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## Some remarks on compactifications of commutative algebraic groups

F. KNOP and H. LANGE

### Introduction

In the theory of transcendental numbers commutative algebraic groups play an important role. In fact if such a group  $E$  is embedded in some projective space, its exponential map can be described by holomorphic functions. The values of these functions at algebraic points are good candidates for transcendency (cf. [8], [9]). In order to embed  $E$  into a projective space, it is convenient to compactify  $E$  and embed the compactification  $\bar{E}$ . The main method to compactify  $E$  (due to Serre [7]) is the following:

Let  $E$  be a connected commutative algebraic group over an algebraically closed field  $k$ . Then there is a canonical exact sequence

$$0 \rightarrow G \rightarrow E \xrightarrow{p} A \rightarrow 0 \quad (1)$$

with  $G$  a connected linear group and  $A$  an abelian variety over  $k$ . Given a projective  $G$ -variety  $P$  over  $\mathbb{P}^n$ , every open  $G$ -equivariant immersion  $G \rightarrow P$  induces in a natural way a compactification  $\bar{E}$  of  $E$ , namely the fibre bundle  $E(P) = E \times^G P$  with fibre  $P$  over  $A$ , associated to the  $G$ -principal bundle  $E \rightarrow A$  with respect to the given  $G$ -action of  $P$  (cf. [4]).

In this note we shall study the following questions: Are there other compactifications (section 1)? Which are the  $G$ -equivariant embeddings of  $\bar{E}$  into  $\mathbb{P}^n$  (section 2)? How many compactifications of the above type are there (section 3)? And by forms of which degrees is the homogeneous ideal of  $\bar{E}$  in  $\mathbb{P}^n$  generated (section 4)?

To be more precise, in section 1 we shall prove that if the linear part  $G$  of  $E$  is a torus or one dimensional, then there are no other normal compactifications of  $E$  than those described above (Theorem 1.1) and we give a counterexample to this statement in the non-normal case. In section 2 we investigate the group  $\text{Pic}_G(E(P))$  of  $G$ -line bundles on the compactification  $E(P)$  and show (Theorem 2.1) that it is isomorphic to  $\text{Pic}_G(P) \times \text{Pic}(A)$ . Hence every  $G$ -embedding of

$E(P)$  into projective space comes from a pair consisting of a projective  $G$ -embedding of  $P$  and a projective embedding of  $A$ .

In section 3 we show that Serre's original compactifications of  $E$  depend on the splitting of  $G$  into one dimensional groups. Finally in section 4 we show that in the most important cases (with respect to applications in transcendence theory) the homogeneous ideal of  $E(P)$  in  $\mathbb{P}^n$  is generated by forms of degree  $\leq \dim A + 3$ .

## 1. Compactifications of $E$

Let  $E$  denote a connected commutative algebraic group over a field  $k$  (algebraically closed for simplicity) with canonical exact sequence (1). Let  $X$  be an  $E$ -variety (not necessarily projective) and  $i: E \rightarrow X$  an equivariant open immersion. Let  $\bar{G} = \overline{Gi(0)}$  be the closure of the orbit of  $i(0)$  under  $G$  in  $X$ . Then there is a canonical  $E$ -equivariant morphism

$$\psi: E(\bar{G}) \rightarrow X$$

defined by  $\psi(h, x) = h \cdot x$  for  $h \in E$  and  $x \in \bar{G} \subset X$ . Here  $E(G)$  denotes as usual the fibre bundle  $E \times^G \bar{G}$  associated to the principal bundle. It is the compactification of  $E$  corresponding to the embedding  $G \rightarrow \bar{G}$  (if  $\bar{G}$  is proper over  $k$ , c.f. [4]). The following theorem in particular gives conditions under which the compactifications of type  $E(G)$  are the only  $E$ -equivariant compactifications of  $E$ .

**THEOREM 1.1.** (a)  $\psi$  is birational and proper, and in particular surjective.

(b) Suppose (i)  $X$  is normal and (ii)  $\bar{G}$  consists of finitely many  $G$ -orbits. Then  $\psi$  is an isomorphism.

Condition (ii) is always fulfilled if  $G$  is a torus or one dimensional (cf. [5]).

*Proof.* 1. Since  $\psi$  restricted to the open subset  $E \times^G G$  of  $E(\bar{G})$  is an isomorphism onto the open subset  $E$  of  $X$ ,  $\psi$  is birational.

2. Let  $\pi: E(\bar{G}) \rightarrow A$  denote the canonical projection map. We claim that  $(\pi, \psi): E(\bar{G}) \rightarrow A \times X$  is a closed immersion.

Since  $E \rightarrow A$  is faithfully flat, it suffices to show that  $id_E \times_A (\pi, \psi): E \times_A E(\bar{G}) \rightarrow E \times_A (A \times X)$  is a closed immersion. Now  $\phi_1: E \times \bar{G} \rightarrow E \times_A (E \times^G \bar{G}): (h, x) \rightarrow (h, (h, x))$  is an isomorphism (with inverse map  $\phi_1^{-1}(h, (h', x)) = (h, (h^{-1}h')x)$ ; note that  $h^{-1}h' \in G$ ). If  $\phi_2: E \times_A (A \times X) \rightarrow E \times X$  denotes the canonical isomorphism and  $\phi_3: E \times X \rightarrow E \times X$  the isomorphism

$(h, x) \rightarrow (h, h^{-1}x)$ , then it suffices to show that the composed map

$$\phi_3\phi_2(id_E \times_A (\pi, \psi))\phi_1: E \times \bar{G} \rightarrow E \times X$$

is a closed immersion. But

$$\begin{aligned} \phi_3\phi_2(id_E \times_A (\pi, \psi))\phi_1(h, x) &= \phi_3\phi_2(id_E \times_A (\pi, \psi))(h, (h, x)) \\ &= \phi_3\phi_2(h, (\pi(h), hx)) \\ &= \phi_3(h, hx) = (h, x) \end{aligned}$$

which proves the assertion. As composition of the closed immersion  $E(\bar{G}) \rightarrow A \times X$  and the projection  $A \times X \rightarrow X$  the map  $\psi$  is proper. (Note that if  $X$  is itself proper over  $k$ , assertion 2 follows immediately from EGA II, 5.4.3).

3. To complete the proof of the theorem, by Zariski's Theorem (EGA III 4.4.9) it suffices to show that  $\psi$  has finite fibres under the assumptions (i) and (ii).

Suppose  $x \in X$ . Since  $\psi$  is surjective there are points  $h \in E$  and  $x_0 \in \bar{G}$  with  $x = hx_0$ . Since  $\psi$  is  $E$ -equivariant, it suffices to show that  $\psi^{-1}(x_0)$  is finite. We have

$$\begin{aligned} \psi^{-1}(x_0) &= \{(h, x) \in E(\bar{G}) \mid hx = x_0\} \\ &= \{(h, h^{-1}x_0) \in E(\bar{G}) \mid h^{-1}x_0 \in \bar{G}\}. \end{aligned}$$

Denote  $E_0 = \{h \in E \mid h^{-1}x_0 \in \bar{G}\}$ .  $E_0$  in general is not a subgroup of  $E$ , however it is  $G$ -stable. Obviously we have a bijection  $E_0/G \xrightarrow{\sim} \psi^{-1}(x_0)$ . Let  $E_{x_0}$  denote the stabilizer of  $x_0$  in  $E$ .  $E_0$  is also  $E_{x_0}$ -stable and we claim:

Suppose  $h, \bar{h} \in E_0$ . Then  $h^{-1}x_0$  and  $\bar{h}^{-1}x_0$  are in the same  $G$ -orbit of  $\bar{G}$  if and only if  $h\bar{h}^{-1} \in GE_{x_0}$ . In fact for  $g \in G$  we have

$$g\bar{h}^{-1}x_0 = h^{-1}x_0 \Leftrightarrow gh\bar{h}^{-1}x_0 = x_0 \Leftrightarrow gh\bar{h}^{-1} \in E_{x_0} \Leftrightarrow h\bar{h}^{-1} \in GE_{x_0}.$$

Hence there is an injective map of  $E_0/GE_{x_0}$  into the set of  $G$ -orbits of  $\bar{G}$ . It follows from assumption (ii) that  $E_0/GE_{x_0}$  is a finite set. It remains to show that  $GE_{x_0}/G = p(E_{x_0}) \subseteq A$  ( $p: E \rightarrow A$  the canonical map) is finite. But by the following lemma  $E_{x_0}$  is a linear group. Hence  $p(E_{x_0})$  is finite as a linear subgroup of an abelian variety. This completes the proof of Theorem 1.1. It remains to show:

**LEMMA 1.2.** *Let  $E$  be an algebraic group acting effectively on a variety  $X$  (i.e.  $E \rightarrow \text{Aut}(X)$  is injective). Then for every  $x \in X$  the stabilizer  $E_x$  of  $x$  in  $E$  is linear.*

*Proof.* We have the following canonical inclusions

$$E_x \hookrightarrow \text{Aut } \mathcal{O}_{X,x} \hookrightarrow \text{Aut } \hat{\mathcal{O}}_{X,x} = \varprojlim_n \text{Aut } (\mathcal{O}_{X,x}/m_x^n).$$

Hence there is an integer  $n$ , such that the canonical map  $E \rightarrow \text{Aut } (\mathcal{O}_{X,x}/m_x^n)$  is injective. Since  $\text{Aut } (\mathcal{O}_{X,x}/m_x^n)$  as a subgroup of the group of automorphisms of a finite dimensional vector space is linear, the same is true for  $E_x$ .

**EXAMPLE 1.3.** We want to give an example showing that Theorem 1.1 is not correct without the assumption that  $X$  is normal.

Let  $l \neq \text{char } k$  be a prime number,  $k$  algebraically closed,  $E$  as above with canonical exact sequence (1) with the additional assumption that  $G = \mathbb{G}_m$ . Let  $\bar{G} = \mathbb{P}^1$  be the canonical compactification and  $\Gamma \subseteq A$  be a subgroup with  $\Gamma \simeq \mathbb{Z}/l\mathbb{Z}$ . The exact sequence

$$0 \rightarrow G \rightarrow p^{-1}(\Gamma) \rightarrow \Gamma \rightarrow 0$$

splits (cf. [2] Theorem 16.2). Since  $\text{Hom}(\Gamma, G) \neq 0$ , there are 2 sections  $s_1, s_2: \Gamma \rightarrow p^{-1}(\Gamma) \subseteq E$  such that  $s_1(\Gamma) \cap s_2(\Gamma) = \{0\}$  and  $s_1(\Gamma) \cap G = s_2(\Gamma) \cap G = \{0\}$ .

Define  $E_1 = E/s_1(\Gamma)$  and  $E_2 = E/s_2(\Gamma)$ . Then the sequences

$$0 \rightarrow G \rightarrow E_i \rightarrow A/\Gamma \rightarrow 0$$

are exact. Moreover for  $i=1, 2$  there is a natural  $E$ -equivariant morphism  $\phi_i: E(\bar{G}) \rightarrow E_i(\bar{G})$ .

Let

$$X := \text{Im}(\phi_1, \phi_2) \subseteq E_1(\bar{G}) \times E_2(\bar{G}).$$

Then  $i: E \rightarrow E(\bar{G}) \rightarrow X$  is an open imbedding, since  $s_1(\Gamma) \cap s_2(\Gamma) = \{0\}$ . On the other hand  $i|_{\bar{G}}$  is an isomorphism, since  $s_1(\Gamma) \cap G = s_2(\Gamma) \cap G = \{0\}$ . But  $A \simeq E(\infty) \subseteq E(\bar{G})$  is mapped onto  $A/\Gamma$ . It follows  $p(E_{x_0}) = \Gamma \neq \{0\}$ , where  $x_0$  denotes the image of  $(0, \infty) \in E(\bar{G})$  in  $X$ , which means that  $\psi: E(\bar{G}) \rightarrow X$  is not an isomorphism.

## 2. $\text{Pic}_G(E(P))$

Let  $E$  be as in section 1 with canonical exact sequence (1). Let  $P$  be a complete  $G$ -variety over  $k$  and  $i: G \rightarrow P$  an equivariant open immersion. Then  $E(P) = E \times^G P$  is a compactification of  $E$ . Moreover if  $L$  is a  $G$ -linearized line bundle in  $P$ , then  $E(L) = E \times^G L$  is a  $G$ -linearized (even  $E$ -linearized) line bundle on  $E(P)$  (cf. [4], Lemma 1.2). If as usual  $\text{Pic}(\ )$  (resp.  $\text{Pic}_G(\ )$ ) denotes the group of line bundles (resp.  $G$ -linearized line bundles), there is a canonical map

$$\Phi: \begin{cases} \text{Pic}_G(P) \times \text{Pic}(A) \rightarrow \text{Pic}_G(E(P)) \\ (L, M) \mapsto E(L) \otimes \pi^* M. \end{cases}$$

(Note that, whereas  $E(L)$  is  $E$ -linearized,  $\pi^* M$  is only  $G$ -linearized on  $E(P)$ ). The aim of this section is to prove

**THEOREM 2.1.** *Given an exact sequence (1) and let  $P$  denote a complete  $G$ -variety over  $k$ . Then the canonical map*

$$\Phi: \text{Pic}_G(P) \times \text{Pic}(A) \rightarrow \text{Pic}_G(E(P))$$

*is an isomorphism of groups.*

*Proof.* We shall construct an inverse map  $\Psi: \text{Pic}_G(E(P)) \rightarrow \text{Pic}_G(P) \times \text{Pic}(A)$ . Suppose  $N \in \text{Pic}_G(E(P))$ . Let  $\pi: E(P) \rightarrow A$  denote the projection map.  $\pi^{-1}(0)$  is canonically isomorphic to  $P$ . We identify both and consider the closed embedding  $j: P \hookrightarrow E(P)$ . Define  $L = j^* N$  with the induced  $G$ -action. We claim that  $E(L)^{-1} \otimes N|_{\pi^{-1}(a)} \simeq \mathcal{O}_{\pi^{-1}(a)}$  (without  $G$ -action) for every point  $a \in A$ .

First of all  $E(L)|_{\pi^{-1}(0)} = L$  (even as  $G$ -line bundles) that is  $E(L)^{-1} \otimes N|_{\pi^{-1}(0)} \simeq \mathcal{O}_{\pi^{-1}(0)}$  with trivial  $G$ -action. Since  $E(L)^{-1} \otimes N$  may be considered as a family of line bundles on  $P$  parametrized by  $A$ , it suffices to show that the Picard variety  $\text{Pic}^0(P)$  of  $P$  is zerodimensional, since then every deformation of a line bundle on  $P$  is trivial. But  $P$  is as compactification of a connected linear group a rational variety which implies  $\dim \text{Pic}^0(P) \leq \dim H^1(P, G_P) = 0$ .

Applying Grauert's theorem (cf. [1], III, 12.9), we get that  $M := \pi_*(E(L)^{-1} \otimes N)$  is a line bundle on  $A$ . We claim that the natural map

$$\sigma: \pi^* M = \pi^* \pi_*(E(L)^{-1} \otimes N) \rightarrow E(L)^{-1} \otimes N$$

is an isomorphism of line bundles. Since  $\sigma$  is a homomorphism of line bundles we have only to show that  $\sigma$  is surjective.

For this is sufficient to show that for every point  $a \in A$  there is a neighbourhood  $U$  in  $A$  such that  $E(L)^{-1} \otimes N | \pi^{-1}(U)$  admits a nowhere vanishing section. But since  $\pi_*(E(L)^{-1} \otimes N)$  is a line bundle on  $A$ , we may take for  $U$  any trivializing open set in  $A$ . Hence  $1 \times \sigma : E(L) \otimes \pi^*M \rightarrow N$  is an isomorphism of line bundles and it remains to show that it is compatible with the  $G$ -actions. Since any 2  $G$ -linearizations of a given line bundle on  $E(P)$  differ by a character on  $G$ , it suffices to check this on the restrictions to the fibre  $\pi^{-1}(0)$ . But we noted already that  $E(L) | \pi^{-1}(0) \simeq N | \pi^{-1}(0)$  as  $G$ -line bundles.

Now define  $\Psi : \text{Pic}_G(E(P)) \rightarrow \text{Pic}_G(P) \times \text{Pic}(A)$  by  $\Psi(N) = (L, M)$  with  $L$  and  $M$  as above. It is easy to see that the maps  $\Phi$  and  $\Psi$  are inverse to each other.

### 3. Serre-compactifications of $E$

In this section let  $k$  denote the field of complex numbers. The group  $G$  in the exact sequence (1) then is of the form

$$G = \prod_{i=1}^r \mathbb{G}_m \times \prod_{j=1}^s \mathbb{G}_a. \tag{2}$$

In [7] Serre constructed a compactification of  $E$  as follows: Consider for each factor  $\mathbb{G}_m$  and  $\mathbb{G}_a$  of  $G$  in (2) the natural embedding into  $\mathbb{P}^1$ . This gives a  $G$ -equivariant embedding of  $G$  into  $(\mathbb{P}^1)^{r+s}$ . The compactification of  $E$  is defined to be the associated bundle  $X = E((\mathbb{P}^1)^{r+s})$ . We want to show by an example that this compactification heavily depends on the splitting (2) of  $G$ .

Start with an abelian variety  $A$  and  $G = \mathbb{G}_m^2$ . Let  $G = \mathbb{G}_m \times \mathbb{G}_m$  be a given decomposition. The isomorphism  $\Phi : G \rightarrow G$ ,  $\Phi(a_1, a_2) = (a_1^{-1}a_2, a_2)$  yields another decomposition of  $G$ .

Now let  $E_1 = \mathbb{G}_m \times A$  be the trivial bundle and  $E_2 \rightarrow A$  be an arbitrary principal  $\mathbb{G}_m$ -bundle over  $A$ , whose associated line bundle  $M_2$  on  $A$  is algebraically equivalent to zero. By the theorem of Weil–Rosenlicht–Serre (cf. [6], p. 184 Théorème 6)  $E = E_1 \times_A E_2$  is a group extension of  $A$  by  $G$ .

If  $(U_i, \alpha_{ij})_{i,j}$  is a description of  $E_2$  by open sets  $U_i$  in  $A$  and transition morphisms  $\alpha_{ij}$ , then  $(U_i, (1, \alpha_{ij}))_{i,j}$  is a description of the principal  $G$ -bundle  $E = E_1 \times_A E_2$  over  $A$ . On the other hand any principal  $G$ -bundle over  $A$  may be considered as an element of  $H^1(A, G)$ . The element of  $H^1(A, G)$  corresponding to  $E$  over  $A$  does not reflect the decomposition of  $G$  which however the description  $(U_i, (1, \alpha_{ij}))_{i,j}$  does. Applying the isomorphism  $\Phi : G \rightarrow G$  we get

$$E \hat{=} (U_i, (1, \alpha_{ij}))_{i,j} \hat{=} (U_i, \Phi((1, \alpha_{ij})))_{i,j} = (U_i, (\alpha_{ij}, \alpha_{ij}))_{i,j}$$

which means

$$E \simeq E_1 \times_A E_2 \simeq E_2 \times_A E_1.$$

If  $X$  (resp.  $\bar{X}$ ) denotes the compactification of  $E$  corresponding to the given decomposition of  $G$  (resp. the decomposition of  $G$  given by applying  $\Phi$ ), then we have according to the definitions

$$X \simeq P(\mathcal{O}_A^2) \times_A P(\mathcal{O}_A \oplus M_2)$$

and

$$\bar{X} \simeq P(\mathcal{O}_A \oplus M_2) \times_A P(\mathcal{O}_A \oplus M_2)$$

where  $P(\cdot)$  denotes the projective bundle associated to the vector bundle  $(\cdot)$ .

We claim that  $X$  and  $\bar{X}$  are not isomorphic in general. For this it suffices to compute the canonical line bundles  $K_X$  and  $K_{\bar{X}}$ . We get

$$K_X = M_2 \otimes_A \mathcal{O}_{P(\mathcal{O}_A^2)}(-2) \otimes_A \mathcal{O}_{P(\mathcal{O}_A \oplus M_2)}(-2)$$

and

$$K_{\bar{X}} = M_2^2 \otimes_A \mathcal{O}_{P(\mathcal{O}_A \oplus M_2)}(-2) \otimes_A \mathcal{O}_{P(\mathcal{O}_A \oplus M_2)}(-2)$$

which obviously are nonisomorphic for  $M_2 \neq \mathcal{O}_A$ . (Consider  $X$  and  $\bar{X}$  as projective bundles over  $P(\mathcal{O}_A \oplus M_2)$ ).

#### 4. Projective embeddings

We want to study the projective embeddings of the compactifications  $\bar{E}$  of  $E$ , and in particular the question: by forms of which degrees is the homogeneous ideal of  $\bar{E}$  in  $\mathbb{P}^N$  generated? For this we need a slight generalization of a criterium of Mumford (cf. [5], pp. 39–40) which we shall prove first.

If  $X$  is any projective variety embedded in  $\mathbb{P}^N$  by the complete linear system of a very ample line bundle  $L$ , denote by  $I := \bigoplus_{k \geq 0} I_k$  its homogenous ideal in  $\mathbb{P}^N$ . For any  $i \geq 1$  define

$$\mathcal{R}(L^i, L) = \text{Ker}(H^0(L^i) \otimes H^0(L) \rightarrow H^0(L^{i+1}))$$

and

$$\mathcal{R}_i(L) = \text{Ker}(H^0(L)^i \rightarrow H^0(L^i)).$$

If moreover for a vector space  $V$  we denote by  $S^n(V)$  its  $n$ -th symmetric product, we have

LEMMA 4.1. *If  $L$  is normally generated on  $X$ , then for any  $k \geq 1$  the following conditions are equivalent*

(1) *The canonical map  $H^0(L) \otimes \mathcal{R}(L^i, L) \rightarrow \mathcal{R}(L^{i+1}, L)$  is surjective for every  $i \geq k$ .*

(2) *The canonical map  $I_{k+1} \otimes S^{i-k}H^0(L) \rightarrow I_{i+1}$  is surjective for every  $i \geq k$ .*

In other words: condition (1) is equivalent to the fact that the ideal  $I$  is generated by forms of degree  $\leq k + 1$ . For  $k = 1$  this is just Mumford's result.

*Proof.* We shall prove only the implication (1)  $\Rightarrow$  (2), since we do not need the converse. Consider the condition (2'): The canonical map

$$\phi = \sum_v \phi_v : \bigoplus_v \mathcal{R}_{k+1}(L) \otimes H^0(L)^{i-k} \rightarrow \mathcal{R}_{i+1}(L)$$

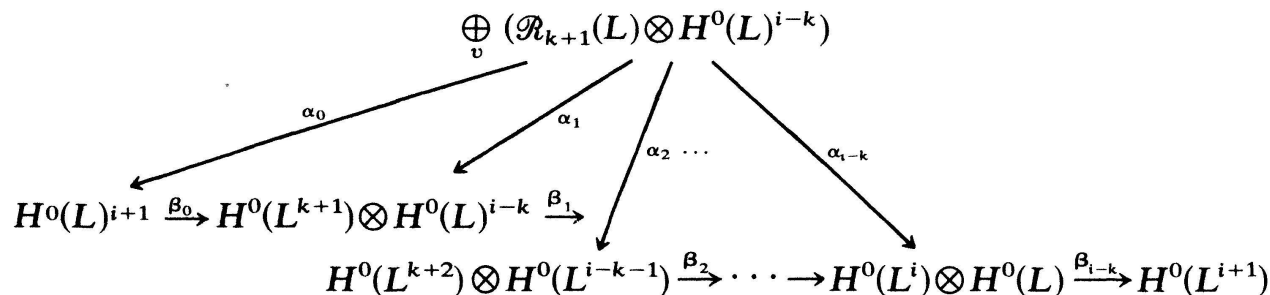
is surjective, where the direct sum is to be taken over all  $v = (v_1, \dots, v_{k+1})$  with  $1 \leq v_1 < \dots < v_{k+1} \leq i + 1$ . Here the map  $\phi_v$  on the direct summand with index  $v = (v_1, \dots, v_{k+1})$  is given by

$$a_1 \otimes \dots \otimes a_{k+1} \otimes b_1 \otimes \dots \otimes b_{i-k}$$

$$\mapsto b_1 \otimes \dots \otimes \underset{v_1}{\downarrow} a_1 \otimes \dots \otimes \underset{v_2}{\downarrow} a_2 \otimes \dots \otimes \underset{v_{k+1}}{\downarrow} a_{k+1} \otimes \dots \otimes b_{i-k}$$

that is  $a_i$  is inserted in the  $v_i$ -th place. It is easy to see that (2') is just the desymmetrization of (2). Hence it suffices to show that (1) implies (2').

Consider the commutative diagram



with  $\alpha_0 = \phi$ , considered as a map into  $H^0(L)^{i+1}$ , with  $\beta_i$  the canonical maps, and

$\alpha_j = \beta_j \cdot \alpha_{j-1}$ . We have to show:

$$\text{Ker}(\beta_{i-k} \cdots \beta_1 \cdot \beta_0) \subseteq \text{Im} \alpha_0.$$

Since  $L$  is normally generated,  $\beta_j$  is surjective for every  $j = 0, \dots, i - k$  (c.f. [5]). Hence it suffices to show

$$\text{Ker}(\beta_j) \subseteq \text{Im} \alpha_j \quad \text{for } j = 0, \dots, i - k.$$

This is true for  $j = 0$  by definition of the maps. For  $j = 1, \dots, i - k$  we have

$$\text{Ker}(\beta_j) = \mathcal{R}(L^{k+j}, L) \otimes H^0(L^{i-k-j}).$$

By restriction to a suitable direct summand of  $\bigoplus_v (\mathcal{R}_{k+1}(L) \otimes H^0(L)^{i-k})$  and omission of some tensor factors  $H^0(L)$  we see, that it suffices to show that the canonical map

$$\tilde{\alpha}_j : H^0(L)^j \otimes \mathcal{R}_{k+1}(L) \rightarrow \mathcal{R}(L^{k+j}, L)$$

is surjective for  $j = 1, \dots, k - i$ . But  $\tilde{\alpha}_j$  factorizes canonically as follows

$$\begin{array}{ccc} H^0(L)^j \otimes \mathcal{R}_{k+1}(L) & \xrightarrow{\tilde{\alpha}_j} & \mathcal{R}(L^{k+j}, L) \\ & \searrow 1 \otimes \gamma_j & \nearrow \delta_j \\ & & H^0(L)^j \otimes \mathcal{R}(L^k, L) \end{array}$$

$\delta_j$  is surjective according to the assumption (1). To show the surjectivity of  $\gamma_j$  consider the diagram

$$\begin{array}{ccccccc} & & & & & 0 & \\ & & & & & \downarrow & \\ & & & & & \mathcal{R}(L^k, L) & \\ & & & & & \downarrow & \\ 0 & \longrightarrow & \text{Ker } \phi & \longrightarrow & H^0(L)^k \otimes H^0(L) & \xrightarrow{\phi} & H^0(L^k) \otimes H^0(L) \longrightarrow 0 \\ & & \downarrow \psi & & \parallel & & \downarrow & \\ 0 & \longrightarrow & \mathcal{R}_{k+1}(L) & \longrightarrow & H^0(L)^{k+1} & \longrightarrow & H^0(L^{k+1}) \longrightarrow 0 \\ & & \downarrow \pi & & \downarrow & & & \\ & & \text{Coker } \psi & & 0 & & & \\ & & \downarrow & & & & & \\ & & 0 & & & & & \end{array}$$

According to the serpent lemma  $\mathcal{R}(L^k, L) \simeq \text{Coker } \psi$  canonically, and under this isomorphism  $\pi$  identifies with  $\gamma_j$  which completes the proof of the lemma.

In order to apply Lemma 4.1 suppose we are given an exact sequence (1). Let  $P$  be a  $G$ -equivariant compactification of  $G$ . It induces a compactification  $E(P)$  of  $E$ . Denote by  $\pi: E(P) \rightarrow A$  the natural projection. In [4] the following result was proved (cf. [4], pp. 564–567).

**THEOREM 4.2.** *Let  $L \in \text{Pic}(P)$  be normally presented,  $G$ -linearized,  $M \in \text{Pic}(A)$  ample, generated by its global sections, and  $F = E(L) \otimes \pi^*M$ . Then for every  $i \geq \dim A + 2$  the canonical map*

$$H^0(F) \otimes \mathcal{R}(F^i, F) \rightarrow \mathcal{R}(F^{i+1}, F)$$

*is surjective.*

If moreover  $M$  is normally generated on  $A$ , the methods of [4], section 3 show, that  $F = E(L) \otimes \pi^*M$  is normally generated on  $E(P)$  and we may apply Lemma 4.1 to get:

**COROLLARY 4.3.** *Let  $L \in \text{Pic}(P)$  be normally presented,  $G$ -linearized, and  $M \in \text{Pic}(A)$  normally generated. Then  $F = E(L) \otimes \pi^*M$  is normally generated on  $E(P)$  and the homogeneous ideal of the corresponding projective embedding  $E(P) \hookrightarrow \mathbb{P}^N$  is generated by forms of degree  $\leq \dim A + 3$ .*

The most important compactifications of  $E$  are those where  $P$  is a multiprojective space. Since for such a  $P$  a line bundle is normally presented if and only if it is very ample (or even ample) (cf. [4] section 6) we get

**COROLLARY 4.4.** *Let  $P$  be a multiprojective  $G$ -equivariant compactification of  $G$ ,  $L \in \text{Pic}(P)$  very ample,  $G$ -linearized and  $M \in \text{Pic}(A)$  normally generated. Let  $E(P) \hookrightarrow \mathbb{P}^N$  be the projective embedding associated to the line bundle  $F = E(L) \otimes \pi^*M$ . Then the homogeneous ideal of  $E(P)$  in  $\mathbb{P}^N$  is generated by its forms of degree  $\leq \dim A + 3$ .*

Since for any ample line bundle  $M$  on  $A$  the third power  $M^3$  is normally generated, one can even give a bound in case  $M$  is only very ample. We omit this.

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