancing and retreating minimal energy configurations x of atomic mean distance p/q such that $x^- < x < x^+$. For the former (resp. latter), we have $x_i^+ - x_i \to 0$ as $i - i \to +\infty$ (resp. as $i \to -\infty$) and $x_i - x_i^-$ as $i \to -\infty$ (resp. as $i \to +\infty$). A closely related result as proved by Katok [14].

§14. Peierls's energy barrier

Let ω be irrational. We define a non-negative real-valued function P_ω on $\mathbb{R}$, as follows. Let $\xi \in \mathbb{R}$. There exists a unique $t \in \mathbb{R}$ such that $\phi_\omega(t-) \leq \xi \leq \phi_\omega(t+)$ since ϕ_ω is strictly order preserving, according to Addendum 2 to the Theorem in [16] (cf. §10). We set

$$P_\omega(\xi) = \min\{G_{\omega t}(x) : x \in \mathscr{X}_{\omega t} \text{ and } x_0 = \xi\}.$$

Since $\mathscr{X}_{\omega t}$ is compact and $G_{\omega t} : X_{\omega t} \to \mathbb{R}$ is continuous, this minimum value is actually achieved (cf. §12.)

Let $pr_1 : \mathbb{R}^2 \to \mathbb{R}$ be the projection of $\mathbb{R}^2$ on its first factor. When t is a point of continuity of ϕ_ω, we have that $\mathscr{X}_{\omega t}$ is reduced to the one point x^-, and $G_{\omega t}(x^-) = 0$, $x_0^- = \xi$. Therefore, $P_\omega(\xi) = 0$, for $\xi = \phi_\omega(t)$.

More generally, we have $P_\omega(\xi) = 0$, if $\xi \in pr_1 M'_\omega$. For, if $\xi \in pr_1 M'_{\omega t}$, then $\xi = x_0$, where $(\dots, (x_i, y_i) \dots)$ is an orbit in $M'_{\omega t}$. By definition of $M'_{\omega t}$, we have $G_{\omega t}(x) = 0$, where $x = (\dots, x_i, \dots)$ Therefore, $P_\omega(\xi) = 0$, as asserted.

Conversely, we have $P_\omega(\xi) > 0$ if $\xi \notin pr_1 M'_\omega$. For, let t be such that $\phi_\omega(t-) \leq \xi \leq \phi_\omega(t+)$. Since $G_{\omega t} \geq 0$, we have $P_\omega(\xi) \geq 0$. If $P_\omega(\xi) = 0$, then by definition of $P_\omega(\xi)$, there exists $x = (\dots, x_i \dots) \in \mathscr{X}_{\omega t}$ such that $G_{\omega t}(x) = 0$ and $\xi = x_0$. Then $(x_0, y_0) \in M'_\omega$, where $y_0 = -h_1(x_0, x_1)$, by definition of M'_ω. So, $\xi = x_0 \in pr_1 M'_\omega$, contrary to our own assumption. This contradiction shows that $P_\omega(\xi) > 0$. We have thus shown:

PROPOSITION 14.1. *$P_\omega(\xi) \geq 0$, for all $\xi \in \mathbb{R}$, and $P_\omega(\xi) = 0$ if and only if $\xi \in pr_1 M'_\omega$.* □

This result was obtained independently by Aubry, Le Daeron, and André on the one hand and by Katok and the author on the other hand. It was announced in [5] and a closely related result was proved in [18]. The proof in [18] was based on joint research of Katok and the author.

The following result is a slight variant of Lemma 20.1 in [18].

PROPOSITION 14.2. *There exists an $\bar{f}$-invariant circle S_ω of angular rotation number ω if and only if $pr_1 M'_\omega = \mathbb{R}$. If such a circle exists, then M'_ω is the inverse image of S_ω under the projection $\mathbb{R}^2 \to (\mathbb{R}/\mathbb{Z}) \times \mathbb{R}$.*

Proof. If $pr_1 M'_\omega = \mathbb{R}$, then, clearly, the image of M'_ω in $(\mathbb{R}/\mathbb{Z}) \times \mathbb{R}$ under the projection of $\mathbb{R}^2$ on $(\mathbb{R}/\mathbb{Z}) \times \mathbb{R}$ is an $\bar{f}$-invariant circle.

Conversely, suppose an $\bar{f}$-invariant circle S_ω of angular rotation number ω exists. Lemma 20.1 of [18] says that M'_ω is the inverse image of S_ω under the projection $\mathbb{R}^2 \to (\mathbb{R}/\mathbb{Z}) \times \mathbb{R}$. (The hypothesis which we have imposed on f in [18] are slightly different from those which we have imposed on f in this paper. But, that makes no difference in the proof of Lemma 20.1). It follows immediately that if there is an invariant circle, then $pr_1 M'_\omega = \mathbb{R}$. □

COROLLARY 14. *There exists an $\bar{f}$-invariant circle of angular rotation number ω if and only if P_ω vanishes everywhere.* □

The result was announced, but not proved in [5]. Theorem 5.2 of [18] is closely related; it states that there exists an $\bar{f}$-invariant circle of angular rotation number ω if and only if $\Delta W_\omega = 0$. We refer to [18] for the definition of ΔW_ω. We have $P_\omega(\xi) \leq \Delta W_\omega$. A discussion of the relation between the two results is contained in [18, §25].

The quantity $P_\omega(\xi)$ appears to be what Aubry, Le Daeron, and André call Peierls's energy barrier in [5].

§15. The main theorem

Throughout the rest of this paper, we let ω and ξ be fixed real numbers, and n a fixed positive integer. In this section, we let Δ be a function of $\mathbb{Z}$ to $\mathbb{R}$ of period n.

DEFINITION. We denote by $\mathcal{A}_{n\xi\Delta}$ the set of all measurable mappings $\phi : \mathbb{R} \to \mathbb{R}$ with the following three properties:

1) $\phi(t+n) = \phi(t) + n$,
2) $\phi(t) \leq \xi + j$, if $t \leq j + \Delta(j)$, and
3) $\phi(t) \geq \xi + j$, if $t > j + \Delta(j)$,

for all $j \in \mathbb{Z}$.

DEFINITION. We denote by $\mathcal{B}_{n\xi\Delta}$ the subset of $\mathcal{A}_{n\xi\Delta}$ consisting of those ϕ which are weakly order preserving and continuous from the left.

In other words, if $\phi \in \mathcal{A}_{n\xi\Delta}$, then it is in $\mathcal{B}_{n\xi\Delta}$ if and only if $s \leq t$ implies $\phi(s) \leq \phi(t)$ and, for all $t \in \mathbb{R}$, we have $\phi(t) = \phi(t-)$. (Recall, from §5, that $\phi(t-) = \lim_{s\uparrow t} \phi(s)$ and $\phi(t+) = \lim_{s\downarrow t} \phi(s)$.)

The condition of continuity from the left is imposed only for reasons of technical convenience; we could just as well require continuity from the right.

It is easy to see that the necessary and sufficient condition for $\mathcal{A}_{n\xi\Delta}$ to be non-empty is that $j + \Delta(j) \leq j + 1 + \Delta(j+1)$, for all $j \in \mathbb{Z}$. This is also the condition for $\mathcal{B}_{n\xi\Delta}$ to be non-empty.

We set $\|\Delta\| = \max_{i,j\in\mathbb{Z}} |\Delta(i) - \Delta(j)|$.

THEOREM 15. *If $\mathcal{A}_{n\xi\Delta}$ is non-empty, then there exists $\phi_{\omega n\xi\Delta} \in \mathcal{B}_{n\xi\Delta}$ which minimizes F_ω over $\mathcal{A}_{n\xi\Delta}$.*

If ω is irrational, $P_\omega(\xi) > 0$, and $\|\Delta\|$ is sufficiently small, then the inequalities 2) *and* 3) *are strict for $\phi = \phi_{\omega n\xi\Delta}$. More precisely, we have that if $\phi \varepsilon \mathcal{B}_\Delta = \mathcal{B}_{n\xi\Delta}$ minimizes F_ω over $\mathcal{B}_\Delta$, then*

2′) $\phi(t-) < \xi + j$, if $t \leq j + \Delta(j)$, *and*

3′) $\phi(t+) > \xi + j$, *if* $t \geq j + \Delta(j)$,

for all $j \in \mathbb{Z}$.

Moreover, we have the following form of uniqueness: Suppose there exists $\phi_\Delta \in \mathcal{B}_\Delta$ which minimizes F_ω over $\mathcal{A}_\Delta = \mathcal{A}_{n\xi\Delta}$ and which satisfies 2′) *and* 3′). *Suppose, in addition, that ω is irrational. Then any member of $\mathcal{A}_\Delta$ which minimizes F_ω over $\mathcal{A}_\Delta$ differs from ϕ_Δ at most on a set of zero measure.*

We may summarize Theorem 15, as follows: When ω is irrational, $P_\omega(\xi) > 0$, and $\|\Delta\|$ is sufficiently small, then F_ω takes its minimum value at a unique point in $\mathcal{A}_{n\xi\Delta}$ and that point is in $\mathcal{B}_{n\xi\Delta}$ and satisfies 2′) and 3′).

How small $\|\Delta\|$ has to be in order for 2′) and 3′) to hold depends on the Diophantine properties of ω, on how large $P_\omega(\xi)$ is, and on the size of the first derivative of the generating function h. An explicit estimate can be given in terms of the following quantities: We let

$$C(h, \omega, n) = \sup\{|h_1(x, x')| + |h_2(x, x')| : n[\omega/n] - 1 \leq x' - x \leq n[\omega/n] + n + 1\},$$

where $[\omega/n]$ denotes the greatest integer $\leq \omega/n$, and h_1 and h_2 denote the first partial derivatives of h with respect to the first and second variables, resp. Note that $C(h, \omega, n) < \infty$, since $h(x+1, x'+1) = h(x, x')$ and h is C^1.

For a positive number P, we let $N(h, \omega, n, P)$ be the least integer $> 4C(h, \omega, n)/P$. We let

$$\delta(h, \omega, n, P) = \min\{|i\omega + j| : i, j \in \mathbb{Z} \quad \text{and} \quad 0 < i \leq N(h, \omega, n, P)\}.$$

Obviously, the size of $\delta(h, \omega, n, P)$ depends on the Diophantine properties of ω, as well as on P and the size of the first derivative of h. When ω is irrational and $P > 0$, we have $\delta(h, \omega, n, P) > 0$.

ADDENDUM 1. *If ω is irrational and $P_\omega(\xi) > 0$, then $\mathcal{A}_{n\xi\Delta}$ is non-empty and* 2′) *and* 3′) *hold when $\|\Delta\| < \delta(h, \omega, n, P_\omega(\xi))$.*

The next two addenda are analogous to Addenda 1 and 2 of [16]:

ADDENDUM 2. *Suppose $\phi_{\omega n\xi\Delta}$ satisfies* 2′) *and* 3′). *If t is a point of continuity of $\phi_{\omega n\xi\Delta}$, then so are $t + \omega$ and $t - \omega$.*

ADDENDUM 3. *If ω is irrational and $\phi_{\omega n\xi\Delta}$ satisfies* 2′) *and* 3′), *then $\phi_{\omega n\xi\Delta}$ is not constant on any interval.*

In what follows, we will often use the abbreviations $\mathcal{A}_\Delta$, $\mathcal{B}_\Delta$, and ϕ_Δ for $\mathcal{A}_{n\xi\Delta}$, $\mathcal{B}_{n\xi\Delta}$, and $\phi_{\omega n\xi\Delta}$.

We will begin the proof of Theorem 15 in §17, where we will show that there exists $\phi_\Delta \in \mathcal{B}_\Delta$ which minimizes F_ω over $\mathcal{B}_\Delta$, provided that $\mathcal{B}_\Delta$ is not empty. This follows the method of [16], as outlined and modified in §5 of this paper. Then we will prove (§§18–20) a relative version of the first assertion in Theorem 6: ϕ_Δ minimizes F_ω over $\mathcal{A}_\Delta$. The proof of this is a slight modification of the argument in §§7–9. At this point we will have the proof of existence, i.e. the first assertion of Theorem 15. Uniqueness will be proved in §22, by a slight modification of the argument in §10. The inequalities 2′) and 3′) and Addendum 1 will be proved in §25. Addendum 2 will be proved in §26 and Addendum 3 in §21.

Before beginning the proof of Theorem 15 we discuss several applications of it in §16. Proposition 16 leads to the result announced in the abstract of this paper. See the discussion following Proposition 16, where we describe an $n-1$ dimensional disk which has the properties announced in the abstract of this paper. We will not actually prove that it is a topological $(n-1)$-disk until §29. See Theorem 29 and the discussion following it.

§16. Application of the main theorem: existence of many Denjoy minimal sets of angular rotation number ω and intrinsic rotation number $(\omega+R)/n$

In this section, we will assume that ω is irrational and that there is no invariant circle of angular rotation number ω for $\bar{f}$. We let R be an integer. We will show that Theorem 15 implies the existence of many Denjoy minimal sets of angular rotation number ω and intrinsic rotation number $(\omega+R)/n$ for $\bar{f}$.

By Proposition 14.1, $P_\omega \geqslant 0$. By our assumption that there is no invariant circle of angular rotation number ω for $\bar{f}$ and Corollary 14, it follows that P_ω doesn't vanish identically. We will assume throughout this section that $P_\omega(\xi)>0$; we may suppose this without loss of generality since ξ is an arbitrary real number.

We let $\mathscr{D}=\mathscr{D}_{\omega n \xi}$ denote the set of all mappings $\Delta:\mathbb{Z}\to\mathbb{R}$ of period n for which there exists $\phi\in\mathscr{B}_\Delta$ which minimizes F_ω over $\mathscr{A}_\Delta$ and which satisfies inequalities 2′) and 3′) in Theorem 15. By Theorem 15 and Addendum 1, $\Delta\in\mathscr{D}$ if $\|\Delta\|<\delta(h,\omega,n,P_\omega(\xi))$. Note that $\delta(h,\omega,n,P_\omega(\xi))>0$ by our assumptions that ω is irrational and $P_\omega(\xi)>0$.

If $\Delta\in\mathscr{D}$, we will define, in this section, a Denjoy minimal set Σ_Δ whose angular rotation number is ω and whose intrinsic rotation number is ω/k (mod. 1), where k is the (minimum) period of Δ. Moreover, we will show that if $\Delta,\Delta'\in\mathscr{D}$, then $\Sigma_\Delta=\Sigma_{\Delta'}$ if and only if $\Delta'-\Delta:\mathbb{Z}\to\mathbb{R}$ is constant. This will have the consequence that $\{\Sigma_\Delta\}$ is an $n-1$ parameter family.

The definition and properties of Σ_Δ depend on the existence of ϕ_Δ and the inequalities 2′) and 3′) of Theorem 15.

LEMMA 16.1. *Suppose $\Delta\in\mathscr{D}$. Then $\phi=\phi_\Delta$ satisfies the Euler–Lagrange equation*

$$h_2(\phi(t-\omega),\phi(t))+h_1(\phi(t),\phi(t+\omega))=0.$$

Proof. Of course, we use the usual method of computing $dF_\omega(\phi_\tau)/d\tau|_{\tau=0}$ along suitable test curves ϕ_τ. The test curves which we consider are of the form $\phi_\tau(t)=\phi(t)+\tau\dot{\phi}(t)$, where $\dot{\phi}$ is an arbitrary measurable bounded function satisfying $\dot{\phi}(t+n)=\dot{\phi}(t)$. To apply the usual method, we have to verify that $\phi_\tau\in\mathscr{A}_\Delta$ if $|\tau|$ is sufficiently small. It is obvious that ϕ_τ is measurable and satisfies equation 1) in the definition of $\mathscr{A}_\Delta=\mathscr{A}_{n\xi\Delta}$ (§15). The fact that inequalities 2) and 3) are satisfied for $|\tau|$ small enough follows from the fact that there exists $\delta>0$ such that

$$\phi(t-)<\xi+j-\delta,\quad \text{if}\quad t\leqslant j+\Delta(j), \text{ and}$$

$$\phi(t+)>\xi+j+\delta,\quad \text{if}\quad t\geqslant j+\Delta(j).$$

This, in turn, is a consequence of the fact that 2′) and 3′) in Theorem 15 hold, the fact that $\phi = \phi_\Delta$ is weakly order preserving (since it is in $\mathcal{B}_\Delta$ by Theorem 15), and the fact that $\phi(t+n) = \phi(t) + n$.

Since ϕ minimizes F_ω over $\mathcal{A}_\Delta$ and $\phi_\tau \in \mathcal{A}_\Delta$ for $|\tau|$ sufficiently small, $dF_\omega(\phi_\tau)/d\tau|_{\tau=0} = 0$, or

$$n^{-1}\int_0^n [h_1(\phi(t), \phi(t+\omega)) + h_2(\phi(t-\omega), \phi(t))]\dot{\phi}(t)dt = 0.$$

Since $\dot{\phi}$ is an arbitrary bounded measurable function of period n and ϕ is left continuous, the Euler–Lagrange equation follows. □

LEMMA 16.2. *Suppose $\Delta, \Delta' \in \mathcal{D}$ and $a \in \mathbb{R}$. We have $\phi_{\Delta'} = \phi_\Delta T_a$ if and only if $\Delta - \Delta' = a$.*

Recall that $T_a(t) = t + a$.

Proof. Since ϕ_Δ satisfies 2′) and 3′) of Theorem 15, we have that $\phi_\Delta T_a \in \mathcal{A}_{\Delta'}$ if and only if $\Delta - \Delta' = a$. Since $\phi_{\Delta'} \in \mathcal{A}_{\Delta'}$, this shows that if $\phi_{\Delta'} = \phi_\Delta T_a$, then $\Delta - \Delta' = a$.

Conversely, if $\Delta - \Delta' = a$, then $\phi \mapsto \phi T_a$ maps $\mathcal{A}_\Delta$ bijectively onto $\mathcal{A}_{\Delta'}$. Since F_ω is translation invariant, it follows from the uniqueness of the minimizing element in Theorem 15 that $\phi_{\Delta'} = \phi_\Delta T_a$. □

Consider $\Delta \in \mathcal{D}$. Let $\eta_\Delta(t) = -h_1(\phi_\Delta(t), \phi_\Delta(t+\omega)) = h_2(\phi_\Delta(t-\omega), \phi_\Delta(t))$. Let

$$M_\Delta = \overline{\{(\phi_\Delta(t), \eta_\Delta(t)) : t \in \mathbb{R}\}}.$$

From the Euler–Lagrange equation (Lemma 16.1), and the fact that h is a generating function for f, it follows that M_Δ is f-invariant, if $\phi \in \mathcal{D}$. Since $\phi_\Delta(t+n) = \phi_\Delta(t) + n$, it follows that M_Δ is T^n invariant, where $T(x, y) = (x+1, y)$.

LEMMA 16.3. *Suppose $\Delta, \Delta' \in \mathcal{D}$. Then $M_\Delta = M_{\Delta'}$ if and only if $\Delta' - \Delta$ is constant.*

Proof. First suppose that $\Delta - \Delta'$ is a constant, a. By Lemma 16.3, $\phi_{\Delta'} = \phi_\Delta T_a$. The equality $M_{\Delta'} = M_\Delta$ then follows immediately from the definition of M_Δ.

Conversely, suppose $M_\Delta = M_{\Delta'}$. By Theorem 15, $\phi_\Delta \in \mathcal{B}_\Delta$. It follows from this and Addendum 3 to Theorem 15 that ϕ_Δ is strictly order preserving and continuous from the left. From this, and the definition of M_Δ, it follows that a point $(x, y) \in M_\Delta$ has the form $(\phi_\Delta(t), \eta_\Delta(t))$ if and only if x is not the right endpoint of a complementary interval to $pr_1 M_\Delta$ if $\mathbb{R}$ (where pr_1 denotes the projection of $\mathbb{R}^2$ on its first factor). The same remarks applies with Δ' in the place of Δ.

Choose a point $(x, y) \in M_\Delta$ such that x is not the right endpoint of a complementary interval to $pr_1 M_\Delta$ in $\mathbb{R}$. Then $(x, y) = (\phi_\Delta(t_0), \eta_\Delta(t_0))$, for some $t_0 \in \mathbb{R}$. Since $M_\Delta = M_{\Delta'}$, we also have $(x, y) = (\phi_\Delta(t_0 - a), \eta_\Delta(t_0 - a))$, for some $a \in \mathbb{R}$.

From the fact that ϕ_Δ and $\phi_{\Delta'}$ satisfy the Euler–Lagrange equation (Lemma 16.1), the fact that h is a generating function for f, and the fact that ϕ_Δ and $\phi_{\Delta'}$ satisfy the periodicity condition $\phi(t+n) = \phi(t) + n$, it follows that

$$\begin{aligned} T^{kn} f^j(x, y) &= (\phi_\Delta(t_0 + j\omega + kn), \eta_\Delta(t_0 + j\omega + kn)) \\ &= (\phi_{\Delta'}(t_0 + j\omega + kn - a), \eta_{\Delta'}(t_0 + j\omega + kn - a)), \end{aligned}$$

for all integers j and k. Since ω is irrational and n is an integer, $\{t_0 + j\omega + kn : j, k \in \mathbb{Z}\}$ is dense in $\mathbb{R}$.

Thus, we have shown that $\phi_\Delta(t) = \phi_{\Delta'}(t - a)$, for a dense set of $t \in \mathbb{R}$. Since ϕ_Δ and $\phi_{\Delta'}$ are continuous from the left, it follows that we have this equation for all $t \in \mathbb{R}$. In other words, $\phi_{\Delta'} = \phi_\Delta T_a$. It then follows from Lemma 16.2 that $\Delta - \Delta' = a$. □

Let $j \in \mathbb{Z}$, $\Delta, \Delta' \in \mathscr{D}$. Suppose $\Delta'(i) = \Delta(i + j)$, for all $i \in \mathbb{Z}$. It follows immediately from the definitions and the uniqueness assertion in Theorem 15 that $\phi_{\Delta'}(t) + j = \phi_\Delta(t + j)$ and, consequently, $T^j M_{\Delta'} = M_\Delta$, where T is the Deck transformation $T(x, y) = (x + 1, y)$. We therefore obtain from Lemma 16.3:

LEMMA 16.4. *Suppose $\Delta, \Delta' \in \mathscr{D}$. Then $T^j M_{\Delta'} = M_\Delta$ if and only if $\Delta(i + j) - \Delta'(i)$ is independent of i.* □

When $\Delta \in \mathscr{D}$, we define $\Sigma_\Delta^{(k)} = M_\Delta / T^k \subset \mathbb{R}^2 / T^k = (\mathbb{R}/k\mathbb{Z}) \times \mathbb{R}$ and define $\Sigma_\Delta = \Sigma_\Delta^{(1)}$. We let $\bar{f}^{(k)}$ denote the homeomorphism of $\mathbb{R}^2/T^k$ induced by f.

In order to define the angular rotation number of $\Sigma_\Delta^{(k)}$ with respect to $\bar{f}^{(k)}$, it will be convenient to identify $\mathbb{R}^2/T^k$ with $\mathbb{R}^2/T$ by the homeomorphism $(x, y) \mapsto (x/k, y)$, and then use the previous definition. The resulting number is k^{-1} times the angular rotation number of Σ_Δ with respect to $\bar{f}$.

Suppose $\Delta \in \mathscr{D}$. Since every element of $\mathscr{D}$ is periodic of period n, the (minimum) period k of Δ divides n. From the uniqueness property of ϕ_Δ in Theorem 15 and the translation invarince of F_ω, it follows that ϕ_Δ commutes with

$t \mapsto t+k$. It follows that M_Δ is invariant under T^k, where T is the Deck transformation $T(x, y) = (x+1, y)$. Because ϕ_Δ is order preserving, $\Sigma_\Delta^{(k)}$ is a Denjoy minimal set for the homeomorphism of $\mathbb{R}^2/T^k$ induced by f, of angular rotation number ω/k and intrinsic rotation number congruent to ω/k (mod. 1).

We continue to denote by T the homeomorphism of $\mathbb{R}^2/T^k$ induced by T. Suppose $\Delta, \Delta' \in \mathcal{D}$. By Lemma 16.4, we have that $T^j\Sigma_{\Delta'}^{(n)} = \Sigma_\Delta^{(n)}$ if and only if $\Delta_{i+j} - \Delta'_i$ is independent of i. If $T^j\Sigma_{\Delta'}^{(n)} \neq \Sigma_\Delta^{(n)}$, then these two sets must be disjoint, since they are both minimal sets for the homeomorphism $\bar{f}^{(n)}$.

Taking $\Delta = \Delta'$ of period k, we obtain that the sets $T^j\Sigma_\Delta^{(k)}$ are disjoint for $j = 0, \ldots, k-1$. Consequently, the projection $\Sigma_\Delta^{(k)} \to \Sigma_\Delta$ (the restriction of $\mathbb{R}^2/T^k \to \mathbb{R}^2/T$) is a homeomorphism. It follows that the intrinsic rotation number of Σ_Δ with respect to $\bar{f}$ is the same as that of $\Sigma_\Delta^{(k)}$ for the homeomorphism of $\mathbb{R}^2/T^k$ induced by f, namely ω/k.

Summarizing, we have shown that Theorem 15 implies:

PROPOSITION 16. *If $\Delta \in \mathcal{D}$, then Σ_Δ is a Denjoy minimal set for $\bar{f}$, whose angular rotation number is ω, and whose intrinsic rotation number is congruent* (mod. 1) *to ω/k, where k is the* (*minimum*) *period of Δ.*

Moreover, if Δ' is a second member of $\mathcal{D}$, then $\Sigma_\Delta = \Sigma_{\Delta'}$ if and only if $\Delta' - \Delta$ is constant. If $\Delta' - \Delta$ is not constant, then Σ_Δ is disjoint from $\Sigma_{\Delta'}$. $\square$

The set of periodic mappings $\mathbb{Z} \to \mathbb{R}$ of period n may be identified with $\mathbb{R}^n$. We provide $\mathbb{R}^n$ with its standard topology. Thinking of $\mathcal{D}$ as a subset of $\mathbb{R}^n$, we provide it with the induced topology. By Theorem 15, $\mathcal{D}$ contains a neighborhood of the origin. On the other hand, among all $\Delta : \mathbb{Z} \to \mathbb{R}$ of period n, those whose minimum period is n form an open dense subset with respect to the standard topology on $\mathbb{R}^n$. Thus, Proposition 16 provides us with an $(n-1)$ parameter family of Denjoy minimal sets of angular rotation number ω and intrinsic rotation number ω/n.

Applying Proposition 16 to fT^R in place of f, we obtain Denjoy minimal sets which have angular rotation number $\omega + R$, with respect to fT^R, and intrinsic rotation number (mod. 1) $(\omega+R)/n$. Recall that the intrinsic rotation number doesn't depend on the lift (f or fT^R) of $\bar{f}$. On the other hand, the angular rotation number (defined as a real number) depends on the lift of $\bar{f}$; since it is $\omega + R$ for fT^R, it is ω for f.

Thus, we have found an $n-1$ parameter family of Denjoy minimal sets for $\bar{f}$ of angular rotation number ω and intrinsic rotation number congruent to $(\omega+R)/n$ (mod. 1). In §29, we will show that this family is a disk in the vague topology.

§17. Existence of an ordered relatively minimizing element

Throughout §§17–26, we let $\Delta:\mathbb{Z}\to\mathbb{R}$ be a fixed function of period n. We suppose $\mathscr{B}_\Delta$ is non-empty, or, equivalently (as we have remarked just before Theorem 15) that $\mathscr{A}_\Delta$ is non-empty. We will begin the proof of Theorem 15 by showing in this section that there exists $\phi_\Delta=\phi_{\omega n\xi\Delta}\in\mathscr{B}_\Delta$ which minimizes F_ω over $\mathscr{B}_\Delta$. In other words, ϕ_Δ minimizes F_ω relative to the constraints and the condition of f being ordered.

Let Y_n denote the set of weakly order preserving, left continuous mappings $\phi:\mathbb{R}\to\mathbb{R}$ which satisfy $\phi(t+n)=\phi(t)+n$. Obviously,

$$\mathscr{B}_\Delta=\mathscr{B}_{n\xi\Delta}\subset Y_n\subset Y_n^*,$$

and $Y=Y_1$. We define a metric d on Y_n by the formula in §5 which was already used to define d on Y. The function $F_\omega: Y_n\to\mathbb{R}$ is continuous with respect to the metric d: the proof in §5 applies with no essential change.

Moreover, $\mathscr{B}_\Delta$ is compact with respect to d. For, let K be a large integer, and set $R_K=[-K+\Delta(-K), K+\Delta(K)]\times[-K+\xi, K+\xi]$. If K is large enough, then the mapping $\phi\mapsto(\text{graph }\phi)\cap R_K$ is an isometry of $\mathscr{B}_\Delta$ onto a closed subset of the space of closed subsets of R_K with the Hausdorff metric. It is well known that the space of closed subsets of a compact metric space with the Hausdorff metric is compact; consequently $\mathscr{B}_\Delta$ is compact, as asserted.

Since $\mathscr{B}_\Delta$ is compact and non-empty and F_ω is continuous, it follows that there exists $\phi_\Delta\in\mathscr{B}_\Delta$ which minimizes F_ω over $\mathscr{B}_\Delta$.

§18. The existence of a relatively minimizing element implies ϕ_Δ minimizes relative to the constraints

Recall that $\mathscr{B}_\Delta\subset\mathscr{A}_\Delta$ and we have proved in §17 that there exists $\phi_\Delta\in\mathscr{B}_\Delta$ which minimizes F_ω over $\mathscr{B}_\Delta$. In this section, we will show that if there is an element in $\mathscr{A}_\Delta$ which minimizes F_ω over $\mathscr{A}_\Delta$, then ϕ_Δ is such an element.

This is a relativized version of Lemma 7.3. The proof follows the reasoning in §7 closely and we will only show how to modify the argument given there in order to apply it to our present circumstances. The result we want follows directly from the following analogue of Lemma 7.2:

LEMMA 18. *Let $\phi\in\mathscr{A}_\Delta$ and suppose ϕ minimizes F_ω over $\mathscr{A}_\Delta$. Let $\psi(t)=$ ess.* $\inf_{s\geq t}\phi(s)$. *Then $F_\omega(\psi)=F_\omega(\phi)$.*

Proof. Let $a>0$. It follows from the definition of $\mathscr{A}_\Delta$ that $\phi\wedge\phi T_a\in\mathscr{A}_\Delta$ and $(\phi\vee\phi T_a)T_{-a}=\phi T_{-a}\vee\phi\in\mathscr{A}_\Delta$. Thus, $F_\omega(\phi\wedge\phi T_a)\geq F_\omega(\phi)$ and $F_\omega(\phi\vee\phi T_a)=F_\omega(\phi T_{-a}\vee\phi)\geq F_\omega(\phi)$ by the translation invariance of F_ω and the hypothesis that ϕ minimizes F_ω over $\mathscr{A}_\Delta$. By Lemma 7.1 and the translation invariance of F_ω we then have

$$F_\omega(\phi\wedge\phi T_a)=F_\omega(\phi\vee\phi T_a)=F_\omega(\phi)=F_\omega(\phi T_a).$$

Having established these equations, we can finish the proof of Lemma 18 by repeating word for word the part of the proof of Lemma 7.2 which follows these equations. $\square$

Obviously, ψ is weakly order preserving. Let $\psi_-(t)=\psi(t-)$. It is easy to see that $\psi_-\in\mathscr{B}_\Delta$. Since ϕ minimizes F_ω over $\mathscr{A}_\Delta$ and $F_\omega(\psi_-)=F_\omega(\psi)=F_\omega(\phi)$, and ϕ_Δ minimizes F_ω over $\mathscr{B}_\Delta$, it follows that ϕ_Δ minimizes F_ω over $\mathscr{A}_\Delta$.

§19. Existence of a relatively minimizing element on the case that ω is rational

In this section, we will show that F_ω takes a minimum value over $\mathscr{A}_\Delta$ when ω is rational. This is a relativized version of Lemma 8.2. The proof follows the method of §8 with only slight changes. Throughout this section we suppose ω is rational. We let $\omega=p/q$ in lowest terms, with $q>0$.

Given $\phi\in Y_n^*$, $\omega\in\mathbb{R}$, and $t\in\mathbb{R}$, we define $x=x_{\phi\omega t}$ as in §8, i.e. $x_i=\phi(\omega_i+t)$. Note that $\mathscr{A}_\Delta\subset Y_n^*$. If $\phi\in\mathscr{A}_\Delta$, then $x=x_{\phi\omega t}$ satisfies $x_{i+qn/r}=x_i+pn/r$, where r is the greatest common divisor of n and p, and x also satisfies

$$\begin{aligned} x_i&\leq\xi+j, \quad \text{if} \quad t+\omega i\leq j+\Delta(j)\\ x_i&\geq\xi+j, \quad \text{if} \quad t+\omega i> j+\Delta(j). \end{aligned}$$

We let $\mathscr{X}_{pqn\Delta t}$ denote the set of all configurations which satisfy these conditions. Since ω is rational and Δ is periodic, there exists $t_i\in\mathbb{R}$, $t_i<t_{i+1}$, $t_i\to\pm\infty$ as $i\to\pm\infty$ such that if $t_{i-1}<s\leq t\leq t_i$ then $\mathscr{X}_{pqn\Delta s}=\mathscr{X}_{pqn\Delta t}$. For simplicity, we denote this space by $\mathscr{X}_{pqn\Delta i}$. In other words, $\mathscr{X}_{pqn\Delta i}=\mathscr{X}_{pqn\Delta t}$ for $t_{i-1}<t\leq t_i$. At the beginning of §15, we remarked that the necessary and sufficient condition for $\mathscr{A}_\Delta\neq\varnothing$ is $j+\Delta(j)\leq j+1+\Delta(j+1)$ for all j. This is also the necessary and sufficient condition that $\mathscr{X}_{pqn\Delta i}\neq\varnothing$, for all $i\in\mathbb{Z}$.

Choose $a\in\mathbb{R}$ and $j,k\in\mathbb{Z}$ such that $t_j\leq a$ and $a+rq^{-1}\leq t_k$. Suppose that for each i satisfying $j<i\leq k$ an $x^i\in\mathscr{X}_{pqn\Delta i}$ is given. Then it follows immediately from

the definitions that there exists a unique $\phi \in \mathscr{A}_\Delta$ such that $x_{\phi\omega t} = x^i$ when both $a \leq t < a + rq^{-1}$ and $t_{i-1} < t \leq t_i$ are satisfied.

It follows from the formula $F_\omega(\phi) = r^{-1} \int_a^{a+rq^{-1}} W(x_{\phi\omega t})\, dt$ of §8 that if x^i minimizes W over $\mathscr{X}_{pqn\Delta i}$ for $j < i \leq k$, then ϕ minimizes F_ω over $\mathscr{A}_\Delta$. Note that $\mathscr{X}_{pqn\Delta_i} \subset \mathscr{X}_{pqn}$, so that the function W which was defined on $\mathscr{X}_{pqn}$ in §8 is defined on $\mathscr{X}_{pqn\Delta i}$.

Consequently, in order to prove that there exists $\phi \in \mathscr{A}_\Delta$ which minimizes F_ω over $\mathscr{A}_\Delta$, it is enough to prove that there exists $x^i \in \mathscr{X}_{pqn\Delta i}$ which minimizes W over $\mathscr{X}_{pqn\Delta i}$ for each $i \in \mathbb{Z}$. It is clear that the projection mapping $\mathscr{X}_{pqn} \to \mathscr{X}_{pqn}/T$ is one-one on $\mathscr{X}_{pqn\Delta i}$. Moreover, the image of $\mathscr{X}_{pqn\Delta i}$ in $\mathscr{X}_{pqn}/T$ is closed. Consequently, Lemma 8.1 implies that for any $a \in \mathbb{R}$, $\{W \leq a\} \cap \mathscr{X}_{pqn\Delta i}$ is compact. Since W is obviously continuous on $\mathscr{X}_{pqn}$, we obtain that the desired x^i exists.

This shows that F_ω takes a minimum value on $\mathscr{A}_\Delta$. By the result of §18, it takes the minimum value in $\mathscr{B}_\Delta$.

§20. Existence of a relatively minimizing element in general

In this section, we will show that F_ω takes a minimum value over $\mathscr{A}_\Delta$, for any $\omega \in \mathbb{R}$. In §19, we proved this result when ω is rational. The deduction of this result for irrational ω from the case of rational ω follows the method of §9 closely.

Suppose ω is irrational and let ϕ_Δ be an element of $\mathscr{B}_\Delta$ which minimizes F_ω over $\mathscr{B}_\Delta$. We have proved the existence of such an element in §17. In this section, we will prove that ϕ_Δ minimizes F_ω over the larger space $\mathscr{A}_\Delta$.

Let $\phi \in \mathscr{A}_\Delta$. In §18, we have proved that if ϕ minimizes F_ω over $\mathscr{A}_\Delta$, then ϕ_Δ minimizes F_ω over $\mathscr{A}_\Delta$. Consequently, we may suppose that ϕ does not minimize; then, there exists $\phi' \in \mathscr{A}_\Delta$ such that $F_\omega(\phi') < F_\omega(\phi)$.

Let $\varepsilon = F_\omega(\phi) - F_\omega(\phi')$. Let K be a compact subset of $\mathbb{R}$ which contains ω in its interior. Note that Lemma 9.2 is still true if Y is replaced by Y_n in its statement, where Y_n is defined as in §17; the proof requires only slight modification. Let C be the constant given by this modified form of Lemma 9.2 (with Y replaced by Y_n). Let p/q be a rational number in K such that $C\,|\omega - p/q| < \varepsilon/2$ and $|F_\omega(\phi') - F_{p/q}(\phi')| < \varepsilon/2$. Such a number exists by Lemma 9.1, since $\phi' \in \mathscr{A}_\Delta \subset Y_n^*$. In §19, we have proved that $F_{p/q}$ has a minimum value over $\mathscr{A}_\Delta$ and that it takes its minimum value at a point in $\mathscr{B}_\Delta$. Since, by definition (§17), $\phi_{p/q,n\xi\Delta}$ minimizes $F_{p/q}$ over $\mathscr{B}_\Delta$, it follows that $F_{p/q}$ takes its minimum value (over $\mathscr{A}_\Delta$) at $\phi_{p/q,n\xi\Delta}$. Therefore $F_{p/q}(\phi_{p/q,n\xi\Delta}) \leq F_{p/q}(\phi')$. Since $\phi_{p/q,n\xi\Delta} \in \mathscr{B}_\Delta \subset Y_n$, it follows from Lemma

9.2 (with Y replaced by Y_n) that $|F_\omega(\phi_{p/q,n\xi\Delta}) - F_{p/q}(\phi_{p/q,n\xi\Delta})| < \varepsilon/2$. Hence,

$$\begin{aligned} F_\omega(\phi_{p/q,n\xi\Delta}) &< F_{p/q}(\phi_{p/q,\xi n\Delta}) + \varepsilon/2 \leq F_{p/q}(\phi') + \varepsilon/2 \\ &< F_\omega(\phi') + \varepsilon = F_\omega(\phi). \end{aligned}$$

Since $\phi_{p/q,n\xi\Delta} \in \mathscr{B}_\Delta$, we have $F_\omega(\phi_{\omega n\xi\Delta}) \leq F_\omega(\phi_{p/q,n\xi\Delta})$, by the definition (§17) of $\phi_{\omega n\xi\Delta}$.

Therefore, $F_\omega(\phi_{\omega n\xi\Delta}) < F_\omega(\phi)$, and we have shown that $\phi_{\omega n\xi\Delta}$ minimizes F_ω over $\mathscr{A}_\Delta$. This completes the proof of existence in Theorem 15.

§21. Proof of Addendum 3 to Theorem 15

We give the proof now because we will need it in the proof that $\phi_\Delta = \phi_{\omega n\xi\Delta}$ is the unique element of $\mathscr{A}_\Delta = \mathscr{A}_{n\xi\Delta}$ which minimizes F_ω over $\mathscr{A}_\Delta$ (next section). The proof is essentially the same as the proof of Addendum 2 of the Theorem in [16], which was given in [16, §12]. For the convenience of the reader, we repeat it here.

As in [16], we set

$$V(\phi, t) = \frac{\partial}{\partial x}[h(\bar{x}, x) + h(x, x')],$$

evaluated at

$$\bar{x} = \phi(t-\omega), \qquad x = \phi(t), \qquad x' = \phi(t+\omega).$$

Suppose ϕ is weakly order preserving. If, in addition, ϕ is constant in an interval (α, β), then $V(\phi, t)$ is non-increasing in that interval and is constant there if and only if ϕ is constant in each of $(\alpha-\omega, \beta-\omega)$ and $(\alpha+\omega, \beta+\omega)$. This is an immediate consequence of the fact that $h_{12} < 0$. (Note that it is non-increasing, rather than non-decreasing as in [16], because of the change of sign convention.)

If $\phi = \phi_{\omega n\xi\Delta}$ and 2′) and 3′) of Theorem 15 are satisfied, then ϕ satisfies the Euler–Lagrange equation $V(\phi, t) = 0$ and, consequently, if ϕ is constant on (α, β), it is constant also in $(\alpha-\omega, \beta-\omega)$ and $(\alpha+\omega, \beta+\omega)$ by the discussion in the previous paragraph. Using the periodicity condition $\phi(t+n) = \phi(t) + n$ and induction we then obtain that ϕ is constant on each interval $(\alpha + k\omega + ln, \beta + k\omega + ln)$, $k, l \in \mathbb{Z}$. Since the union of these intervals is $\mathbb{R}$, it follows that ϕ is constant, a contradiction to $\phi(t+n) = \phi(t) + n$.

This contradiction shows that $\phi_{\omega n\xi\Delta}$ is not constant on any interval, when ω is irrational, and 2′) and 3′) of Theorem 15 are satisfied.

§22. Uniqueness of the relatively minimizing element

In this section, we suppose that there exists $\phi_\Delta = \phi_{\omega n\xi\Delta}$ which minimizes F_ω over $\mathscr{B}_\Delta = \mathscr{B}_{n\xi\Delta}$ and also satisfies 2′) and 3′) of Theorem 15. In §20, we proved that ϕ_Δ necessarily minimizes F_ω over the larger space $\mathscr{A}_\Delta = \mathscr{A}_{n\xi\Delta}$. In this section, we will show that if ϕ is any element of $\mathscr{A}_\Delta$ which minimizes F_ω over $\mathscr{A}_\Delta$, then $\phi = \phi_\Delta$ almost everywhere. This is the content of the last assertion in Theorem 15.

The proof is based on the following trick: If $a \in \mathbb{R}$, let $\Delta + a$ have its usual meaning, viz. $(\Delta + a)(i) : \Delta(i) + a$. Let $\mathscr{A}'_{n\xi\Delta} = \mathscr{A}'_\Delta = \bigcup \{\mathscr{A}_{\Delta+a} : a \in \mathbb{R})$. We will show that if $\phi \in \mathscr{A}'_\Delta$ minimizes F_ω over $\mathscr{A}'_\Delta$, then $\phi = \phi_\Delta T_a$ almost everywhere for some $a \in \mathbb{R}$. This will be enough because if $\phi_\Delta T_a \in \mathscr{A}_\Delta$, then $a = 0$, since ϕ_Δ satisfies 2′) and 3′) of Theorem 15.

To show that $\phi = \phi_\Delta T_a$ almost everywhere, we may use the proof in §10, word for word, except for the following changes: ϕ_Δ in place of ϕ_ω, $\mathscr{A}'_\Delta$ in place of Y^*_n, and Addendum 3 to Theorem 15 in place of Addendum 2 to the Theorem in [16]. □

§23. Locating a relatively minimizing element

As we have pointed out in §5, the argument of [16] shows the existence of $\phi'_\omega \in Y$ which minimizes F_ω over Y. By the translation invariance of F_ω, we have that for any $a \in \mathbb{R}$, the element $\phi_\omega = \phi'_\omega T_a$ of Y also minimizes F_ω over Y. If we choose $a = \sup\{t : \phi'_\omega(t) \leq \xi\}$, then we have $\phi_\omega(t) \leq \xi$, for $t \leq 0$ and $\phi_\omega(t) > \xi$, for $t > 0$. Since $\phi_\omega(t+1) = \phi_\omega(t) + 1$, this implies

$$\phi_\omega(t) \leq \xi + j, \quad \text{for} \quad t \leq j \tag{23.1}$$

$$\phi_\omega(t) > \xi + j, \quad \text{for} \quad t > j. \tag{23.2}$$

Briefly put, the above argument shows that the element ϕ_ω of Y which minimizes F_ω can always be normalized so that (23.1) and (23.2) are satisfied. In the case that ω is irrational, the normalized minimizing element is unique by the result proved in §10, but there is no such uniqueness in the case that ω is rational.

In this section, we will prove:

PROPOSITION 23. *Suppose ϕ_ω is normalized so that* (23.1) *and* (23.2) *hold. If $\mathcal{A}_\Delta = \mathcal{A}_{n\xi\Delta}$ is non-empty, then there exists $\phi_\Delta = \phi_{\omega n\xi\Delta} \in \mathcal{B}_\Delta = \mathcal{B}_{n\xi\Delta}$ which not only minimizes F_ω over $\mathcal{A}_\Delta$ but also satisfies the following inequalities:*

$$\phi_\omega(t-\Delta_+) \leq \phi_\Delta(t) \leq \phi_\omega(t-\Delta_-),$$

for all $t \in \mathbb{R}$, where $\Delta_- = \min\{\Delta(i) : i \in \mathbb{Z}\}$ and $\Delta_+ = \max\{\Delta(i) : i \in \mathbb{Z}\}$.

These inequalities are a key step in proving 2′) and 3′) in Theorem 15.

Proof. In §20, we have finished the proof of existence in Theorem 15. This asserts that there exists $\phi'_\Delta \in \mathcal{B}_\Delta$ which minimizes F_ω over $\mathcal{A}_\Delta$. For brevity, we set $\phi_+(t) = \phi_\omega(t-\Delta_+)$ and $\phi_-(t) = \phi_\omega(t-\Delta_-)$. We set $\phi_\Delta = (\phi'_\Delta \vee \phi_+) \wedge \phi_- = (\phi'_\Delta \wedge \phi_-) \vee \phi_+$. Since $\phi'_\Delta, \phi_\pm \in Y_n$, we have $\phi_\Delta \in Y_n$. In fact, $\phi_\Delta \in \mathcal{B}_\Delta$. The inequalities 2) and 3) of §15 follow from the fact that $\phi'_\Delta \in \mathcal{B}_\Delta$ and the following inequalities, which are a consequence of (23.1) and (23.2):

$$\phi_+(t) \leq \xi + j, \quad \text{if} \quad t \leq j + \Delta_+, \quad \text{and}$$

$$\phi_-(t) \geq \xi + j, \quad \text{if} \quad t > j + \Delta_-.$$

The same reasoning shows that $\phi'_\Delta \vee \phi_+ \in \mathcal{B}_\Delta$. Clearly, $\phi'_\Delta \wedge \phi_+ \in Y^*_n$. Since ϕ'_Δ minimizes F_ω over $\mathcal{A}_\Delta$ and ϕ_+ minimizes F_ω over Y^*_n, we may conclude from Lemma 7.1 that

$$F_\omega(\phi'_\Delta \vee \phi_+) = F_\omega(\phi'_\Delta), \qquad F_\omega(\phi_\Delta \wedge \phi_+) = F_\omega(\phi_+).$$

In particular, $\phi'_\Delta \vee \phi_+$ minimizes F_ω over $\mathcal{A}_\Delta$. A similar argument, applied to $\phi_\Delta = (\phi'_\Delta \vee \phi_+) \wedge \phi_- \in \mathcal{B}_\Delta$ and $(\phi'_\Delta \vee \phi_+) \vee \phi_- \in Y_n$ shows that $F_\omega(\phi_\Delta) = F_\omega(\phi'_\Delta \vee \phi_+) = F_\omega(\phi'_\Delta)$. Consequently, ϕ_Δ minimizes F_ω over $\mathcal{A}_\Delta$.

Finally, we have that ϕ_Δ satisfies the inequalities in Proposition 23, by its definition. □

§24. Relatively minimal energy configurations

In this section, we prove the analogue of Proposition 11.1 for $\phi_{\omega n\xi\Delta}$ in place of ϕ_ω. We will discuss configurations and not orbits as in Proposition 11.1, because we do not wish to assume that 2′) and 3′) of Theorem 15 are satisfied. We will use the results of this section in our proof that 2′) and 3′) of Theorem 15 are satisfied. Because we do not assume these condition, there is no reason why $\phi_{\omega n\xi\Delta}$ should satisfy the Euler–Lagrange equation, and consequently no reason why the configurations associated to $\phi_{\omega n\xi\Delta}$ should be equilibrium configurations.

Given $\phi \in Y_n^*$ and $t \in \mathbb{R}$, we define $x = x_{\phi\omega t}$ as in §8, i.e. $x_i = \phi(t+\omega i)$ and if $\phi \in Y_n$, we define $x = x_{\phi\omega t\pm}$ as in §11, i.e. $x_i = \phi(t+\omega i\pm)$. If $\phi \in \mathcal{A}_{n\xi\Delta}$ and $x = x_{\phi\omega t}$ then

$$x_i \leq \xi + j, \quad \text{if} \quad t+\omega i \leq j + \Delta(j),$$
$$x_i \geq \xi + j, \quad \text{if} \quad t+\omega i > j + \Delta(j).$$

We let $\mathscr{X}_{\omega n\Delta t-}$ denote the set of configurations $x = (\ldots, x_i, \ldots)$ which satisfy these conditions. If $\phi \in \mathscr{B}_{n\xi\Delta}$ and $x = x_{\phi\omega t+}$, then

$$x_i \leq \xi + j, \quad \text{if} \quad t+\omega i < j + \Delta(j),$$
$$x_i \geq \xi + j, \quad \text{if} \quad t+\omega i \geq j + \Delta(j).$$

We let $\mathscr{X}_{\omega n\Delta t+}$ denote the set of configurations $x = (\ldots, x_i, \ldots)$ which satisfy these conditions. Note that for $\phi \in \mathscr{B}_{n\xi\Delta}$, we have $x_{\phi\omega t-} = x_{\phi\omega t} \in \mathscr{X}_{\omega n\Delta t-}$.

We will say that a configuration $x \in \mathscr{X}_{\omega n\Delta t+}$ *has minimal energy relative to* $\mathscr{X}_{\omega n\Delta t\pm}$ if for any pair of integers $m<n$ and any $x' \in \mathscr{X}_{\omega n\Delta t}$ such that $x'_m = x_m$ and $x'_n = x_n$, we have $W_{mn}(x) \leq W_{mn}(x')$, where W_{mn} is the function defined in §11.

We will call such an x a *relatively minimal energy configuration for* $\mathscr{X}_{\omega n\Delta t\pm}$. Its definition differs from that of a minimal energy configuration (cf. §11) only in that x' is constrained to be in $\mathscr{X}_{\omega n\Delta t\pm}$. In this section, we will prove:

PROPOSITION 24. *Suppose ω is irrational. Let $\phi \in \mathscr{B}_{n\xi\Delta}$ and suppose ϕ minimizes F_ω over $\mathcal{A}_{n\xi\Delta}$. Let $t\in\mathbb{R}$. Then $x_{\phi\omega t\pm}$ is a relatively minimal energy configuration for $\mathscr{X}_{\omega n\Delta t\pm}$.*

The hypothesis that ω is irrational is unnecessary, but we will not need the more general result obtained by dropping this hypothesis, and the proof in the case of rational ω is slightly different from the proof we give now.

Proof. Suppose, for example, that $x_{\phi\omega t+}$ is not a relatively minimal energy configuration for $\mathscr{X}_{\omega n\Delta t+}$ so there exists $x' \in \mathscr{X}_{\omega n\Delta t+}$ and integers $m<n$ such that $x'_m = (x_{\phi\omega t+})_m$, $x'_n = (x_{\phi\omega t+})_n$, and $W_{mn}(x') < W_{mn}(x_{\phi\omega t+})$. Let $\delta>0$ and define ϕ' in the same way as in the proof of Proposition 11.1, replacing ϕ_ω by ϕ. Then $F_\omega(\phi) > F_\omega(\phi')$ if $\delta>0$ is small enough by the argument in the end of the proof of Proposition 11.1. A straightforward verification shows that if $\delta>0$ is sufficiently small, then $\phi' \in \mathscr{A}_{n\xi\Delta}$. Thus, we obtain a contradiction to the assumption that ϕ minimizes F_ω over $\mathscr{A}_{n\xi\Delta}$. This contradiction shows that $x_{\phi\omega t+}$ is a relatively minimal energy configuration for $\mathscr{X}_{\omega n\Delta t+}$.

A similar argument shows that $x_{\phi\omega t-}$ is a relatively minimal energy configuration for $\mathscr{X}_{\omega n\Delta t-}$. □

Note that it is not necessarily the case that $\phi' \in \mathscr{B}_{n\xi\Delta}$. It is for this reason that we went to all the bother of verifying that ϕ_Δ minimizes F_ω not only over $\mathscr{B}_\Delta$. but also over the larger space $\mathscr{A}_\Delta$.

§25. Proof of the rest of Theorem 15 and Addendum 1

The only thing which remains to prove in Theorem 15 is the second assertion, namely that 2′) and 3′) hold under suitable hypotheses. Addendum 1 is a more precise version of the assertion. We will prove Addendum 1 in this section, thereby also completing the proof of Theorem 15. Throughout this section, we suppose that ω is irrational and $P_\omega(\xi) > 0$. We will also suppose that the hypothesis of Addendum 1 is satisfied, namely, $\|\Delta\| < \delta(h, \omega, n, P_\omega(\xi))$.

By its definition, $\delta(h, \omega, n, P_\omega(\xi)) < \frac{1}{2}$. Since $\|\Delta\| < \frac{1}{2}$, we have that $\mathscr{A}_{n\xi\Delta}$ and $\mathscr{B}_{n\xi\Delta}$ are non-empty. By Proposition 23, there exists $\phi_\Delta \in \mathscr{B}_\Delta = \mathscr{B}_{n\xi\Delta}$ which not only minimizes F_ω over $\mathscr{A}_\Delta = \mathscr{A}_{n\xi\Delta}$, but also satisfies $\phi_\omega(t - \Delta_+) \leq \phi_\Delta(t) \leq \phi_\omega(t - \Delta_-)$, if we assume (as we may) that ϕ_ω is normalized so that (23.1) and (23.2) hold.

Let $N(h, \omega, n, P)$ be as defined in §15 (preceding the statement of Addendum 1). Let $N = N(h, \omega, n, P_\omega(\xi))$. In view of the hypothesis of Addendum 1 and the definition of $N(h, \omega, n, P)$, we have

$$\Delta_+ - \Delta_- = \|\Delta\| < \min\{|i\omega + j^*| : i, j^* \in \mathbb{Z} \text{ and } 0 < i \leq N\}.$$

Consequently, the intervals $[t - \Delta_+ + i\omega + j^*, t - \Delta_- + i\omega + j^*]$ for $i, j^* \in \mathbb{Z}$ and $0 \leq i \leq N$ are mutually disjoint.

We have pointed out in §10 that a slight modification of the proof of Addendum 2 of the Theorem in [16] shows that ϕ_ω is strictly order preserving. Consequently, the intervals $[\phi_\omega(t - \Delta_+ + i\omega + j^* -), \phi_\omega(t - \Delta_- + i\omega + j^* +)]$ for $i, j^* \in \mathbb{Z}$ and $0 \leq i \leq N$ are also mutually disjoint. Since $\phi_\omega(t+1) = \phi_\omega(t) + 1$, this implies that the projections in $\mathbb{R}/\mathbb{Z}$ of $[\phi_\omega(t - \Delta_+ + i\omega -), \phi_\omega(t - \Delta_- + i\omega +)]$, $i = 0, \ldots, N$ are mutually disjoint. We may therefore choose an integer i_+ satisfying $1 \leq i_+ \leq N$ such that

$$\phi_\omega(t - \Delta_- + i_+\omega +) - \phi_\omega(t - \Delta_+ + i_+\omega -) \leq N^{-1}. \tag{25.1}$$

A similar argument shows that there exists i_- satisfying $-N \leq i_- \leq -1$ such that

$$\phi_\omega(t-\Delta_- + i_-\omega+) - \phi_\omega(t-\Delta_+ + i_-\omega-) \leq N^{-1}. \tag{25.2}$$

Now we will prove that ϕ_Δ satisfies inequality 3′) of Theorem 15 if $t \geq j + \Delta(j)$.

We consider two cases: If $t \geq j + \Delta_+$, then we have $\phi_\Delta(t+) \geq \phi_\omega(t-\Delta_+ +) \geq \xi + j$, since ϕ_Δ is chosen so that the conclusion of Proposition 23 holds and ϕ_ω is normalized so that (23.1) and (23.2) hold. Moreover, the last inequality is strict, since $P_\omega(\xi) > 0$, so $\xi \notin pr_1 M_\omega = \overline{\text{image } \phi_\omega}$ and consequently $\xi + j \notin \overline{\text{image } \phi_\omega}$. Thus, $\phi_\Delta(t+) > \xi + j$, in the first case.

The second case is less trivial: We suppose $t < j + \Delta_+$. We then have $t - \Delta_+ < j \leq t - \Delta(j) \leq t - \Delta_-$. Since ϕ_ω is order preserving, it follows that

$$\phi_\omega(t-\Delta_+ + i\omega+) \leq \phi_\omega(j + i\omega+) \leq \phi_\omega(t - \Delta_- + i\omega+),$$

for any $i \in \mathbb{Z}$. By Proposition 23,

$$\phi_\omega(t-\Delta_+ + i\omega+) \leq \phi_\Delta(t + i\omega+) \leq \phi_\omega(t - \Delta_- + i\omega+),$$

for any $i \in \mathbb{Z}$. Combining these last two displayed formulas and (25.1) and (25.2), we see that

$$|\phi_\Delta(t + i_\pm\omega+) - \phi_\omega(j + i_\pm\omega+)| < N^{-1}. \tag{25.3}$$

Set $\phi = \phi_\Delta$ and let $x = x_{\phi\omega t+}$, in the notation of §24, i.e. $x_i = \phi_\Delta(t + \omega i+)$. Since $\phi \in \mathcal{B}_{n\xi\Delta}$, we have $x \in \mathcal{X}_{\omega n\Delta t+}$. In particular, since $t \geq j + \Delta(j)$, we have $x_0 \geq \xi + j$, by the definition of $\mathcal{X}_{\omega n\Delta t+}$. The inequality 3′), which we are proving, is equivalent to $x_0 > \xi + j$. We will suppose $x_0 = \xi + j$ and obtain a contradiction, thereby proving 3′).

Since ϕ_Δ is a member of $\mathcal{B}_{n\xi\Delta}$ and minimizes F_ω over $\mathcal{A}_{n\xi\Delta}$, it follows from Proposition 24 that $x = x_{\phi\omega t+}$ is a relatively minimal energy configuration for $\mathcal{X}_{\omega n\Delta t+}$. Supposing $x_0 = \xi + j$, we will show that this is not the case, thereby obtaining the desired contradiction.

Let $x'_i = x''_i = x_i$, for $i < i_-$ or $i > i_+$. Let $x'_i = x''_i = \phi_\omega(j + \omega i+)$, for $i = i_-$ or $i = i_+$. Let $x'_i = \phi_\omega(j + \omega i+)$ and $x''_i = x_i$, for $i_- < i < i_+$. These conditions specify configurations x' and x''.

By (25.3), $|x_i - x''_i| < N^{-1}$, for $i = i_-$ or $i = i_+$. By the fact that ϕ_Δ is weakly order preserving and the fact that $\phi_\Delta(t+n) = \phi_\Delta(t) + n$, we have

$$n[\omega/n] \leq x_{i+1} - x_i \leq n[\omega/n] + n,$$

for all $i\in\mathbb{Z}$. It then follows from the mean value theorem and the definition of $C(h,\omega,n)$ (cf. §15) that

$$|W_{lm}(x'')-W_{lm}(x)|\leq 4C(h,\omega,n)N^{-1}$$

if $l<i_-$ and $m>i_+$.

Let $(x_+)_i=\phi_\omega(j+\omega i+)$, $(x_-)_i=\phi_\omega(j+\omega i-)$. Let $y_i=(x_+)_i$, for $i\leq i_-$ or $i\geq i_+$, $y_i=((x\wedge x_+)\vee x_-)_i=((x\vee x_-)\wedge x_+)_i$ for $i_-<i<i_+$. For $l\leq i_-$ and $m\geq i_+$, we have

$$W_{lm}(y)-W_{lm}(x_+)=G_{\omega j}(y)\geq P_\omega(\xi+j)=P_\omega(\xi),$$

where the first equation is a consequence of the fact that $y\in\mathscr{X}_{\omega j}$ and the definitions, and the inequality is a consequence of the fact that $y_0=x_0=\xi+j$. Obviously, $W_{i_-i_+}(x')=W_{i_-i_+}(x_+)$, since $x'_i=(x_+)_i=\phi_\omega(j+\omega i+)$ for $i_-\leq i\leq i_+$. Thus,

$$W_{i_-i_+}(y)-W_{i_-i_+}(x')\geq P_\omega(\xi).$$

The argument which proves Lemma 7.1 also shows

$$W_{i_-i_+}(x''\wedge x_+)+W_{i_-i_+}(x''\vee x_+)\leq W_{i_-i_+}(x'')+W_{i_-i_+}(x_+).$$

In proving Proposition 11.1, we showed that x_+ is a minimal energy configuration. (Note that x_+ is $x_{\phi\omega j+}$ in the notation which was used in Proposition 11.1.) Since $(x''\vee x_+)_i=x''_i=(x_+)_i$, for $i=i_-$ or i_+, we have $W_{i_-i_+}(x''\vee x_+)\geq W_{i_-i_+}(x_+)$. Consequently, the above inequality implies $W_{i_-i_+}(x''\wedge x_+)\leq W_{i_-i_+}(x'')$.

Note that $y_i=((x''\wedge x_+)\vee x_-)_i$, for $i_-\leq i\leq i_+$, so

$$W_{i_-i_+}(y)+W_{i_-i_+}((x''\wedge x_+)\wedge x_-)\leq W_{i_-i_+}(x''\wedge x_+)+W_{i_-i_+}(x_-).$$

The proof of Proposition 11.1 also shows that x_- is a minimal energy configuration. Since $x''_i=(x_+)_i$ for $i=i_-$ or i_+, we have $(x''\wedge x_+\wedge x_-)_i=(x_-)_i$ for $i=i_-$ or i_+ and consequently $W_{i_-i_+}(x''\wedge x_+\wedge x_-)\geq W_{i_-i_+}(x_-)$. Together with the above inequalities, this implies that

$$W_{i_-i_+}(y)\leq W_{i_-i_+}(x''\wedge x_+)\leq W_{i_-i_+}(x'').$$

Thus,

$$\begin{aligned}W_{lm}(x)-W_{lm}(x')&\geq W_{lm}(x'')-W_{lm}(x')-|W_{lm}(x'')-W_{lm}(x)|\\&=W_{i_-i_+}(x'')-W_{i_-i_+}(x')-|W_{lm}(x'')-W_{lm}(x)|\\&\geq W_{i_-i_+}(y)-W_{i_-i_+}(x')-|W_{lm}(x'')-W_{lm}(x)|\\&\geq P_\omega(\xi)-4C(h,\omega,n)N^{-1}>0,\end{aligned}$$

if $l<i_-$ and $m>i_+$. The last inequality is a consequence of the fact that $N=N(h,\omega,n,P_\omega(\xi))>4C(h,\omega,n)/P_\omega(\xi)$, by the definition of $N(h,\omega,n,P)$ (§15). The equality is a consequence of the fact that $x_i'=x_i''$ for $i\leqslant i_-$ and $i\geqslant i_+$.

We have $x'\in\mathscr{X}_{\omega n\Delta t+}$, i.e.

$$x_i'\leqslant\xi+j^*,\quad\text{if}\quad t+\omega i<j^*+\Delta(j^*),$$

$$x_i'\geqslant\xi+j^*,\quad\text{if}\quad t+\omega i\geqslant j^*+\Delta(j^*),$$

where i and j^* run over all the integers. For $i<i_-$ or $i>i_+$, this is obvious, because $x\in\mathscr{X}_{\omega n\Delta t+}$ and $x_i'=x_i$. For $i_-\leqslant i\leqslant i_+$, it is enough to show that $t+\omega i<j^*+\Delta(j^*)\Leftrightarrow j+\omega i<j^*$, i.e. that $t+\omega i\geqslant j^*+\Delta(j^*)\Leftrightarrow j+\omega i\geqslant j^*$, i.e. that $t+\omega i$ is in the interval $[j^*+\Delta(j^*),j^*+1+\Delta(j^*+1))$ if and only if $j+\omega i$ is in the interval $[j^*,j^*+1)$.

We prove the last statement, as follows: First, note that $\Delta_+<\Delta_-+1$, since $\|\Delta\|\leqslant 1/2$. In particular, $j+\Delta_+<j+1+\Delta_-\leqslant j+1+\Delta(j+1)$. We have assumed at the outset that $j+\Delta(j)\leqslant t$, and we are considering the case that $t<j+\Delta_+$. Thus,

$$j+\Delta(j)\leqslant t<j+\Delta_+<j+1+\Delta(j+1),$$

so the case $i=0$ holds. For the case $i>0$, we use the fact that the intervals $[t-\Delta_++\omega i+j^*,t-\Delta_-+\omega i+j^*]$ are mutually disjoint for $i,j^*\in\mathbb{Z}$ and $0\leqslant i\leqslant N$. Since $j\in[t-\Delta_+,t-\Delta_-]$, this implies that none of the intervals $[t-\Delta_++\omega i,t-\Delta_-+\omega i]$, $i=1,\ldots,i_+\leqslant N$ contains an integer; moreover, each of these contains $j+\omega i$. Consequently, if $j+\omega i\in[j^*,j^*+1)$, then $[t-\Delta_++\omega i,t-\Delta_-+\omega i]\subset(j^*,j^*+1)$, and so, $t+i\omega$ is in the interval $(j^*+\Delta_+,j^*+1+\Delta_-)$, which is a subinterval of $(j^*+\Delta(j^*),j^*+1+\Delta(j^*+1))$. For the case $i<0$, we use an argument similar to that for $i>0$.

Thus, $x'\in\mathscr{X}_{\omega n\Delta t+}$. We have previously proved that $W_{lm}(x)-W_{lm}(x')>0$. Still earlier, we had proved that x is a relatively minimal energy configuration for $\mathscr{X}_{\omega n\Delta t+}$, so we have obtained a contradiction. This contradiction was obtained under the assumption that $x_0=\xi+j$, i.e. that inequality 3′) in Theorem 15 was not satisfied.

Thus, we have proved that the inequality 3′) of Theorem 15 holds if $t\geqslant j+\Delta(j)$. The proof that 2′) holds if $t\leqslant j+\Delta(j)$ is similar. □

§26. Proof of Addendum 2

Set

$$V_\phi(t)=h_2(\phi(t-\omega),\phi(t))+h_1(\phi(t),\phi(t+\omega)).$$

If ϕ is continuous at t, then $V_\phi(t-) \geq V_\phi(t+)$, because $h_{12} < 0$. Moreover, we have equality if and only if ϕ is continuous at both $t-\omega$ and $t+\omega$. In the case that $\phi = \phi_{\omega n \xi \Delta}$ and 2′) and 3′) of Theorem 15 are satisfied, the Euler–Lagrange equation holds, i.e. $V_\phi(t-) = V_\phi(t+) = 0$. Since we have equality in this case, the continuity of ϕ at t implies the continuity of ϕ at $t-\omega$ and at $t+\omega$. □

§27. Continuous dependence of the relatively minimizing element on the constraints

As in §16, we let $\mathscr{D}$ denote the set of all mappings $\Delta : \mathbb{Z} \to \mathbb{R}$ of period n for which $\mathscr{A}_\Delta$ is non-empty and inequalities 2′) and 3′) in Theorem 15 hold for ϕ_Δ. In §17, we defined a set Y_n and a metric d on it. In this section, we will show:

PROPOSITION 27. *The mapping $\Delta \mapsto \phi_\Delta$ is a continuous mapping of $\mathscr{D}$ into the metric space (Y_n, d).*

Here, we provide $\mathscr{D}$ with the induced topology associated to the inclusion

$$\mathscr{D} \subset \mathbb{R}^n : \phi \mapsto (\phi(1), \ldots, \phi(n)).$$

Proof. As we remarked in §17, $\mathscr{B}_\Delta$ is a compact subset of Y_n. Moreover, it is easily seen that $\Delta \mapsto \mathscr{B}_\Delta$ is a continuous mapping of $\mathscr{D}$ into the space of compact subsets of Y_n, where the latter is provided with the Hausdorff metric on compact subsets of (Y_n, d). The continuity of $\Delta \mapsto \phi_\Delta$ then follows from the uniqueness assertion in Theorem 15. □

§28. The vague topology on Denjoy minimal sets (definitions)

In this section, we give a detailed definition of the vague topology on the set of Denjoy minimal sets for $\bar{f}$ in $(\mathbb{R}/\mathbb{Z}) \times \mathbb{R}$. We have already briefly described this topology in the introduction.

First, we need the notion of the vague topology on the set of Radon measures on $(\mathbb{R}/\mathbb{Z}) \times \mathbb{R}$. This is a special case of a notion which may be found in Bourbaki [8, Chapt. III §1.9]. We recall the Bourbaki notion: Let X be a locally compact, Hausdorff space. We let $\mathscr{K}(X)$ denote the real vector space of real-valued continuous functions on X of compact support. If K is a compact subset of X, we let $\mathscr{K}(X, K)$ denote the vector subspace of $\mathscr{K}(X)$ consisting of all real-valued continuous functions on X having support in K. We provide $\mathscr{K}(X, K)$ with the topology of uniform convergence. The space $\mathscr{K}(X)$ is the union of the subsets $\mathscr{K}(X, K)$, where K runs over all compact subsets of X. We provide $\mathscr{K}(X)$ with the inductive limit (i.e. direct limit) topology of the topologies on $\mathscr{K}(X, K)$. By definition, a *Radon measure* on X is a real-valued continuous linear functional on $\mathscr{K}(X)$. The real vector space of Radon measures on X is denoted $\mathscr{M}(X)$. The *vague topology* on $\mathscr{M}(X)$ is the topology which is variously called the topology of simple convergence on $\mathscr{K}(X)$ (Bourbaki), the topology of pointwise convergence on $\mathscr{K}(X)$ (Kelley [15, Chapt. 7, p. 217]) or the $\mathscr{K}(X)$-weak topology (by functional analysts; see, e.g., Reed and Simon [24, IV.5, bottom of p. 119].)

Second, we need the fact that any Denjoy minimal set supports a unique invariant probability measure. We recall that a Radon measure μ is said to be *positive* if for every $f \in \mathscr{K}(X)$ such that $f \geqslant 0$, we have $\mu(f) \geqslant 0$; it is said to be a *probability measure* if $\|\mu\| = 1$, where, following Bourbaki [8, Chapt. III §1.8], we define

$$\|\mu\| = \sup\{|\mu(f)| : f \in \mathscr{K}(X), \|f\| \leqslant 1\},$$

and $\|f\| = \sup\{|f(x)| : x \in X\}$. A proof that a Denjoy minimal set has a unique invariant probability measure is given, for example, in [12, II, 8.5]; a Denjoy minimal set is said to be uniquely ergodic because it has this property. (See, e.g., [12, II, 8].)

Since every Denjoy minimal set is uniquely ergodic, we have an inclusion

$$\{\text{Denjoy minimal sets for } \bar{f}\} \subset \{\bar{f}\text{-invariant probability measures}\},$$

obtained by associating to a Denjoy minimal set the unique invariant probability measure which it supports. We define the *vague topology* on the set of Denjoy minimal sets to be the topology induced from the vague topology on Radon measures.

§29. Continuous dependence of the Denjoy minimal set Σ_Δ on the constraint Δ

For $\Delta \in \mathscr{D}$, we defined the Denjoy minimal set $\Sigma_\Delta \subset (\mathbb{R}/\mathbb{Z}) \times \mathbb{R}$ of $\bar{f}$ after Lemma 16.4.

THEOREM 29. *The mapping $\Delta \to \Sigma_\Delta$ of $\mathscr{D}$ into the space of Denjoy minimal sets of $\bar{f}$, provided with the vague topology, is continuous.*

Proof. Consider a compactly supported function $\bar{u}$ on $(\mathbb{R}/\mathbb{Z}) \times \mathbb{R}$. For $\Delta \in \mathscr{D}$, let

I_Δ denote the integral of $\bar{u}$ with respect to the unique $\bar{f}$-invariant probability measure on Σ_Δ. In view of the definition of the vague topology, it is enough to prove that $\Delta \to I_\Delta$ is a continuous mapping from $\mathcal{D}$ to $\mathbb{R}$.

Let u be the composition of the projection $\mathbb{R}^2 \to (\mathbb{R}/\mathbb{Z}) \times \mathbb{R}$ and the function $\bar{u}$. Clearly,

$$I_\Delta = n^{-1} \int_0^n u(\phi_\Delta(t), \eta_\Delta(t))\, dt,$$

where η_Δ is as defined in §16 (preceding Lemma 16.3). Let $\delta > 0$ be a positive number. The argument given in [16, §6] shows that there exists $\delta_1 > 0$ such that if $\Delta, \Delta' \in \mathcal{D}$ and $d(\phi_\Delta, \phi_{\Delta'}) < \delta_1$, then $|\phi_\Delta(t) - \phi_{\Delta'}(t)| < \delta$ fails to hold for $t \in [0, n]$ at most on a set of measure δ. Since $\eta_\Delta(t) = -h_1(\phi_\Delta(t), \phi_\Delta(t+\omega))$ and there is a uniform bound on $|\phi_\Delta(t+\omega) - \phi_\Delta(t)|$ for $\Delta \in \mathcal{D}$ and $t \in \mathbb{R}$, the same statement holds for η_Δ in place of ϕ_Δ.

The continuity of I_Δ as a function of Δ follows immediately.

It follows from Theorem 29 that the $(n-1)$ parameter family of Denjoy minimal sets for $\bar{f}$ of angular rotation number ω and intrinsic rotation number congruent to $(\omega + R)/n$ (mod. 1) which was described at the end of §16 is in fact a topological $(n-1)$ disk with respect to the vague topology. Thus, we have proved the result announced in the abstract of this paper.

Acknowledgements

I would like to thank S. Aubry for explaining to me that whenever Peierls's energy barrier is positive, there should be local minimal energy states which are not minimal energy states, in a conversation we had in August, 1983. After I wrote [19], I had considered the possibility that the result of this paper might be true, but I had been unable to carry through the proof, since I was unable to show that the relatively minimizing element does not bump up against the constraint. Aubry's explanation of why there should be local minimal energy states which are not minimal energy states suggested the method of proof which I have used in this paper. I would also like to thank Aubry for a second conversation we had in the Spring of 1984 in which he told me that the result of this paper concerning Denjoy minimal sets was new to him and encouraged me to write it up.

The main result of this paper answers a question which A. Katok posed after a talk I gave at Berkeley in January 1984. I would like to thank A. Katok for posing his question and encouraging me to write up this paper. I would like to thank J. Bellisard for listening to an oral presentation of the result of this paper during the Spring semester of 1984 and for pointing out a number of mistakes, which I have corrected in this written account. His comments greatly aided me in writing up this paper. Naturally, any remaining mistakes are my responsibility. I would like to thank E. Stein and M. Herman for helpful comments and encouragement. I would like to thank J. Moser for inviting me to ETH and for his interest in this work and his encouragement.

I would like to thank Janet Goldberg and Maureen Kirkham for typing this paper.

REFERENCES

[1] R. Abraham and J. Marsden, *Foundations of Mechanics*, 2nd ed. Benjamin Cummings, Reading, Mass., 1978.
[2] V. I. Arnold and A. Avez, *Ergodic Problems of Classical Mechanics*. Benjamin, NY, 1968.
[3] S. Aubry, *The Twist Map, the Extended Frenkel–Kontorova Model, and the Devil's Staircase*. Physica *7D* (1983) 240–258.
[4] S. Aubry and P. Y. Le Daeron, *The Discrete Frenkel–Kontorova Model and its Extensions*. Physica *8D* (1983) 381–422.
[5] S. Aubry, P. Y. Le Daeron, and G. André, *Classical Ground-States of a One-Dimensional Model for Incommensurate Structures*. Preprint (1982).
[6] G. D. Birkhoff, *Proof of Poincaré's Geometric Theorem*. Trans. AMS *14*, 14–22. Reprinted in *Collected Works*, vol. I, AMS, New York, (1950) 673–681.
[7] G. D. Birkhoff, *Sur quelques courbes fermees remarquables*. Bull. Soc. Math. France *60*, 1–26. Reprinted in Collected Works, vol. II, AMS, (1950) 418–443.
[8] N. Bourbaki, *Elements de mathematique*, 6, *Intégration*, 2. éd. Paris, Hermann (1965).
[9] A. Chenciner, *La dynamique au voisinage d'un point fixe elliptique, conservatif: de Poincaré et Birkhoff à Aubry et Mather*. Séminaire N. Bourbaki, *1983–1984*, exposé 622 (1984).
[10] R. Douady, *Applications du théorème des tores invariants*. Thèse 3ème cycle, Université Paris VII (1982).
[11] G. A. Hedlund, *Geodesics on a two-dimensional Riemannian manifold with periodic coefficients*. Annals of Math. *33* (1932) 719–739.
[12] M. R. Herman, *Sur la conjugaison differentiable des difféomorphismes du cercle à des rotations*. Publications Maths. de l'I.H.E.S. *49* (1979) 5–234.
[13] M. R. Herman, *Sur les courbes invariantes par les difféomorphismes de l'anneau*, vol. 1. astérisque, *103–104* (1983).
[14] A. Katok, *Periodic and Quasi-periodic Orbits for Twist Maps. Dynamical Systems and Chaos*, Proceedings, Sitges, 1982, L. Garrido, ed. Springer-Verlag, Berlin (1983).
[15] J. L. Kelley, *General Topology*, New York, Van Nostrand (1955).
[16] J. N. Mather, *Existence of Quasi-Periodic Orbits for Twist Homeomorphisms of the Annulus*. Topology *21* (1982) 457–467.
[17] J. N. Mather, *Concavity of the Lagrangian for Quasi-periodic Orbits*. Comment. Math. Helvetici *57* (1982) 356–376.
[18] J. N. Mather, *A Criterion for the Non-existence of Invariant Circles*. Preprint (1982).
[19] J. N. Mather, *Non-uniqueness of Solutions of Percival's Euler–Lagrange Equation*. Commun. Math. Phys *86* (1982) 465–476.
[20] J. N. Mather, *Amount of Rotation about a Point and the Morse Index*. Commun. Math. Physics *94* (1984), 141–153.
[21] M. Morse, *A fundamental class of geodesics on any closed surface of genus greater than one*, Trans. AMS, *26* (1924) 25–60.

[22] I. C. Percival, *Variational principles for invariant tori and cantori. Symp. on Nonlinear Dynamics and Beam–Beam Interactions.* (M. Month and J. C. Herrara, eds.) No. 57, American Institute of Physics Conf. Proc. (1980) 310–320.
[23] I. C. Percival, *A rational principle for invariant tori of fixed frequence* J. Phys. A. Math. and Gen. *12* (1979) L.57.
[24] M. Reed and B. Simon, *Methods of modern mathematical physics*, vol. 1. New York, Academic Press (1980).

Dept of Mathematics
Princeton University
Fine Hall
Washington Road
Princeton N.J. 08544/USA

Received December 5, 1984